MATHEMATICAL COMMUNICATIONS
Math. Commun. **22**(2017), 1–11

Symmetry analysis of time-fractional potential Burgers' equation

MANOJ GAUR AND KARANJEET SINGH*

Department of Mathematics, Jaypee University of Information Technology Waknaghat, Solan (H.P.) 173 234, *India*

Received December 5, 2015; accepted April 14, 2016

Abstract. Lie point symmetries of time-fractional potential Burgers' equation are presented. Using these symmetries fractional potential Burgers' equation has been transformed into an ordinary differential equation of fractional order corresponding to the Erdélyi-Kober fractional derivative. Further, an analytic solution is furnished by means of the invariant subspace method.

AMS subject classifications: 35B06, 35R11, 34A08

Key words: Fractional differential equations, fractional Lie group method, Erdélyi-Kober fractional derivative, fractional integration, fractional invariant subspace method

1. Introduction

There are numerous physical phenomena with an inherent fractional order description. In particular, with the help of fractional derivatives, nonlinear oscillation of earthquakes, phenomena in material science, acoustics, electrochemistry and electromagnetics can be elegantly described [13, 24, 26, 28, 16]. The subject of fractional calculus has caught attention of many mathematicians, who contributed to its development. In 1819, Lacroix published the first paper that mentioned a fractional derivative [25]. In the last few decades, a slight revitalisation of interest in fractional calculus has taken place, but the application of fractional derivatives and fractional integrals has not yet been fully uncovered; mainly because of their unusualness. Fractional calculus has applications in many diverse areas, such as mathematical physics, viscoelasticity, transmission theory, electric conductance of biological systems, modelling of neurons, diffusion processes, damping laws, and growth of intergranular grooves on metal surfaces [1, 22, 4]. It has been revealed that non-conservative forces can be described by fractional differential equations. Therefore, as most of the processes in real physical world are non-conservative, fractional calculus can be used to describe them. Fractional integrals and derivatives also appear in the theory of control of dynamical systems, when the controlled system and/or the controller is described by a fractional differential equation [24]. In recent years, some analytical and numerical methods [3, 1, 10, 22, 6, 21] have been introduced to solve a fractional order differential equation. It is very well known that the Lie group

*Corresponding author. *Email addresses:* `karan_jeet@yahoo.com` (K. Singh), `manojgaur22@gmail.com` (M. Gaur)

method is the most effective technique in the field of applied mathematics to find exact solutions of ordinary and partial differential equations [23]. However, this approach has not been applied much to investigate symmetry properties of fractional differential equations (FDEs). Concerning symmetry analysis of fractional partial differential equations, the recognized results in the literature are very few and the method is still not sufficiently developed. To the best of our knowledge, there are only few papers (for example [9, 10, 27, 5, 2, 17, 11]) in which Lie symmetries and similarity solutions of some fractional differential equations have been discussed by some researchers. In this paper, by means of the Lie group method we consider the following time-fractional potential Burgers' equation of the form

$$u_t^{(\alpha)} = Au_{xx} + B(u_x)^2, \quad x \in (0,\infty),\ t > 0,\ 0 < \alpha < 1, \tag{1}$$

where A and B are real constant parameters. This paper is based on some basic elements of fractional calculus, with special emphasis on the Riemann-Liouville type and modified Riemann-Liouville type derivatives [14]. We use Lie symmetries with the prolongation formula given by Gazizov et al. [10]. The paper is organized as follows. In Section 2, we briefly provide some definitions and properties of fractional calculus. In Section 3, we obtain the symmetries for the fractional potential Burgers' equation having three-dimensional Lie algebra. In Section 4, we reduce equation (1) into an ordinary differential equation (ODE) and find some exact solutions of equation (1) using the invariant subspace method in Section 5. Finally, a conclusion is given in Section 6.

2. Some definitions in fractional calculus

In this section, some definitions and basic properties about fractional calculus are given, which have been used throughout this paper.

