ROBUSTNESS ANALYSIS OF MAGNETIC TORQUER CONTROLLED SPACECRAFT ATTITUDE DYNAMICS

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***Abstract*— This paper describes a systematic approach to the robustness analysis of linear periodically time-varying (LPTV) systems. The method uses the technique known as Lifting to transform the original time-varying uncertain system into linear fractional transformation (LFT) form. The stability and performance robustness of the system to structured parametric uncertainty can then be analysed non-conservatively using the structured singular value μ. The method is applied to analyse the stability robustness of an attitude control law for a spacecraft controlled by magnetic torquer bars, whose linearised dynamics can naturally be written in linear periodically time-varying form. The proposed method allows maximum allowable levels of uncertainty, as well as worst-case uncertainty combinations to be computed. The destabilising effect of these uncertain parameter combinations is verified in time-domain simulations.**

I. INTRODUCTION

Many different engineering applications, from helicopter rotors and the dynamics of beams and plates, to the motion of ships and spacecraft, may be modelled as LPTV (Linear Periodically Time-Varying) systems, (Dugundji & Wendell 1983, Spyrou 1997, Verdult, et al. 2004, Psiaki 2001). Rigorous analysis of the stability and robustness properties of such systems is thus an important problem, which requires dedicated system theoretic tools. In the literature, Floquet theory has frequently been used to study the *nominal* stability of LPTV systems (Chen 1984, Khalil 2002, Rugh 1996), see for example Dugundji & Wendell (1983). Previous work on *robust* stability analysis of LPTV systems has considered robustness to unstructured complex uncertainty using the ν-gap metric, (Cantoni 1998, Cantoni & Glover 1998, Cantoni & Glover 2000) and to complex and nonlinear/time varying uncertainty using IQC's (Integral Quadratic Constraints), (Kao, et al. 2001). In this paper, we consider the problem of analysing the robustness of LPTV systems to structured (i.e diagonal) LTI (Linear Time-Invariant) uncertainty, which may be real, complex or mixed. We show how a technique known as Lifting, (Chen & Francis 1995, Ma & Iglesias 2002a), can be used to cast the original uncertain LPTV system in the form of an LFT (Linear Fractional Transformation). Stability and performance robustness of the resulting system can then be analysed non-conservatively using standard μ-analysis methods. A significant advantage of the proposed approach over methods based on the ν-gap metric and IQC is that a worst-case destabilising uncertainty may be computed from the μ lower bound which can be used to check the (possible) conservatism of the analysis results. For the case of purely real parametric uncertainty, this means that the original time-varying system may be simulated with the worst-case uncertain parameters to verify the predicted instability or performance degradation. We note that the idea of using the Lifting technique in the context of robustness analysis first appeared in Ma & Iglesias (2002b) where it was applied to the problem of analysing the robustness of limit cycles in nonlinear (but time-invariant) models of biochemical networks, see also Ma & Iglesias (2002a) and Kim, et al. (2006b). The application of Lifting described in this paper was first proposed and applied to a simple numerical example in Kim, et al. (2006a).

In contrast to the extremely simple numerical examples considered in Cantoni (1998), Cantoni & Glover (1998), Cantoni & Glover (2000), Kao et al. (2001) and Kim et al. (2006a), in this paper the proposed analysis method is applied to a highly realistic attitude control system for a small satellite. In recent years, the demand for small satellites has increased significantly due to their cheaper cost to build, relatively smaller launcher requirements, and potential for high performance when appropriately coordinated (Carpenter, et al. 2003). For this type of spacecraft, actuation based on magnetic torque is especially attractive due to its reduced mass and power consumption when compared to wheel based actuators or gas jet thrusters, (Psiaki 2001). Interestingly, the resulting linearised equations of motion for this type of spacecraft can be naturally written as a periodically time-varying system. In the recent literature several papers have addressed the resulting attitude controller design problem (Psiaki 2001, Alfriend 1975, Junkins, et al. 1981, Wiśniewski & Blanke 1999, Lovera, et al. 2002, Silani & Lovera 2005). In Psiaki (2001), a magnetic torquer attitude controller was designed for a small spacecraft using time-varying linear quadratic full state feedback. Robustness of the control law to parametric model uncertainties was evaluated using a trial-and-error simulation based approach. In this paper we perform a systematic robustness analysis of this controller to multiple uncertain parameters such as inertia, inclination angles and altitude. The results of our study show that the original robustness properties claimed for the controller in Psiaki (2001) are rather optimistic.

The paper is organised as follows: In section II, the magnetic torque attitude controller for the spacecraft, and the resulting robustness analysis problem are described. In Section III, the linearised closed-loop dynamics of the system are written in standard linear fractional form using Lifting techniques. The robustness analysis of the attitude controller is described in detail in Section IV. Finally, Section V presents some conclusions.

II. Problem Formulation

Spacecraft attitude is generally defined relative to a set of local-level coordinates. The local-level reference frame is defined as follows: the $+z$ axis points toward nadir, the y axis is perpendicular to both the nadir vector and the orbital velocity at each instant, the positive direction of y is given by the negative orbit normal, and the $+x$ axis is defined by the right-hand rule. The other reference frame of interest is the body frame which is fixed to the spacecraft - the attitude is then defined as the relative angle from the local-level coordinates to the body frame (Psiaki 2001).

