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Collocation method based on modified cubic B-spline for option pricing models

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Abstract. A collocation method based on modified cubic B-spline functions has been developed for the valuation of European, American and barrier options of a single asset. The new approach contains discretization of temporal derivative using finite difference approximation of and approximating the option price with the modified B-spline functions. Stability of this method has been discussed and it is shown that it is unconditionally stable. The efficiency of the proposed method is tested by different examples.
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1. Introduction

The past few decades have witnessed a revolution in the trading of derivative securities in the world financial markets. A derivative security, or contingent claim, is a financial contract whose value at its expiry date T is completely determined by the prices of an underlying asset in a fixed range of times within the interval $[0, T]$.
One of important financial derivatives is a contract between two parties about trading the asset at a certain future time called the option. One party is the writer, for example, a bank, that fixes the terms of the option contract and sells the option. The other party is the holder, that purchases the option, paying the market price, which is called the premium. Options have a limited life time, the maturity date T fixes the time horizon. On this date the rights of the holder expire, and for later times ($t > T$) the option is worthless. There are two basic types of the option: call and put. The call option gives the holder the right to buy the underlying asset for an agreed price E by the date T. The put option gives the holder the right to sell the underlying asset for the price E by the date T. The previously agreed price E of the contract is called the strike or exercise price [38]. The option is called a European option if the exercise is only permitted at the expiration date T, and an American option if it can be exercised at any time up to and including the expiration date t. There are also different kinds of barrier options. In general, such contracts specify various payoffs if the underlying asset price reaches certain levels. For example, an up-and-out call option is like a standard call provided that the underlying asset price remains below a barrier level for the duration of the contract. Should the barrier

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level be reached, the contract is canceled and the options payoff will become zero, i.e., the option will be worthless [23].

Option pricing theory has made a great leap forward since the development of the Black-Scholes option pricing model by Fischer Black and Myron Scholes in [1], and by Robert Merton in [28]. In an idealized financial market, the prices of European, American and Barrier options are governed by the Black-Scholes equation.

We consider the dividend-free Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{1}$$

where $V(S,t)$ is the option price, r the risk free interest rate, σ the volatility and S the stock price, associated with a final condition $V(S,T) = V_g(S)$ and boundary conditions of the form

$$V_a(a,t) = \alpha(t), \quad V_b(b,t) = \beta(t), \tag{2}$$

where T is the expiry time, and we consider a truncated domain $\Omega = [a,b] \times [0,T]$. Following [14], a simple transformation $S = e^x$ changes the Black-Scholes equation into a constant-coefficient partial differential equation in the domain $\Omega = [x \times t]$, $x \in [\log(a), \log(b)]$, $t \in [0,T]$,

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial u}{\partial x} - ru = 0, \tag{3}$$

with the final condition $u(x,T) = g(x)$ and boundary conditions

$$u(\log(a),t) = \alpha(t), \quad u(\log(b),t) = \beta(t). \tag{4}$$

Numerical solutions to several mathematical models arising in financial economics for the valuation of the European options on different types of assets are considered. Brennan and Schwartz [2] were the first to describe finite-difference methods for option pricing. Geske and Shastri [13] compared the efficiency of various finite-difference and other numerical methods for option pricing. Vazquez [34] presented an upwind scheme for solving the backward parabolic partial differential equation problem in the case of European options. Chawla et al. [7] presented high-accuracy finite-difference methods for the Black-Scholes equation, in which they employed the fourth-order L-stable time integration schemes (LSIMP) developed in Chawla et al. [6] and the well-known Numerov method for discretization in the asset direction. Company et al. [10] constructed a finite difference scheme and numerical analysis of its solution for a nonlinear Black-Scholes equation modeling stock option prices in the realistic case when transaction costs arising in the hedging of portfolios are taken into account. The most common numerical method for pricing American options are binomial methods ([11]), where the price process of the underlying asset is approximated by a binomial lattice (see [30]). Another approach to computing the expectation (as mentioned in [30]) is to represent the price as the sum of the European option price and an early exercise premium (see [24, 19, 5]) using an integral equation. Cho et al. [8] considered a free boundary problem arising in the pricing of an American call option. The free boundary represents the optimal exercise price as a function of time before a maturity date. They developed a parameter estimation technique to obtain the optimal exercise curve of an American call option and its price.

