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# A NON-COMMUTATIVE SEMI-DISCRETE TODA EQUATION AND ITS QUASI-DETERMINANT SOLUTIONS

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**Abstract.** A non-commutative version of the semi-discrete Toda equation is considered. A Lax pair and its Darboux transformations and binary Darboux transformations are found and they are used to construct two families of quasi-determinant solutions.

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**1. Introduction.** Recently, there has been a lot of interest in non-commutative versions of some well-known soliton equations, such as the KP equation, the KdV equation, the Hirota–Miwa equation, the modified KP equation and the two-dimensional Toda lattice [1,5–9,12,14,16,18,19,21–23]. There are a number of reasons for this lack of commutativity. For example, the variables might be square matrices or quaternions and so on. Another natural way in which the variables fail to commute is because of a quantization of the phase space resulting in the normal product being replaced by a Moyal product. In the approach taken here, it is not necessary to specify the reason for the lack of commutivity. Often, the non-commutative version is obtained simply by assuming that the coefficients in the Lax pair of the commutative equation do not commute.

Quasi-determinants [2–4] play the almost similar role in non-commutative algebra as determinants do in standard, commutative algebra. They arise in many situations where determinants appear in commutative algebra and this suggests that they are a very natural structure to use when working with non-commutative integrable systems. In particular, determinants are ubiquitous as solutions of commutative integrable systems and one way to obtain these is through the use of Darboux or binary Darboux transformations. Quasi-determinant solutions of non-commutative versions of these integrable systems arise in the same way. The reader is referred to the original papers [2–4] for a detailed and general treatment of quasi-determinants (see also [14] for a summary of the key results used in the current paper).

The semi-discrete Toda equation

$$\frac{d}{dt} \log \left( \frac{v_n^{k+1}}{v_n^k} \right) = v_{n+1}^{k+1} + v_{n-1}^k - v_n^k - v_n^{k+1}, \quad (1.1)$$

was first considered in [11]. Some other related results were presented in [13,20,24].

If one writes

$$v_n^k = \frac{\tau_{n+1}^{k+1} \tau_{n-1}^k}{\tau_n^{k+1} \tau_n^k},$$

and  $\tau_n^k$  satisfies the Hirota bilinear equation

$$D_t \tau_n^{k+1} \cdot \tau_n^k - \tau_{n+1}^{k+1} \tau_{n-1}^k + \tau_n^{k+1} \tau_n^k = 0, \quad (1.2)$$

then  $v_n^k$  satisfies (1.1). In (1.2)  $D_t$  denotes Hirota's bilinear operator defined by [10]

$$D_t^n a(t) \cdot b(t) \equiv \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n a(t) b(t')|_{t'=t}.$$

This paper is concerned with the following non-commutative generalization of (1.1):

$$v_n^k u_n^k = u_n^{k+1} v_{n+1}^k, \quad (1.3)$$

$$v_{n+1,t}^k + u_{n+1}^{k+1} - u_n^k = 0, \quad (1.4)$$

in which  $u_n$  and  $v_n$  and their derivatives do not commute in general. By introducing new variable  $X_n^k$  where

$$u_n^k = X_n^k (X_{n+1}^k)^{-1}, \quad v_n^k = X_n^{k+1} (X_n^k)^{-1},$$

(1.3) is satisfied identically and (1.4) becomes

$$(X_{n+1}^{k+1} (X_{n+1}^k)^{-1})_t + X_{n+1}^{k+1} (X_{n+2}^{k+1})^{-1} - X_n^k (X_{n+1}^k)^{-1} = 0. \quad (1.5)$$