With the above conventions, the spacecraft attitude kinematics and dynamics are given by (Psiaki 2001, Shuster 1993, Wertz 1986)

$$\dot{\mathbf{q}} = \frac{1}{2}\Omega\left(\boldsymbol{\omega}_{LL}^{B}\right)\mathbf{q} \tag{1a}$$

$$I\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times (I\boldsymbol{\omega}) + \mathbf{m} \times \mathbf{b} + \mathbf{n}_{gg} + \mathbf{n}_{d} \tag{1b}$$

where $\mathbf{q}$ is the quaternion with respect to the local level coordinates, and

$$\Omega\left(\boldsymbol{\omega}_{LL}^{B}\right) = \begin{bmatrix} 0 & \omega_{LL_3}^{B} & -\omega_{LL_2}^{B} & \omega_{LL_1}^{B} \\ -\omega_{LL_3}^{B} & 0 & \omega_{LL_1}^{B} & \omega_{LL_2}^{B} \\ \omega_{LL_2}^{B} & -\omega_{LL_1}^{B} & 0 & \omega_{LL_3}^{B} \\ -\omega_{LL_1}^{B} & -\omega_{LL_2}^{B} & -\omega_{LL_3}^{B} & 0 \end{bmatrix}, \tag{2}$$

$\omega_{LL_i}^{B}$ is the i^{th} component of $\boldsymbol{\omega}_{LL}^{B}$ for $i = 1, 2, 3$, $\boldsymbol{\omega}_{LL}^{B}$ is the angular rate of spacecraft with respect to the local frame described in the body frame, $\boldsymbol{\omega}$ is the spacecraft angular rate with respect to the inertia frame described in the body frame, I is the inertia matrix, $\mathbf{m}$ is the magnetic dipole moment of the torque rods (the control input) and $\mathbf{b}$ is the Earth's magnetic field written in the body frame. $\mathbf{b}$ can also be written in the local-level frame, (Psiaki 2001), as follows :

$$\mathbf{b}(t) := \begin{bmatrix} b_1(t) \\ b_2(t) \\ b_3(t) \end{bmatrix} = \frac{\mu_f}{a^3}\begin{bmatrix} \cos(\omega_0 t)\sin(i_m) \\ -\cos(i_m) \\ 2\sin(\omega_0 t)\sin(i_m) \end{bmatrix} \tag{3}$$

where μ_f is the dipole strength of the earth's magnetic field and is equal to 7.9×10^{15} Wb-m, a is the semi-major axis of the orbit, ω_0 is the orbital rate, t is time which is measured from zero at the ascending node crossing of the magnetic equator, i_m is the orbit inclination angle, $\mathbf{n}_{gg}$ is the gravity gradient torque, and $\mathbf{n}_d$ is the summation of all other external disturbances such as aerodynamic drag torque, solar radiation torque, higher frequencies in the Earth's magnetic field, etc.

Since 1 is nonlinear and rather complicated for the purposes of control law design, it is usually linearised with the assumption of a circular-orbit around the equilibrium nadir-pointing attitude. The other assumptions required for validity of the linearised model are as follows: roll (ϕ), pitch (θ) and yaw (ψ) attitude angles are small, the corresponding angular rates, i.e. $\dot{\phi}$, $\dot{\theta}$ and $\dot{\theta}$, are small, and the control input $\mathbf{m}$ is small. With these assumptions, and including the nominal term of the gravity gradient torque, the nonlinear equation of motion can be linearised as follows (Wertz 1986):

$$\dot{\mathbf{x}}_s = A_s\mathbf{x}_s + B_s(t)\mathbf{u} \tag{4}$$

where

$$\mathbf{x}_s := \begin{bmatrix} \phi & \theta & \psi & \dot{\phi} & \dot{\theta} & \dot{\psi} \end{bmatrix}^T \tag{5a}$$

$$\mathbf{u} := \mathbf{m} \tag{5b}$$

$$A_s := \begin{bmatrix} 0_{3\times 3}, & I_{3\times 3} \\ \text{diag}[-4\omega_0^2\sigma_1, 3\omega_0^2\sigma_2, \omega_0^2\sigma_3], & \begin{matrix} 0 & 0 & \omega_0(1-\sigma_1) \\ 0 & 0 & 0 \\ -\omega_0(1+\sigma_3) & 0 & 0 \end{matrix} \end{bmatrix} \tag{5c}$$

$$B_s(t) := \begin{bmatrix} 0_{3\times 3} \\ B_2(t) \end{bmatrix} \tag{5d}$$

$$B_2(t) := \begin{bmatrix} 0 & b_3(t)/I_1 & -b_2(t)/I_1 \\ -b_3(t)/I_2 & 0 & b_1(t)/I_2 \\ b_2(t)/I_3 & -b_1(t)/I_3 & 0 \end{bmatrix} \tag{5e}$$

In the above, $0_{3\times 3}$ is a 3×3 zero matrix, the inertia matrix is about the principal axis, hence, $I := \text{diag}[I_1\ I_2\ I_3]$, $\sigma_i := (I_j - I_k)/I_i$ for $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$, ω_0 is the orbital rate which for a circular orbit is given by

$$\omega_0 = \sqrt{\frac{\mu_{\oplus}}{a^3}} \tag{6}$$

In the above, $\mu_{\oplus}$ is the Earth's gravitation constant and is equal to 3.986005×10^{14} m^3/s^2 while a is equal to $R_{\oplus} + h$, where $R_{\oplus}$ is the radius of the Earth, 6378.14 km, and h is the altitude of the spacecraft. Note that $B_s(t)$ is a periodic matrix with period T equal to $2\pi/\omega_0$, i.e., $B_s(t) = B_s(t + T)$ for all $t \in [0, \infty)$. To add integral action in the control law, the integrals of the attitude errors are defined by

$$\mathbf{z} := \begin{bmatrix} \int_0^t \phi(\tau)d\tau & \int_0^t \theta(\tau)d\tau & \int_0^t \psi(\tau)d\tau \end{bmatrix}^T \tag{7}$$

and augmented with the $\mathbf{x}_s$ as follows:

$$\dot{\mathbf{x}} = A\mathbf{x} + B(t)\mathbf{u} \tag{8}$$

where

$$\mathbf{x} := \begin{bmatrix} \mathbf{z}^T & \mathbf{x}_s^T \end{bmatrix}^T \tag{9a}$$

$$A := \begin{bmatrix} 0_{3\times 3} & [I_{3\times 3}\ 0_{3\times 3}] \\ 0_{3\times 3} & A_s \end{bmatrix} \tag{9b}$$

$$B(t) := \begin{bmatrix} 0_{3\times 3} \\ B_s(t) \end{bmatrix} \tag{9c}$$

and $I_{3\times 3}$ is a 3×3 identity matrix. To avoid confusion between the identity and inertia matrices, in this paper $I_{k\times k}$ with a positive integer k represents a $k \times k$ identity matrix, while I is the inertia matrix and I_i for $i = 1, 2, 3$ is the diagonal element of the inertia matrix. A periodic linear quadratic regulator (LQR) controller was designed for (8) in (Psiaki 2001). The control law is given by:

$$\mathbf{u}(t) = -\alpha_0 R^{-1}\bar{B}^T(t)P_{\text{ss}}\mathbf{x}(t) \tag{10}$$

where α_0 is a positive real scalar, R is a positive definite weighting matrix for the control input in the LQR cost function, $\bar{B}(t)$ is the same as $B(t)$ except that the inertia is replaced by the nominal value, i.e., each I_i in (5e) is replaced by $\bar{I}_i$ for $i = 1, 2, 3$, the relation between the actual and the nominal inertia is given by $I_i := \bar{I}_i(1+\delta_{I_i})$, where δ_{I_i} is the uncertainty in each term of the inertia matrix for $i = 1, 2, 3$. The control gain, P_{ss}, is the solution of the following algebraic Riccati equation:

$$\bar{A}^T P_{\text{ss}} + P_{\text{ss}}\bar{A} - P_{\text{ss}}B_{\text{avg}}P_{\text{ss}} + Q = 0 \tag{11}$$

where

$$B_{\text{avg}} = \frac{1}{T}\int_0^T \bar{B}(\tau)R^{-1}\bar{B}^T(\tau)d\tau \tag{12}$$

$\bar{A}$ is also of the same form of A with the nominal values of the inertia matrix, the inclination angle and the altitude, and Q is a positive semi-definite matrix weight on the state. As a result, the controller gain P_{ss} is a constant matrix based on the fixed nominal altitude, inclination angle and the inertia matrix. More details of the controller design can be found in Psiaki (2001). The final closed loop system is given by

$$\dot{\mathbf{x}} = A_{\text{cl}}(t)\mathbf{x} \tag{13}$$

where

$$A_{\text{cl}}(t) = \left[A - \alpha_0 B(t)R^{-1}\bar{B}^T(t)P_{\text{ss}}\right] \tag{14}$$

Note that $A_{\text{cl}}(t) = A_{\text{cl}}(t+T)$, i.e., it is a periodic matrix with a period of T.

III. Linear Fractional Transformation

A. Transformation to Time-Varying LFT-Form

For brevity in the exposition, the weighing matrix for the control input, R, is assumed to be a positive scalar multiplied by the identity matrix, i.e. $R = rI_{3\times3}$. Then, the closed loop system is given by

$$\dot{\mathbf{x}}(t) = \left[A(\delta_I) - \frac{\alpha_0}{r}B(t;\delta_I,\delta_{i_m})\bar{B}^T(t;\delta_{i_m})P_{\text{ss}}\right]\mathbf{x}(t) \tag{15}$$

where each matrix is written with arguments to emphasise its dependencies on time and on the uncertain parameters, δ_I and δ_{i_m}. δ_I represents the uncertainty in the diagonal elements of the inertia matrix (δ_{I_1}, δ_{I_2}, and δ_{I_3}), while δ_{i_m} represents the uncertainty in the spacecraft's orbit inclination angle i_m. Note that for physical reasons the inclination angle uncertainty is modelled as additive uncertainty ($i_m = \bar{i}_m + \delta_{i_m}$ where $\bar{i}_m$ is the nominal value of the inclination angle) whereas the inertia uncertainty is multiplicative (i.e., $I_i = \bar{I}_i(1+\delta_{I_i})$).

Transformation of (15) to the form of a time-varying LFT is straightforward, except for the second term in the bracket which contains sinusoidal terms in i_m. To see this, note that

$$B_2(t)\bar{B}_2^T(t) = \begin{bmatrix} \frac{b_2^2(t)+b_3^2(t)}{I_1\bar{I}_1} & -\frac{b_1(t)b_2(t)}{I_1\bar{I}_2} & -\frac{b_1(t)b_3(t)}{I_1\bar{I}_3} \\ -\frac{b_1(t)b_2(t)}{\bar{I}_1 I_2} & \frac{b_1^2(t)+b_3^2(t)}{I_2\bar{I}_2} & -\frac{b_2(t)b_3(t)}{I_2\bar{I}_3} \\ -\frac{b_1(t)b_3(t)}{I_3\bar{I}_1} & -\frac{b_2(t)b_3(t)}{I_3\bar{I}_2} & \frac{b_1^2(t)+b_2^2(t)}{I_3\bar{I}_3} \end{bmatrix} \tag{16}$$

To write the closed loop system in an LFT form the uncertain parameters in each element of (16) must be identified and isolated. To do this, we first of all note that the numerator of the first element in the diagonal can be simplified as follows:

$$\begin{aligned} b_2^2(t) + b_3^2(t) &= \left(\frac{\mu_f}{a^3}\right)^2\left[\cos^2 i_m + 4\sin^2(\omega_0 t)\sin^2 i_m\right] \\ &= \left(\frac{\mu_f}{a^3}\right)^2\left\{\left[4\sin^2(\omega_0 t) - 1\right]\sin^2 i_m + 1\right\} \\ &= \left(\frac{\mu_f}{a^3}\right)^2\left\{\left[2\sin^2(\omega_0 t) + \frac{1}{2}\right]\right. \\ &\quad \left. - \left[2\sin^2(\omega_0 t) - \frac{1}{2}\right]\cos(2i_m)\right\} \end{aligned} \tag{17}$$

Similarly, the numerators of the other two diagonal terms are

$$\begin{aligned} b_1^2(t) + b_3^2(t) &= \left(\frac{\mu_f}{a^3}\right)^2\left\{\frac{1}{2}\left[3\sin^2(\omega_0 t) + 1\right]\right. \\ &\quad \left. -\frac{1}{2}\left[3\sin^2(\omega_0 t) + 1\right]\cos(2i_m)\right\} \end{aligned} \tag{18a}$$

$$\begin{aligned} b_1^2(t) + b_2^2(t) &= \left(\frac{\mu_f}{a^3}\right)^2 \\ &\quad \times\left[1 + \frac{1}{2}\sin^2(\omega_0 t)(-1+\cos(2i_m))\right] \end{aligned} \tag{18b}$$