Kadalbajoo et al. [20] applied the uniform cubic B-spline collocation method to find the numerical solution of a generalized Black-Scholes partial differential equation; the method is shown to be unconditionally stable. Jiang Huang et al. [17] developed a numerical method based on cubic polynomial spline approximation to solve a generalized Black-Scholes equation using the implicit Euler method for time discretization. Further, Kadalbajoo et al. [21] gave a numerical method for solving the generalized Black-Scholes equation which is second-order convergent with respect to both variables, it approximates not only the option value but also some of its important 'Greeks' (Delta and Gamma) at the same time without any extra effort. Hon [15] developed a numerical method for solving the Black-Scholes equation for the valuation of American options where he used the concept of quasi-interpolation and radial basis functions (RBFs) approximation. Figlewski and Gao [12] illustrated the application of an adaptive mesh technique to the case of barrier options. Zvan et al. [39] proposed to use an implicit method, which has superior convergence (when the barrier is close to the region of interest), and stability properties as well as offered additional flexibility in terms of constructing the spatial grid. For some further reading on Barrier options, the reader may refer to [3, 4, 16, 18, 26, 27, 32, 33, 35, 36].

In this paper, the collocation method based on a modified cubic B-spline has been developed. In our approach, in the first step, the time derivative is approximated by the backward difference and in the second step, the option price is approximated by modified cubic B-spline functions. In Section 2, a description of the method is given. The stability of the constructed method has been proved in Section 3 and numerical solutions are presented in Section 4.

2. Description of the method

We consider a uniform mesh Δ with grid points $\lambda_{j,n}$ to discretize the region $\Omega = [x \times t]$. Each $\lambda_{j,n}$ is the vertex of grid points (x_j, t_n), where $x_j = \log(a) + jh$, $j = 0, \dots, N$ and $t_n = T - nk$, $n = 1, 2, \dots, M$.

Following [31], we define the cubic B-spline for $j = -1, 0, \dots, N, N+1$ as

$$B_{3,j} = \frac{1}{h^3} \begin{cases} (x - x_{j-2})^3, & x \in [x_{j-2}, x_{j-1}], \\ h^3 + 3h^2(x - x_{j-1}) + 3h(x - x_{j-1})^2 - 3(x - x_{j-1})^3, & x \in [x_{j-1}, x_j], \\ h^3 + 3h^2(x_{j+1} - x) + 3h(x_{j+1} - x)^2 - 3(x_{j+1} - x)^3, & x \in [x_j, x_{j+1}], \\ (x_{j+2} - x)^3, & x \in [x_{j+1}, x_{j+2}], \\ 0, & \text{otherwise}. \end{cases} \tag{5}$$

Our numerical treatment for solving equation (3) using the collocation method with modified cubic B-splines is to find an approximate solution $U(x)$ to the exact solution $u(x,t)$ in the form

$$U(x)=\sum_{j=-1}^{N+1}\hat{c_j}(t)B_j(x), \tag{6}$$

where $\hat{c_j}(t)$ are unknown time-dependent parameters that need to be determined.

Using approximate solution (6) and the cubic B-spline, the approximate values at the knots of $U(x)$ and theirs derivatives are determined in terms of the time-dependent parameters $\hat{c_j}(t)$ as

$$\begin{aligned} U(x) &= \hat{c}_{j-1}+4\hat{c}_j+\hat{c}_{j+1}, \\ hU'(x) &= 3(\hat{c}_{j+1}-\hat{c}_{j-1}), \\ h^2U''(x) &= 6(\hat{c}_{j-1}-2\hat{c}_j+\hat{c}_{j+1}). \end{aligned} \tag{7}$$

