Kakei, S. and Nimmo, J.J.C. and Willox, R. (2009) *Yang-Baxter maps and the discrete KP hierarchy*. Glasgow Mathematical Journal, 51 (A). pp. 107-119. ISSN 0017-0895

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Deposited on: 12 February 2010

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Glasgow Math. J. **51A** (2009) 107–119. © 2009 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089508004825. Printed in the United Kingdom

YANG–BAXTER MAPS AND THE DISCRETE KP HIERARCHY

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Abstract. We present a systematic construction of the discrete KP hierarchy in terms of Sato–Wilson-type shift operators. Reductions of the equations in this hierarchy to 1+1-dimensional integrable lattice systems are considered, and the problems that arise with regard to the symmetry algebra underlying the reduced systems as well as the ultradiscretizability of these systems are discussed. A scheme for constructing ultradiscretizable reductions that give rise to Yang–Baxter maps is explained in two explicit examples.

2000 *Mathematics Subject Classification.* 39A10, 39A12.

1. Introduction. A problem widely recognized as one of the most fundamental open problems in the theory of integrable systems is that of the relation between classical and quantum integrable systems with infinitely many degrees of freedom. Recently a possible route emerged through which a firm connection between the classical and quantum settings might be established: the *ultradiscretization* of (classical) discrete integrable systems [**8**, **22**]. Crystal bases, arising in the zero-temperature limit of quantum enveloping algebras, have been shown to play an important rôle in the description of the dynamical properties of so-called box and ball systems (BBSs) [**20**], which in turn can be obtained from 1+1-dimensional discrete integrable systems through a special limiting procedure: the *ultradiscrete limit* [**22**]. On the other hand, geometric crystals [**1**] are classical analogues of such crystals. These are such that (quantum) crystal bases can be obtained from them through a limiting procedure very much like the ultradiscrete limit. Geometric crystals have been studied extensively in connection with the combinatorial properties of BBSs [**10**, **11**], but, most importantly, they also offer prime examples of Yang–Baxter (YB) maps [**7**], i.e. of set-theoretical solutions to the YB equation [**6**, **26**]. In particular, it can be shown that the R-matrix associated to such a YB map (or with the *tropical* version thereof, obtained from the geometric crystal) in the ultradiscrete limit turns into the combinatorial R-matrix that governs the time evolution of the BBS [**8**]. This R-matrix, in turn, is directly related to the crystal associated with that BBS. The question that remains however,

is how these R-matrices are related to the time evolutions described by the (classical) discrete integrable systems, in particular discrete soliton systems, that allow for these particular ultradiscrete limits. An especially interesting problem is to understand how it is exactly that geometric crystals arise in this context, i.e. how the algebraic structure of discrete 1+1-dimensional integrable systems gives rise to that of a geometric crystal. Furthermore, one would like to be able to describe this relationship in terms of the fundamental symmetries of the discrete system, preferably in a general framework in order to cover as many examples as possible.

In this paper we shall describe a rather general method for constructing ultradiscretizable YB maps from 1+1-dimensional discrete (integrable) lattice equations. The general framework which allows us to do so is that of the discrete KP hierarchy. We therefore start by giving an explicit construction of this hierarchy and its associated linear problem, in terms of Sato–Wilson-type shift operators. We then discuss the problem of the reduction of the integrable lattice equations that make up this hierarchy to 1+1-dimensional ones, such that the underlying symmetry algebra of these lattices is of a given type. In the last section of this paper we describe the procedure that allows us to express the dynamics of such reductions as YB maps. Different types of reductions, related to the same symmetry algebra but giving rise to different YB maps, will be discussed.

2. The discrete KP hierarchy. Over the years there have been several descriptions of hierarchies of higher order lattice equations that, for the right set of continuous variables, have the equations of the KP hierarchy as their continuum limits. The 'standard' approach [**5**] would rely on introducing discrete variables–through a Miwa-type transformation [**12**] – in the KP bilinear identity, the expansion of which then yields lattice equations of an arbitrarily high order. An explicit determinantal form of the hierarchy thus obtained was given in [**17**]. A different but not entirely unrelated approach was introduced in [**14**, **15**], where such higher order lattice equations were obtained as non-linear relations that the solutions of certain linear integral equations have to satisfy. Our definition of the discrete KP hierarchy and its associated linear problem relies instead on an appropriate discretization of the original Sato approach to the continuous KP hierarchy. Similar attempts can be found in [**2**] and [**23**], but the present one is probably the first to exploit two different sets of Sato–Wilson shift operators, in close analogy to the case of the continuous Toda hierarchy [**25**].

We define the following Sato–Wilson-type operators:

$$\left.\begin{aligned} W(\ell,\mathbf{m}) &:= I + w_1 T_\ell^{-1} + w_2 T_\ell^{-2} + w_3 T_\ell^{-3} + \cdots , \\ \overline{W}(\ell,\mathbf{m}) &:= \overline{w}_0 + \overline{w}_1 T_\ell + \overline{w}_2 T_\ell^{2} + \overline{w}_3 T_\ell^{3} + \cdots \end{aligned}\right\} \tag{1}$$

in terms of functions $w_i(\ell,\mathbf{m})$, $\overline{w}_i(\ell,\mathbf{m})$, $\overline{w_0} \neq 0$ which, for the time being, only depend on $M+1$-independent variables ($M \in \mathbb{N}$), $(\ell,\mathbf{m}) = (\ell, m_1, \ldots, m_M)$; the $M \to \infty$ limit will be discussed later on. The T_ℓ operators are shift operators acting on the ℓ variable, $T_\ell f(\ell,\mathbf{m}) = f(\ell+1,\mathbf{m})$, and we define shift operators T_{m_j} acting on the m_j variable ($j = 1, \ldots, M$) analogously; I stands for the identity (shift) operator. We also define the projection of operators (1) onto their non-negative part as

$$A(T_\ell) = \sum_{n\in\mathbb{Z}} a_n T_\ell^n , \qquad [A(T_\ell)]_{\geq 0} := \sum_{n\geq 0} a_n T_\ell^n . \tag{2}$$