The off-diagonal numerator terms are given by

$$b_1(t)b_2(t) = -\left(\frac{\mu_f}{a^3}\right)^2\frac{1}{2}\cos(\omega_0 t)\sin(2i_m) \tag{19a}$$

$$\begin{aligned} b_1(t)b_3(t) &= \left(\frac{\mu_f}{a^3}\right)^2\left[\frac{1}{2}\sin(2\omega_0 t)\right. \\ &\quad \left. -\frac{1}{2}\sin(2\omega_0 t)\cos(2i_m)\right] \end{aligned} \tag{19b}$$

$$b_2(t)b_3(t) = -\left(\frac{\mu_f}{a^3}\right)^2\sin(\omega_0 t)\sin(2i_m) \tag{19c}$$

In (17), (18), and (19), for a fixed ω_0, i.e. a fixed altitude, the only uncertain term is the inclination angle, i.e. i_m. Note that in (17), (18), and (19) the inclination angle appears as either $\sin(2i_m)$ or $\cos(2i_m)$, which can be expanded as follows:

$$\begin{aligned} \sin\left[2\left(\bar{i}_m + \delta_{i_m}\right)\right] &= \sin(2\bar{i}_m)\cos(2\delta_{i_m}) \\ &\quad + \cos(2\bar{i}_m)\sin(2\delta_{i_m}) \end{aligned} \tag{20a}$$

$$\begin{aligned} \cos\left[2\left(\bar{i}_m + \delta_{i_m}\right)\right] &= \cos(2\bar{i}_m)\cos(2\delta_{i_m}) \\ &\quad - \sin(2\bar{i}_m)\sin(2\delta_{i_m}) \end{aligned} \tag{20b}$$

where the magnitude of the uncertainty in the inclination angle, i.e. $|\delta_{i_m}|$, is assumed to be strictly less than 45 degrees, which is large enough to cover all practical cases. By this assumption the cosine of the uncertainty is non-negative as follows:

$$\cos(2\delta_{i_m}) = \sqrt{1 - \sin^2(2\delta_{i_m})} \tag{21}$$

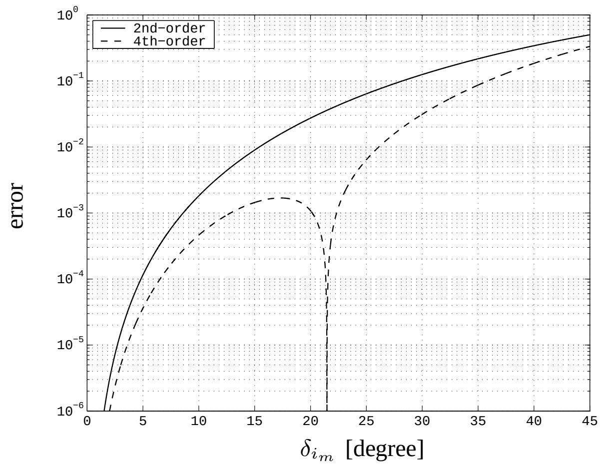

Fig. 1. Magnitude of the $\cos(2\delta_{i_m})$ approximation error for the second and the fourth order polynomials as a function of the inclination angle.

To allow the uncertainties in (20) to be written in the form of polynomials, we define:

$$\delta_s \equiv \sin(2\delta_{i_m}) \tag{22}$$

Then, by the binomial expansion (21) can be written as

$$\cos(2\delta_{i_m}) = \sqrt{1-\delta_s^2} = 1 - \frac{1}{2}\delta_s^2 - \frac{1}{6}\delta_s^4 - \frac{1}{16}\delta_s^6 + \ldots \tag{23}$$

In (Psiaki 2001), the robustness of the controller was evaluated by trial and error for inclination angles with uncertainty of up to 30 degrees. As shown in Figure 1, the approximation errors at $\delta_{i_m} = 30$ degrees for the second-order and the fourth-order polynomials of (23) are approximately 0.15 and 0.03, respectively. Note that, while the higher order polynomial makes the approximation error smaller, the price that has to be paid for this is an increase in the computation time required to calculate the bounds on μ. For this problem, the approximation for (23) was chosen as:

$$\cos(2\delta_{i_m}) \approx 1 - \frac{1}{2}\delta_s^2 \tag{24}$$

which gives an approximation error of less than 0.01 up to 15 degrees. Substituting (22) and (24) into (20) gives

$$\sin(2\bar{i}_m) + \cos(2\bar{i}_m)\,\delta_s - \frac{1}{2}\sin(2\bar{i}_m)\,\delta_s^2 \tag{25a}$$

$$\cos(2\bar{i}_m) - \sin(2\bar{i}_m)\,\delta_s - \frac{1}{2}\cos(2\bar{i}_m)\,\delta_s^2 \tag{25b}$$

With the uncertain parameters written in the above form, all sources of uncertainty can now be ”pulled out” of the closed loop equation of motion and placed in a diagonal uncertainty matrix Δ given by

$$\Delta = \text{diag}\left[\delta_{I_1} I_{4\times4},\ \delta_{I_2} I_{4\times4},\ \delta_{I_3} I_{4\times4},\ \delta_s I_{9\times9}\right] \tag{26}$$

Then, the uncertain closed loop system (15) can be written in standard LFT form as:

$$\dot{\mathbf{x}} = A_{\text{LFT}}(t)\mathbf{x} + B_{\text{LFT}}(t)\mathbf{w} \tag{27a}$$

$$\mathbf{z} = C_{\text{LFT}}(t)\mathbf{x} + D_{\text{LFT}}(t)\mathbf{w} \tag{27b}$$

$$\mathbf{w} = \Delta\mathbf{z} \tag{27c}$$

where $A_{\text{LFT}}(t) = A_{\text{LFT}}(t+T)$, $B_{\text{LFT}}(t) = B_{\text{LFT}}(t+T)$, $C_{\text{LFT}}(t) = C_{\text{LFT}}(t+T)$, and $D_{\text{LFT}}(t) = D_{\text{LFT}}(t+T)$ for all $t \in [0,\infty)$.

Remark 3.1: Recall the definition of the inertia element: $I_i = \bar{I}_i(1+\delta_{I_i})$ for $i = 1, 2, 3$. Since the inertia matrix must always be positive definite, the inertia uncertainty parameters cannot be smaller than -1 and also the following inequality has to be satisfied: $I_i \leq I_j + I_k$, where (i,j,k) are (1, 2, 3), (2, 3, 1) and (3, 2, 1).