In the commutative reduction, it is easy to show, by writing  $X_n^k = \tau_{n-1}^k / \tau_n^k$ , that (1.5) becomes

$$\begin{aligned} & \tau_{n+1}^{k+1} \tau_{n+1}^k (D_t \tau_n^{k+1} \cdot \tau_n^k - \tau_{n+1}^{k+1} \tau_{n-1}^k + \tau_n^{k+1} \tau_n^k) \\ & - \tau_n^k \tau_n^{k+1} (D_t \tau_{n+1}^{k+1} \cdot \tau_{n+1}^k - \tau_{n+2}^{k+1} \tau_n^k + \tau_{n+1}^{k+1} \tau_{n+1}^k) = 0, \end{aligned}$$

which is satisfied whenever the bilinear semi-discrete Toda equation (1.2) is satisfied, thus verifying that indeed (1.3) and (1.4) is a non-commutative generalization of (1.1). For the rest of this paper, we will refer to (1.5) as the non-commutative semi-discrete Toda equation.

The main results of this paper are to show that this non-commutative system is integrable in the sense that it has a Lax pair and the associated Darboux and binary Darboux transformations may be iterated to construct families of exact solutions. We show how these solutions may be expressed in terms of quasi-determinants of two different types.

The paper is organized as follows. In Section 2, we present a Lax pair and its Darboux transformation and describe how iteration of this transformation gives quasi-casoratian solutions. An adjoint linear problem and binary Darboux transformations are discussed in Section 3, and quasi-grammian solutions are obtained. Conclusions are given in Section 4.

**2. Quasi-casoratian solutions obtained by Darboux transformations.** The non-commutativesemi-discrete Toda lattice (1.5) has Lax pair

$$\phi_{n,t}^k = X_n^k (X_{n+1}^k)^{-1} \phi_{n+1}^k, \tag{2.1}$$

$$\phi_{n+1}^{k+1} = \phi_n^k - X_{n+1}^{k+1} (X_{n+1}^k)^{-1} \phi_{n+1}^k. \tag{2.2}$$

This was obtained by discretizing the Lax pair of the non-Abelian Toda lattice [15,17] and simplifying the Lax pair of the semi-discrete Toda equation with self-consistent sources [20].

Let  $\theta_{n,i}^k, i = 1, \dots, N$  be a particular set of eigenfunctions of the linear system and from these define the row vector  $\Theta_n^k = (\theta_{n,1}^k, \dots, \theta_{n,N}^k)$ . The Darboux transformation, determined by the particular solution  $\theta_n^k$ , for the non-commutative semi-discrete Toda lattice is

$$\begin{aligned} \tilde{\phi}_n^k &= \phi_n^k - \theta_n^k (\theta_{n+1}^k)^{-1} \phi_{n+1}^k, \\ \tilde{X}_n^k &= \theta_n^k (\theta_{n+1}^k)^{-1} X_{n+1}^k. \end{aligned}$$

This transformation may be iterated by defining

$$\phi_n^k[l+1] = \phi_n^k[l] - \theta_n^k[l] (\theta_{n+1}^k[l])^{-1} \phi_{n+1}^k[l], \tag{2.3}$$

$$X_n^k[l+1] = \theta_n^k[l] (\theta_{n+1}^k[l])^{-1} X_{n+1}^k[l], \tag{2.4}$$

where  $\phi_n^k[1] = \phi_n^k, X_n^k[1] = X_n^k$  and

$$\theta_n^k[l] = \phi_n^k[l] |_{\phi_n^k \rightarrow \theta_n^k}. \tag{2.5}$$

In particular,

$$\phi_n^k[2] = \phi_n^k - \theta_{n,1}^k (\theta_{n+1,1}^k)^{-1} \phi_{n+1}^k, \tag{2.6}$$

$$X_n^k[2] = \theta_{n,1}^k (\theta_{n+1,1}^k)^{-1} X_{n+1}^k. \tag{2.7}$$

In what follows, we shall show by induction that the results of  $N$  repeated Darboux transformations,  $\phi_n^k[N+1]$  and  $X_n^k[N+1]$ , can be expressed in closed form as quasi-determinants

$$\phi_n^k[N+1] = \begin{vmatrix} \Theta_n^k & \boxed{\phi_n^k} \\ \Theta_{n+1}^k & \phi_{n+1}^k \\ \vdots & \vdots \\ \Theta_{n+N}^k & \phi_{n+N}^k \end{vmatrix}, \quad X_n^k[N+1] = (-1)^N \begin{vmatrix} \Theta_n^k & \boxed{0} \\ \Theta_{n+1}^k & 0 \\ \vdots & \vdots \\ \Theta_{n+N}^k & 1 \end{vmatrix} X_{n+N}^k. \tag{2.8}$$

