# ON THE RESTRICTION OF THE FOURIER TRANSFORM TO POLYNOMIAL CURVES

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**Abstract.** We prove a Fourier restriction theorem on curves parametrised by the mapping $t \mapsto P(t) = (P_1(t), \dots, P_n(t))$, where each of the $P_1, \dots, P_n$ is a real-valued polynomial and $t$ belongs to an interval on which each of the $P_1, \dots, P_n$ "resembles" a monomial.

## 1 INTRODUCTION

*Fourier restriction theorems* are results of the form

$$\int_M |\widehat{f}(\xi)|^q d\sigma \leq C\|f\|^q_{L^p(\mathbf{R}^n)}, \tag{1}$$

where $f$ is in the Schwartz class $S(\mathbf{R}^n)$, $\widehat{f}$ denotes the Fourier transform of $f$ and $\sigma$ is a measure on a manifold $M$ in $\mathbf{R}^n$. Since, for $f \in L^p$, with $p > 1$, $\widehat{f}$ does not make sense pointwise, it is natural to introduce a measure on the manifold $M$ and ask for such results.

Fourier restriction has played an important role in Harmonic Analysis over the last 30 years. Interest in this area is largely due to its intimate connections with Bochner-Riesz multipliers (see [3], [9]), while, through Strichartz and dispersive estimates, Fourier restriction inequalities are used to study the regularity and uniqueness of solutions to hyperbolic partial differential equations (see e.g. [12]).

Fourier restriction on curves in $\mathbf{R}^n$, has been studied by many authors; the papers [11], [4], [5], [7], [8] and [6] are particularly notable. Common to all of these works is the interplay between the curvature properties of the curves and the sharp $L^p$ to $L^q$ boundedness properties of the corresponding Fourier restriction operator.

Drury and Marshall in [7], [8] and [6], introduced the affine arclength measure in the study of Fourier restriction theorems (it had previously occured in a disguised form in Sjölin [11]). The affine arclength measure on a polynomial curve $P(t) = (P_1(t), \dots, P_n(t))$ in $\mathbf{R}^n$ is defined by $d\sigma = |L|^{2/n(n+1)}dt$, where

$$L(t) = \det(P'(t), P''(t), \dots, P^{(n)}(t)).$$

The mapping properties of the Fourier restriction operator with respect to Euclidean arclength measure degenerate when there are points where the curvature vanishes. However, the mapping properties with respect to the affine arclength measure do not degenerate because the affine arclength measure has correspondingly little mass near these points.

In this article, we consider inequality (1), where the manifold $M$ is a polynomial curve and the measure $\sigma$ is the affine arclength measure. In addition, we aim to obtain Fourier restriction theorems with the characteristic that the constant $C$ in (1) is uniform over all polynomial curves of a given degree. The latter should be possible given the choice of the measure. The following conjecture seems reasonable:

*Let* $P(t) = (P_1(t), \ldots, P_n(t))$, *where each of the* $P_i, 1 \le i \le n$, *is a real-valued polynomial with degree* $d_i$, *and let* $L(t) = \det(P'(t), P''(t), \ldots, P^{(n)}(t))$. *Then, for* $f \in S(\mathbf{R}^n)$,

$$\int_{\mathbf{R}} |\widehat{f}(P(t))|^q |L(t)|^{2/n(n+1)} dt \le C_{n,\mathbf{d}} \|f\|_p^q \,, \tag{2}$$

*where* $\frac{1}{q} = \frac{n(n+1)}{2}\frac{1}{p'}$, $1 \le p < \frac{n^2+n+2}{n^2+n}$, *and the constant* $C_{n,\mathbf{d}}$ *only depends on* $n$ *and the degree of* $P$, $\mathbf{d} = (d_1, \ldots, d_n)$, *and in particular not on the coefficients of* $P$.

The condition $1 \le p < (n^2+n+2)/(n^2+n)$ is suggested by considering the curve $P(t) = (t, t^2, \ldots, t^n)$. The sufficiency of the condition for this curve is a result of Drury [5] and the necessity follows from the work of Arkhipov, Chubarikov and Karatsuba [1].

The conjecture can be shown to be true for the case $n = 2$, by the result in Sjölin [11]. Sjölin's method, however, does not appear to generalise to higher dimensions. The conjecture for $n \ge 3$ is open. We prove a weaker version of the conjecture in this article. First, in inequality (2), we restrict the Fourier transform to a certain "large" portion of the polynomial curve. We do this by restricting the integration over $\mathbf{R}$ to certain intervals $I$ on which each $P_i$ "resembles" a monomial, i.e., $P_i \sim c_i t^{j_i}$. The intervals $I$ will lie far from the roots of the polynomials. The entire real line can be covered by a bounded number of such intervals together with a finite number of dyadic intervals (see Section 2 below for details). We impose the additional condition that all the $j_i$ are distinct positive integers. Second, our result concerns the smaller range of $p$, $1 \le p < (n^2+2n)/(n^2+2n-2)$. This is because of our method of proof, which uses a similar strategy to the one in Christ [4].

Our main result is the following:

**Theorem 1.1.** *With* $P(t) = (P_1(t), \ldots, P_n(t))$, $L(t)$ *and* $I$ *as above, we have*

$$\left( \int_I |\widehat{f}(P(t))|^q |L(t)|^{2/n(n+1)} dt \right)^{\frac{1}{q}} \le C_{n,\mathbf{d}} \|f\|_p \,, \tag{3}$$

*for* $f \in S(\mathbf{R}^n)$, *where* $\frac{1}{q} = \frac{n(n+1)}{2}\frac{1}{p'}$ *and* $1 \le p < \frac{n(2+n)}{n(2+n)-2}$.

An important paper in this area, which presents several useful ideas, is that of Christ [4]. There, he considers inequalities of the form

$$\int_{-\delta}^{\delta} |\widehat{f}(\psi(t))|^q dt \le C \|f\|_p^q, \tag{4}$$

for some sufficiently small $\delta > 0$, which depends on the curve $\psi(t)$. The inequality (4) differs from (2), since the integration on the curve in (4) takes place on a very small interval $(-\delta, \delta)$ on which the components of $\psi(t)$ are approximated by monomials. In contrast, for the curve given by $P(t)$ in (2), the integration is over the whole of $\mathbf{R}$ and consequently includes all the competing homogeneities that exist in a polynomial. Christ's argument can be extended to unbounded curves $\psi(t)$, when the components of $\psi(t)$ are pure monomials with distinct powers (e.g. $\psi(t) = (t, t^2, t^3, \ldots)$). Another difference is that the Euclidean arclength measure is used in (4) as opposed to the affine arclength measure, which is used in (2). Nondegenerate results can be obtained using the latter measure, which also allows one to obtain uniform estimates for certain families of curves.

This article is organised as follows. In Section 2 we prove some lemmas which are used to analyse the behaviour of a polynomial of a single variable and describe the interval $I$ on which we restrict the Fourier transform. In Section 3 we show how the quantity $L(t)$, used in the definition of the affine arclength measure, behaves on $I$. Finally, in Section 4 we prove the Fourier restriction theorem in this setting.

This paper is part of the author's doctoral thesis which was partially funded by the Alexander S. Onassis Foundation. The author wishes to thank Professor James Wright for all his support.

*Notation:* For the rest of this paper we denote by $\beta \lesssim \gamma$ or $\beta = \mathrm{O}(\gamma)$ that there exists a constant $C = C_{n,\mathbf{d}}$ only depending on the degree $\mathbf{d}$ and the dimension $n$, such that $|\beta| \leq C|\gamma|$. By $\beta \sim \gamma$, we mean that $\beta \lesssim \gamma \lesssim \beta$. Also, when we say that $A$ is sufficiently large, we mean that there exists a constant $K(\mathbf{d})$ only depending on the degree such that $A > K(\mathbf{d})$.

## 2 ANALYSIS OF POLYNOMIALS OF A SINGLE VARIABLE

In this section we concentrate on the analysis of the behaviour of polynomials of a single variable. We describe a decomposition of the positive real axis into a number of intervals, some of which we call *gaps* and others *dyadic intervals*. Exactly symmetrical intervals to these also exist for the negative real axis, but without loss of generality we can restrict our attention to the positive one. The decomposition is achieved by a couple of lemmas. We start by quoting Lemma 2.5 of [2]. We then prove a generalisation. After we have established this we will proceed to a number of results that will be needed in Section 3.

**Lemma 2.1.** *Let $t_1, \ldots, t_d$ be the complex roots of a polynomial*

$$R(t) = \sum_{m=0}^{d} r_m t^m = r_d \prod_{m=1}^{d} (t - t_m)$$

*of degree $d$, ordered so that $|t_1| \leq |t_2| \leq \ldots \leq |t_d|$. Then, there exist positive constants $K(d)$ and $\epsilon(d)$ such that if $A > K(d)$ and $t$ satisfies $A|t_k| < t < A^{-1}|t_{k+1}|$, for some $0 \leq k \leq d$ (let $t_0 = 0$ and $t_{d+1} = \infty$), then*

*a)* $R(t) \sim r_k t^k$,

*b)* $\left|\frac{R'(t)}{R(t)}\right| \geq \frac{\epsilon(d)}{t}$ *for* $k \geq 1$,

*c)* $|R(t)|$ *is strictly increasing on* $[A|t_k|, A^{-1}|t_{k+1}|]$.

REMARK. Strictly speaking the lemma in [2] only shows that $R(t) \sim c_k t^k$, where $c_k = r_d t_{k+1} \ldots t_d$. However it was shown in [10] that $r_k \sim r_d t_{k+1} \ldots t_d$ if $A|t_k| < A^{-1}|t_{k+1}|$ for sufficiently large $A$.