2.1. Fractional Riemann-Liouville integral

The fractional Riemann-Liouville integral of a continuous (but not necessarily differentiable) real valued function $f(x)$ with respect to $(dx)^\alpha$ is defined as [15, 22]

$$\begin{aligned} {}_0I_x^\alpha\ f(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\,dt \\ &= \frac{x^\alpha}{\Gamma(\alpha)}\int_0^1 (1-\sigma)^{\alpha-1} f(x\sigma)\,d\sigma\ ,\ 0<\alpha\le 1. \end{aligned} \tag{2}$$

2.2. Riemann-Liouville Fractional derivative

The fractional Riemann-Liouville derivative of $f(x)$ is defined as [22]

$$ {}_0D_x^\alpha\, f(x) = \begin{cases} \frac{\partial^n f}{\partial x^n}, \quad \alpha = n \in \mathbb{N} \\ \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int\limits_0^x (x-t)^{n-\alpha-1} f(t)\, dt\,, \quad n-1 < \alpha < n,\ n \in \mathbb{N}. \end{cases} \tag{3} $$

2.3. Modified Riemann-Liouville derivative

Through the fractional Riemann-Liouville integral, Jumarie [14] proposed the modified Riemann-Liouville derivative of $f(x)$ as

$$ {}_0D_x^\alpha\, f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int\limits_0^x (x-t)^{n-\alpha-1} (f(t) - f(0))\, dt\,, \quad n-1 < \alpha < n,\ n \in \mathbb{N} \tag{4} $$

2.4. Some useful formulae

Here, some properties of modified Riemann-Liouville derivative are given which have been used in this paper

(i) $df(x) = \dfrac{D_x^\alpha\, f(x)(dx)^\alpha}{\Gamma(1+\alpha)}, \quad \alpha > 0.$

(ii) $D_t^\alpha (u(t)v(t)) = \sum\limits_{n=0}^{\infty} \dbinom{\alpha}{n} D_t^{\alpha-n} u(t) D_t^n v(t), \quad \alpha > 0,$
where $\dbinom{\alpha}{n} = \dfrac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}.$

(iii) $D_t^\alpha f(x(t)) = \dfrac{df}{dx} D_t^\alpha\, x(t), \;\; 0 < \alpha < 1$, given $\dfrac{df}{dx}$ exists.

(iv) $D_x^\alpha x^\beta = \dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}$, where x^β is α-differentiable.

(v) $\displaystyle\int (dx)^\beta = x^\beta.$

(vi) $\Gamma(1+\beta)dx = (dx)^\beta.$

For a complete description of these formulae and their scope of applications and limitations, the reader may refer to [14],[15].

3. Symmetry classification of time-fractional potential Burgers' equation

Herein, we investigate the symmetries and reductions of time-fractional potential Burgers' equation (1). We assume that equation (1) admits the Lie symmetries of the form

$$\tilde{x} = x + \epsilon\xi(x,t,u) + o(\epsilon^2) \tag{5}$$

$$\tilde{t} = t + \epsilon\tau(x,t,u) + o(\epsilon^2) \tag{6}$$

$$\tilde{u} = u + \epsilon\eta(x,t,u) + o(\epsilon^2), \tag{7}$$

where ϵ is the group parameter and ξ, τ and η are the infinitesimals of the transformations for the independent and dependent variables, respectively. The associated Lie algebra of infinitesimal symmetries of equation (1) is then the vector field of the form

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{8}$$

The fractional second order prolongation [10] of (1) is

$$\begin{aligned} pr^{(2)}V =& \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} + \eta^t\frac{\partial}{\partial u_t} + \eta^x\frac{\partial}{\partial u_x} \\ &+ \eta_\alpha^0\frac{\partial}{\partial u_t^{(\alpha)}} + \eta^{tt}\frac{\partial}{\partial u_{tt}} + \eta^{xx}\frac{\partial}{\partial u_{xx}}. \end{aligned} \tag{9}$$

Now for the invariance of equation (1) under equations(5)-(7), we must have

$$pr^{(2)}V([\Delta u])|_{([\Delta u])=0} = 0, \tag{10}$$

where $[\Delta u] = u_t^{(\alpha)} - Au_{xx} - B(u_x)^2$, or equivalently, if

$$(\eta_\alpha^0 - 2Bu_x\eta^x - A\eta^{xx})|_{([\Delta u])=0} = 0. \tag{11}$$

The generalised fractional prolongation vector fields [10], η^x, η_α^0 and η^{xx} are given by