We then fix the dynamics of the coefficients $w_i(\ell, \mathbf{m})$ and $\overline{w}_i(\ell, \mathbf{m})$ with respect to the m_j variables, by imposing the discrete Sato equations ($^\forall j = 1 \cdots M$)

$$\begin{aligned} \left(T_{m_j} W\right)(1 - \alpha_j + \alpha_j T_\ell) &= B_j W, \\ \left(T_{m_j} \overline{W}\right)(1 - \alpha_j + \alpha_j T_\ell) &= B_j \overline{W}\,. \end{aligned} \tag{3}$$

In these equations, α_j are arbitrary constants, and the shift operators B_j are defined as

$$B_j := \left[\left(T_{m_j} W\right)(1 - \alpha_j + \alpha_j T_\ell) W^{-1}\right]_{\geq 0} \equiv \alpha_j T_\ell + (1 - \alpha_j) u_j, \tag{4}$$

in terms of a new set of dependent variables:

$$u_j = \frac{T_{m_j} \overline{w}_0}{\overline{w}_0}\,. \tag{5}$$

It should be remarked that were it not for the introduction of the second set of Sato–Wilson operators $\overline{W}$, it would be impossible to determine the constant term (w.r.t. T_ℓ) in (4) in terms of a single, well-defined variable. Conversely, the order of the operators B_j is fixed because of equation (3) for W. Moreover, it is easily seen that the lowest order term in $\overline{W}$, $\overline{w}_0$, has a natural expression in terms of Casorati determinants of arbitrary size (n),

$$\overline{w}_0 = \frac{T_\ell \tau}{\tau}\,, \qquad \tau(\ell, \mathbf{m}) = \begin{vmatrix} \phi^{(1)} & \cdots & \Delta_\ell^{n-1}\phi^{(1)} \\ \vdots & \ddots & \vdots \\ \phi^{(n)} & \cdots & \Delta_\ell^{n-1}\phi^{(n)} \end{vmatrix}\,, \tag{6}$$

defined in terms of functions $\phi^{(j)}$ that are only required to satisfy the linear equations

$$\Delta_\ell \phi = \Delta_{m_k} \phi, \tag{7}$$

for all k. The difference operators Δ used in these relations are defined as $\Delta_\ell := a(T_\ell - 1)$ and $\Delta_{m_j} := b_j(T_{m_j} - 1)$, for some lattice parameters $a, b_j \neq 0$, related to the constants α_j in the Sato equations (3) by $\alpha_j = a/b_j$. Note that the dispersion relations (7) ensure that the tau functions $\tau(\ell, \mathbf{m})$ defined in (6) remain invariant if one changes the difference operators Δ_ℓ in their entries to any of the Δ_{m_j} operators. Elementary solutions to (7), depending on an arbitrary parameter λ, take the form

$$\varphi_\lambda := \left(1 - \frac{\lambda}{a}\right)^\ell \prod_{j=1}^{M} \left(1 - \frac{\lambda}{b_j}\right)^{m_j} \tag{8}$$

and can be used to define two sets of wave functions by means of the formal expansions

$$\begin{aligned} \Psi_\lambda(\ell, \mathbf{m}) := W(\ell, \mathbf{m})\varphi_\lambda &= \left[1 + \sum_{k=1}^{+\infty} w_k \left(1 - \frac{\lambda}{a}\right)^{-k}\right]\varphi_\lambda, \\ \overline{\Psi}_\lambda(\ell, \mathbf{m}) := \overline{W}(\ell, \mathbf{m})\varphi_\lambda &= \left[\overline{w}_0 + \sum_{k=1}^{+\infty} \overline{w}_k \left(1 - \frac{\lambda}{a}\right)^{k}\right]\varphi_\lambda. \end{aligned} \tag{9}$$

As a direct consequence of the Sato equations (3), these wave functions satisfy linear equations of the type $T_{m_j}\Phi = B_j\Phi$, or equivalently,

$$\Phi = \frac{1}{a-b_j}\frac{1}{u_j}\left[a\,(T_\ell\,\Phi) - b_j\,(T_{m_j}\Phi)\right] \qquad {}^\forall j = 1\cdots M\,, \tag{10}$$

with u_j given by

$$u_j = \frac{\tau\,(T_\ell T_{m_j}\tau)}{(T_\ell\,\tau)(T_{m_j}\tau)}\,. \tag{11}$$

The compatibility conditions of the linear system (10), $(T_{m_i}B_j)B_i = (T_{m_j}B_i)B_j$, take the form of a system of non-linear difference equations ($\forall i, j = 1, \ldots, M;\ i \neq j$),

$$\left.\begin{array}{l} u_i(T_{m_i}u_j) = u_j(T_{m_j}u_i), \\ \alpha_i(1-\alpha_j)(T_{m_i}u_j) + \alpha_j(1-\alpha_i)(T_\ell\,u_i) = \alpha_j(1-\alpha_i)(T_{m_j}u_i) + \alpha_i(1-\alpha_j)(T_\ell\,u_j), \end{array}\right\} \tag{12}$$

the first equation of which is trivially satisfied if one adopts the tau-function representation (11) for the u_i. This system comprises $M(M-1)/2$ equations belonging to the discrete KP hierarchy. The tau-function representation (11) allows one to rewrite (12) in a bilinear form,

$$(b_i - b_j)(T_\ell\,\tau)(T_{m_i}T_{m_j}\tau) + (b_j - a)(T_{m_i}\tau)(T_\ell T_{m_j}\tau) + (a - b_i)(T_{m_j}\tau)(T_\ell T_{m_i}\tau) = 0, \tag{13}$$

which is of course nothing but the Hirota–Miwa (HM) equation [**9**, **12**]. (In fact, the coefficients in the HM equation (13) depend on the choice of a specific gauge for the tau function, i.e. $\tau \to C^{m_i m_j}\tau$, which leaves $\overline{w}_0$ (6) invariant). It should be noted also that both the HM equation (13) and the tau-function expression (11) for u_j are invariant under a simple gauge with $\varphi_\lambda\colon \tau \to \tau\varphi_\lambda$.