Before continuing we shall consider some of the consequences of Lemma 2.1. For a polynomial whose roots are ordered by $|t_1| \leq |t_2| \leq \ldots |t_d|$ we consider a *dyadic interval* $[A^{-1}|t_k|, A|t_k|]$ associated to each root $t_k$, whose logarithmic measure is bounded above by $2\log A$. The complement of the union of the dyadic intervals is a disjoint union of possibly very long intervals which we call *gaps*. It is on the gaps that we focus our attention. According to Lemma 2.1, on the gaps the polynomial "behaves" like a monomial and in particular if there is a gap between $|t_1|$ and $|t_2|$ the polynomial behaves there like $t$, if there is a gap between $|t_2|$ and $|t_3|$ it behaves like $t^2$ etc.; of course some roots might not be seperated enough to guarantee the existence of a gap "between" the roots.

Part *b)* of Lemma 2.1 says that on the interval $[A|t_k|, A^{-1}|t_{k+1}|]$, the first derivative of the polynomial behaves like that of a monomial (it is one power lower). We extend this to certain higher derivatives. To accomplish this we will need the following formula.

**Lemma 2.2.** *Let $R(t)$ be a polynomial of degree $d$ and let $t_1, \ldots, t_d$ be its complex roots. Then for any $r \geq 1$,*

$$\frac{R^{(r)}}{R}(t) = r! \sum_{1 \leq l_1 < \ldots < l_r \leq d} \prod_{i=1}^{r} \frac{1}{t - t_{l_i}}. \tag{5}$$

*Proof.* One can easily verify (5) for $r = 1$. The rest of the lemma can then be proved by induction on $r$. □

We are now in a position to extend part *b)* of Lemma 2.1 to higher derivatives.

**Lemma 2.3.** *Using the notation of Lemma 2.1, there exist constants $\epsilon_1(d)$ and $\epsilon_2(d)$ such that if $t$ satisfies $A|t_k| < t < A^{-1}|t_{k+1}|$, for $A$ sufficiently large and some $0 \leq k \leq d$, then for any $0 \leq r \leq k$,*

$$\frac{\epsilon_1(d)}{t^r} \geq \left| \frac{R^{(r)}(t)}{R(t)} \right| \geq \frac{\epsilon_2(d)}{t^r}.$$

*Proof.* The upper bound follows immediately from (5) and the fact that $A|t_k| < t < A^{-1}|t_{k+1}|$. Thus we concentrate on the lower bound. We use (5) from Lemma 2.2 to write

$$\frac{R^{(r)}}{r!R}(t) = \sum_{1 \leq l_1 < \ldots < l_r \leq d} \prod_{i=1}^{r} \frac{1}{t - t_{l_i}}.$$

By the triangle inequality we have

$$\begin{aligned} \left| \frac{R^{(r)}}{r!R}(t) \right| \quad &\geq \quad \left| \sum_{1 \leq l_1 < \ldots < l_r \leq k} \prod_{i=1}^{r} \frac{1}{t - t_{l_i}} \right| - \sum_{q=1}^{r} \left| \sum_{\substack{1 \leq l_1 < \ldots < l_r \leq d \\ k+1 \leq l_q \leq d}} \prod_{i=1}^{r} \frac{1}{t - t_{l_i}} \right| \\ &=: \quad \mathrm{I} \quad - \quad \sum_{q=1}^{r} \mathrm{II}_q \quad . \end{aligned}$$

For the $\mathrm{II}_q$'s we can bound each term from above by $\mathrm{O}(A^{-1}t^{-r})$, since for each $q$ the corresponding $l_q$ satisfies $k+1 \le l_q \le d$. For I we have

$$\begin{aligned}
\mathrm{I} &= \left| \sum_{1\le l_1<\ldots<l_r\le k} \prod_{i=1}^{r} \frac{1}{t-t_{l_i}} \right| \\
&\ge \mathrm{Re}\left( \sum_{1\le l_1<\ldots<l_r\le k} \prod_{i=1}^{r} \frac{t-\bar{t}_{l_i}}{|t-t_{l_i}|^2} \right) \\
&= \sum_{1\le l_1<\ldots<l_r\le k} \frac{t^r \pm \mathrm{Re}(\sum_{i=1}^{r} \bar{t}_{l_i})t^{r-1} \pm \ldots \pm \mathrm{Re}\prod_{i=1}^{r} \bar{t}_{l_i}}{\prod_{i=1}^{r} |t-t_{l_i}|^2}.
\end{aligned}$$

We note that unless $r \le k$ the sum in I is empty. Hence

$$\begin{aligned}
\left| \frac{R^{(r)}}{R}(t) \right| &\ge \sum_{1\le l_1<\ldots<l_r\le k} \frac{t^r \pm \mathrm{Re}(\sum_{i=1}^{r} \bar{t}_{l_i})t^{r-1} \pm \ldots \pm \mathrm{Re}\prod_{i=1}^{r} \bar{t}_{l_i}}{\prod_{i=1}^{r} |t-t_{l_i}|^2} \\
&\quad -\mathrm{O}(A^{-1}t^{-r}) \\
&\gtrsim \frac{1}{t^r},
\end{aligned}$$

since for each $l_i \le k$, $|t_{l_i}| \le A^{-1}t$. This ends the proof of Lemma 2.3. □

We formally record the estimate derived near the end of the above proof in the following lemma.

**Lemma 2.4.** *Let $\alpha \in \mathbf{N}$, $\alpha = \mathrm{O}(1)$ and $L$ any index set such that $\sharp(L) = \mathrm{O}(1)$. Consider any arbitrary set of complex numbers $\{t_{l,i}\}_{\substack{1\le i\le \alpha \\ l\in L}}$ satisfying $|t_{l,i}| \le A^{-1}t$ for some $t > 0$. Then, for sufficiently large $A$,*

$$\sum_{l\in L} \prod_{i=1}^{\alpha} \frac{1}{t-t_{l,i}} \sim \frac{1}{t^\alpha}.$$

*Proof.* The bounds from above are trivial and so we concentrate on the lower bounds.

$$\begin{aligned}
&\left| \sum_{l\in L} \prod_{i=1}^{\alpha} \frac{1}{t-t_{l,i}} \right| \\
&\ge \mathrm{Re} \sum_{l\in L} \prod_{i=1}^{\alpha} \frac{1}{t-t_{l,i}} \ge \sum_{l\in L} \mathrm{Re} \prod_{i=1}^{\alpha} \frac{1}{t-t_{l,i}} \\
&= \sum_{l\in L} \mathrm{Re} \prod_{i=1}^{\alpha} \frac{t-\bar{t}_{l,i}}{|t-t_{l,i}|^2} \\
&\ge C \sum_{l\in L} \frac{1}{t^{2\alpha}} \mathrm{Re} \left[ t^\alpha - \left( \sum_{i=1}^{\alpha} \bar{t}_{l,i} \right) t^{\alpha-1} + \ldots + (-1)^\alpha \prod_{i=1}^{\alpha} \bar{t}_{l,i} \right],
\end{aligned}$$

with $C$ an absolute constant, since $|t - t_{l,i}| \le 2t$ for sufficiently large $A$ and all $i$,$l$. Finally, again since each $|t_{l,i}| \le A^{-1}t$, the last expression is greater than

$$C \sum_{l\in L} \frac{1}{t^{2\alpha}} \left( t^\alpha - \frac{D}{A} t^\alpha \right) \gtrsim \frac{1}{t^\alpha},$$

where $D$ is an absolute constant, thus completing the proof of Lemma 2.4. □

The last lemma of this section is about the difference of two $\alpha$-fold products as considered in Lemma 2.4.

**Lemma 2.5.** *Let* $\{t_{l,i}\}_{\substack{1\le i\le\alpha \\ l\in\{1,2\}}}$ *and* $t>0$ *be as in Lemma 2.4. Then, for* $A$ *sufficiently large,*

$$\prod_{i=1}^{\alpha}\frac{1}{t-t_{1,i}}-\prod_{i=1}^{\alpha}\frac{1}{t-t_{2,i}}=\mathrm{O}\left(\frac{1}{At^{\alpha}}\right).$$

*Proof.*

$$\begin{aligned}
&\left|\prod_{i=1}^{\alpha}\frac{1}{t-t_{1,i}}-\prod_{i=1}^{\alpha}\frac{1}{t-t_{2,i}}\right| \\
=\ &\left|\frac{\prod_{i=1}^{\alpha}(t-t_{2,i})-\prod_{i=1}^{\alpha}(t-t_{1,i})}{\prod_{i=1}^{\alpha}(t-t_{1,i})\prod_{i=1}^{\alpha}(t-t_{2,i})}\right| \\
\le\ &\frac{\left|\left(\sum_{i=1}^{\alpha}t_{1,i}-\sum_{i=1}^{\alpha}t_{2,i}\right)t^{\alpha-1}+\ldots+(-1)^{\alpha}\left(\prod_{i=1}^{\alpha}t_{2,i}-\prod_{i=1}^{\alpha}t_{1,i}\right)\right|}{\left(1-\frac{1}{A}\right)t^{2\alpha}} \\
\le\ &\frac{\frac{1}{A}t^{\alpha}}{\left(1-\frac{1}{A}\right)t^{2\alpha}}\ \le\ \frac{C}{At^{\alpha}},
\end{aligned}$$

for $C$ an absolute constant and for sufficiently large $A$. This completes the proof of Lemma 2.5. □

Let us now consider the mapping $P:\mathbf{R}^{+}\to\mathbf{R}^{n}$ given by $P(t)=(P_1(t),\ldots,P_n(t))$, where each $P_i$ is a real-valued polynomial of a single variable. Then for each $P_i$ we have a corresponding splitting of $\mathbf{R}^{+}$ into gaps and dyadic intervals (following the discussion after Lemma 2.1). We shall consider an interval $I$ which lies inside a $n$-fold intersection of gaps; each gap corresponding to a different $P_i$. Therefore, on $I$ the components of $P(t)=(P_1(t),\ldots,P_n(t))$ look like various monomials according to Lemma 2.1. Specifically if $P_i(t)=\sum_{m=1}^{d_i}p_{i,m}t^{m}$, then on $I$,

$$P_i(t)\sim p_{i,j_i}t^{j_i},$$

for some $j_i$. We impose the additional condition that on $I$ all the $j_i$ are distinct.