$$\begin{aligned} \eta^x =& \eta_x + u_x\eta_u - (\xi_x + u_x\xi_u)u_x - (\tau_x + u_x\tau_u)u_t \\ \eta_\alpha^0 =& \eta_t^\alpha + (\eta_u - \alpha(\tau_t + u_t\tau_u))u_t^\alpha - u(\eta_u)_t^\alpha - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x) \\ &+ \sum_{n=1}^{\infty}[\binom{\alpha}{n}\partial_t^n\eta_u - \binom{\alpha}{n+1}D_t^{n+1}(\tau)]D_t^{\alpha-n}(u) + \mu, \end{aligned}$$

where $\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}(-u)^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}$, and

$$\begin{aligned} \eta^{xx} =& \eta_{xx} + u_x\eta_{ux} - (\xi_{xx} + u_x\xi_{ux})u_x - (\tau_{xx} + u_x\tau_{ux})u_t - 2(\tau_x + u_x\tau_u)u_{xt} \\ &+ [\eta_{xu} + u_x\eta_{uu} - (\xi_{xu} + u_x\xi_{uu})u_x - (\tau_{xu} + u_x\tau_{uu})u_t]u_x \\ &+ u_{xx}(\eta_u - u_x\xi_u - u_t\tau_u) - 2(\xi_x + u_x\xi_u)u_{xx}. \end{aligned}$$

Now using the above generalised prolongation vector fields in equation (11) and equating the coefficient of various derivative terms to zero, we get the following simplified set of determining equations

$$\tau_u = 0 \tag{12}$$

$$\tau_x = 0 \tag{13}$$

$$\xi_u = 0 \tag{14}$$

$$\binom{\alpha}{n}\partial_t^n \eta - \binom{\alpha}{n+1} D_t^{n+1}(\tau) = 0, \quad n = 1, 2, 3, \dots \tag{15}$$

$$2\xi_x - \alpha\tau_u = 0 \tag{16}$$

$$2B\xi_x - B\alpha\tau_u - B\tau_u = 0 \tag{17}$$

$$D_t^n(\xi) = 0, \quad n = 1, 2, 3, \dots \tag{18}$$

On solving the above equations (12)-(18), we obtain the infinitesimals as

$$\xi = c_1 x + c_2 \tag{19}$$

$$\tau = \frac{2c_1 t}{\alpha} \tag{20}$$

$$\eta = c_3, \tag{21}$$

where c_1,c_2,c_3 are arbitrary parameters. The point symmetry generators admitted by equation (1) are given by

$$V_1 = x\frac{\partial}{\partial x} + \frac{2t}{\alpha}\frac{\partial}{\partial t} \tag{22}$$

$$V_2 = \frac{\partial}{\partial x} \tag{23}$$

$$V_3 = \frac{\partial}{\partial u}. \tag{24}$$

Hence, the infinitesimal operator (8) becomes $V = (c_1 x + c_2)\frac{\partial}{\partial x} + \frac{2t}{\alpha}\frac{\partial}{\partial t} + c_3\frac{\partial}{\partial u}$. Further, these infinitesimal generators (22-24) can be used to determine a three-parameter fractional Lie group of point transformations acting on (x, t, u)-space which is fewer than those for the standard Burgers' equation [19]. It can be verified easily that the set $\{V_1, V_2, V_3\}$ forms a three-dimensional Lie algebra under the Lie bracket $[X, Y] = XY - YX$ and its commutator table is given as below:

	V_1	V_2	V_3
V_1	0	-V_2	0
V_2	V_2	0	0
V_3	0	0	0

Further, from the commutator table it can be seen that V_3 forms a solvable sub-algebra. Also V_3 is the centre of the three-dimensional Lie algebra as it commutes with every element of the Lie algebra. The group transformation generated by the infinitesimal generators $V_i, i = 1, 2, 3$ is obtained by solving the system of ordinary differential equations

$$\frac{d\tilde{x}}{d\epsilon} = \xi(\tilde{x}, \tilde{t}, \tilde{u}) \tag{25}$$

$$\frac{d\tilde{t}}{d\epsilon} = \tau(\tilde{x}, \tilde{t}, \tilde{u}) \tag{26}$$

$$\frac{d\tilde{u}}{d\epsilon} = \eta(\tilde{x}, \tilde{t}, \tilde{u}) \tag{27}$$

with the initial conditions

$$\tilde{x}|_{\epsilon=0} = x \tag{28}$$

$$\tilde{t}|_{\epsilon=0} = t \tag{29}$$

$$\tilde{u}|_{\epsilon=0} = u. \tag{30}$$

Exponentiating the infinitesimal symmetries of equation (1), we get the one-parameter groups $g_i(\epsilon)$ generated by $V_i, i = 1, 2, 3$