One of the most noteworthy features of the dKP hierarchy is the remarkable isotropy of the associated lattice (here, $M+1$-dimensional). If we denote the HM equation involving independent variables (ℓ, m_i, m_j) as $\mathbf{E}(\ell, m_i, m_j)$, the following statements are readily proven:

(i) The equations in $\mathbb{E} = \{\mathbf{E}(\ell, m_j, m_k)\big|\, m_j, m_k \in \{m_1, \ldots, m_M\},\ m_j \neq m_k\}$ generate all bilinear equations in $\overline{\mathbb{E}} = \{\mathbf{E}(\ell_1, \ell_2, \ell_3)\big|\ \ell_1, \ell_2, \ell_3 \in \{\ell, \mathbf{m}\},\ \ell_1 \neq \ell_2 \neq \ell_3\}$.

(ii) The equations in $\mathbb{E}$ (or $\overline{\mathbb{E}}$) generate all equations of the discrete KP hierarchy proposed in [**17**]:

$${}^\forall N = 3, \ldots, M+1\ :\qquad \begin{vmatrix} \tau_{\ell_1}\tau_{\widehat{\ell_1}} & \tau_{\ell_2}\tau_{\widehat{\ell_2}} & \cdots & \tau_{\ell_N}\tau_{\widehat{\ell_N}} \\ 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_N \\ \vdots & \vdots & & \vdots \\ a_1^{N-2} & a_2^{N-2} & \cdots & a_N^{N-2} \end{vmatrix} = 0, \tag{14}$$

where $\tau_{\ell_k} = T_{\ell_k}\tau$, $\tau_{\widehat{\ell_k}} = \left(\prod_{j\neq k} T_{\ell_j}\right)\tau$, for $(\ell_1, \ldots, \ell_n) \equiv (\ell, m_1, \ldots, m_{N-1})$ and $a_j = a$ when $j = 1$ and $a_j = b_{j-1}$ $(j \geq 2)$ otherwise.

These results basically derive from the fact that the linear system (10), taken together with the HM equation (13), is equivalent to the fully symmetric system

$$\Phi = \frac{1}{a_i - a_j} \frac{(T_{\ell_i}\tau)(T_{\ell_j}\tau)}{\tau\,(T_{\ell_i}T_{\ell_j}\tau)} \left[a_i\,(T_{\ell_i}\Phi) - a_j\,(T_{\ell_j}\Phi)\right] \qquad ({}^\forall i \neq j)\,. \tag{15}$$

That is, although the ℓ lattice direction was singled out as special in our construction, the actual set of equations $\overline{\mathbb{E}}$ the tau functions for the discrete KP hierarchy must satisfy is fully isotropic. We therefore define the discrete KP hierarchy to be the infinite set of HM equations obtained from $\overline{\mathbb{E}}$ as $M \to \infty$. Locally, i.e. in three dimensions, this infinite lattice is always governed by the HM equation (14) at $N = 3$.

Furthermore, the equations of order $M+1$ in (14) take the form $\sum_{n=1}^{M+1} \frac{\tau_{\ell_n}\,\tau_{\hat{\ell}_n}}{\prod_{k\neq n}(a_n - a_k)} = 0$, which is equivalent to the (formal) residue formula

$$\sum_{n=1}^{M+1} \operatorname*{Res}_{\lambda=a_n} \left[\tau(\mathbf{x} - \boldsymbol{\varepsilon}[\lambda^{-1}])\,\tau(\mathbf{x}' + \boldsymbol{\varepsilon}[\lambda^{-1}]) \prod_{k=1}^{M+1} (a_k - \lambda)^{-1}\right] = 0, \tag{16}$$

where $\mathbf{x} := \sum_{n=1}^{M+1} \ell_n \boldsymbol{\varepsilon}[a_n^{-1}]$ for $\boldsymbol{\varepsilon}[\zeta] \equiv (\zeta, \frac{\zeta^2}{2}, \ldots, \frac{\zeta^{M+1}}{M+1})$ and $\mathbf{x}' = \left(\prod_{n=1}^{M+1} T_{\ell_n}^{-1}\right)\mathbf{x}$. When $M \to \infty$, this relation is nothing but the KP bilinear identity

$$\oint_{\lambda=\infty} \frac{\mathrm{d}\lambda}{2\pi i}\, \tau(\mathbf{x} - \boldsymbol{\varepsilon}[\lambda^{-1}])\tau(\mathbf{x}' + \boldsymbol{\varepsilon}[\lambda^{-1}])\, e^{\sum_{n=1}^{+\infty}(x_n - x'_n)\lambda^n} = 0 \tag{17}$$

subject to the Miwa transformation [**12**]

$$\mathbf{x} \equiv (x_1, x_2, x_3, \ldots) = \mathbf{x}_0 + \sum_{n=1}^{\infty} \ell_n \boldsymbol{\varepsilon}[a_n^{-1}], \qquad \boldsymbol{\varepsilon}[\lambda^{-1}] = \left(\lambda^{-1}, \frac{\lambda^{-2}}{2}, \ldots, \frac{\lambda^{-n}}{n}, \ldots\right) \tag{18}$$