In the following sections we will use the functions, $L_{P_1\ldots P_\mu}(t)$, defined by

$$L_{P_1\ldots P_\mu}(t)=\det(P'(t),P''(t),\ldots,P^{(\mu)}(t)), \tag{6}$$

for $1\le\mu\le n$, where $P(t)=(P_1(t),\ldots,P_\mu(t))$.

## 3 ESTIMATING $L_{P_1\ldots P_n}(t)$

**Proposition 3.1.** *Let* $I$ *and* $L_{P_1\ldots P_n}(t)$ *be defined as above. Recall that for* $t\in I$, $P_i(t)\sim p_{i,j_i}t^{j_i}$ *and we may suppose* $0<j_1<j_2<\ldots<j_n$. *Then, for* $t\in I$,

$$L_{P_1\ldots P_n}(t)\sim\left(\prod_{i=1}^{n}p_{i,j_i}\right)t^{\sum_{i=1}^{n}j_i-\frac{n(n+1)}{2}}.$$

*Proof.* First, let us denote by $d_i$ the degree of the polynomial $P_i$, by $\sigma$ a permutation of $\{1,\ldots,n\}$ and by $t_{i,k}$ the (complex) roots of $P_i$ ordered so that $|t_{i,k_1}| \leq |t_{i,k_2}|$ if $k_1 \leq k_2$. Then by expanding the determinant $L_{P_1\ldots P_n}$, we have

$$\frac{L_{P_1\ldots P_n}}{P_1\cdots P_n}(t) = \sum_{\sigma \text{ even}} \frac{P_1^{(\sigma(1))}\ldots P_n^{(\sigma(n))}}{P_1\cdots P_n}(t) - \sum_{\sigma \text{ odd}} \frac{P_1^{(\sigma(1))}\ldots P_n^{(\sigma(n))}}{P_1\cdots P_n}(t).$$

We then use Lemma 2.2 to express the derivatives of polynomials in terms of the roots $t_{i,k}$. Thus

$$\begin{aligned}\left(\prod_{i=1}^{n} i!\right)^{-1} \frac{L_{P_1\ldots P_n}}{P_1\ldots P_n}(t) &= \sum_{\sigma \text{ even}} \prod_{i=1}^{n} \sum_{1\leq k_1<\ldots<k_{\sigma(i)}\leq d_i} \prod_{l=1}^{\sigma(i)} \frac{1}{t-t_{i,k_l}} \\ &\quad - \sum_{\sigma \text{ odd}} \prod_{i=1}^{n} \sum_{1\leq k_1<\ldots<k_{\sigma(i)}\leq d_i} \prod_{l=1}^{\sigma(i)} \frac{1}{t-t_{i,k_l}} \\ &= \sum_{\sigma \text{ even}} \prod_{i=1}^{n} \sum_{1\leq k_1<\ldots<k_{\sigma(i)}\leq j_i} \prod_{l=1}^{\sigma(i)} \frac{1}{t-t_{i,k_l}} \\ &\quad - \sum_{\sigma \text{ odd}} \prod_{i=1}^{n} \sum_{1\leq k_1<\ldots<k_{\sigma(i)}\leq j_i} \prod_{l=1}^{\sigma(i)} \frac{1}{t-t_{i,k_l}} \\ &\quad + \mathrm{O}\left(\frac{1}{At^{\frac{n(n+1)}{2}}}\right).\end{aligned}$$

When $\sigma(i) > j_i$, the sum over $k_1 < \ldots < k_{\sigma(i)}$ is empty and interpreted as zero. We then proceed to interchange the order of the middle product and sum. That is we can express $L_{P_1\ldots P_n}(t)/P_1(t)\cdots P_n(t)$ as a difference,

$$\begin{aligned}\left(\prod_{i=1}^{n} i!\right)^{-1} \frac{L_{P_1\ldots P_n}}{P_1\cdots P_n}(t) &= \sum_{E_+} \prod_{i=1}^{n} \prod_{l=1}^{\sigma(i)} \frac{1}{t-t_{i,k_{i,l}}} \\ &\quad - \sum_{E_-} \prod_{i=1}^{n} \prod_{l=1}^{\sigma(i)} \frac{1}{t-t_{i,k_{i,l}}} + \mathrm{O}\left(\frac{1}{At^{\frac{n(n+1)}{2}}}\right),\end{aligned} \tag{7}$$

where

$$\begin{aligned}E_+ &= \Big\{\sigma, \{k_{i,l}\}_{\substack{1\leq i\leq n\\ 1\leq l\leq\sigma(i)}} : \sigma(i) \leq j_i \text{ for all } i, \sigma \text{ even}, \\ &\qquad 1 \leq k_{i,1} < \ldots < k_{i,\sigma(i)} \leq j_i \text{ for all } i\Big\}\end{aligned}$$

and

$$\begin{aligned}E_- &= \Big\{\sigma, \{k_{i,l}\}_{\substack{1\leq i\leq n\\ 1\leq l\leq\sigma(i)}} : \sigma(i) \leq j_i \text{ for all } i, \sigma \text{ odd}, \\ &\qquad 1 \leq k_{i,1} < \ldots < k_{i,\sigma(i)} \leq j_i \text{ for all } i\Big\}.\end{aligned}$$

We observe that both sums in (7) are sums of $\frac{n(n+1)}{2}$-fold products. This allows us to use Lemma 2.5 to compare a term from $E_+$ with a term from $E_-$, creating an error

$O(A^{-1}t^{-\frac{n(n+1)}{2}})$. Hence if $\sharp E_+ \neq \sharp E_-$, we have

$$\left(\prod_{i=1}^{n} i!\right)^{-1} \frac{L_{P_1\ldots P_n}}{P_1\cdots P_n}(t) = \pm \sum_S \prod_{i=1}^{n} \prod_{l=1}^{\sigma(i)} \frac{1}{t - t_{i,k_{i,l}}} + O(A^{-1}t^{-\frac{n(n+1)}{2}})$$

where either $S$ is a nonempty subset of $E_+$ (if $\sharp E_+ > \sharp E_-$) or a nonempty subset of $E_-$ (if $\sharp E_- > \sharp E_+$). Now Lemma 2.4 can be employed to obtain the desired bounds for $L_{P_1\ldots P_n}(t)$. It only remains to verify $\sharp E_+ \neq \sharp E_-$. This is done by counting as follows. Using the notation $\left(\begin{array}{c} j \\ k \end{array}\right) = (k!)^{-1}\prod_{i=1}^{k}(j-i+1)$, we have

$$\begin{aligned}
&\sharp E_+ - \sharp E_- \\
= &\sum_{\sigma \text{ even}} \left(\begin{array}{c} j_1 \\ \sigma(1) \end{array}\right)\left(\begin{array}{c} j_2 \\ \sigma(2) \end{array}\right)\cdots\left(\begin{array}{c} j_n \\ \sigma(n) \end{array}\right) - \sum_{\sigma \text{ odd}} \left(\begin{array}{c} j_1 \\ \sigma(1) \end{array}\right)\cdots\left(\begin{array}{c} j_n \\ \sigma(n) \end{array}\right)
\end{aligned}$$

$$\begin{aligned}
= &\left(\prod_{r=1}^{n} i!\right)^{-1} \Bigg[\sum_{\sigma \text{ even}} \prod_{i=1}^{n}(j_i(j_i-1)\ldots(j_i-\sigma(i)+1)) \\
&- \sum_{\sigma \text{ odd}} \prod_{i=1}^{n}(j_i(j_i-1)\ldots(j_i-\sigma(i)+1))\Bigg] \\
= &\left(\prod_{r=1}^{n} i!\right)^{-1} \left|\begin{array}{ccc} j_1 & \cdots & j_n \\ j_1(j_1-1) & \cdots & j_n(j_n-1) \\ \vdots & & \vdots \\ j_1\ldots(j_1-n+1) & \cdots & j_n\ldots(j_n-n+1) \end{array}\right|.
\end{aligned}$$

Then by expanding the products and performing row operations the determinant above is equal to

$$\left|\begin{array}{cccc} j_1 & j_2 & \cdots & j_n \\ j_1^2 & j_2^2 & \cdots & j_n^2 \\ \vdots & \vdots & & \vdots \\ j_1^n & j_2^n & \cdots & j_n^n \end{array}\right| = \prod_{i=1}^{n} j_i \left|\begin{array}{ccc} 1 & \cdots & 1 \\ j_1 & \cdots & j_n \\ \vdots & & \vdots \\ j_1^{n-1} & \cdots & j_n^{n-1} \end{array}\right|.$$

The last determinant is a Vandermonde determinant and so the last expression is equal to

$$\prod_{i=1}^{n} j_i \prod_{1\le l<k\le n} (j_k - j_l),$$

which is nonzero since $j_k \neq j_l$ for all $1 \le l < k \le n$ and $j_i > 0$ for all $1 \le i \le n$. This completes the proof of Proposition 3.1. □

REMARK. It is easy to see that Proposition 3.1 still holds with $P_1, \ldots P_n$ replaced by any $P_{\xi(1)}, \ldots, P_{\xi(\mu)}$ with $1 \le \mu \le n$ and $\xi$ a one-to-one function from $1, \ldots, \mu$ to $1, \ldots, n$.

## 4 PROOF OF THEOREM 1.1

We now proceed with the proof of Theorem 1.1. We will need a few preliminary results. We note that the condition that all the $j_i$'s are distinct is crucial for the proofs.