Using (7) and boundary conditions (4), the approximate solutions at the boundary points can be obtained in the following form

$$U_0(x_0,t)=\sum_{j=-1}^{1}\hat{c}_jB_j(x_0)=\alpha(t), \tag{8}$$

and

$$U_N(x_N,t)=\sum_{j=N-1}^{N+1}\hat{c}_jB_j(x_N)=\beta(t), \tag{9}$$

2.1. Redefined cubic B-spline method

Following [29], the procedure for modifying the basis functions is as follows: Our numerical treatment for solving equation (3) with (4) using the cubic B-splines collocation method with redefined basis functions is to find an approximate solution $U^N(x,t)$ to the exact solution $u(x,t)$ by eliminating $\hat{c}_{-1}$ and $\hat{c}_{N+1}$ from (6) and (8, 9); we get the approximate solution in the form

$$U^N(x,t)=\Omega(x,t)+\sum_{j=0}^{N}\hat{c}_j\tilde{B}j(x), \tag{10}$$

where

$$\Omega(x,t)=\frac{B_{-1}(x)}{B_{-1}(x_0)}\alpha(t)+\frac{B_{N+1}(x)}{B_{N+1}(x_{N+1})}\beta(t), \tag{11}$$

$$\begin{cases} \tilde{B_0}(x) = B_0(x)-\dfrac{B_0(x_0)}{B_{-1}(x_0)}B_{-1}(x) \\ \tilde{B_1}(x) = B_1(x)-\dfrac{B_1(x_0)}{B_{-1}(x_0)}B_{-1}(x) \\ \tilde{B_j}(x) = B_j(x), j=2,\dots,N-2 \\ B_{\tilde{N}-1}(x) = B_{N-1}(x)-\dfrac{B_{N-1}(x_N)}{B_{N+1}(x_N)}B_{N+1}(x) \\ \tilde{B_N}(x) = B_N(x)-\dfrac{B_N(x_N)}{B_{N+1}(x_N)}B_{N+1}(x). \end{cases} \tag{12}$$

Here the new set of basis functions is $\{\tilde{B}_j(x), j = 0, 1, 2, \ldots, N-1, N\}$ and they vanish on the boundary. The function $\Omega(x,t)$ defined in (11) takes care of Dirichlet type boundary conditions. Applying the modified set of basis functions $\tilde{B}_j(x), j = 0, 1, 2, \ldots, N-1, N$ into equation (3) we get the required approximate solution.

To apply the proposed method we discretize the problem in time variable using the backward finite difference approximation and the Crank-Nicolson scheme to space derivatives and we get

$$\begin{aligned}\frac{u_j^n - u_j^{n-1}}{k} &+ \frac{1}{2}\,\sigma^2 \frac{(u_{xx})_j^n + (u_{xx})_j^{n-1}}{2} \\ &+ (r - \frac{1}{2}\,\sigma^2)\frac{(u_x)_j^n + (u_x)_j^{n-1}}{2} - r\frac{u_j^n + u_j^{n-1}}{2} = 0. \qquad (13)\end{aligned}$$

Substituting the approximate solution U^N and its derivatives using relations (7) we have

$$\begin{aligned}(-1-\frac{kr}{2}\,)\sum_{j=0}^{N}\alpha_j^{n-1}\tilde{B}_j(x) &+ \frac{k}{2}\,(r-\frac{1}{2}\,\sigma^2)\sum_{j=0}^{N}\alpha_j^{n-1}\tilde{B'}_i(x) + \frac{k\sigma^2}{4}\sum_{j=0}^{N}\alpha_j^{n-1}\tilde{B''}_i(x) \\ =(-1+\frac{kr}{2}\,)\sum_{j=0}^{N}\alpha_j^{n}\tilde{B}_j(x) &- \frac{k}{2}\,(r-\frac{1}{2}\,\sigma^2)\sum_{j=0}^{N}\alpha_j^{n}\tilde{B'}_i(x) - \frac{k\sigma^2}{4}\sum_{j=0}^{N}\alpha_j^{n}\tilde{B''}_i(x) \\ &+ \{(-1+\frac{kr}{2}\,)\Omega(x_j,t^n) - \frac{k}{2}\,(r-\frac{1}{2}\,\sigma^2)\Omega_x(x_j,t^n) - \frac{k\sigma^2}{4}\Omega_{xx}(x_j,t^n)\} \\ &- \{(-1-\frac{kr}{2}\,)\Omega(x_j,t^{n-1}) + \frac{k}{2}\,(r-\frac{1}{2}\,\sigma^2)\Omega_x(x_j,t^{n-1}) + \frac{k\sigma^2}{4}\Omega_{xx}(x_j,t^{n-1})\}. \qquad (14)\end{aligned}$$