and with $\mathbf{x} - \mathbf{x}' = \sum_{n=1}^{\infty} \boldsymbol{\varepsilon}[a_n^{-1}]$, as explained in [**5**]. In that paper, the tau functions satisfying the above bilinear equations are described by means of a representation of the $gl(\infty)$ algebra in terms of free fermions, in complete analogy to the case of the continous KP hierarchy. The algebra $gl(\infty)$ can therefore be regarded the fundamental symmetry algebra underlying the discrete KP hierarchy. In fact, the invariance of the discrete KP hierarchy (and of the HM equation in particular) under Darboux transformations [**16**] is a prime example of the action of this symmetry algebra. Given a particular solution Φ to the linear system (15) corresponding to a given tau function τ_{vac}, we define the Darboux transformation of this tau function as $\tau^{\text{vac}} \to \tau^{\text{vac}}\Phi$; solutions of the linear system are transformed by the operator $G_\Phi := a(T_\ell - \frac{T_\ell \Phi}{\Phi})$; i.e. if ψ is also a solution of the linear system (15) for τ_{vac}, then $G_\Phi \psi \equiv [(T_\ell \psi)\Phi - \psi(T_\ell \Phi)]/\Phi$ will be a solution of the linear system (15) corresponding to the new tau function $\tau^{\text{vac}}\Phi$. Iteration of this procedure starting from $\tau^{\text{vac}} = 1$, yields Casorati-type tau functions as those in (6). (Note that one recovers the dispersion relations (7) from the linear system (15) if $\tau = 1$.)

Furthermore, it is easily seen that a shift in the ℓ-direction can, itself, be interpreted as a special Darboux transformation: $\tau \to (\tau \overline{\Psi}_\lambda \varphi_\lambda^{-1})|_{\lambda=a} \equiv T_\ell\, \tau$. This offers yet another perspective on the discrete KP hierarchy: one can think of the entire hierarchy as arising from a single copy of the HM equation, augmented with time evolutions

(in fact symmetries) that correspond to all Darboux transformations of that equation. A final comment concerns the characterization of the tau functions as solutions to a linear integral equation [**28**]:

$$\tau(\mathbf{x}-\boldsymbol{\varepsilon}[k^{-1}]) = \tau(\mathbf{x}) - \int_{\mathcal{C}_\lambda} \frac{\mathrm{d}\lambda}{2\pi i} \int_{\mathcal{C}_\mu} \frac{\mathrm{d}\mu}{2\pi i}\, h(\lambda,\mu) \frac{e^{\sum_{n=1}^{+\infty} x_n(\lambda^n-\mu^n)}}{k-\mu}\, \tau(\mathbf{x}-\boldsymbol{\varepsilon}[\lambda^{-1}])\,. \tag{19}$$

For a given kernel $h(\lambda,\mu)$ and for $\mathbf{x}$ as defined above, any $\tau(\mathbf{x})$ satisfying (19) will be a tau function for the discrete KP hierarchy. This offers a direct link to the construction found in [**13**], where it was shown that such linear integral equations can be used to generate linear problems and Bäcklund transformations for integrable lattice equations, including the discrete KP lattice and several of its 1+1-dimensional reductions.

3. Reductions of the discrete KP hierarchy. Soliton solutions for the discrete KP hierarchy can be expressed in the form of Casorati determinants (6), with entries that are appropriate linear combinations of the elementary solutions φ_λ to the vacuum linear problem (15) for $\tau = 1$. In particular, defining

$$\varphi_{p,q} := \prod_{n=1}^{\infty} \left(\frac{a_n - q}{a_n - p}\right)^{\ell_n} \qquad p, q \in \mathbb{C}, \tag{20}$$

one has 1- and 2-soliton solutions

$$\begin{aligned} \tau_{1-\mathrm{sol}} &= 1 + \varphi_{p,q}, \\ \tau_{2-\mathrm{sol}} &= 1 + \varphi_{p_1,q_1} + \varphi_{p_2,q_2} + \frac{(p_1-p_2)(q_1-q_2)}{(p_1-q_2)(q_1-p_2)} \varphi_{p_1,q_1}\varphi_{p_2,q_2}. \end{aligned} \tag{21}$$

Under the Miwa transformation (18) and at the limit $a_n \to \infty$, the functions $\varphi_{p,q}$ tend to $\exp \sum_{n=1}^{\infty} x_n(p^n - q^n)$, and the soliton solutions tend to those of the continuous KP hierarchy. More precisely, due to the Miwa transformation (18), for $|a_n| \gg 1$ one can write

$$T_{\ell_n}\tau = \sum_{j=0}^{\infty} a_n^{-j}[p_j(\tilde{\partial})\tau(\mathbf{x})], \tag{22}$$

where $p_j(\tilde{\partial})$ denotes for the Schur polynomial of weight j, $p_j(\partial_{x_1}, \frac{1}{2}\partial_{x_2}, \ldots, \frac{1}{n}\partial_{x_n}, \ldots)$.

It is worth emphasizing that when considering dimensional reductions of the discrete KP hierarchy, one should study both the behaviour of the equations under this natural continuum limit as well as that of their solutions. For example, the simplest possible, non-trivial reduction is the one in which the tau functions that solve the HM equation (13) have to satisfy the constraint $T_m T_n \tau = \tau$ (where for simplicity we have chosen m and n to represent the two distinct directions m_i and m_j). In this case the HM equation $\mathbf{E}(\ell, m, n)$ reduces to

$$(1-\delta)\tau(T_\ell\tau) + \delta(T_m\tau)\big(T_\ell T_m^{-1}\tau\big) = (T_\ell T_m\tau)\big(T_m^{-1}\tau\big) \tag{23}$$

with $\delta = \frac{a-c}{a-b}$ ($b \equiv b_i$, $c \equiv b_j$), and the corresponding discrete KP equation (12) yields

$$\frac{1}{(T_\ell T_m u)} - \frac{1}{u} = \delta[(T_\ell u) - (T_m u)] \tag{24}$$

in the variable $u = \frac{\tau\,(T_\ell T_m^{-1}\tau)}{(T_\ell\,\tau)(T_m^{-1}\tau)}$. This equation is often referred to as the discrete KdV (dKdV) equation, as consecutive continuum limits $a \to \infty, b \to \infty$ indeed yield the KdV equation after an appropriate change of (dependent) variables: $u = 1 - U/(2ab)$. In addition to the constraint on the tau functions, one can also impose the following reduction condition on the solutions of the linear problem (15): $T_m T_n \psi = (1 - \lambda/b)(1 - \lambda/c)\psi$, for some spectral parameter λ. The resulting linear system is