$$g_1 : (x, t, u) \to (e^{\epsilon}x, e^{\frac{2}{\alpha}}t, u) \tag{31}$$

$$g_2 : (x, t, u) \to (x + \epsilon, t, u) \tag{32}$$

$$g_3 : (x, t, u) \to (x, t, u + \epsilon). \tag{33}$$

Now, since g_i is a symmetry, if $u = f(x, t)$ is a solution of equation (1) the following u_i are also solutions of equation (1)

$$u_1 = f(e^{\epsilon}x, e^{\frac{2}{\alpha}}t) \tag{34}$$

$$u_2 = f(x + \epsilon, t) \tag{35}$$

$$u_3 = f(x, t) - \epsilon. \tag{36}$$

4. Reducion to ODE

Herein, we reduce the fractional potential Burgers' equation (1) to an ODE with the Erdélyi-Kober fractional differential operator [17]. For the infinitesimal generator V_1 the characteristic equations are

$$\frac{dx}{x} = \frac{\alpha dt}{2t} = \frac{du}{0}, \tag{37}$$

which give the invariants as $u(x, t) = f(z)$, $z = xt^{\frac{-\alpha}{2}}$. Corresponding to these invariants, we can reduce equation (1) to an ODE of fractional order. We summarize the result in the following theorem:

Theorem 1. *The similarity transformation $u(x, t) = f(z)$ along with the similarity variable $z = xt^{\frac{-\alpha}{2}}$ reduces the time-fractional potential Burgers' equation (1) to the ordinary differential equation of fractional order of the form*

$$(P_{\frac{2}{\alpha}}^{1-\alpha,\alpha} f)(z) = A\frac{d^2 f}{dz^2} + B(\frac{df}{dz})^2 \tag{38}$$

with the Erdélyi-Kober fractional differential operator [17]

$$(P_{\delta}^{\tau,\alpha}f)(z)=\prod_{j=0}^{m-1}(\tau+j-\frac{1}{\delta}z\frac{d}{dz})(K_{\delta}^{\tau+\alpha,m-\alpha}f)(z),\ z>0,\ \delta>0,\ \alpha>0, \tag{39}$$

$m=\begin{cases}[\alpha]+1, & \alpha\notin\mathbb{N}\\ \alpha, & \alpha\in\mathbb{N}.\end{cases}$, *where*

$$(K_{\delta}^{\tau,\alpha}f)\ (z)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int\limits_{0}^{\infty}(\nu-1)^{\alpha-1}\nu^{-(\tau+\alpha)}f(z\nu^{\frac{1}{\delta}})\,d\nu\,, & \alpha>0;\\ f(z), & \alpha=0\end{cases} \tag{40}$$

is the Erdélyi-Kober fractional integral operator.

Proof. Let $n-1<\alpha<n$, $n=1,2,3,\ldots$ Then the Riemann-Liouville fractional derivative for the similarity transformation $u(x,t)=f(z)$ with the similarity variable $z=xt^{\frac{-\alpha}{2}}$ becomes

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^{n}}{\partial t^{n}}[\frac{1}{\Gamma(n-\alpha)}\int\limits_{0}^{t}(t-s)^{n-\alpha-1}f(xs^{\frac{-\alpha}{2}})\,ds].$$

Let $\nu=\frac{t}{s}$. Then the above equation can be written as

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^{n}}{\partial t^{n}}[t^{n-\alpha}\frac{1}{\Gamma(n-\alpha)}\int\limits_{1}^{\infty}(\nu-1)^{n-\alpha-1}\nu^{-(n-\alpha+1)}f(z\nu^{\frac{\alpha}{2}})\,d\nu].$$

Following the definition of the Erdélyi-Kober fractional integral operator given in equation (40), we have

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^{n}}{\partial t^{n}}[t^{n-\alpha}(K_{\frac{2}{\alpha}}^{1,n-\alpha}f)(z)]. \tag{41}$$