Associating the boundary conditions with (14), this can be written in matrix form as

$$\mathbf{A}\hat{\mathbf{c}}^{n-1} = \mathbf{B}\hat{\mathbf{c}}^{n} + \mathbf{b}, \qquad (15)$$

where

$$\mathbf{A} = \begin{bmatrix} w_0 & z_1 & & & & \\ r_1 & r_2 & r_3 & & & \\ & \cdots & \cdots & \cdots & & \\ & & \cdots & \cdots & \cdots & \\ & & & r_1 & r_2 & r_3 \\ & & & & z_{N-1} & w_N \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} p_0 & q_1 & & & & \\ s_1 & s_2 & s_3 & & & \\ & \cdots & \cdots & \cdots & & \\ & & \cdots & \cdots & \cdots & \\ & & & s_1 & s_2 & s_3 \\ & & & & q_{N-1} & p_N \end{bmatrix},$$

$$\mathbf{b}=\begin{bmatrix} b_0 \\ 0 \\ \vdots \\ 0 \\ b_N \end{bmatrix}, \quad \hat{\mathbf{c}}^{n-1}=\begin{bmatrix} \hat{c}_0^{n-1} \\ \hat{c}_1^{n-1} \\ \vdots \\ \hat{c}_N^{n-1} \end{bmatrix}, \quad \hat{\mathbf{c}}^{n}=\begin{bmatrix} \hat{c}_0^{n} \\ \hat{c}_1^{n} \\ \vdots \\ \hat{c}_N^{n} \end{bmatrix},$$

$$\begin{aligned}
w_0 =&\frac{k}{2}(r-\frac{1}{2}\ \sigma^2)(\frac{12}{h})+\frac{k\sigma^2}{4}(-\frac{36}{h^2}), \quad z_1=\frac{k}{2}(r-\frac{1}{2}\ \sigma^2)(\frac{6}{h}),\\
r_1 =&1+\frac{k}{2}(r-\frac{1}{2}\ \sigma^2)(\frac{-3}{h})+\frac{k\sigma^2}{4}(\frac{6}{h^2}),\\
r_2 =&4+\frac{k\sigma^2}{4}(-\frac{12}{h^2}),\\
r_3 =&1+\frac{k}{2}(r-\frac{1}{2}\ \sigma^2)(\frac{3}{h})+\frac{k\sigma^2}{4}(\frac{6}{h^2}),\\
z_{N-1} =&\frac{k}{2}(r-\frac{1}{2}\ \sigma^2)(-\frac{6}{h}),\\
w_N =&\frac{k}{2}(r-\frac{1}{2}\ \sigma^2)(-\frac{12}{h})+\frac{k\sigma^2}{4}(-\frac{36}{h^2}),\\
p_0 =&-\frac{k}{2}(r-\frac{1}{2}\ \sigma^2)(\frac{12}{h})-\frac{k\sigma^2}{4}(-\frac{36}{h^2}),\\
q_1 =&-\frac{k}{2}(r-\frac{1}{2}\ \sigma^2)(\frac{6}{h}),\\
s_1 =&1-\frac{k}{2}(r-\frac{1}{2}\ \sigma^2)(-\frac{3}{h})-\frac{k\sigma^2}{4}(\frac{6}{h^2}),\\
s_2 =&4-\frac{k\sigma^2}{4}(-\frac{12}{h^2}),\\
s_3 =&1-\frac{k}{2}(r-\frac{1}{2}\ \sigma^2)(\frac{3}{h})-\frac{k\sigma^2}{4}(\frac{6}{h^2}),\\
q_{N-1} =&-\frac{k}{2}(r-\frac{1}{2}\ \sigma^2)(-\frac{6}{h}),\\
p_N =&-\frac{k}{2}(r-\frac{1}{2}\ \sigma^2)(-\frac{12}{h})-\frac{k\sigma^2}{4}(-\frac{36}{h^2}),\\
b_0 =&\alpha(t^n)\{1-\frac{k}{2}(r-\frac{1}{2}\ \sigma^2)(-\frac{3}{h})-\frac{k\sigma^2}{4}(\frac{6}{h^2})\}\\
&-\alpha(t^{n-1})\{1+\frac{k}{2}(r-\frac{1}{2}\ \sigma^2)(-\frac{3}{h})-\frac{k\sigma^2}{4}(\frac{6}{h^2}),\\
b_N =&\beta(t^n)\{1-\frac{k}{2}(r-\frac{1}{2}\ \sigma^2)(\frac{3}{h})-\frac{k\sigma^2}{4}(\frac{6}{h^2})\}\\
&-\beta(t^{n-1})\{1+\frac{k}{2}(r-\frac{1}{2}\ \sigma^2)(\frac{3}{h})+\frac{k\sigma^2}{4}(\frac{6}{h^2}).
\end{aligned}$$