**Proposition 4.1.** *With*

$$J_{P_1\ldots P_n}(t_1,\ldots,t_n) = \begin{vmatrix} P_1'(t_1) & \cdots & P_1'(t_n) \\ \vdots & & \vdots \\ P_n'(t_1) & \cdots & P_n'(t_n) \end{vmatrix},$$

*the Jacobian of the mapping* $t \mapsto x(t) = (x_1(t),\ldots,x_n(t))$, *where*

$$x_k(t) = \sum_{i=1}^{n} P_k(t_i),$$

$1 \le k \le n$ *and* $t = (t_1,\ldots t_n)$, *the following lower bound holds for* $0 \le t_1 \le t_2 \le \ldots \le t_n$ *and for* $[t_1,t_n] \subseteq I$*:*

$$\begin{aligned} J_{P_1\ldots P_n}(t_1,\ldots,t_n) &\gtrsim \left(\prod_{i=1}^{n} p_{i,j_i}\right) t_1^{j_1-1} t_2^{j_2-2}\ldots t_n^{j_n-n} \prod_{1\le k<l\le n}(t_l-t_k) \\ &\gtrsim \prod_{i=1}^{n} |L_{P_1\ldots P_n}(t_i)|^{1/n} \prod_{1\le k<l\le n}(t_l-t_k)\,. \end{aligned}$$

The proof will be carried out in several steps. We start by establishing the second inequality first. In view of Proposition 3.1, it suffices to show the inequality

$$t_1^{j_1-1} t_2^{j_2-2}\ldots t_n^{j_n-n} \ge \prod_{i=1}^{n} t_i^{\frac{1}{n}(\sum_{k=1}^{n} j_k)-\frac{n+1}{2}}. \tag{8}$$

Inequality (8) can be easily seen by taking logs on both sides and using the fact that $0 \le t_1 \le t_2 \le \ldots \le t_n$ and $j_1 < j_2 < \ldots < j_n$. For the first inequality of Proposition 4.1, we will express $J_{P_1,\ldots,P_n}(t_1,\ldots,t_n)$ in terms of the $L_{P_1,\ldots,P_m}$'s, $1 \le m \le n$, for $t_1 \le \ldots \le t_n$ and $[t_1,t_n] \subseteq I$. This will be accomplished by the following two lemmas.

**Lemma 4.2.** *Let* $f_i = \frac{g_i'}{g_1'}$, $1 \le i \le n$, *and assume that* $g_i$ *and* $f_i$ *are differentiable functions in* $[t_1,t_n]$ *for all* $i$. *Then*

$$\begin{aligned} &\begin{vmatrix} g_1'(t_1) & \cdots & g_1'(t_n) \\ \vdots & & \vdots \\ g_n'(t_1) & \cdots & g_n'(t_n) \end{vmatrix} \\ &= \prod_{i=1}^{n} g_1'(t_i) \int_{t_1}^{t_2} dx_1 \ldots \int_{t_{n-1}}^{t_n} dx_{n-1} \begin{vmatrix} f_2'(x_1) & \cdots & f_2'(x_{n-1}) \\ \vdots & & \vdots \\ f_n'(x_1) & \cdots & f_n'(x_{n-1}) \end{vmatrix}. \end{aligned} \tag{9}$$

*Proof.* By factoring $g_1'(t_i)$ out of every column we write

$$\begin{vmatrix} g_1'(t_1) & \cdots & g_1'(t_n) \\ \vdots & & \vdots \\ g_n'(t_1) & \cdots & g_n'(t_n) \end{vmatrix} = \prod_{i=1}^{n} g_1'(t_i) \begin{vmatrix} 1 & \cdots & 1 \\ f_2(t_1) & \cdots & f_2(t_n) \\ \vdots & & \vdots \\ f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix}.$$

Then by conducting column operations the determinant involving the $f_i$'s is equal to

$$\begin{vmatrix} 1 & 0 & \cdots & 0 \\ f_2(t_1) & f_2(t_2)-f_2(t_1) & \cdots & f_2(t_n)-f_2(t_1) \\ \vdots & \vdots & & \vdots \\ f_n(t_1) & f_n(t_2)-f_n(t_1) & \cdots & f_n(t_n)-f_n(t_1) \end{vmatrix}$$

$$= \int_{t_1}^{t_2} dx_1 \ldots \int_{t_1}^{t_n} dx_{n-1} \begin{vmatrix} f_2'(x_1) & \cdots & f_2'(x_{n-1}) \\ \vdots & & \vdots \\ f_n'(x_1) & \cdots & f_n'(x_{n-1}) \end{vmatrix}.$$

For fixed $x_1, x_2, \ldots, x_{n-1}$ except $x_l$ and $x_m$ with $1 \le l < m \le n-1$, consider

$$I_k := \int_{t_k}^{t_{k+1}} dx_l \int_{t_k}^{t_{k+1}} dx_m \begin{vmatrix} f_2'(x_1) & \cdots & f_2'(x_{n-1}) \\ \vdots & & \vdots \\ f_n'(x_1) & \cdots & f_n'(x_{n-1}) \end{vmatrix}.$$

By interchanging the $l$th with the $m$th column we have

$$\begin{aligned} & I_k \\ = & \int_{t_k}^{t_{k+1}} dx_l \int_{t_k}^{t_{k+1}} dx_m \begin{vmatrix} f_2'(x_1) & \cdots & f_2'(x_l) & \cdots & f_2'(x_m) & \cdots & f_2'(x_{n-1}) \\ \vdots & & \vdots & & \vdots & & \vdots \\ f_n'(x_1) & \cdots & f_n'(x_l) & \cdots & f_n'(x_m) & \cdots & f_n'(x_{n-1}) \end{vmatrix} \\ = & -\int_{t_k}^{t_{k+1}} dx_l \int_{t_k}^{t_{k+1}} dx_m \begin{vmatrix} f_2'(x_1) & \cdots & f_2'(x_m) & \cdots & f_2'(x_l) & \cdots & f_2'(x_{n-1}) \\ \vdots & & \vdots & & \vdots & & \vdots \\ f_n'(x_1) & \cdots & f_n'(x_m) & \cdots & f_n'(x_l) & \cdots & f_n'(x_{n-1}) \end{vmatrix} \\ = & -\int_{t_k}^{t_{k+1}} dx_m \int_{t_k}^{t_{k+1}} dx_l \begin{vmatrix} f_2'(x_1) & \cdots & f_2'(x_l) & \cdots & f_2'(x_m) & \cdots & f_2'(x_{n-1}) \\ \vdots & & \vdots & & \vdots & & \vdots \\ f_n'(x_1) & \cdots & f_n'(x_l) & \cdots & f_n'(x_m) & \cdots & f_n'(x_{n-1}) \end{vmatrix}, \end{aligned}$$

the last equality follows by changing the variables of integration. Thus $I_k = -I_k$ and so $I_k = 0$. So finally

$$\begin{aligned} & \prod_{i=1}^{n} g_1'(t_i) \int_{t_1}^{t_2} dx_1 \ldots \int_{t_1}^{t_n} dx_{n-1} \begin{vmatrix} f_2'(x_1) & \cdots & f_2'(x_{n-1}) \\ \vdots & & \vdots \\ f_n'(x_1) & \cdots & f_n'(x_{n-1}) \end{vmatrix} \\ = & \prod_{i=1}^{n} g_1'(t_i) \int_{t_1}^{t_2} dx_1 \ldots \int_{t_{n-1}}^{t_n} dx_{n-1} \begin{vmatrix} f_2'(x_1) & \cdots & f_2'(x_{n-1}) \\ \vdots & & \vdots \\ f_n'(x_1) & \cdots & f_n'(x_{n-1}) \end{vmatrix}, \end{aligned}$$

concluding the proof of Lemma 4.2. □

We aim to use Lemma 4.2 inductively to obtain an expression of $J_{P_1\ldots P_n}$ in terms of the $L_{P_1\ldots P_m}$'s with $1 \le m \le n$. We shall do this by the following lemma.

**Lemma 4.3.**

$$\left(\frac{L_{P_1\ldots P_m Q}}{L_{P_1\ldots P_m R}}\right)' = \frac{L_{P_1\ldots P_m RQ} L_{P_1\ldots P_m}}{L^2_{P_1\ldots P_m R}} \tag{10}$$

*Proof.* The proof will be by induction on $m$. The statement is true for $m = 0$ because

$$\left(\frac{L_Q}{L_R}\right)' = \frac{L'_Q L_R - L'_R L_Q}{L_R^2} = \frac{Q''R' - R''Q'}{L_R^2} = \frac{L_{RQ}}{L_R^2}.$$

Now suppose the statement is true for $m = k - 1$. Then for $m = k$ we have,

$$\begin{aligned}\left(\frac{L_{P_1\ldots P_k Q}}{L_{P_1\ldots P_k R}}\right)' &= \frac{L'_{P_1\ldots P_k Q} L_{P_1\ldots P_k R} - L'_{P_1\ldots P_k R} L_{P_1\ldots P_k Q}}{L^2_{P_1\ldots P_k R}} \\ &= \frac{1}{L^2_{P_1\ldots P_k R}} \left( \begin{vmatrix} P'_1 & \cdots & P_1^{(k)} & P_1^{(k+2)} \\ \vdots & & \vdots & \vdots \\ P'_k & \cdots & P_k^{(k)} & P_k^{(k+2)} \\ Q' & \cdots & Q^{(k)} & Q^{(k+2)} \end{vmatrix} L_{P_1\ldots P_k R} \right. \\ &\quad \left. - \begin{vmatrix} P'_1 & \cdots & P_1^{(k)} & P_1^{(k+2)} \\ \vdots & & \vdots & \vdots \\ P'_k & \cdots & P_k^{(k)} & P_k^{(k+2)} \\ R' & \cdots & R^{(k)} & R^{(k+2)} \end{vmatrix} L_{P_1\ldots P_k Q} \right).\end{aligned}$$