$$\left.\begin{aligned} &\left[T_\ell^2 + \left[\left(\frac{b}{a} - 1\right)\frac{1}{T_\ell T_m u} + \left(\frac{c}{a} - 1\right)(T_m u)\right] T_\ell\right]\psi = \frac{(\lambda - a)(\lambda + a - b - c)}{a^2}\,\psi, \\ &T_m \psi = \left[\frac{a}{b} T_\ell + \left(1 - \frac{a}{b}\right)\frac{1}{T_m u}\right]\psi, \end{aligned}\right\} \tag{25}$$

which yields the KdV Lax pair in the above continuum limit. However, if one takes a closer look at the behaviour of the constraint $T_m T_n \tau = \tau$ at the continuum limit, it is immediately clear that extra care is needed if one wishes to obtain non-trivial limits for the solutions to the discrete equation (24). Substituting the expansion (22) in the reduction constraint, one obtains

$$\tau = \tau + \left(\frac{1}{b} + \frac{1}{c}\right)\tau_{x_1} + \frac{1}{2}\left(\frac{1}{b^2} + \frac{1}{c^2}\right)\tau_{x_2} + \frac{1}{2}\left(\frac{1}{b} + \frac{1}{c}\right)^2 \tau_{2x_1} + \mathcal{O}(3); \tag{26}$$

i.e. unless $b + c = 0$, the tau functions for the continuous version of the reduced equation (24) will not depend on the lowest weight variable x_1 and thus will all be trivial. If on the contrary, one imposes the condition $b + c$ on the solutions of (24), the reduction condition on the continuous tau functions obtained from (26) is $\tau_{x_2} = 0$, exactly as one would expect when reducing the continuous KP equation to the KdV equation. Furthermore, for the soliton solutions to satisfy the reduction constraint, it is necessary and sufficient for the functions $\varphi_{p,q}$ to be invariant under combined shifts: $T_m T_n\, \varphi_{p,q} = \varphi_{p,q}$. This yields a condition on the spectral parameters p and q, $p + q = b + c$, which under the previous assumptions yields $p = -q$. This is of course nothing but the reduction condition required to reduce $gl(\infty)$ to the affine Lie algebra $A_1^{(1)}$, i.e. to the symmetry algebra underlying the KdV hierarchy (cf. [**4**] for a detailed discussion of such reductions). If of course, one chooses to study the discrete lattices obtained through such reductions for their own sake, i.e. if one admits the possibility that their solutions might only have trivial continuum limits, it might seem there is no compelling reason to impose any restrictions on the lattice parameters themselves, were it not for the problems the algebraic interpretation of such reductions pose. Reduction conditions (on the soliton solutions of the KP equation) of the type $p^n - q^n = \mathrm{c}\,(p - q),\ p \neq q$ for some non-zero constant c are referred to as 'pseudo-reductions', and their geometric interpretation has been clarified in [**19**]: the integrable equations found in a pseudo-reduced KP hierarchy are mere recombinations of the flows in the hierarchy one would obtain if the constant that appears in the reduction condition were exactly equal to zero (hence, in this case, mere recombinations of KdV and its higher order flows). However, in the discrete case there is no such clear-cut answer, and the question of what the underlying symmetry algebras of such 'pseudo-reduced' discrete KP hierarchies are remains an intriguing problem.

Another interesting problem is that of the classification of possible reductions of the discrete KP hierarchy, according to their corresponding reduction constraints. Imposing the above KdV-type constraint on the Sato–Wilson operator W, we obviously have

$$W(\ell, m_1 + 1, m_2 + 1) = W(\ell, m_1, m_2), \tag{27}$$

where we only indicate those independent variables that are used in the reduction and/or the reduced system (equation (24) was expressed in terms of ℓ and m_1). Note also that under this condition, the corresponding wave functions (9) automatically satisfy (25). However, retaining the same independent variables ℓ and m_1, there is a second possible reduction condition,

$$W(\ell, m_1, m_2 + 1, m_3 + 1) = W(\ell, m_1, m_2, m_3), \tag{28}$$

which now involves a fourth direction on the original discrete KP lattice. Under the above assumptions, both reductions clearly correspond to the same symmetry algebra, $A_1^{(1)}$, but as we shall see in the next section, they describe quite different dynamics.

We shall also briefly discuss the three possible reductions involving three shifts that result in reduced lattices expressible in terms of the ℓ and m_1 variables and which are related to the $A_2^{(1)}$ algebra:

$$\text{(i)} \qquad W(\ell + 1, m_1 + 1, m_2 + 1) = W(\ell, m_1, m_2), \tag{29}$$

$$\text{(ii)} \qquad W(\ell, m_1 + 1, m_2 + 1, m_3 + 1) = W(\ell, m_1, m_2, m_3), \tag{30}$$

$$\text{(iii)} \qquad W(\ell, m_1, m_2 + 1, m_3 + 1, m_4 + 1) = W(\ell, m_1, m_2, m_3, m_4). \tag{31}$$