In order to simplify the right-hand side of equation (41), we consider the relation $z=xt^{\frac{-\alpha}{2}}$, $f\in C^{1}(0,\infty)$,

$$\begin{aligned}t\frac{\partial}{\partial t}f(z)&=tx(-\frac{\alpha}{2})t^{-\frac{\alpha}{2}-1}f'(z)\\&=-\frac{\alpha}{2}z\frac{d}{dz}f(z)\end{aligned}$$

and thus, we get

$$\begin{aligned}\frac{\partial^{n}}{\partial t^{n}}[t^{n-\alpha}(K_{\frac{2}{\alpha}}^{1,n-\alpha}f)(z)]&=\frac{\partial^{n-1}}{\partial t^{n-1}}[\frac{\partial}{\partial t}(t^{n-\alpha}(K_{\frac{2}{\alpha}}^{1,n-\alpha}f)(z))]\\&=\frac{\partial^{n-1}}{\partial t^{n-1}}[t^{n-\alpha-1}(n-\alpha-\frac{\alpha}{2}z\frac{d}{dz}(K_{\frac{2}{\alpha}}^{1,n-\alpha}f)(z))].\end{aligned}$$

Repeating the similar procedure for $n-1$ times, we have

$$\frac{\partial^n}{\partial t^n}[t^{n-\alpha}(K^{1,n-\alpha}_{\frac{2}{\alpha}}f)(z)] = t^{-\alpha}\prod_{j=0}^{n-1}(1-\alpha+j-\frac{\alpha}{2}z\frac{d}{dz})(K^{1,n-\alpha}_{\frac{2}{\alpha}}f)(z).$$

Now using the definition of Erdélyi-Kober fractional differential operator given in equation (39), the above equation can be written as

$$\frac{\partial^n}{\partial t^n}[t^{n-\alpha}(K^{1,n-\alpha}_{\frac{2}{\alpha}}f)(z)] = t^{-\alpha}(P^{1-\alpha,\alpha}_{\frac{2}{\alpha}}f)(z).$$

We obtain an expression for the time-fractional derivative

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\alpha}(P^{1-\alpha,\alpha}_{\frac{2}{\alpha}}f)(z).$$

Continuing further we find that the time-fractional potential Burgers' equation (1) reduces to an ordinary differential equation of fractional order

$$(P^{1-\alpha,\alpha}_{\frac{2}{\alpha}}f)(z) = A\frac{d^2f}{dz^2} + B(\frac{df}{dz})^2.$$

□

As the order $0<\alpha<1$ of the above equation is arbitrary, there is no existing method to solve the above differential equation of fractional order in general. However, for some special cases, such as the initial value problems and the linear equations, the solutions can be furnished by the power series method with the Mittag-Leffler function and Wright and the generalised Wright functions [7, 20, 2]. Further, the transmutation method [18] can be utilised to solve the Erdélyi-Kober fractional differential equations of the form $x^{-\beta\delta}(P^{\alpha,\delta}_{\beta}y)(x) - \lambda y(x) = f(x)$. In particular, when $B=0$, the procedure given in [2] can be followed to derive two independent solutions as $W(-\frac{xt^{-\frac{\alpha}{2}}}{\sqrt{A}};-\frac{\alpha}{2},1)$ and $W(\frac{xt^{-\frac{\alpha}{2}}}{\sqrt{A}};-\frac{\alpha}{2},1)$, where $W(z;\lambda,\mu)$ is the Wright function [29] given by $W(z;\lambda,\mu)=\sum_{i=0}^{\infty}\frac{z^i}{i!\Gamma(\lambda i+\mu)}$.
Consequently, the group invariant solution of equation (1),when $B=0$, has the form $u(x,t) = K_1W(-\frac{xt^{-\frac{\alpha}{2}}}{\sqrt{A}};-\frac{\alpha}{2},1)+K_2W(\frac{xt^{-\frac{\alpha}{2}}}{\sqrt{A}};-\frac{\alpha}{2},1)$, where K_1 and K_2 are arbitrary parameters.

5. Some exact solutions of fractional potential Burgers' equation by the invariant subspace method.

The invariant subspace method was introduced by Galaktionov [8] in order to discover exact solutions of nonlinear partial differential equations. The method was further applied by Gazizov and Kasatkin [12] to some fractional order differential equations. Here we give a brief description of the method.
Consider the fractional evolution equation $\frac{\partial^\alpha u}{\partial t^\alpha} = F[u]$, where $u=u(x,t)$ and $F[u]$ is a nonlinear differential operator.