The initial case  $N = 1$  follows directly from (2.6) and (2.7). Also, using the non-commutative Jacobi identity, row homological relations and definition of quasi-Plücker

coordinates (see (2.3)–(2.5) in [14] for example), we have

$$\begin{aligned}
& \phi_n^k[N+2] \\
&= \phi_n^k[N+1] - \theta_n^k[N+1]\theta_{n+1}^k[N+1]^{-1}\phi_{n+1}^k[N+1] \\
&= \begin{vmatrix} \Theta_n^k & \boxed{\phi_n^k} \\ \Theta_{n+1}^k & \phi_{n+1}^k \\ \vdots & \vdots \\ \Theta_{n+N}^k & \phi_{n+N}^k \end{vmatrix} - \begin{vmatrix} \Theta_n^k & \boxed{\theta_{n,N+1}^k} \\ \Theta_{n+1}^k & \theta_{n+1,N+1}^k \\ \vdots & \vdots \\ \Theta_{n+N}^k & \theta_{n+N,N+1}^k \end{vmatrix} \begin{vmatrix} \Theta_{n+1}^k & \boxed{\theta_{n+1,N+1}^k} \\ \Theta_{n+2}^k & \theta_{n+2,N+1}^k \\ \vdots & \vdots \\ \Theta_{n+N+1}^k & \theta_{n+N+1,N+1}^k \end{vmatrix}^{-1} \begin{vmatrix} \Theta_{n+1}^k & \boxed{\phi_{n+1}^k} \\ \Theta_{n+2}^k & \phi_{n+2}^k \\ \vdots & \vdots \\ \Theta_{n+N+1}^k & \phi_{n+N+1}^k \end{vmatrix} \\
&= \begin{vmatrix} \Theta_n^k & \boxed{\phi_n^k} \\ \Theta_{n+1}^k & \phi_{n+1}^k \\ \vdots & \vdots \\ \Theta_{n+N}^k & \phi_{n+N}^k \end{vmatrix} - \begin{vmatrix} \Theta_n^k & \boxed{\theta_{n,N+1}^k} \\ \Theta_{n+1}^k & \theta_{n+1,N+1}^k \\ \vdots & \vdots \\ \Theta_{n+N}^k & \theta_{n+N,N+1}^k \end{vmatrix} \begin{vmatrix} \Theta_{n+1}^k & \theta_{n+1,N+1}^k \\ \Theta_{n+2}^k & \theta_{n+2,N+1}^k \\ \vdots & \vdots \\ \Theta_{n+N+1}^k & \boxed{\theta_{n+N+1,N+1}^k} \end{vmatrix}^{-1} \begin{vmatrix} \Theta_{n+1}^k & \phi_{n+1}^k \\ \Theta_{n+2}^k & \phi_{n+2}^k \\ \vdots & \vdots \\ \Theta_{n+N+1}^k & \boxed{\phi_{n+N+1}^k} \end{vmatrix} \\
&= \begin{vmatrix} \Theta_n^k & \theta_{n,N+1}^k & \boxed{\phi_n^k} \\ \Theta_{n+1}^k & \theta_{n+1,N+1}^k & \phi_{n+1}^k \\ \vdots & \vdots & \vdots \\ \Theta_{n+N+1}^k & \theta_{n+N+1,N+1}^k & \phi_{n+N+1}^k \end{vmatrix}.
\end{aligned}$$