4. BBSs and YB maps. The classical BBs due to Takahashi and Satsuma [**21**]

$$U_\ell^{t+1} = \min\left[1 - U_\ell^t, \sum_{j=\infty}^{\ell-1}\left(U_j^t - U_j^{t+1}\right)\right], \tag{32}$$

where $U_\ell^t \in \{0, 1\}$ and with boundary conditions $U_\ell^t \equiv 0$ when $|\ell| \gg 1$, is known to be related to the dKdV equation through ultradiscretization [**22**, **24**]. This procedure relies on the positivity of the (dependent) variables and parameters of a given lattice equation and involves introducing a small positive parameter (ϵ) in the solutions of these equations, which are then subjected to a limiting procedure. For example, rewriting the dKdV equation (24) as

$$\frac{1}{(T_\ell T_m u)} + \delta(T_m u) = \frac{1}{u} + \delta(T_\ell u) \tag{33}$$

and assuming positivity of the solutions $u(\ell, m)$ and δ, one can introduce a small ϵ dependence by writing $u = \exp U/\epsilon$, $\delta = \exp \Delta/\epsilon$. Taking the limit

$$\lim_{\epsilon\to 0^+} \epsilon \log\left(e^{\frac{-(T_\ell T_m U)}{\epsilon}} + e^{\frac{\Delta+(T_m U)}{\epsilon}}\right) = \lim_{\epsilon\to 0^+} \epsilon \log\left(e^{\frac{-U}{\epsilon}} + e^{\frac{\Delta+(T_\ell U)}{\epsilon}}\right), \tag{34}$$

of (33) one obtains $\max[-(T_\ell T_m U), (T_m U) + \Delta] = \max[-U, (T_\ell U) + \Delta]$. Unfortunately, since the max operation is present on both its left- and right-hand sides, this relation does not yield an evolution equation for the variable $U(\ell, m)$, and hence the

dKdV equation is not 'ultradiscretizable' in this naive sense. In fact, a rather clever separation of variables is needed to overcome this problem [**24**]. Here we wish to propose a systematic procedure for constructing ultradiscretizable 1+1-dimensional integrable lattices. First, we implement the constraint which led to the linear problem for the dKdV equation (27), $\Psi_\zeta(\ell, m_1+1, m_2+1) = \zeta\Psi_\zeta(\ell, m_1, m_2)$, by means of the auxiliary variable $\Phi_\zeta(\ell, m_1, m_2) := \Psi_\zeta(\ell, m_1, m_2+1)$. We thus obtain the 2×2 linear problem

$$\begin{aligned} T_\ell \begin{bmatrix} \Psi_\zeta \\ \Phi_\zeta \end{bmatrix} &= \begin{bmatrix} (1-\alpha_2^{-1})u & \alpha_2^{-1} \\ \alpha_1^{-1}\zeta & (1-\alpha_1^{-1})u^{-1} \end{bmatrix} \begin{bmatrix} \Psi_\zeta \\ \Phi_\zeta \end{bmatrix}, \\ T_{m_1} \begin{bmatrix} \Psi_\zeta \\ \Phi_\zeta \end{bmatrix} &= \begin{bmatrix} (1-\alpha_1)v^{-1} & \alpha_1\alpha_2^{-1} \\ \zeta & 0 \end{bmatrix} \begin{bmatrix} \Psi_\zeta \\ \Phi_\zeta \end{bmatrix}, \end{aligned} \tag{35}$$

where u is defined as before and $v^{-1} := \frac{1}{T_{m_1}u} - \frac{\alpha_1(1-\alpha_2)}{\alpha_2(1-\alpha_1)}u$. The compatibility condition of this sytem is

$$T_{m_1}u = \frac{v}{1+\delta uv}, \qquad T_\ell v = u(1+\delta uv), \tag{36}$$

with δ as before. This system obviously implies that $u = \delta^{-1}(\frac{1}{T_{m_1}u} - \frac{1}{v})$ and $T_\ell v = \frac{uv}{T_{m_1}u}$, which after a change of variables $m_1 = -t$ (i.e. $T_{m_1} \equiv T_t^{-1}$) can be rewritten as

$$u_\ell^{t+1} = \frac{1}{\delta}\left(\frac{1}{u_\ell^t} - \frac{1}{v_\ell^{t+1}}\right), \qquad v_\ell^{t+1} = \frac{u_{\ell-1}^{t+1}}{u_{\ell-1}^t} v_{\ell-1}^{t+1}. \tag{37}$$

The last equation in this system can be solved explicitly for the boundary conditions $u_\ell^t = 1$ and $v_\ell^t = \frac{1}{1-\delta}$ for $\ell \ll 0$, yielding $v_\ell^{t+1} \equiv \frac{1}{1-\delta}\prod_{j=-\infty}^{\ell-1}\frac{u_j^{t+1}}{u_j^t}$ and, most importantly,

$$u_\ell^{t+1} = \frac{1}{\delta}\frac{1}{u_\ell^t} + \left(1-\frac{1}{\delta}\right)\prod_{j=-\infty}^{\ell-1}\frac{u_j^t}{u_j^{t+1}}. \tag{38}$$

Assuming overall positivity, one can put $u_\ell^t = \exp(-U_\ell^t/\epsilon)$, $\delta = \exp(1/\epsilon) > 1$, upon which the ultradiscrete limit of (38) yields $U_\ell^{t+1} = -\max[U_\ell^t - 1, \sum_{j=-\infty}^{\ell-1}(U_j^{t+1} - U_j^t)]$ which is equivalent to the BBS (32).

This alternative form (36) of the dKdV lattice is not only crucial for the ultradiscretization to yield a meaningful (evolutive) system, but it also defines a *YB map* [**26**], i.e. a set theoretical solution to the YB equation in the sense of Drinfeld [**6**]. Following [**27**] we consider the general, spectral-parameter-dependent YB equation

$$R_{12}(\lambda_1,\lambda_2)R_{13}(\lambda_1,\lambda_3)R_{23}(\lambda_2,\lambda_3) = R_{23}(\lambda_2,\lambda_3)R_{13}(\lambda_1,\lambda_3)R_{12}(\lambda_1,\lambda_2), \tag{39}$$

where R is a map from the square of some set X to itself

$$\begin{array}{rccc} R: & X\times X & \longrightarrow & X\times X \\ & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & (x,y) & \mapsto & (f(x,y), g(x,y)) \end{array} \tag{40}$$

and R_{ij} is the corresponding action on X^n:

$$\begin{array}{cccc} R_{ij}: & X^n & \longrightarrow & X^n \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & (\ldots, x_i, \ldots, x_j, \ldots) & \mapsto & (\ldots, f(x_i, x_j), \ldots, g(x_i, x_j), \ldots). \end{array} \tag{41}$$

Solutions of this type to the YB equation are called YB maps and a plethora of examples related to classical integrable systems are known; an excellent review of YB maps, including several examples, can be found in [**27**].