This equation can be written in terms of the $L$'s by expanding the determinants using the last column:

$$\begin{aligned}\left(\frac{L_{P_1\ldots P_k Q}}{L_{P_1\ldots P_k R}}\right)' = \; & \frac{1}{L^2_{P_1\ldots P_k R}}[(L_{P_1\ldots P_k} Q^{(k+2)} L_{P_1\ldots P_k R} \\ & -L_{P_1\ldots P_{k-1} Q} P_k^{(k+2)} L_{P_1\ldots P_k R} \\ & \vdots \\ & (-1)^k L_{P_2\ldots P_k Q} P_1^{(k+2)} L_{P_1\ldots P_k R}) \\ & - (L_{P_1\ldots P_k} R^{(k+2)} L_{P_1\ldots P_k Q} \\ & -L_{P_1\ldots P_{k-1} R} P_k^{(k+2)} L_{P_1\ldots P_k Q} \\ & \vdots \\ & (-1)^k L_{P_2\ldots P_k Q} P_1^{(k+2)} L_{P_1\ldots P_k Q})],\end{aligned}$$

so grouping the terms appropriately we obtain

$$\begin{aligned}&\left(\frac{L_{P_1\ldots P_k Q}}{L_{P_1\ldots P_k R}}\right)' \\ = \; & \frac{1}{L^2_{P_1\ldots P_k R}}(L_{P_1\ldots P_k} Q^{(k+2)} L_{P_1\ldots P_k R} - L_{P_1\ldots P_k} R^{(k+2)} L_{P_1\ldots P_k Q} \\ & -L_{P_1\ldots P_{k-1} Q} P_k^{(k+2)} L_{P_1\ldots P_k R} + L_{P_1\ldots P_{k-1} R} P_k^{(k+2)} L_{P_1\ldots P_k Q} \\ & \vdots \\ & (-1)^k L_{P_2\ldots P_k Q} P_1^{(k+2)} L_{P_1\ldots P_k R} - (-1)^k L_{P_2\ldots P_k Q} P_1^{(k+2)} L_{P_1\ldots P_k Q}). \qquad (11)\end{aligned}$$

All the terms in (11), except the first two, can be combined in pairs. We make the claim,

$$-L_{P_1\ldots P_{k-1} Q} L_{P_1\ldots P_k R} + L_{P_1\ldots P_{k-1} R} L_{P_1\ldots P_k Q} = L_{P_1\ldots P_k} L_{P_1\ldots P_{k-1} RQ}, \qquad (12)$$

with similar claims for the rest of the pairs in (11). If the claim is true then by substituting (12) in (11) we obtain an expansion for $L_{P_1\ldots P_k RQ}$ using the last column. This would then complete the proof of Lemma 4.3. To show (12) we use the induction hypothesis to obtain

$$\begin{aligned}
L_{P_1\ldots P_{k-1}R}L_{P_1\ldots P_k Q} &= L^2_{P_1\ldots P_k}L_{P_1\ldots P_{k-1}R}\left(\frac{L_{P_1\ldots P_{k-1}Q}}{L_{P_1\ldots P_k}}\right)' \\
&= L^2_{P_1\ldots P_k}L_{P_1\ldots P_{k-1}R}\left(\frac{L_{P_1\ldots P_{k-1}Q}}{L_{P_1\ldots P_{k-1}R}}\frac{L_{P_1\ldots P_{k-1}R}}{L_{P_1\ldots P_k}}\right)' \\
&= L^2_{P_1\ldots P_k-1R}L_{P_1\ldots P_k}\left(\frac{L_{P_1\ldots P_{k-1}Q}}{L_{P_1\ldots P_{k-1}R}}\right)' \\
&\quad + L^2_{P_1\ldots P_k}L_{P_1\ldots P_{k-1}Q}\left(\frac{L_{P_1\ldots P_{k-1}R}}{L_{P_1\ldots P_k}}\right)' \\
&= L_{P_1\ldots P_k}L_{P_1\ldots P_{k-1}RQ} + L_{P_1\ldots P_{k-1}Q}L_{P_1\ldots P_k R}.
\end{aligned}$$

This proves the claim in (12) and consequently Lemma 4.3. □

We are now in a position to express $J_{P_1\ldots P_n}(t_1,\ldots,t_n)$ in terms of the $L_{P_1\ldots P_m}$'s with $1 \leq m \leq n$, for $t_1 < \ldots < t_n$ and $[t_1,t_n] \subseteq I$. Let us define inductively in $k$, $1 \leq k \leq n$,

$$F_{i,1} = \frac{P_i'}{P_1'}, F_{i,k} = \frac{F'_{i,k-1}}{F'_{k,k-1}},$$

for $i$ in $k \leq i \leq n$. Then by repeated applications of Lemma 4.2 we obtain

$$\begin{aligned}
&J_{P_1\ldots P_n}(t_1,\ldots,t_n) \\
= &\prod_{i=1}^{n} P_1'(t_i)\int_{t_1}^{t_2} dx_{1,1}\ldots\int_{t_{n-1}}^{t_n} dx_{n-1,1}\prod_{i=1}^{n-1} F'_{2,1}(x_{i,1})\int_{x_{1,1}}^{x_{2,1}} dx_{1,2}\ldots \\
&\int_{x_{n-2,1}}^{x_{n-1,1}} dx_{n-2,2}\prod_{i=1}^{n-2} F'_{3,2}(x_{i,2})\int_{x_{1,2}}^{x_{2,2}} dx_{1,3}\ldots\int_{x_{n-3,2}}^{x_{n-2,2}} dx_{n-3,3}\ \ldots \\
&\ldots\ \prod_{i=1}^{2} F'_{n-1,n-2}(x_{i,n-2})\int_{x_{1,n-2}}^{x_{2,n-2}} dx_{1,n-1}F'_{n,n-1}(x_{1,n-1}),
\end{aligned} \quad (13)$$

where in the applications of Lemma 4.2 we make sure that the $F_{i,k}$ are differentiable. In fact, using Lemma 4.3, one can show that

$$F'_{i,k} = \left(\frac{L_{P_1\ldots P_{k-1}P_i}}{L_{P_1\ldots P_k}}\right)' = \frac{L_{P_1\ldots P_{k-1}}L_{P_1\ldots P_k P_i}}{L^2_{P_1\ldots P_k}} \quad (14)$$

for $k \leq i \leq n$. This would then imply that the $F_{i,k}$ are differentiable on $[t_1, tn]$ by Proposition 3.1. We prove (14) by induction on $k$. For $k = 1$

$$F'_{i,1} = \left(\frac{L_{P_i}}{L_{P_1}}\right)' = \frac{L_{P_1 P_i}}{L^2_{P_1}}.$$

If (14) is true for $k = m-1$ then for $k = m$ we have

$$\begin{aligned} F'_{i,m} &= \left(\frac{F'_{i,m-1}}{F'_{m,m-1}}\right)' = \left(\frac{L_{P_1\ldots P_{m-2}} L_{P_1\ldots P_{m-1}P_i}/L^2_{P_1\ldots P_{m-1}}}{L_{P_1\ldots P_{m-2}} L_{P_1\ldots P_m}/L^2_{P_1\ldots P_{m-1}}}\right)' \\ &= \left(\frac{L_{P_1\ldots P_{m-1}P_i}}{L_{P_1\ldots P_m}}\right)' = \frac{L_{P_1\ldots P_{m-1}} L_{P_1\ldots P_m P_i}}{L^2_{P_1\ldots P_m}}, \end{aligned}$$

where the last inequality follows by Lemma 4.3. This completes the proof of (14). We are now in a position to substitute (14) into (13) to express $J_{P_1\ldots P_n}(t_1,\ldots,t_n)$ in terms of $L_{P_1\ldots P_m}$'s with $1 \le m \le n$. Precisely

$$\begin{aligned} & J_{P_1\ldots P_n}(t_1,\ldots,t_n) \\ = & \prod_{i=1}^n L_{P_1}(t_i) \int_{t_1}^{t_2} dx_{1,1} \ldots \int_{t_{n-1}}^{t_n} dx_{n-1,1} \prod_{i=1}^{n-1} \frac{L_{P_1P_2}}{L^2_{P_1}}(x_{i,1}) \int_{x_{1,1}}^{x_{2,1}} dx_{1,2} \ldots \\ & \int_{x_{n-2,1}}^{x_{n-1,1}} dx_{n-2,2} \prod_{i=1}^{n-2} \frac{L_{P_1} L_{P_1P_2P_3}}{L^2_{P_1P_2}}(x_{i,2}) \int_{x_{1,2}}^{x_{2,2}} dx_{1,3} \ldots \int_{x_{n-3,2}}^{x_{n-2,2}} dx_{n-3,3} \ldots \\ & \ldots \int_{x_{1,n-3}}^{x_{2,n-3}} dx_{1,n-2} \int_{x_{2,n-3}}^{x_{3,n-3}} dx_{2,n-2} \prod_{i=1}^{2} \frac{L_{P_1\ldots P_{n-3}} L_{P_1\ldots P_{n-1}}}{L^2_{P_1\ldots P_{n-2}}}(x_{i,n-2}) \\ & \int_{x_{1,n-2}}^{x_{2,n-2}} dx_{1,n-1} \frac{L_{P_1\ldots P_{n-2}} L_{P_1\ldots P_n}}{L^2_{P_1\ldots P_{n-1}}}(x_{1,n-1}). \end{aligned} \tag{15}$$