Remark 3.2: By definition, the magnitude of the inclination angle error, δ_s, has to be less than 1. In addition, because of the approximation of trigonometric functions in the inclination angle uncertainty, it cannot be greater than a certain number which is strictly less than 1. For the first-order approximation, it should be less than 0.5, which corresponds to a δ_{i_m} equal to 15 degrees, so that the magnitude of the error remains less than 0.01.

B. *Transformation to Time Invariant LFT-Form*

Although (27) is in the form of an LFT, it is still a time-varying system. To transform this periodically time-varying system into a time-invariant form, a procedure known as *Lifting*, employed. Firstly, (27) is sampled with the sampling time, Δt, where Δt is equal to T/n_h and n_h is a positive integer. Then, a linear time-invariant switching system is obtained as follows:

$$\dot{\mathbf{x}}_d(t) = A_{\text{LFT}}(t_k)\mathbf{x}_d(t) + B_{\text{LFT}}(t_k)\mathbf{w}_d(t_k) \tag{28a}$$

$$\mathbf{z}_d(t_k) = C(t_k)\mathbf{x}_d(t_k) + D(t_k)\mathbf{w}_d(t_k) \tag{28b}$$

$$\mathbf{w}_d(t_k) = \Delta\mathbf{z}_d(t_k) \tag{28c}$$

for t in the interval of $[t_k, t_k+\Delta t)$ and k a non-negative integer. Secondly, (28) is discretised using a sample and hold for each time interval as follows:

$$\mathbf{x}_d(k+1) = \Phi\mathbf{x}_d(k) + \Gamma(k)\mathbf{w}_d(k) \tag{29a}$$

$$\mathbf{z}_d(k) = H(k)\mathbf{x}_d(k) + J(k)\mathbf{w}_d(k) \tag{29b}$$

$$\mathbf{w}_d(k) = \Delta\mathbf{z}_d(k) \tag{29c}$$

where each matrix is defined appropriately, and $\mathbf{w}_d(k)$ and $\mathbf{z}_d(k)$ are equal to $\mathbf{w}_d(t_k)$ and $\mathbf{z}_d(t_k)$, respectively. Because of the periodicity of the system, the following is satisfied: $\Gamma(k+n_h) = \Gamma(k)$, $H(k+n_h) = H(k)$, and $J(k+n_h) = J(k)$. Thirdly, the technique known as Lifting is applied to the above discrete-time system. Setting the initial k for (29) equal to 0 without loss of generality, the input and the output are redefined as follows:

$$\underline{\mathbf{w}_d} = \left\{\underline{\mathbf{w}_d}(0),\ \underline{\mathbf{w}_d}(1),\ \underline{\mathbf{w}_d}(2),\ \ldots\right\} \tag{30a}$$

$$\underline{\mathbf{z}_d} = \left\{\underline{\mathbf{z}_d}(0),\ \underline{\mathbf{z}_d}(1),\ \underline{\mathbf{z}_d}(2),\ \ldots\right\} \tag{30b}$$

TABLE I

SPACECRAFT CONFIGURATION PARAMETERS : ALL OF THEM ARE TAKEN FROM Psiaki (2001)

Nominal Values	Configuration A	Configuration B
Eccentricity	0	0
Altitude ($\bar{h}$) [km]	600	657
Inclination ($\bar{i}_m$) [degree]	90	57
$\bar{I} = \text{diag}[\bar{I}_1,\ \bar{I}_2,\ \bar{I}_3]$ [kg·m^2]	[8.7, 10, 6.5]	[250, 250, 10]
State Weighting (Q)	$\text{diag}[1.5\times10^{-8}, 1.5\times10^{-7}, 1.5\times10^{-8}, 0.1, 1.0, 0.1, 1.0, 1.0, 0.1]$	$\text{diag}[1.5\times10^{-8}I_{3\times3}, 0.01I_{3\times3}, 1.0I_{3\times3}]$
Control Weighting (R)	$6.2\times10^{7}I_{3\times3}$	$4.9\times10^{4}I_{3\times3}$
Control Scaling Factor (α_0)	2500	8130
Maximum Control Input (u_{max})	0.03	0.1

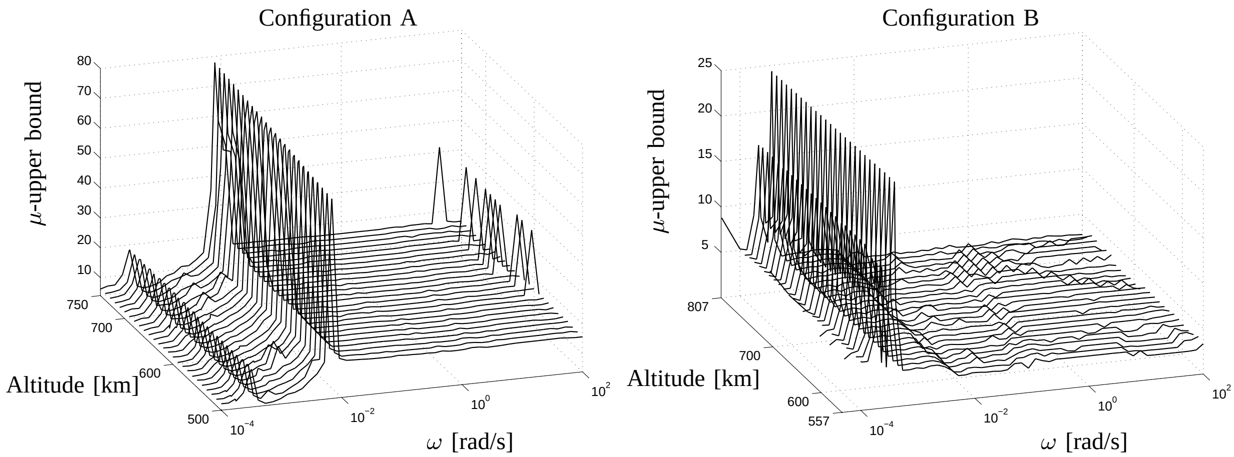

Fig. 2. μ upper bounds for configurations A and B at each altitude. The maximum among the μ-upper bound peaks at each altitude is 77.93 at 750 km for configuration A and 24.73 at 557 km for configuration B.