The n-dimensional linear space $W_n = \langle f_1(x), \ldots, f_n(x)\rangle$ is called invariant under the operator $F[u]$, iff $F[u] \in W_n$ for any $u \in W_n$, which means there exist n functions $\phi_1, \ldots, \phi_n$ such that

$$F[C_1 f_1(x) + \ldots + C_n f_n(x)] = \phi_1(C_1, \ldots, C_n) f_1(x) + \ldots + \phi_n(C_1, \ldots, C_n) f_n(x),$$

where $C_1, \ldots, C_n$ are arbitrary constants. The exact solution of a fractional evolution equation can be obtained as

$$u(x,t) = \sum_{i=1}^{n} a_i(t) f_i(x).$$

For equation (1) $F[u] = Au_{xx} + Bu_x{}^2$. We have the space $W_3 = \langle 1, x, x^2\rangle$ as invariant under $F[u]$, since

$$\begin{aligned} F[C_1 + C_2 x + C_3 x^2] &= 2C_3 A + B(C_2 + 2C_3 x)^2 \\ &= b_1 + b_2 x + b_3 x^2 \in W_3 \end{aligned}$$

where b_1,b_2 and b_3 are arbitrary constants given by

$$\begin{aligned} 2C_3 A + BC_2^2 &= b_1 \\ 4BC_2C_3 &= b_2, \text{ and} \\ 4BC_3^2 &= b_3. \end{aligned}$$

This allows us to consider an exact solution of equation (1) as

$$u(x,t) = a_1(t) + a_2(t)x + a_3(t)x^2. \tag{42}$$

Substituting the value of $u(x,t)$ from equation (42) into equation (1) and equating the coefficients of x^j, $j = 0, 1, 2$, we get the following system of fractional differential equations

$$\frac{d^\alpha a_3(t)}{dt^\alpha} = 4B(a_3(t))^2, \tag{43}$$

$$\frac{d^\alpha a_2(t)}{dt^\alpha} = 4Ba_2(t)a_3(t) \tag{44}$$

and

$$\frac{d^\alpha a_1(t)}{dt^\alpha} = 2Aa_3(t) + B(a_2(t))^2. \tag{45}$$

Equations (43)-(45) can be readily solved to yield

$$-\frac{1}{a_3(t)} = \frac{2}{\Gamma(1+\alpha)} \int B dt^\alpha + s_1, \tag{46}$$

$$\log a_2(t) = \frac{4}{\Gamma(1+\alpha)} \int a_3(t) B dt^\alpha + s_2 \tag{47}$$

and

$$a_1(t) = \frac{1}{\Gamma(1+\alpha)}[\int (2a_3(t)A + (a_2(t))^2 B) dt^\alpha + s_3], \tag{48}$$

where s_1,s_2 and s_3 are arbitrary constants.
Using equation (42) and equations (46)-(48) one can easily obtain an exact solution of equation (1).

6. Conclusion

In this paper, the authors have attempted to illustrate the application of the Lie symmetry method to time-fractional partial differential equations. More precisely, we considered time-fractional potential Burgers' equation and derived their Lie point symmetries. The Lie symmetry analysis shows that the underlying symmetry algebra of the equation is three-dimensional unlike the six-dimensional Lie algebra for the standard potential Burgers' equation. The reduction of dimension in the symmetry algebra is due to the fact that the time-fractional equation is not invariant under time translation symmetry. It is appropriate to mention here that the fractional order significantly affects the properties of the equation. The main reason is that the fractional order $0 < \alpha < 1$ is an arbitrary parameter in the studied fractional model. Using the Lie point symmetries, we have shown that the equation can be transformed into an ODE of fractional order with the Erdélyi-Kober fractional derivative. Finally, we furnish some exact solutions to the fractional potential Burgers' equation by means of the fractional invariant subspace method.

Acknowledgments

The authors express their sincere thanks to the anonymous reviewers for their careful review and their useful suggestions. One of the authors (Manoj Gaur) would like to thank the University Grants Commission (UGC) New Delhi, India, for providing a Research Fellowship under the scheme UGC-CSIR NET JRF in Science, Humanities & Social Sciences.