To complete the proof of Proposition 4.1, we will need to make use of the consequence of the remark after Proposition 3.1, that on the interval $I$ each of the $L_{P_1\ldots P_m}$ is either positive or negative. Hence on $I$ we have

$$\begin{aligned} & |J_{P_1\ldots P_n}(t_1,\ldots,t_n)| \\ = & \prod_{i=1}^n |L_{P_1}(t_i)| \int_{t_1}^{t_2} dx_{1,1} \ldots \int_{t_{n-1}}^{t_n} dx_{n-1,1} \prod_{i=1}^{n-1} \left|\frac{L_{P_1P_2}}{L^2_{P_1}}(x_{i,1})\right| \\ & \int_{x_{1,1}}^{x_{2,1}} dx_{1,2} \ldots \int_{x_{n-2,1}}^{x_{n-1,1}} dx_{n-2,2} \prod_{i=1}^{n-2} \left|\frac{L_{P_1} L_{P_1P_2P_3}}{L^2_{P_1P_2}}(x_{i,2})\right| \quad \ldots \\ & \ldots \int_{x_{1,n-2}}^{x_{2,n-2}} dx_{1,n-1} \left|\frac{L_{P_1\ldots P_{n-2}} L_{P_1\ldots P_n}}{L^2_{P_1\ldots P_{n-1}}}(x_{1,n-1})\right|, \end{aligned} \tag{16}$$

so we can substitute the estimate from Proposition 3.1 to obtain

$$\begin{aligned} & J_{P_1\ldots P_n}(t_1,\ldots,t_n) \\ \gtrsim & \prod_{i=1}^n p_{i,j_i} \prod_{i=1}^n t_i^{j_1-1} \int_{t_1}^{t_2} dx_{1,1} \ldots \int_{t_{n-1}}^{t_n} dx_{n-1,1} \prod_{i=1}^{n-1} x_{i,1}^{j_2-j_1-1} \\ & \int_{x_{1,1}}^{x_{2,1}} dx_{1,2} \ldots \int_{x_{n-2,1}}^{x_{n-1,1}} dx_{n-2,2} \prod_{i=1}^{n-2} x_{i,2}^{j_3-j_2-1} \quad \ldots \\ & \ldots \int_{x_{1,n-2}}^{x_{2,n-2}} dx_{1,n-1} x_{1,n-1}^{j_n-j_{n-1}-1}. \end{aligned} \tag{17}$$

We finally need to bound from below the multiple integral in (17). This will be done through the following two lemmas.

**Lemma 4.4.** *With* $s_1 \leq s_2 \leq \ldots \leq s_m$*,*

$$\begin{aligned} & \int_{s_1}^{s_2} dy_{1,1} \ldots \int_{s_{m-1}}^{s_m} dy_{m-1,1} \int_{y_{1,1}}^{y_{2,1}} dy_{1,2} \ldots \int_{y_{m-2,1}}^{y_{m-1,1}} dy_{m-2,2} \quad \ldots \\ & \quad \int_{y_{1,m-2}}^{y_{2,m-2}} dy_{1,m-1} \\ = & \prod_{1 \leq q \leq m} ((q-1)!)^{-1} \prod_{1 \leq k < l \leq m} (s_l - s_k). \end{aligned} \tag{18}$$

*Proof.* For the proof of this lemma it is useful to keep in mind the following diagram which shows the ranges of the various variables in (18).

$$\begin{array}{ccccccccccc} s_1 & & s_2 & & s_3 & & \cdots & & s_{n-1} & & s_n \\ & y_{1,1} & & y_{2,1} & & y_{3,1} & & \cdots & & y_{n-1,1} & \\ & & y_{1,2} & & y_{2,2} & & \cdots & & y_{n-2,2} & & \\ & & & & & \vdots & & & & & \\ & & & & & y_{1,n-1} & & & & & \end{array}$$

We prove Lemma 4.4 by induction on $m$. For $m = 2$ we just have $s_2 - s_1 = \int_{s_1}^{s_2} dy_{1,1}$. Then, assuming (18) for $m = p-1$, we can use the Vandermonde determinant to write

$$\prod_{1 \leq k < l \leq p} (s_l - s_k) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ s_1 & s_2 & \cdots & s_p \\ s_1^2 & s_2^2 & \cdots & s_p^2 \\ \vdots & \vdots & & \vdots \\ s_1^{p-1} & s_2^{p-1} & \cdots & s_p^{p-1} \end{vmatrix}.$$

Subtracting the first column from the second, the second from the third and so on, we have

$$\begin{aligned} \prod_{1 \leq k < l \leq p} (s_l - s_k) &= \begin{vmatrix} 1 & 0 & \cdots & 0 \\ s_1 & s_2 - s_1 & \cdots & s_p - s_{p-1} \\ s_1^2 & s_2^2 - s_1^2 & \cdots & s_p^2 - s_{p-1}^2 \\ \vdots & \vdots & & \vdots \\ s_1^{p-1} & s_2^{p-1} - s_1^{p-1} & \cdots & s_p^{p-1} - s_{p-1}^{p-1} \end{vmatrix} \\ &= \begin{vmatrix} s_2 - s_1 & \cdots & s_p - s_{p-1} \\ s_2^2 - s_1^2 & \cdots & s_p^2 - s_{p-1}^2 \\ \vdots & & \vdots \\ s_2^{p-1} - s_1^{p-1} & \cdots & s_p^{p-1} - s_{p-1}^{p-1} \end{vmatrix} \\ &= \begin{vmatrix} \int_{s_1}^{s_2} dy_{1,1} & \cdots & \int_{s_{p-1}}^{s_p} dy_{p-1,1} \\ 2\int_{s_1}^{s_2} y_{1,1} dy_{1,1} & \cdots & 2\int_{s_{p-1}}^{s_p} y_{p-1,1} dy_{p-1,1} \\ \vdots & & \vdots \\ (p-1)\int_{s_1}^{s_2} y_{1,1}^{p-2} dy_{1,1} & \cdots & (p-1)\int_{s_{p-1}}^{s_p} y_{p-1,1}^{p-2} dy_{p-1,1} \end{vmatrix}, \end{aligned}$$

leading to

$$\begin{aligned}
\prod_{1\le k<l\le p}(s_l-s_k) &= (p-1)!\int_{s_1}^{s_2}dy_{1,1}\dots\int_{s_{p-1}}^{s_p}dy_{p-1,1}\begin{vmatrix} 1 & \cdots & 1\\ y_{1,1} & \cdots & y_{p-1,1}\\ \vdots & & \vdots\\ y_{1,1}^{p-2} & \cdots & y_{p-1,1}^{p-2}\end{vmatrix}\\
&= (p-1)!\int_{s_1}^{s_2}dy_{1,1}\dots\int_{s_{p-1}}^{s_p}dy_{p-1,1}\prod_{1\le k'<l'\le p-1}(y_{l',1}-y_{k',1})\\
&= (p-1)!\int_{s_1}^{s_2}dy_{1,1}\dots\int_{s_{p-1}}^{s_p}dy_{p-1,1}\prod_{q=1}^{p-1}(q-1)!\\
&\quad \int_{y_{1,1}}^{y_{2,1}}dy_{1,2}\dots\int_{y_{p-2,1}}^{y_{p-1,1}}dy_{p-2,2}\quad\dots\quad\int_{y_{1,p-2}}^{y_{2,p-2}}dy_{1,p-1},
\end{aligned}$$

proving (18) for $m=p$ and completing the proof of Lemma 4.4. □

**Lemma 4.5.** *With $0\le t_1\le t_2\le\dots\le t_n$ and $\alpha_{i,j}\in\mathbf{N}$, $1\le j\le n$ and $1\le i\le n-j+1$, we have*

$$\begin{aligned}
&\prod_{i=1}^{n}t_i^{\alpha_{i,1}}\int_{t_1}^{t_2}dx_{1,1}\dots\int_{t_{n-1}}^{t_n}dx_{n-1,1}\prod_{i=1}^{n-1}x_{i,1}^{\alpha_{i,2}}\int_{x_{1,1}}^{x_{2,1}}dx_{1,2}\dots\int_{x_{n-2,1}}^{x_{n-1,1}}dx_{n-2,2}\\
&\prod_{i=1}^{n-2}x_{i,2}^{\alpha_{i,3}}\quad\dots\quad\int_{x_{1,n-2}}^{x_{2,n-2}}dx_{1,n-1}x_{1,n-1}^{\alpha_{1,n}}\\
\gtrsim\ &\prod_{i=1}^{n}t_i^{A_i}\prod_{1\le k<l\le n}(t_l-t_k), \qquad (19)
\end{aligned}$$

*where $A_i=\sum_{r=1}^{i}\alpha_{i-r+1,r}$ and the constant involved in the $\gtrsim$ sign only depends on the $\alpha_{i,j}$ and $n$.*

*Proof.* In the proof of this lemma, it is worth having in mind the following diagram, similar to the one in Lemma 4.4, which shows not only the ranges of the variables, but also the powers that they are raised to in the integrands in (19).

$$\begin{array}{ccccccccccc}
t_1^{\alpha_{1,1}} & & t_2^{\alpha_{2,1}} & & t_3^{\alpha_{3,1}} & & \dots & & t_{n-1}^{\alpha_{n-1,1}} & & t_n^{\alpha_{n,1}}\\
& x_{1,1}^{\alpha_{1,2}} & & x_{2,1}^{\alpha_{2,2}} & & x_{3,1}^{\alpha_{3,2}} & & \dots & & x_{n-1,1}^{\alpha_{n-1,2}} & \\
& & x_{1,2}^{\alpha_{1,3}} & & x_{2,2}^{\alpha_{2,3}} & & \dots & & x_{n-2,2}^{\alpha_{n-2,3}} & & \\
& & & & & \vdots & & & & & \\
& & & & & x_{1,n-1}^{\alpha_{1,n}} & & & & &
\end{array}$$

We prove Lemma 4.5 by induction on $n$. For $n=2$, (19) becomes

$$t_1^{\alpha_{1,1}}t_2^{\alpha_{2,1}}\int_{t_1}^{t_2}x_{1,1}^{\alpha_{1,2}}dx_{1,1}\gtrsim t_1^{\alpha_{1,1}}t_2^{\alpha_{2,1}+\alpha_{1,2}}(t_2-t_1),$$

which is equivalent to

$$\int_{t_1}^{t_2}x_{1,1}^{\alpha_{1,2}}dx_{1,1}\gtrsim t_2^{\alpha_{1,2}}(t_2-t_1). \qquad (20)$$