Inspired by the form of (36), we define the map

$$R(\delta): \quad (u, v) \mapsto \left(\frac{v}{1+\delta uv},\ u(1+\delta uv)\right), \tag{42}$$

such that the dKdV evolution takes the form $(T_{m_1}u, T_\ell v) = R(\delta)(u, v)$. This map, as it stands, is not a YB map. It does however satisfy a YB equation when thought of as a degenerate version of the map

$$R(\kappa, \mu): \quad (u, v) \mapsto \left(\frac{v(1+\mu uv)}{1+\kappa uv}, \frac{u(1+\kappa uv)}{1+\mu uv}\right), \tag{43}$$

which is a 'symmetrized' version of the dKdV lattice, such that $R(\delta, 0) = R(\delta)$. A straightforward calculation shows that the map (43) satisfies the YB equation (39) and hence that the map $R(\delta)$ itself satisfies

$$R_{12}(\delta_1, \delta_2)R_{13}(\delta_1)R_{23}(\delta_2) = R_{23}(\delta_2)R_{13}(\delta_1)R_{12}(\delta_1, \delta_2). \tag{44}$$

The map (43) is identical to the Harrison-type map (47) in [**18**], up to rescaling of the variables. This map is in fact related to the $A_1^{(1)}$ reduction of type (28). Using Sato–Wilson operators that satisfy (28) to define wave functions Ψ_ζ, we have $\Psi_\zeta(\ell, m_1, m_2+1, m_3+1) = \zeta\Psi_\zeta(\ell, m_1, m_2, m_3)$. Introducing the auxiliary variable $\Phi_\zeta(\ell, m_1, m_2, m_3) := \Psi_\zeta(\ell, m_1, m_2+1, m_3)$ we obtain the 2×2 linear problem

$$\begin{aligned} T_\ell \begin{bmatrix} \Psi_\zeta \\ \Phi_\zeta \end{bmatrix} &= \begin{bmatrix} (1-\alpha_2^{-1})u & \alpha_2^{-1} \\ \alpha_3^{-1}\zeta & (1-\alpha_3^{-1})u^{-1} \end{bmatrix} \begin{bmatrix} \Psi_\zeta \\ \Phi_\zeta \end{bmatrix}, \\ T_{m_1} \begin{bmatrix} \Psi_\zeta \\ \Phi_\zeta \end{bmatrix} &= \begin{bmatrix} (1-\alpha_1\alpha_2^{-1})v^{-1} & \alpha_1\alpha_2^{-1} \\ \alpha_1\alpha_3^{-1}\zeta & (1-\alpha_1\alpha_3^{-1})v \end{bmatrix} \begin{bmatrix} \Psi_\zeta \\ \Phi_\zeta \end{bmatrix} \end{aligned} \tag{45}$$

the compatibility condition of which is

$$T_{m_1}u = v\frac{\alpha_1(1-\alpha_3^{-1}) + (\alpha_1\alpha_3^{-1}-1)uv}{(\alpha_1\alpha_2^{-1}-1) + \alpha_1(1-\alpha_2^{-1})uv}, \quad T_\ell v = \frac{uv}{T_{m_1}u}. \tag{46}$$

This system is equivalent to the action of the YB map (43),

$$(T_{m_1}u, T_\ell v) = R\left(\frac{\alpha_3(\alpha_2-1)}{\alpha_2(\alpha_3-1)}, \frac{\alpha_3(\alpha_1-\alpha_3)(\alpha_1-\alpha_2)}{\alpha_1^2\alpha_2(\alpha_3-1)^2}\right)(u, v), \tag{47}$$

up to rescaling of the field v: $v \to \frac{\alpha_3(\alpha_1-\alpha_2)}{\alpha_1\alpha_2(\alpha_3-1)}v$. However, not only is the discrete system obtained from this reduction associated with a YB map, but it is also ultradiscretizable for an appropriate choice of the parameters (so as to ensure positivity). In fact, its ultradiscrete limit can be shown to be equivalent to the BBS with carrier introduced in [**20**], which can be interpreted as the soliton cellular automaton that corresponds to the crystal obtained from the symmetric tensor representation of $U_q'(A_1^{(1)})$ in the limit $q \to 0$ [**11**].

The above procedure also generalizes to higher orders. If we consider the reduction (31), the wave functions obtained from such Sato–Wilson operators satisfy $\Psi_\zeta(\ell, m_1, m_2+1, m_3+1, m_4+1) = \zeta\Psi_\zeta(\ell, m_1, m_2, m_3, m_4)$. Introducing new variables $\Phi_\zeta(\ell, m_1, m_2, m_3, m_4) := \Psi_\zeta(\ell, m_1, m_2+1, m_3, m_4)$ and $\Xi_\zeta(\ell, m_1, m_2, m_3, m_4) := \Psi_\zeta(\ell, m_1, m_2+1, m_3+1, m_4)$, we obtain the following 3×3 linear problem:

$$T_\ell \begin{bmatrix} \Psi_\zeta \\ \Phi_\zeta \\ \Xi_\zeta \end{bmatrix} = \begin{bmatrix} \left(1-\alpha_2^{-1}\right)u_1 & \alpha_2^{-1} & 0 \\ 0 & (1-\alpha_3^{-1})u_2 & \alpha_3^{-1} \\ \alpha_4^{-1}\zeta & 0 & \left(1-\alpha_4^{-1}\right)u_3 \end{bmatrix} \begin{bmatrix} \Psi_\zeta \\ \Phi_\zeta \\ \Xi_\zeta \end{bmatrix}, \tag{48}$$