and

$$\underline{\mathbf{w}_d}(k) = [\mathbf{w}_d(k),\ \mathbf{w}_d(k+1),\ \dots,\ \dots,\ \mathbf{w}_d(k+n_h-2),\ \mathbf{w}_d(k+n_h-1)]^T \tag{31a}$$

$$\underline{\mathbf{z}_d}(k) = [\mathbf{z}_d(k),\ \mathbf{z}_d(k+1),\ \dots,\ \dots,\ \mathbf{z}_d(k+n_h-2),\ \mathbf{z}_d(k+n_h-1)]^T \tag{31b}$$

Then, the lifted system is given by

$$\mathbf{x}_d(k+n_h) = \tilde{\Phi}\mathbf{x}_d(k) + \tilde{\Gamma}\underline{\mathbf{w}_d}(k) \tag{32a}$$

$$\underline{\mathbf{z}_d}(k) = \tilde{H}\mathbf{x}_d(k) + \tilde{J}\underline{\mathbf{w}_d}(k) \tag{32b}$$

$$\underline{\mathbf{w}_d}(k) = \bar{\Delta}\underline{\mathbf{z}_d}(k) \tag{32c}$$

where and $\bar{\Delta}$ is a block diagonal matrix with Δ repeated n_h-times, as follows:

$$\bar{\Delta} = \text{diag}[\Delta,\ \Delta,\ \dots, \Delta] \tag{33}$$

Since each uncertainty appears in each Δ, by defining the row re-ordering matrix V such that

$$V\bar{\Delta} = \tilde{\Delta}V, \tag{34}$$

and $V^TV = I$, we have

$$\tilde{\Delta} = \text{diag}\left[\delta_{I_1}I_{4n_h\times4n_h},\ \delta_{I_2}I_{4n_h\times4n_h},\ \delta_{I_3}I_{4n_h\times4n_h},\ \delta_s I_{9n_h\times9n_h}\right] \tag{35}$$

Finally, the continuous time-invariant LFT form is obtained by transforming the lifted system, (32), into a corresponding time-invariant continuous system using a standard technique such as zero-order hold or Tustin's method (Franklin, et al. 1994).

Remark 3.3: To reduce the approximation error involved in the lifting procedure, the number of samples, n_h, should be chosen as large as possible. However, the bigger the number of samples, the more times each uncertain parameter is repeated in the Δ matrix, and the more difficult the resulting μ bound calculation becomes. Here, we use a trick - to reduce the approximation error in the system matrix, i.e. $\tilde{\Phi}$, n_h was set equal to 3000 (since this has no impact on the dimension of Δ), while a value of n_h equal to 10 was used for $\tilde{\Gamma}, \tilde{H}$, and $\tilde{J}$ to minimise the dimension of Δ.

IV. ROBUSTNESS ANALYSIS

Having transformed the closed-loop system to the appropriate form, in this section a systematic analysis of the robustness of the control law is performed using the structured singular value μ. The results of this analysis are verified by simulations of the original closed loop system with the uncertain parameters fixed at the "worst-case" values predicted by μ.

A. *μ-Analysis*

Our robustness analysis is applied for the two different configurations of the spacecraft considered in Psiaki (2001) - parameter values for each configuration are shown in Table I. These two configurations were chosen to demonstrate the control design scheme in Psiaki (2001). Configuration A corresponds to the case where the magnetic torquer bar is used for three-axis stabilisation of a small-size spacecraft,

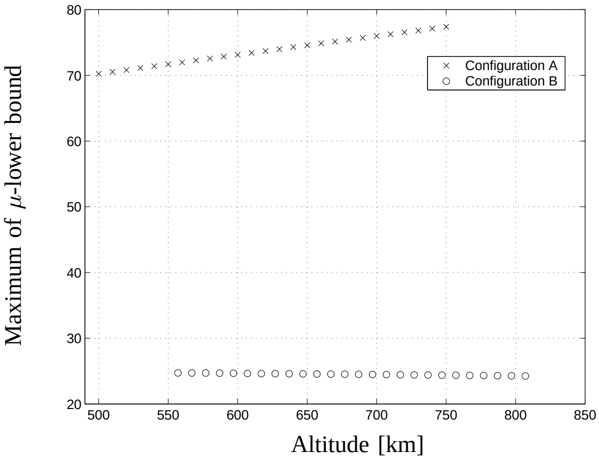

Fig. 3. Maximum peak of μ lower bound for the configuration A and B for each altitude.

which in a practical sense is the most likely application of magnetic torquer control. Configuration B again corresponds to a small-size spacecraft, however, in this case its yaw motion is neutrally stable and the resulting closed loop response is slower than that of Configuration A. More details about each configuration can be found in Psiaki (2001) and the references therein. In order to demonstrate the proposed robustness analysis method and compare the results with the ones from a heuristic search as shown in Psiaki (2001), the same two configurations are used as examples in this paper.

The control gain P_{ss} is obtained by solving (11) with the nominal values given in the table. The robustness analysis is performed with respect to the uncertainties in the elements of the inertia matrix and the inclination angle for a fixed altitude. Then, this analysis is repeated over a range from 100 km below to 150 km above the nominal altitude in a 10 km gridding.