Now A and B are $(N+1)\times(N+1)$ tridiagonal matrices, which depend on boundary conditions. At each time level we solve (15) and recover the solution via (10).

2.2. Final state

In order to start any computation using the above formula we need the values of the final vector $\hat{c}^M$. The final vector $\hat{c}^M$ can be determined from the final condition, which gives $N+1$ equations in $N+1$ unknowns. For the determination of the unknowns, the following relations at the knot are used

$$\begin{aligned} U_x(x_j,0) &= g'(x_j), \qquad j=0,\\ U(x_j,0) &= g(x_j), \qquad j=1,\dots,N-1,\\ U_x(x_j,0) &= g'(x_j), \qquad j=N, \end{aligned}$$

The final vector is then determined as the solution of the matrix equation

$$\mathbf{A}\hat{\mathbf{c}}^M = \mathbf{b}, \tag{16}$$

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & & & \\ 1 & 4 & 1 & & \\ & \cdots & \cdots & \cdots & \\ & & \cdots & \cdots & \cdots \\ & & 1 & 4 & 1 \\ & & & 2 & 4 \end{bmatrix}, \quad \hat{\mathbf{c}}^M = \begin{bmatrix} \hat{\mathbf{c}}_0^M \\ \hat{\mathbf{c}}_1^M \\ \vdots \\ \hat{\mathbf{c}}_N^M \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} g(x_0) + \frac{h}{3}g'(x_0) \\ g(x_1) \\ \vdots \\ g(x_{N-1}) \\ g(x_N) - \frac{h}{3}g'(x_N) \end{bmatrix},$$

3. Stability analysis

We have established stability analysis of the proposed method by using the Von-Neumann s tability method. For stability analysis we should consider equation (14) as follows

$$r_1\hat{c}_{j-1}^{n-1} + r_2\hat{c}_j^{n-1} + r_3\hat{c}_{j+1}^{n-1} = s_1\hat{c}_{j-1}^n + s_2\hat{c}_j^n + s_3\hat{c}_{j+1}^n, \tag{17}$$

where r_1, r_2, r_3, s_1, s_2 and s_3 are given in (15).

Now, it is necessary to assume that the solution of scheme (17) at the mesh point (x_j, t_n) may be written as $\hat{c}_j^n = \xi^n \exp(ij\beta h)$, where ξ is, in general, complex, β is the mode number, h is the element size, and $i = \sqrt{-1}$. Thus using $\hat{c}_j^n = \xi^n \exp(ij\beta h)$ in (17) we obtain the characteristic equation

$$\xi = \frac{r_1\exp(-i\beta h) + r_2 + r_3\exp(i\beta h)}{s_1\exp(-i\beta h) + s_2 + s_3 exp(i\beta h)}. \tag{18}$$

Substituting the values of r_1, r_2, r_3, s_1, s_2 and s_3 from (15) we have

$$\xi = \frac{[2(\cos\beta h + 2) - \frac{3k\sigma^2}{h^2}(1-\cos\beta h)] + i[\frac{3k}{h}(r - \frac{1}{2}\ \sigma^2)\sin\beta h]}{[2(\cos\beta h + 2) + \frac{3k\sigma^2}{h^2}(1-\cos\beta h)] - i[\frac{3k}{h}(r - \frac{1}{2}\ \sigma^2)\sin\beta h]}, \tag{19}$$

i.e.,

$$\xi = \frac{X_1 - iY}{X_2 + iY}, \tag{20}$$

where

$$X_1 = [2(\cos\beta h + 2) - \frac{3k\sigma^2}{h^2}(1-\cos\beta h)],$$
$$X_2 = [2(\cos\beta h + 2) + \frac{3k\sigma^2}{h^2}(1-\cos\beta h)],$$
$$Y = -\frac{3k}{h}\ (r - \frac{1}{2}\ \sigma^2)\sin\beta h.$$