$$T_{m_1} \begin{bmatrix} \Psi_\zeta \\ \Phi_\zeta \\ \Xi_\zeta \end{bmatrix} = \begin{bmatrix} \left(1-\alpha_1\alpha_2^{-1}\right)v_1 & \alpha_1\alpha_2^{-1} & 0 \\ 0 & \left(1-\alpha_1\alpha_3^{-1}\right)v_2 & \alpha_1\alpha_3^{-1} \\ \alpha_1\alpha_4^{-1}\zeta & 0 & \left(1-\alpha_1\alpha_4^{-1}\right)v_3 \end{bmatrix} \begin{bmatrix} \Psi_\zeta \\ \Phi_\zeta \\ \Xi_\zeta \end{bmatrix}, \tag{49}$$

for functions u_i, v_i subject to $u_1u_2u_3 = v_1v_2v_3 = 1$. Further conditions on these variables arise from the compatibility condition of the above linear equations,

$$T_{m_1}u_j = \frac{u_j(\mu_{j+1}u_{j+1} - \nu_{j+1}v_{j+1})}{\mu_j u_j - \nu_j v_j}, \quad T_\ell v_j = \frac{v_j(T_{m_1}u_j)}{u_j} \qquad j = 1, 2, 3, \tag{50}$$

defined cyclically, $(u_4, v_4, \mu_4, \nu_4) \equiv (u_1, v_1, \mu_1, \nu_1)$, for parameters: $\mu_j = 1 - \alpha_{j+1}^{-1}$, $\nu_j = \alpha_1^{-1} - \alpha_{j+1}^{-1}$. This integrable lattice is a discretization of the Boussinesq equation and its Bäcklund transformations. Solving these equations in terms of (u_1, u_2) and (v_1, v_2), we obtain the map

$$\begin{aligned} R(\boldsymbol{\mu}, \boldsymbol{\nu}): \quad ((x_1, x_2), (y_1, y_2)) \mapsto \Bigg(&\left(\frac{x_1(\mu_2x_2 - \nu_2y_2)}{\mu_1x_1 - \nu_1y_1}, \frac{\mu_3y_1y_2 - \nu_3x_1x_2}{x_1y_1y_2(\mu_2x_2 - \nu_2y_2)}\right), \\ &\left(\frac{y_1(\mu_2x_2 - \nu_2y_2)}{\mu_1x_1 - \nu_1y_1}, \frac{\mu_3y_1y_2 - \nu_3x_1x_2}{x_1x_2y_1(\mu_2x_2 - \nu_2y_2)}\right)\Bigg), \end{aligned} \tag{51}$$

such that $R(\boldsymbol{\mu}, \boldsymbol{\nu})(\mathbf{u}, \mathbf{v}) = (T_{m_1}\mathbf{u}, T_\ell\mathbf{v})$. On the other hand, solving in terms of (u_1, u_2) and $(T_\ell v_1, T_\ell v_2)$ we obtain the map $\widetilde{R}(\boldsymbol{\mu}, \boldsymbol{\nu}): \quad \big((x_1, x_2), (y_1, y_2)\big) \mapsto$

$$\begin{aligned} \Bigg(&\left(\frac{x_1\left(\mu_2\mu_3x_1^{-1} + \mu_3\nu_1(x_1x_2)^{-1}y_1 + \nu_1\nu_2y_1y_2\right)}{\mu_1\mu_3x_2^{-1} + \mu_1\nu_2x_1y_2 + \nu_2\nu_3y_1^{-1}}, \frac{x_2\left(\mu_1\mu_3x_2^{-1} + \mu_1\nu_2x_1y_2 + \nu_2\nu_3y_1^{-1}\right)}{\mu_1\mu_2x_1x_2 + \mu_2\nu_3x_2(y_1y_2)^{-1} + \nu_1\nu_3y_2^{-1}}\right), \\ &\left(\frac{y_1\left(\mu_1\mu_3x_2^{-1} + \mu_1\nu_2x_1y_2 + \nu_2\nu_3y_1^{-1}\right)}{\mu_2\mu_3x_1^{-1} + \mu_3\nu_1(x_1x_2)^{-1}y_1 + \nu_1\nu_2y_1y_2}, \frac{y_2\left(\mu_1\mu_2x_1x_2 + \mu_2\nu_3x_2(y_1y_2)^{-1} + \nu_1\nu_3y_2^{-1}\right)}{\mu_1\mu_3x_2^{-1} + \mu_1\nu_2x_1y_2 + \nu_2\nu_3y_1^{-1}}\right)\Bigg). \end{aligned} \tag{52}$$

The map R, taken together with $\widetilde{R}$, satisfies a 'twisted' YB equation,

$$R_{12}(\lambda_1, \lambda_2)R_{13}(\lambda_1, \lambda_3)\widetilde{R}_{23}(\lambda_2, \lambda_3) = \widetilde{R}_{23}(\lambda_2, \lambda_3)R_{13}(\lambda_1, \lambda_3)R_{12}(\lambda_1, \lambda_2), \tag{53}$$

which is equivalent to the 3D consistency condition for the lattice (50), as used in the context of quad-graphs [**3**]. However, the dual map $\widetilde{R}$ itself turns out to be a genuine YB map, and the YB equation it satisfies is again essentially equivalent to the 3D consistency for the lattice equations at hand. Note as well that both YB maps are indeed ultradiscretizable for a suitable choice of the parameters.

The lattice equations and corresponding YB maps obtained from the constraint (30) can be treated as a degenerate case of the present one. However, difficulties arise when one tries to interpret the lattice obtained from (29), and it is not clear if it corresponds to a YB map at all. Interestingly enough, the lattices that do correspond to YB maps bear no obvious relation to those discussed in [**18**] in relation to quad-graphs. These problems, as well as the study of the BBSs related to the above YB maps deserve further attention.

ACKNOWLEDGEMENTS. S. K. wishes to acknowledge support from the Japan Society for the Promotion of Science (JSPS) through a Grant-In-Aid for Scientific Research (no. 19540228); R. W. also acknowledges support by JSPS through a Grant-in-Aid (no. 18540204).

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