In the case that $t_2 < At_1$ for a sufficiently large $A$, we have

$$\int_{t_1}^{t_2} x_{1,1}^{\alpha_{1,2}} dx_{1,1} \geq t_1^{\alpha_{1,2}} \int_{t_1}^{t_2} dx_{1,1} \gtrsim t_2^{\alpha_{1,2}}(t_2 - t_1).$$

Also, in the opposite case $t_2 \geq At_1$,

$$\int_{t_1}^{t_2} x_{1,1}^{\alpha_{1,2}} dx_{1,1} \sim t_2^{\alpha_{1,2}+1} - t_1^{\alpha_{1,2}+1} \gtrsim t_2^{\alpha_{1,2}+1} \geq t_2^{\alpha_{1,2}}(t_2 - t_1),$$

establishing (20) and hence proving (19) for $n = 2$. Now assuming (19) for $n = p - 1$, we have

$$\begin{aligned}
&\prod_{i=1}^{p} t_i^{\alpha_{i,1}} \int_{t_1}^{t_2} dx_{1,1} \ldots \int_{t_{p-1}}^{t_p} dx_{p-1,1} \prod_{i=1}^{p-1} x_{i,1}^{\alpha_{i,2}} \int_{x_{1,1}}^{x_{2,1}} dx_{1,2} \ldots \int_{x_{p-2,1}}^{x_{p-1,1}} dx_{p-2,2} \\
&\prod_{i=1}^{p-2} x_{i,2}^{\alpha_{i,3}} \quad \ldots \quad \int_{x_{1,p-2}}^{x_{2,p-2}} dx_{1,p-1} x_{1,p-1}^{\alpha_{1,p}} \\
\gtrsim\ &\prod_{i=1}^{p} t_i^{\alpha_{i,1}} \int_{t_1}^{t_2} dx_{1,1} \ldots \int_{t_{p-1}}^{t_p} dx_{p-1,1} \prod_{i=1}^{p-1} x_{i,1}^{B_i} \prod_{1 \leq k < l \leq p-1} (x_{l,1} - x_{k,1}), \qquad (21)
\end{aligned}$$

where $B_i = \sum_{r=1}^{i} \alpha_{i-r+1,r+1}$. Lemma 4.5 would then be proved if we showed that

$$\begin{aligned}
&\int_{t_1}^{t_2} dx_{1,1} \ldots \int_{t_{p-1}}^{t_p} dx_{p-1,1} \prod_{i=1}^{p-1} x_{i,1}^{B_i} \prod_{1 \leq k < l \leq p-1} (x_{l,1} - x_{k,1}) \\
\gtrsim\ &\prod_{i=2}^{p} t_i^{B_{i-1}} \int_{t_1}^{t_2} dx_{1,1} \ldots \int_{t_{p-1}}^{t_p} dx_{p-1,1} \prod_{1 \leq k < l \leq p-1} (x_{l,1} - x_{k,1}), \qquad (22)
\end{aligned}$$

because of Lemma 4.4 and because

$$B_{i-1} + \alpha_{i,1} = \sum_{r=1}^{i-1} \alpha_{i-r,r+1} + \alpha_{i,1} = \sum_{r=0}^{i-1} \alpha_{i-r,r+1} = A_i.$$

Inequality (22) essentially asserts that we can take the product of the monomials out of all the integrals evaluating them each time at the highest endpoint. We show (22) using an iterative procedure of which we describe the $q$'th step. After $q - 1$ steps we will have shown that

$$\begin{aligned}
&\int_{t_1}^{t_2} dx_{1,1} \ldots \int_{t_{p-1}}^{t_p} dx_{p-1,1} \prod_{i=1}^{p-1} x_{i,1}^{B_i} \prod_{1 \leq k < l \leq p-1} (x_{l,1} - x_{k,1}) \\
\gtrsim\ &\prod_{i=2}^{q} t_i^{B_{i-1}} \int_{t_1}^{t_2} dx_{1,1} \ldots \int_{t_{p-1}}^{t_p} dx_{p-1,1} \prod_{i=q}^{p-1} x_{i,1}^{B_i} \prod_{1 \leq k < l \leq p-1} (x_{l,1} - x_{k,1}).
\end{aligned}$$

Concentrating now on the $dx_{q,1}$ integration, we have

$$\begin{aligned}
&\int_{t_q}^{t_{q+1}} dx_{q,1} \prod_{i=q}^{p-1} x_{i,1}^{B_i} \prod_{1\le k<l\le p-1} (x_{l,1}-x_{k,1}) \\
= &\prod_{i=q+1}^{p-1} x_{i,1}^{B_i} \prod_{\substack{1\le k<l\le p-1\\ k,l\neq q}} (x_{l,1}-x_{k,1}) \int_{t_q}^{t_{q+1}} dx_{q,1} x_{q,1}^{B_q} \\
&\prod_{1\le k<q} (x_{q,1}-x_{k,1}) \prod_{q<l\le p-1} (x_{l,1}-x_{q,1}).
\end{aligned}$$

In the case where $t_{q+1}\le At_q$ for $A$ sufficiently large, we only have

$$\begin{aligned}
&\prod_{i=q+1}^{p-1} x_{i,1}^{B_i} \prod_{\substack{1\le k<l\le p-1\\ k,l\neq q}} (x_{l,1}-x_{k,1}) \int_{t_q}^{t_{q+1}} dx_{q,1} x_{q,1}^{B_q} \\
&\prod_{1\le k<q} (x_{q,1}-x_{k,1}) \prod_{q<l\le p-1} (x_{l,1}-x_{q,1}) \\
\gtrsim\ & t_{q+1}^{B_q} \int_{t_q}^{t_{q+1}} dx_{q,1} \prod_{i=q+1}^{p-1} x_{i,1}^{B_i} \prod_{1\le k<l\le p-1} (x_{l,1}-x_{k,1}),
\end{aligned}$$

putting us in the right position for the $(q+1)$'th step. In the opposite case $t_{q+1}>At_q$ we have

$$\begin{aligned}
&\prod_{i=q+1}^{p-1} x_{i,1}^{B_i} \prod_{\substack{1\le k<l\le p-1\\ k,l\neq q}} (x_{l,1}-x_{k,1}) \int_{t_q}^{t_{q+1}} dx_{q,1} x_{q,1}^{B_q} \\
&\prod_{1\le k<q} (x_{q,1}-x_{k,1}) \prod_{q<l\le p-1} (x_{l,1}-x_{q,1}) \\
\gtrsim\ &\prod_{i=q+1}^{p-1} x_{i,1}^{B_i} \prod_{\substack{1\le k<l\le p-1\\ k,l\neq q}} (x_{l,1}-x_{k,1}) \int_{\sqrt{A}t_q}^{t_{q+1}/\sqrt{A}} dx_{q,1} x_{q,1}^{B_q} \\
&\prod_{1\le k<q} (x_{q,1}-x_{k,1}) \prod_{q<l\le p-1} (x_{l,1}-x_{q,1}) \\
\gtrsim\ &\prod_{i=q+1}^{p-1} x_{i,1}^{B_i} \prod_{\substack{1\le k<l\le p-1\\ k,l\neq q}} (x_{l,1}-x_{k,1}) \int_{\sqrt{A}t_q}^{t_{q+1}/\sqrt{A}} dx_{q,1} x_{q,1}^{B_q+q-1} \prod_{q<l\le p-1} x_{l,1} \\
\gtrsim\ &\prod_{i=q+1}^{p-1} x_{i,1}^{B_i} \prod_{\substack{1\le k<l\le p-1\\ k,l\neq q}} (x_{l,1}-x_{k,1}) \prod_{q<l\le p-1} x_{l,1} \quad t_{q+1}^{B_q+q} \\
\gtrsim\ & t_{q+1}^{B_q} \int_{t_q}^{t_{q+1}} dx_{q,1} \prod_{i=q+1}^{p-1} x_{i,1}^{B_i} t_{q+1}^{q-1} \prod_{\substack{1\le k<l\le p-1\\ k,l\neq q}} (x_{l,1}-x_{k,1}) \prod_{q<l\le p-1} x_{l,1} \\
\gtrsim\ & t_{q+1}^{B_q} \int_{t_q}^{t_{q+1}} dx_{q,1} \prod_{i=q+1}^{p-1} x_{i,1}^{B_i} \prod_{1\le k<l\le p-1} (x_{l,1}-x_{k,1}),
\end{aligned}$$

again putting us in the right position for the $(q+1)$'th step. This iterative procedure will finish after $p-1$ steps, proving (22) and hence completing the proof of Lemma 4.5. $\square$

For the integral in (17) that we want to estimate, we have $\alpha_{k,l} = j_l - j_{l-1} - 1$. Thus

$$A_i = \sum_{r=1}^{i} \alpha_{i-r+1,r} = \sum_{r=1}^{i} j_r - j_{r-1} - 1 = j_i - i.$$

So from (17) and Lemma 4.5, we have

$$J_{P_1 \ldots P_n}(t_1, \ldots, t_n) \gtrsim \prod_{i=1}^{n} p_{i,j_i} \prod_{i=1}^{n} t_i^{j_i - i} \prod_{1 \le k < l \le n} (t_l - t_k),$$

completing the proof of Proposition 4.1.

REMARK. An analogous estimate to Proposition 4.1 holds for $P_1, \ldots P_n$ replaced by any $P_{\xi(1)}, \ldots, P_{\xi(\mu)}$ with $1 \le \mu \le n$ and $\xi$ a one-to-one function from $1, \ldots, \mu$ to $1, \ldots, n$.

In the proof of Theorem 1.1 we will perform the change of variables $t \mapsto x(t)$, $t = (t_1, \ldots t_n)$, where $x_k(t) = \sum_{i=1}^{n} P_k(t_i)$, $1 \le k \le n$. The following lemma will allow us to perform this change of variables.