In a similar way, we also have

$$\begin{aligned}
& X_n^k[N+2] = \theta_n^k[N+1]\theta_{n+1}^k[N+1]^{-1}X_{n+1}^k[N+1] \\
&= (-1)^N \begin{vmatrix} \Theta_n^k & \boxed{\theta_{n,N+1}^k} \\ \Theta_{n+1}^k & \theta_{n+1,N+1}^k \\ \vdots & \vdots \\ \Theta_{n+N}^k & \theta_{n+N,N+1}^k \end{vmatrix} \begin{vmatrix} \Theta_{n+1}^k & \boxed{\theta_{n+1,N+1}^k} \\ \Theta_{n+2}^k & \theta_{n+2,N+1}^k \\ \vdots & \vdots \\ \Theta_{n+N+1}^k & \theta_{n+N+1,N+1}^k \end{vmatrix}^{-1} \begin{vmatrix} \Theta_{n+1}^k & \boxed{0} \\ \Theta_{n+2}^k & 0 \\ \vdots & \vdots \\ \Theta_{n+N+1}^k & 1 \end{vmatrix} X_{n+N+1}^k \\
&= (-1)^N \begin{vmatrix} \Theta_n^k & \boxed{\theta_{n,N+1}^k} \\ \Theta_{n+1}^k & \theta_{n+1,N+1}^k \\ \vdots & \vdots \\ \Theta_{n+N}^k & \theta_{n+N,N+1}^k \end{vmatrix} \begin{vmatrix} \Theta_{n+1}^k & \theta_{n+1,N+1}^k \\ \vdots & \vdots \\ \Theta_{n+N}^k & \theta_{n+N,N+1}^k \\ \Theta_{n+N+1}^k & \boxed{\theta_{n+N+1,N+1}^k} \end{vmatrix}^{-1} X_{n+N+1}^k \\
&= (-1)^{N+1} \begin{vmatrix} \Theta_n^k & \theta_{n,N+1}^k & \boxed{0} \\ \Theta_{n+1}^k & \theta_{n+1,N+1}^k & 0 \\ \vdots & \vdots & \vdots \\ \Theta_{n+N+1}^k & \theta_{n+N+1,N+1}^k & 1 \end{vmatrix} X_{n+N+1}^k.
\end{aligned}$$

This proves the inductive step and the proof is complete.

**3. Quasi-grammian solutions obtained by binary Darboux transformations.** The linear equations (2.1) and (2.2) have the formal adjoint

$$-\psi_{n,t}^k = u_{n-1}^k \dagger \psi_{n-1}^k, \tag{3.1}$$

$$\psi_{n+1}^{k+1} = \psi_n^k + v_n^{k\dagger} \psi_n^{k+1}. \tag{3.2}$$

As always with discrete Lax equations, some care is needed in defining their adjoint. The key point is that they should be chosen in such a way that the compatibility condition for (3.1) and (3.2) is identical to that of (2.1) and (2.2). In other words, (3.1) and (3.2) also form a Lax pair for (1.1).

Following the standard construction of a binary Darboux transformation, one introduces a potential  $\Omega_n^k = \Omega(\phi_n^k, \psi_n^k)$  satisfying the following three conditions:

$$\Omega_{n,t}^k = -\psi_n^{k\dagger} u_n^k \phi_{n+1}^k, \tag{3.3}$$

$$\Omega_n^{k+1} - \Omega_n^k = \psi_{n+1}^{k+1\dagger} \phi_n^k, \tag{3.4}$$

$$\Omega_{n+1}^k - \Omega_n^k = -\psi_{n+1}^k \dagger \phi_{n+1}^k. \tag{3.5}$$

A binary Darboux transformation is then defined by

$$\phi_n^k[N+1] = \phi_n^k[N] - \theta_n^k[N] \Omega(\theta_n^k[N], \rho_n^k[N])^{-1} \Omega(\phi_n^k[N], \rho_n^k[N]), \tag{3.6}$$