Upper bounds on μ were computed using the standard algorithms in the MATLAB μ-toolbox (Balas, et al. 2001) and are shown in Figure 2 for each configuration. For configurations A and B, the maximum peaks of the upper bounds are 77.93 and 24.73, respectively. That is, for all values of δ_{I_i} and δ_s in the range of ± 0.0128 $(= 1/77.93)$ and ± 0.0404 $(= 1/24.73)$ the closed loop dynamics are stable. These results indicate that the controller may have rather poor robustness properties. To be sure whether this is in fact the case, however, lower bounds on μ must also be computed, for two reasons. Firstly, we need to assess the level of conservatism in the upper bounds, and secondly we need to extract the worst-case destabilising uncertainty combinations from the lower bounds and check whether they satisfy the physical constraints discussed in Remarks 3.1 and 3.2. In this paper we used the real μ lower bound algorithm of Ferreres & Biannic (2001) to calculate the peak of the lower bound over a continuous frequency interval. The peaks of the lower bounds for each configuration at each altitude are shown in Figure 3. Since the peaks of the upper and lower bounds are quite tight, we are thus assured that the analysis results are not conservative. However, we still need to check that the worst-case uncertainty combinations computed by the μ-analysis correspond to physically allowable values. For configuration A, the worst-case (i.e. smallest) destabilising values for the inertia matrix uncertainties are extracted from the peak of the μ lower bound as follows: $\delta_{I_1} = -0.0128, \delta_{I_2} = 0.0128$, and $\delta_{I_3} = -0.0128$. For these values, the inertia matrix is still positive definite and satisfies the inequality condition. Hence, the peak of the lower bound is a physically valid bound. However, for the configuration B, the uncertainties extracted from the μ lower bound are given by $\delta_{I_1} = -0.0404, \delta_{I_2} = 0.0404$, and $\delta_{I_3} = -0.0404$, and thus the inequality condition, $I_2 \leq I_1 + I_3$, is not satisfied. In such cases the result from μ-analysis should be interpreted as providing a worst-case direction in the uncertain parameter space, which should be scaled-down until the parameter values become physically possible, and then checked via time-domain simulations. Similar arguments apply for the inclination angle uncertainty, since in this case, it must be checked whether the magnitude of the uncertainty is within the allowable approximation error specified in the LFT modelling stage. In this case, for each configuration, the worst-case uncertainties in δ_{i_m} are ± 1.47 degrees and ± 4.64 degrees, respectively, and thus they are well within the ± 15 degree range required to guarantee an approximation error smaller than 0.01.

B. *Time-Domain Simulation Results*

In Psiaki (2001), since there is a nonlinear magnitude limitation in the magnetic dipole moment, an anti-windup scheme is implemented in the controller. That is, the control input $\mathbf{u}(t)$ or equivalently $\mathbf{m}(t)$ is equal to the following:

$$\mathbf{u}(t) = \begin{cases} \mathbf{u}_{\text{nominal}}(t) \text{ if } \beta \leq 1 \\ (1/\beta)\, \mathbf{u}_{\text{nominal}}(t) \text{ if } \beta > 1 \end{cases} \tag{36}$$

where $\mathbf{u}_{\text{nominal}}$ is equal to the original control input without saturation, i.e. (10), and

$$\beta = \max_{i=\{1,\,2,\,3\}} \frac{|(u_{\text{nominal}})_i|}{u_{\max}}, \tag{37}$$

$(u_{\text{nominal}})_i$ is the i^{th} component of $\mathbf{u}_{\text{nominal}}$. In this section, we simulate the original linearised time-varying closed-loop system (not the transformed LFT) with the above nonlinear saturation and anti-windup logic and the uncertain parameters fixed at their worst-case values found by μ-analysis. A full nonlinear simulation would require implementation of the full nonlinear equations of motion, together with external disturbance such as the J_2 term of the gravity gradient torque, aerodynamic drag, solar radiation pressure, etc. This was considered unnecessary, however, since uncertainty combinations that destabilise the linearised system will obviously also destabilise the full nonlinear system, at least locally.

The initial roll, pitch and yaw attitude angles for each configuration were fixed at [3, -3, 3] degrees for configuration A and [30, -30, 30] degrees for configuration B. Initial

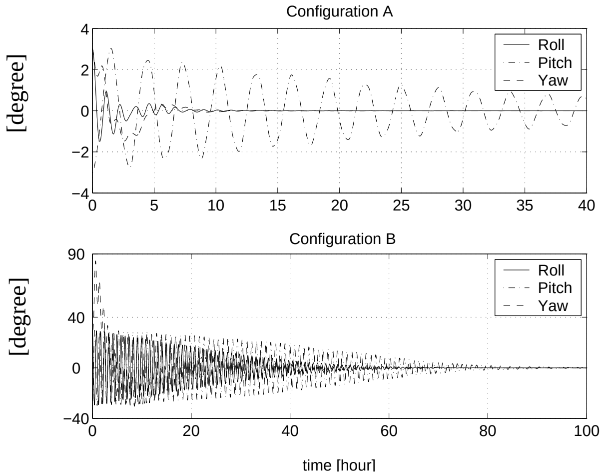

Fig. 4. Attitude angle time histories with no uncertainties for each configuration.

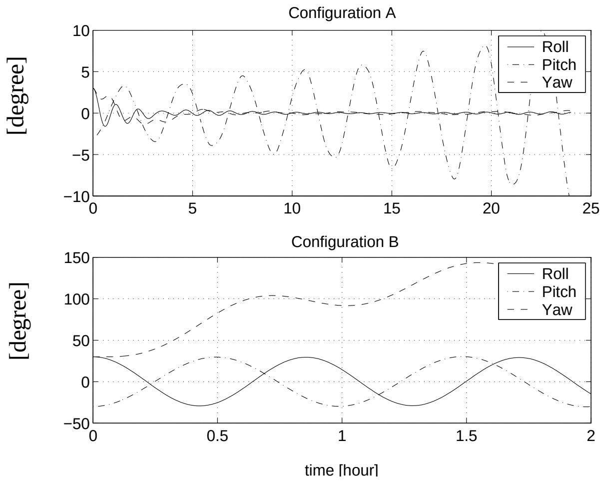

Fig. 5. Attitude angle time histories with the worst perturbation for each configuration.

rates are equal to zero for both cases. Figure 4 shows the stable closed loop responses in the case of zero uncertainty for both configurations A and B. For configuration A, at an altitude of 750 km, the worst uncertainty for the inertia matrix is given by $\delta_{I_1} = -0.0129, \delta_{I_2} = 0.0129$, and $\delta_{I_3} = 0.0129$ and for the inclination angle δ_{i_m} by 1.48 degrees. As shown in Figure 5, the response of the closed loop system is unstable for this uncertainty combination. For configuration B, the worst-case uncertainty at the altitude 807 km, $\delta_{I_1} = 0.0412, \delta_{I_2} = -0.0412$, and $\delta_{I_3} = -0.0412$ is scaled down by $\delta_{I_i} \times 0.04$ to make the inertia terms physically possible. With this level of uncertainty, the time-varying system is still unstable as shown in Figure 5.