Now substituting $\lambda = \frac{k}{h^2}$, $\rho = \lambda\sigma^2$ and $\phi = \cos\beta h$ in equation (20) we have

$$\xi = \frac{[2(\phi+2) - 3\rho(1-\phi)] + i[\frac{3k}{h}(r - \frac{1}{2}\ \sigma^2)\sqrt{1-\phi^2}]}{[2(\phi+2) + 3\rho(1-\phi)] - i[\frac{3k}{h}(r - \frac{1}{2}\ \sigma^2)\sqrt{1-\phi^2}]} \tag{21}$$

and

$$|\xi|^2 = \frac{[2(\phi+2) - 3\rho(1-\phi)]^2 + [\frac{9k^2}{h^2}(r - \frac{1}{2}\ \sigma^2)^2(1-\phi^2)]}{[2(\phi+2) + 3\rho(1-\phi)]^2 + [\frac{9k^2}{h^2}(r - \frac{1}{2}\ \sigma^2)^2(1-\phi^2)]}. \tag{22}$$

This implies $|\xi| \leq 1$, which shows that the proposed scheme is unconditionally stable.

4. Option pricing examples

4.1. European put

We consider the Black-Scholes equation describing a European put option. We want to price a European put option with $T = 1$, $\sigma = 0.30$, $E = 15$ and $r = 0.05$ for a domain $\Omega = [1, 30]$. The appropriate final and boundary conditions for this problem are:

$$g(S) = \max(S - E, 0),$$
$$\alpha(t) = 0,$$
$$\beta(t) = Ee^{-r(T-t)} - S.$$

Table 1 consists of the maximum error in the solution for the European put option. Since $S \in [1, 30]$, hence $x \in [\log(1), \log(30)]$. The analytical solution is given in [14]. The error in the solutions are given in Table (1) for some S. We calculated the computational orders of convergence of the method presented in this article (denoted by C-order) with the following formula:

$$order = \frac{\log(\frac{E_1}{E_2})}{\log(\frac{h_1}{h_2})},$$

in which E_1 and E_2 are maximum errors corresponding to grids, with mesh size h_1 and h_2, respectively. The proposed scheme has been run for various space steps, h, while the time step is fixed as $k = 0.001$.

In Table (2), we have tabulated the maximum error in the solution for the European put option for some S with $E = 10$, $r = 0.05$, $T = 0.5$ and $\sigma = 0.2$. In our computations, we take $k = 0.005$, and the computational orders of convergence for various space steps are given.

Moreover, we have calculated the order of convergence for various time steps, k and space step is fixed as $h = 0.002$, and the results for this case are given in Table (3).

h	Max. error	C- Order
0.1	2.8242e–004	—
0.05	0.7122e–004	1.9941
0.025	0.1798e–004	1.9826
0.0125	0.0474e–004	1.9241

Table 1: *Maximum error and order of convergence of Example (1) with respect to space variable*

h	Max. error	C- Order
0.1	3.4494e–004	—
0.05	0.8812e–004	1.9663
0.025	0.2230e–004	1.9811
0.0125	0.0617e–004	1.8552

Table 2: *Maximum error and order of convergence of Example (2) with respect to space variable*

k	Max. error	C- Order
0.02	0.9944e–004	—
0.01	0.5065e–004	0.9770
0.005	0.2633e–004	0.9471
0.0025	0.1410e–004	0.9041

Table 3: *Maximum error and order of convergence of Example (1) with respect to time variable*

Numerical results shown in Tables (1)-(3) indicate that the B-spline solution provide an accurate solution of the European option. The order of convergence is listed for the proposed scheme in Tables (1) and (2), and is found to be approximately 2 with respect to the space variable. Also, the order of convergence with respect to the time variable is found to be approximately 1 in Table (3).

4.2. American put

Our second pricing example is an American put option. American options allow the holder to exercise the option at any point in time up to and including the expiry. Clearly, this somewhat complicates the problem of pricing, and indeed for American puts and calls, there is no known pricing formula as there is for European options.