$$\psi_n^k[N+1] = \psi_n^k[N] - \rho_n^k[N] \Omega(\theta_{n-1}^k[N], \rho_{n-1}^k[N])^{-1} \Omega(\theta_{n-1}^k[N], \psi_{n-1}^k[N])^\dagger, \tag{3.7}$$

$$X_n^k[N+1] = (I + \theta_n^k[N] \Omega(\theta_n^k[N], \rho_n^k[N])^{-1} \rho_n^k[N]^\dagger) X_n^k[N], \tag{3.8}$$

where  $\phi_n^k[1] = \phi_n^k$ ,  $\psi_n^k[1] = \psi_n^k$ ,  $X_n^k[1] = X_n^k$  and

$$\theta_n^k[N] = \phi_n^k[N] |_{\phi_n^k \rightarrow \theta_n^k}, \quad \rho_n^k[N] = \psi_n^k[N] |_{\psi_n^k \rightarrow \rho_n^k}. \tag{3.9}$$

Using the notation  $\Theta_n^k = (\theta_{n,1}^k, \dots, \theta_{n,N}^k)$  and  $P_n^k = (\rho_{n,1}^k, \dots, \rho_{n,N}^k)$ , it is easy to prove by induction that for  $N \geq 1$ ,

$$\phi_n^k[N+1] = \left| \begin{array}{cc} \Omega(\Theta_n^k, P_n^k) & \Omega(\phi_n^k, P_n^k) \\ \Theta_n^k & \boxed{\phi_n^k} \end{array} \right|, \tag{3.10}$$

$$\psi_n^k[N+1] = \left| \begin{array}{cc} \Omega(\Theta_{n-1}^k, P_{n-1}^k)^\dagger & \Omega(\Theta_{n-1}^k, \psi_{n-1}^k)^\dagger \\ P_n^k & \boxed{\psi_n^k} \end{array} \right| \tag{3.11}$$

and

$$\Omega(\phi_n^k[N+1], \psi_n^k[N+1]) = \left| \begin{array}{cc} \Omega(\Theta_n^k, P_n^k) & \Omega(\phi_n^k, P_n^k) \\ \Omega(\Theta_n^k, \psi_n^k) & \boxed{\Omega(\phi_n^k, \psi_n^k)} \end{array} \right|. \tag{3.12}$$

We may thus after  $N$  binary Darboux transformations obtain

$$X_n^k[N+1] = - \left| \begin{array}{cc} \Omega(\Theta_n^k, P_n^k) & P_n^{k\dagger} \\ \Theta_n^k & \boxed{-I} \end{array} \right| X_n^k. \tag{3.13}$$

In fact, we can prove the above results by induction. Using (3.10)–(3.12), we have

$$\begin{aligned} X_n^k[N+2] &= (I + \Theta_n^k[N+1]\Omega(\Theta_n^k[N+1], P_n^k[N+1])^{-1}P_n^k[N+1]^\dagger)X_n^k[N+1] \\ &= - \left( I + \left| \begin{array}{c} \Omega(\Theta_n^k, P_n^k) \quad \Omega(\theta_{n,N+1}^k, P_n^k) \\ \Theta_n^k \quad \boxed{\theta_{n,N+1}^k} \end{array} \right| \left| \begin{array}{c} \Omega(\Theta_n^k, P_n^k) \quad \Omega(\theta_{n,N+1}^k, P_n^k) \\ \Omega(\Theta_n^k, \rho_{n,N+1}^k) \quad \boxed{\Omega(\theta_{n,N+1}^k, \rho_{n,N+1}^k)} \end{array} \right| \right)^{-1} \\ &\quad \left| \begin{array}{c} \Omega(\Theta_{n-1}^k, P_{n-1}^k) \quad P_n^{k\dagger} \\ \Omega(\Theta_{n-1}^k, \rho_{n-1,N+1}^k) \quad \boxed{\rho_{n,N+1}^k}^\dagger \end{array} \right| \left| \begin{array}{c} \Omega(\Theta_n^k, P_n^k) \quad P_n^{k\dagger} \\ \Theta_n^k \quad \boxed{-I} \end{array} \right| X_n^k. \end{aligned}$$