Finally, it is interesting to note that the above analysis contrasts markedly with the robustness results for this controller presented in Psiaki (2001). In that paper, the authors used a gridding based approach often adopted in the aerospace industry, where a "finite representative set of parameter variations" were investigated, and the authors concluded that, for configuration A, the closed loop system remained stable for variations in inertia levels of $\pm 30\%$ with variations of inclination angle of ± 30 degrees. For configuration B, the authors reported that instability occurred for similar levels of uncertainty only in the case of "very large state perturbations" which caused "extreme levels of saturation to occur". The need for rigorous and reliable methods for the robustness analysis of time-varying feedback control systems is clearly demonstrated by the large discrepancies between these two sets of results.

V. CONCLUSIONS

A systematic approach to the robustness analysis of linear periodically time-varying (LPTV) systems was described. The method uses the technique known as Lifting to transform the original time-varying uncertain system into linear fractional transformation (LFT) form. The stability and performance robustness of the system to structured parametric uncertainty can then be analysed non-conservatively using the structured singular value μ. The method was applied to analyse the robustness of an attitude control law for a spacecraft controlled by magnetic torque rods, whose linearised dynamics can naturally be written in linear periodically time-varying form. The proposed method allows maximum allowable levels of uncertainty, as well as worst-case uncertainty combinations to be computed. The destabilising effect of these uncertain parameter combinations was verified in time-domain simulations.

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REFERENCES

K. T. Alfriend (1975). 'Magnetic Attitude Control System for Dual-Spin Satellites'. *AIAA Journal* **13**(6):817–822.

G. J. Balas, et al. (2001). *μ-Analysis and Synthesis Toolbox: For Use with MATLAB, User's Guide, Version 3*. The MathWorks, Inc.

M. Cantoni (1998). *Linear Periodic Systems: Robustness Analysis and Sampled-Data Control*. Ph.D. thesis, The University of Cambridge, Cambridge, UK.

M. Cantoni & K. Glover (1998). 'Robustness of linear periodically time-varying closed-loop systems'. In *Proceedings of the 37th IEEE Conference on Decision and Control*, pp. 3807–3812, December, Tampa FL, USA.

M. Cantoni & K. Glover (2000). 'Gap-Metric robustness analysis of linear periodically time-varying feedback systems'. *SIAM Journal on Control and Optimization* **38**(3):803–822.

J. R. Carpenter, et al. (2003). 'Benchmark Problems for Spacecraft Formation Flying Missions'. In *AIAA GN&C Conference & Exhibit*, August 11-14, Austin, TX, USA AIAA-2003-5364.

C. T. Chen (1984). *Linear System Theory and Design*. Oxford University Press, Inc.

T. Chen & B. Francis (1995). *Optimal Sampled-Data Control Systems*. Springer-Verlag, 3 edn.

J. Dugundji & J. H. Wendell (1983). 'Some Analysis Methods for Rotating Systems with Periodic Coefficients'. *AIAA Journal* **21**(6):890–897.

G. Ferreres & J.-M. Biannic (2001). 'Reliable computation of the robustness margin for a flexible aircraft'. *Control Engineering Practice* **9**:1267–1278.

G. F. Franklin, et al. (1994). *Feedback Control of Dynamic Systems*. Addison-Wesley Publishing Company Inc., third edn.

J. L. Junkins, et al. (1981). 'Time Optimal Magnetic Attitude Maneuvers'. *Journal of Guidance and Control* **4**(4):363–368.

C.-Y. Kao, et al. (2001). ‘A cutting plane algorithm for robustness analysis of periodically time-varying system’. *IEEE Transactions on Automatic Control* **46**(4):579–592.

H. K. Khalil (2002). *Nonlinear Systems*. Prentice-Hall, Upper Saddle River, NJ, USA, 3 edn.

J. Kim, et al. (2006a). ‘Robustness analysis of linear periodic time-varying systems subject to structured uncertainty’. *Systems & Control Letters* **55**(9):719–725.

J. Kim, et al. (2006b). ‘Robustness analysis of biochemical networks models’. *IEE Proceedings - Systems Biology* **153**(3):96–104.

M. Lovera, et al. (2002). ‘Periodic Attitude Control Techniques for Small Satellites With Magnetic Actuators’. *IEEE Transactions on Control Systems Technology* **10**(1):90–95.

L. Ma & P. A. Iglesias (2002a). ‘Quantifying Robustness Of Biochemical Network Models’. *BMC Bioinformatics* **3**(38).

L. Ma & P. A. Iglesias (2002b). ‘Robustness Analysis of a Self-Oscillating Molecular Network in *Dictyostelium Discoideum*’. In *Proceedings of the joint 41st Conference on Decision and Control and European Control Conference*, Las Vegas, Nevada, USA.

M. L. Psiaki (2001). ‘Magnetic Torquer Attitude Control via Asymptotic Periodic Linear Quadratic Regulation’. *Journal of Guidance, Control, and Dynamics* **24**(2):386–394.

W. J. Rugh (1996). *Linear System Theory*. Prentice-Hall, Upper Saddle River, NJ, USA, 2 edn.

M. D. Shuster (1993). ‘A Survey of Attitude Representations’. *Journal of the Astronautical Sciences* **41**(4):439–517.

E. Silani & M. Lovera (2005). ‘Magnetic spacecraft attitude control: a survey and some new results’. *Control Engineering Practice* **13**:357–371.

K. J. Spyrou (1997). ‘Dynamic instability in quartering seas-Part III: Nonlinear effects on periodic motions’. *Journal of Ship Research* **41**(3):210–213.

V. Verdult, et al. (2004). ‘Identification of linear parameter-varying state-space models with application to helicopter rotor dynamics’. *International Journal of Control* **77**(13):1149–1159.

J. R. Wertz (1986). *Spacecraft Attitude Determination and Control*. D. Reidel Publishing Company.

R. Wiśniewski & M. Blanke (1999). ‘Fully magnetic attitude control for spacecraft subject to gravity gradient’. *Automatica* **35**(7):1201–1214.