References

[1] E. Barkai, *Fractional Fokker-Planck equation, solution, and application*, Physical Review E. **63**(2001), 046118.
[2] E. Buckwar, Y. Luchko, *Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations*, J. Math. Anal. Appl. **227**(1998), 81–97.
[3] Y. Chen, H. L. An, *Numerical solutions of coupled Burgers equations with time-and space-fractional derivatives*, Appl. Math. Comput. **200**(2008), 87–95.
[4] L. Debnath, *Recent applications of fractional calculus to science and engineering*, Int. J. Math. Math. Sci. **54**(2003), 3413–3442.
[5] V. D. Djordjevic, T. M. Atanackovic, *Similarity solutions to nonlinear heat conduction and Burgers/Korteweg-deVries fractional equations*, J. Comput. Appl. Math. **222**(2008), 701–714.
[6] M. Eslami, B. F. Vajargah, M. Mirzazadeh, A. Biswas, *Application of first integral method to fractional partial differential equations*, Indian Journal of Physics **88**(2003), 177–184.
[7] V. A. Galaktionov, S. R. Svirshchevskii, *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*, Springer-Verlag, Berlin, 2010.
[8] V. A. Galaktionov, S. R. Svirshchevskii, *Exact solutions and invariant subspaces of nonlinear partial differential equations in mechanics and physics*, CRC Press, Boca Raton, 2006.

[9] M. Gaur, K. Singh, *On group invariant solutions of fractional order Burgers-Poisson equation*, Appl. Math. Comput. **244**(2014), 870–877.

[10] R. K. Gazizov, A. A. Kasatkin, S. Y. Lukashchuk, *Continuous transformation groups of fractional differential equations*, Vestnik, USATU **9**(2007), 125–135.

[11] R. K. Gazizov, A. A. Kasatkin, S. Y. Lukashchuk, *Group-invariant solutions of fractional differential equations*, in: *Nonlinear Science and Complexity*, (J. A. T. Machado, A. C. J. Luo, R. S. Barbosa, M. F. Silva and L. B. Figueiredo, Eds.), Springer Netherland, 2011, 51–59.

[12] R. K. Gazizov, A. A. Kasatkin, *Construction of exact solutions for fractional order differential equations by the invariant subspace method*, Comput. Math. Appl. **66**(2013), 576–584.

[13] J. H. He, *Some applications of nonlinear fractional differential equations and their approximations*, Bull. Sci. Technol **15**(1999), 86–90.

[14] G. Jumarie, *Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions further results*, Comput. Math. Appl. **51**(2006), 1367–1376.

[15] G. Jumarie, *Fractional Differential Calculus for Non-Differentiable Functions: Mechanics, Geometry, Stochastics, Information Theory*, Lambert Academic Publishing, Saarbrucken, 2013.

[16] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies Vol. 204, Elsevier Science Limited, Amsterdam, 2006.

[17] V. S. Kiryakova, *Generalized fractional calculus and applications*, CRC press, 1993.

[18] V. S. Kiryakova, B. N. Al-Saqabi, *Transmutation method for solving Erdélyi-Kober fractional differintegral equations*, J. Math. Anal. Appl. **211**(1997), 347–364.

[19] H. Liu, J. Li, Q. Zhang, *Lie symmetry analysis and exact explicit solutions for general Burgers'equation*, J. Comput. Appl. Math. **228**(2009), 1–9.

[20] H. Liu, *Complete Group Classifications and Symmetry Reductions of the Fractional Fifth-Order KdV Types of Equations*, Stud. Appl. Math. **131**(2013), 317–330.

[21] B. Lu, *Bäcklund transformation of fractional Riccati equation and its applications to nonlinear fractional partial differential equations*, Physics Letters A **376**(2012), 2045–2048.

[22] K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley, New York, 1993.

[23] P. J. Olver, *Applications of Lie groups to differential equations*, Springer Science & Business Media, 2000.

[24] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Academic Press, 1998.

[25] B. Ross, *A brief history and exposition of the fundamental theory of fractional calculus*, Fractional Calculus and Its Applications, Springer-Verlag, New York, 1975, 1–36.

[26] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives; Theory and Applications*, Gordon and Breach, London, 1993.

[27] G. W. Wang, X. Q. Liu, Y. Y. Zhang, *Lie symmetry analysis to the time fractional generalized fifth-order KdV equation*, Commun. Nonlinear Sci. Numer. Simul. **18**(2013), 2321–2326.

[28] B. West, M. Bologna, P. Grigolini, *Physics of fractal operators*, Springer Science & Business Media, 2012.

[29] E. M. Wright, *The generalized Bessel function of order greater than one*, Q. J. Math. **1**(1940), 36–48.