For some further reading on American options, the reader may refer to [9, 22, 25, 37].
It is to be noted that to solve the American put option we firstly solve a corresponding option pricing problem for European puts and then use an update procedure, in which, at each time-step, the approximated solution is verified to be always larger than the payoff [15]. In other words,

$$U^n(j) = \max\{E - e^{x(j)}, U^n(j)\}.$$

This makes the valuation of American options relatively simple.
Appropriate initial and boundary conditions for this problem are:

$$\begin{aligned} g(S) &= \max(E - S, 0), \\ \alpha(t) &= Ee^{-r(T-t)}, \\ \beta(t) &= 0. \end{aligned}$$

In order to compare our approximation with that from another numerical method, we price the American put option with $E = 100$, $r = 0.1$, $T = 1$ and $\sigma = 0.30$ for a domain $\Omega = [1, e^6]$. A comparison between the binomial and B-spline solutions for the American put option are given in Table (3) with $k = 0.001$ and $h = 0.0001$.

S	Binomial solution	B-spline solution
80	20.2689	20.2683
85	16.3467	16.3425
90	13.1228	13.1278
95	10.4847	10.4916
100	8.3348	8.3362
105	6.6071	6.6107
110	5.2091	5.2080
115	4.0976	4.0954
120	3.2059	3.2079
140	1.1789	1.1843

Table 4: *Comparison of results for American put*

Numerical results in Table (3) indicate that the modified B-spline method provides a resonable approximation to the solution of the American option.

4.3. Barrier Option

Our final example is a continuous down-and-out call barrier option.
Barrier options can be classified into knock-out and knock-in options. Assuming that the barrier price is X, the knock-out option can be exercised unless the asset price S reaches the barrier X during the day of purchase and expiration day. The knock-in option can be exercised if the asset price S overtakes the barrier X. The knock-out options can be classified into up-and-out and down-and-out. The up-and-out option

can be exercised unless the asset price S reaches the barrier X from below the barrier and the down-and-out option can be done unless the asset price reaches the barrier from above the barrier [23]. The value of the down-and-out option, denoted by $V = V(S,t)$, is governed by the equations

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \qquad (S > X),$$
$$V = 0, \qquad (S < X),$$

where S is the current value of the underlying asset at time t and X is the barrier value. The final condition on the expiration day is given by

$$g(S) = \max(S - E, 0),$$

and the boundary conditons are as follows,

$$\alpha(t) = 0, \quad \beta(t) = S.$$

In the case of barrier options the first boundary condition is applied at $S = X$ rather than $S = 0$. If S reaches X, the option is invalid, thus on the line $S = X$ the value of the option is zero.
Now we want to price the down-and-out call options with $T = 0.5$, $E = 10$, $r = 0.05$, $\sigma = 0.2$ and the barrier value 9.0. The maximum errors in the solution of the down-and-out call option is given in Table 4 with $k = 0.0002$ and computational orders of convergence for various h are given.

Moreover, we have calculated the order of convergence for various k and $h = 0.002$ is fixed. Also, the results for this case are shown in Table 5.

h	Max. error	C- order
0.2	2.3377e–003	—
0.1	5.8312e–004	2.004
0.05	1.4557e–004	2.002
0.025	0.3637e–004	2.001
0.0125	0.0909e–004	2.000

Table 5: *Maximum error and order of convergence of down-and-out call option with respect to space variable*

k	Max. error	C- order
0.02	0.6797e–004	—
0.01	0.3401e–004	1.002
0.005	0.1582e–004	1.064
0.0025	0.0794e–004	1.110

Table 6: *Maximum error and order of convergence of down-and-out call option with respect to time variable*

As seen in the tabular results, the B-spline approach gave the results which are in good agreement with the exact solution. Also, in the barrier option case, the order of convergence is shown for the proposed scheme in Table 5, and is found to be approximately 2 with respect to the space variable and 1 with respect to the time variable in Table 6.

5. Conclusion

Option pricing is an important problem in the financial markets. In this study, we develop an unconditional stable method based on modified cubic B-spline functions for the solution of option problems based on the Black scholes equation applied to European, American and Barrier cases. A backward finite difference scheme is used for discretizing the temporal derivative, then the modified B-spline approach is employed for approximating the option prices. As test examples, we applied our method to some benchmarks in literature. As shown, numerical results are in good agreement with exact solutions.

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