**Lemma 4.6.** *If $s'_i, s''_i \in I$ with $I$ as above and $P = (P_1, \ldots, P_n)$, $s'_1 < \ldots < s'_n$, $s''_1 < \ldots < s''_n$, and*

$$\sum_{i=1}^{n} P(s'_i) = \sum_{i=1}^{n} P(s''_i)\,, \tag{23}$$

*then $s'_i = s''_i$ for all $1 \le i \le n$.*

*Proof.* The proof of this lemma makes use of Proposition 4.1 which is also used directly in the proof of Theorem 1.1. Let us assume first that for any $1 \le i, j \le n$, $s'_i \neq s''_j$. The equation

$$\sum_{i=1}^{n} P(s'_i) = \sum_{i=1}^{n} P(s''_i)\,,$$

can be rewritten as

$$\sum_{k=1}^{2n} \epsilon_k P(s_k) = 0\,,$$

where each $s_k$ is one of the $s'_i$ or the $s''_i$ such that $s_1 < \ldots < s_{2n}$ and $\epsilon_k = 1$ if $s_k \in \{s'_1, \ldots, s'_n\}$ and $\epsilon_k = -1$ if $s_k \in \{s''_1, \ldots, s''_n\}$. We observe that $\sum_{k=1}^{2n} \epsilon_k = 0$. Let $\alpha_l = \sum_{k=1}^{l} \epsilon_k$. Then $\alpha_l$ has at most $n-1$ changes of sign. Thus

$$0 = \sum_{k=1}^{2n} \epsilon_k P(s_k) = \sum_{k=1}^{2n-1} \alpha_k (P(s_k) - P(s_{k+1})) = \int_{s_1}^{s_{2n}} \phi(s) P'(s) ds$$

with $\phi(s)$ a step function. Let $\cup_{l=1}^{\mu} I_l$ be a partition of $[s_1, s_{2n}]$ into intervals on which $\phi$ is single-signed. Note that $\mu \le n$ and

$$0 = \sum_{l=1}^{\mu} \int_{I_l} \phi(s) P'(s) ds. \tag{24}$$

Hence we have

$$\begin{vmatrix} \int_{I_1} |\phi(s)|P_1'(s)ds & \cdots & \int_{I_1} |\phi(s)|P_\mu'(s)ds \\ \vdots & & \vdots \\ \int_{I_\mu} |\phi(s)|P_1'(s)ds & \cdots & \int_{I_\mu} |\phi(s)|P_\mu'(s)ds \end{vmatrix} = 0.$$

This in turn implies

$$\int_{u_1 \in I_1} \cdots \int_{u_\mu \in I_\mu} |\phi(u_1)| \ldots |\phi(u_\mu)| J_{P_1 \ldots P_\mu}(u_1, \ldots, u_\mu) du_1 \ldots du_\mu = 0. \qquad (25)$$

But by the remark after Proposition 4.1 we have that

$$J_{P_1 \ldots P_\mu}(u_1, \ldots, u_\mu) \gtrsim \prod_{i=1}^{\mu} p_{i,j_i} \prod_{i=1}^{\mu} u_i^{j_i - i} \prod_{1 \le k < l \le \mu} (u_l - u_k), \qquad (26)$$

which implies that $J_{P_1 \ldots P_\mu}(u_1, \ldots, u_\mu)$ is single signed and because of (25)

$$J_{P_1 \ldots P_\mu}(u_1, \ldots, u_\mu) \equiv 0.$$

This then contradicts (26). If we have that at least some $s_i' \neq s_j''$ for some $1 \le i, j \le n$, but there are some $s_i' = s_j''$, we can still obtain a contradiction by cancelling the corresponding $P(s_i')$'s and $P(s_j'')$'s from either side of (23) and then considering a smaller number of equations. This leaves us with the case that for each $s_i'$ there is a $s_j''$ such that $s_i' = s_j''$. Recalling though that $s_1' < \ldots < s_n'$ and $s_1'' < \ldots < s_n''$, one can realise that the only way this can happen is if $i = j$ for all $1 \le i \le n$. This completes the proof of Lemma 4.6. □

We now conclude with the proof of Theorem 1.1. To prove Theorem 1.1 we see by duality that it suffices to show

$$\|\widehat{gd\sigma}\|_{p'} \lesssim \|g\|_{q'(d\omega)}, \qquad (27)$$

where

$$d\sigma(\phi) = \int_I \phi(P(s))|L(s)|^\alpha ds$$

and

$$d\omega(\phi) = \int_I \phi(s)|L(s)|^\alpha ds,$$

with $\alpha = \frac{2}{n(n+1)}$. Now, with $gd\sigma * \ldots * gd\sigma$ denoting the n-fold convolution of $gd\sigma$ with itself, we have

$$\|\widehat{gd\sigma}\|_{p'}^n = \|\widehat{gd\sigma}^n\|_{p'/n} = \|gd\sigma \widehat{* \ldots *} gd\sigma\|_{p'/n} \le \|gd\sigma * \ldots * gd\sigma\|_r, \qquad (28)$$

where $nr' = p'$ by the Hausdorff-Young inequality. Note that because $1 \le p < \frac{n(n+2)}{n(n+2)-2}$, we have $1 \le r \le 2$. Now

$$gd\sigma * \ldots * gd\sigma(\phi) = \int_{I^n} \phi\left(\sum_{i=1}^n P(t_i)\right) \prod_{i=1}^n g(t_i)|L(t_i)|^\alpha dt,$$

where $t=(t_1,\ldots,t_n)$. For $\pi\in S_n$ a permutation of $\{1,\ldots,n\}$ and writing $x=(x_1,\ldots,x_n)$,

$$\begin{aligned} gd\sigma*\ldots*gd\sigma(\phi) &= \sum_{\pi\in S_n}\int_{\{t_{\pi(1)}<\ldots<t_{\pi(n)}\}\cap I^n}\phi\left(\sum_{i=1}^n P(t_i)\right)\prod_{i=1}^n g(t_i)|L(t_i)|^\alpha dt \\ &= \sum_{\pi\in S_n}\int_{D_\pi}\phi(x)\prod_{i=1}^n g(t_i)|L(t_i)|^\alpha\frac{1}{|J(t)|}dx\,, \end{aligned}$$

where in the second inequality we perform the change of variables

$$x_k=\sum_{i=1}^n P_k(t_i)$$

separately on each region $t_{\pi(1)}<\ldots t_{\pi(n)}$, and which is well defined in each region $t_{\pi(1)}<\ldots<t_{\pi(n)}$ by Lemma 4.6 (note the slight abuse of notation). $D_\pi$ is the image of the region $\{t_{\pi(1)}<\ldots<t_{\pi(n)}\}\cap I^n$ under this transformation and $J(t)=J_{P_1\ldots P_n}(t)$ is the Jacobian of the transformation. Hence

$$gd\sigma*\ldots*gd\sigma=\sum_{\pi\in S_n}\prod_{i=1}^n g(t_i)|L(t_i)|^\alpha\frac{1}{|J(t)|}\chi_{D_\pi}\,.$$

Therefore

$$\begin{aligned} &\|gd\sigma*\ldots*gd\sigma\|_r \\ \leq\ & \sum_{\pi\in S_n}\left\|\prod_{i=1}^n g(t_i)|L(t_i)|^\alpha\frac{1}{|J(t)|}\chi_{D_\pi}\right\|_r \\ =\ & \sum_{\pi\in S_n}\left(\int_{\{t_{\pi(1)}<\ldots<t_{\pi(n)}\}\cap I^n}\prod_{i=1}^n|g(t_i)|^r|L(t_i)|^{r\alpha}\frac{1}{|J(t)|^{r-1}}dt\right)^{\frac{1}{r}}\,, \end{aligned}$$

by changing variables back. ¿From the estimate for the Jacobian in Proposition 4.1 it follows that

$$\begin{aligned} &\|gd\sigma*\ldots*gd\sigma\|_r \\ \leq\ & \sum_{\pi\in S_n}\left(\int_{\{t_{\pi(1)}<\ldots<t_{\pi(n)}\}\cap I^n}\prod_{i=1}^n|g(t_i)|^r|L(t_i)|^{r\alpha-\frac{r-1}{n}}\prod_{k<l}|t_l-t_k|^{1-r}dt\right)^{\frac{1}{r}}\,. \end{aligned}$$

Finally we will need to use a result of M. Christ which is Proposition 2.2 in [4]. Let us state the result as it appears in [4].

**Proposition 4.7.** *If* $0\leq\gamma$ *then*

$$\int\prod_{i=1}^n f(x_i)\prod_{i<j\leq n}|x_i-x_j|^{-\gamma}dx_1\ldots dx_n\leq C\|f\|_p^n,$$

*for all* $f$*, if and only if* $\gamma<2/n$, $1\leq p<n$ *and* $p^{-1}+\gamma(n-1)/2=1$.

We need to use this proposition with $\gamma = r - 1$. One can easily check that $r - 1 < 2/n$ since $nr' = p'$ and $p < \frac{n(n+2)}{n(n+2)-2}$. Using Proposition 4.7, we obtain

$$\|gd\sigma * \ldots * gd\sigma\|_r \lesssim \left( \int (|g(t)|^r |L(t)|^{\frac{1}{n}+r(\alpha-\frac{1}{n})})^{\tilde{p}} dt \right)^{\frac{n}{\tilde{p}r}} ,$$

where

$$\frac{1}{\tilde{p}} + (r-1)\frac{n-1}{2} = 1. \tag{29}$$

By (27) and (28) we see that the required relations for (27) to hold are

$$\tilde{p}r = q' \quad \text{and} \quad \frac{\tilde{p}}{n} + r\tilde{p}\left(\frac{2}{n(n+1)} - \frac{1}{n}\right) = \frac{2}{n(n+1)} = \alpha\,.$$

This can be verified by algebraic calculations, using (29), $nr' = p'$ and $\frac{1}{q} = \frac{n(n+1)}{2}\frac{1}{p'}$.

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