Noticing that

$$\begin{aligned} &\left| \begin{array}{c} \Omega(\Theta_{n-1}^k, P_{n-1}^k) \quad P_n^{k\dagger} \\ \Omega(\Theta_{n-1}^k, \rho_{n-1,N+1}^k) \quad \boxed{\rho_{n,N+1}^k}^\dagger \end{array} \right| \left| \begin{array}{c} \Omega(\Theta_n^k, P_n^k) \quad P_n^{k\dagger} \\ \Theta_n^k \quad \boxed{-I} \end{array} \right| \\ &= -(\rho_{n,N+1}^k{}^\dagger - \Omega(\Theta_{n-1}^k, \rho_{n-1,N+1}^k)\Omega(\Theta_{n-1}^k, P_{n-1}^k)^{-1}P_n^{k\dagger})(I + \Theta_n^k\Omega(\Theta_n^k, P_n^k)^{-1}P_n^{k\dagger}) \\ &= -\rho_{n,N+1}^k{}^\dagger + \Omega(\Theta_{n-1}^k, \rho_{n-1,N+1}^k)\Omega(\Theta_{n-1}^k, P_{n-1}^k)^{-1}P_n^{k\dagger} \\ &\quad + (\Omega(\Theta_n^k, \rho_{n,N+1}^k) - \Omega(\Theta_{n-1}^k, \rho_{n-1,N+1}^k))\Omega(\Theta_n^k, P_n^k)^{-1}P_n^{k\dagger} \\ &\quad + \Omega(\Theta_{n-1}^k, \rho_{n-1,N+1}^k)\Omega(\Theta_{n-1}^k, P_{n-1}^k)^{-1}(\Omega(\Theta_{n-1}^k, P_{n-1}^k) - \Omega(\Theta_n^k, P_n^k))\Omega(\Theta_n^k, P_n^k)^{-1}P_n^{k\dagger} \\ &= -\rho_{n,N+1}^k{}^\dagger + \Omega(\Theta_n^k, \rho_{n,N+1}^k)\Omega(\Theta_n^k, P_n^k)^{-1}P_n^{k\dagger} \\ &= - \left| \begin{array}{c} \Omega(\Theta_n^k, P_n^k) \quad P_n^{k\dagger} \\ \Omega(\Theta_n^k, \rho_{n,N+1}^k) \quad \boxed{\rho_{n,N+1}^k}^\dagger \end{array} \right|, \end{aligned}$$

it follows that

$$X_n^k[N+2] = - \left| \begin{array}{c} \Omega(\Theta_n^k, P_n^k) \quad \Omega(\theta_{n,N+1}^k, P_n^k) \quad P_n^{k\dagger} \\ \Omega(\Theta_n^k, \rho_{n,N+1}^k) \quad \Omega(\theta_{n,N+1}^k, \rho_{n,N+1}^k) \quad \rho_{n,N+1}^k{}^\dagger \\ \Theta_n^k \quad \theta_{n,N+1}^k \quad \boxed{-I} \end{array} \right| X_n^k,$$

as required.

**4. Conclusions.** In this paper, we have described a non-commutative version of the semi-discrete Toda equation. We have obtained quasi-casoratian and quasi-grammian solutions by means of discrete Darboux transformations and binary Darboux transformations, respectively. We have given an inductive proof of the iterated Darboux transformations and solutions by using quasi-determinant identities. Since we have not at any point specified the nature of non-commutativity, the results obtained here are valid whatever the reason for non-commutativity is.

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