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The Area Generating Function for the Column-Convex Polyominoes on the Checkerboard Lattice

Svjetlan Feretić

Šetalište Joakima Rakovca 17, 51000 Rijeka, Croatia

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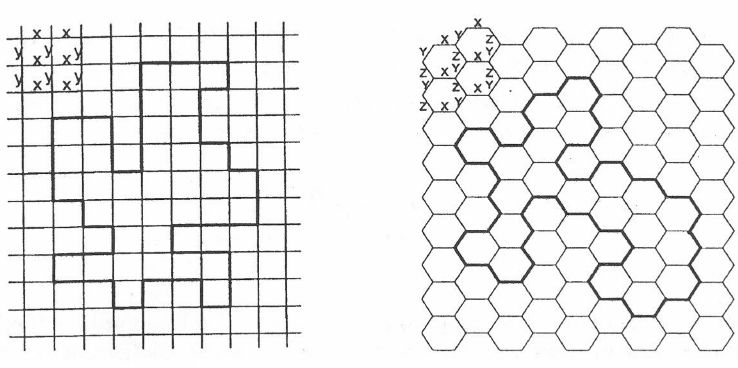
The aim of the present work is to compute the area generating function *(gf)* for the column-convex polyominoes on the checkerboard lattice. It is interesting that this area *gf* includes as two special cases the area *gf*s for the rectangular and honeycomb lattices. The problem treated here is complementary to the problem concerning the perimeter *gf*s, which was suggested by Wu and solved by Tzeng and Lin.

1. PREPARATION

Besides its purely mathematical interest, the computation of the *self-avoiding polygon* (SAP, Figure 1) perimeter and area generating functions would have a significant bearing on the study of chemical problems such as configuration of polymer molecules and gel formation. But despite strenuous efforts over the past 40 years,



a) b)

Figure 1. a) A self-avoiding polygon (SAP) on the rectangular lattice. b) A SAP on the honeycomb lattice.

only some restricted classes of the SAP's have been enumerated so far. Further, in all those enumerations two SAP's are identified iff there is a translation that transforms one into the other (reflections and rotations are not allowed).

An important restricted class of the SAP's arises if we impose convexity in the direction of one of the lattice axes. When this axis is the one perpendicular to the x-edges, the SAP's satisfying the just stated convexity condition are called *column-convex polyominoes* (CCP's, Figure 2).

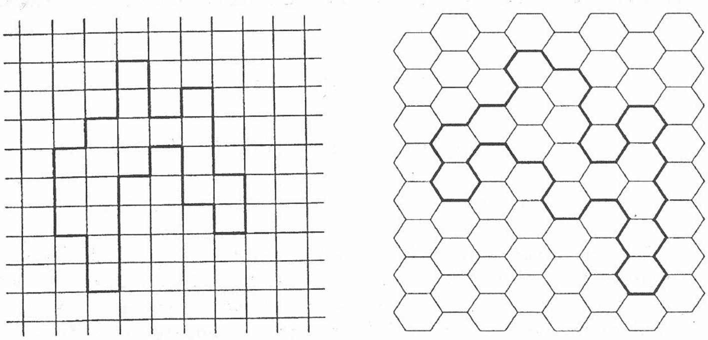


Figure 2. Column-convex polyominoes (CCP's) on the rectangular and honeycomb lattices.

Rectangular lattice. The area generating function for the column-convex polyominoes on the rectangular lattice (RCCP's, for short) is known to be

$$F_{\mathrm{r}}(q,x) = \frac{qx(1-q)^3}{(1-q)^4 - qx(1-q)^2(1+q) - q^3x^2} \tag{1}$$

The above formula was first found by Temperley[1] (1956) and then rederived by different methods by Klarner[2] (case x=1, 1965) and Delest[3] (1988).

The number of RCCP's having the area μ and k columns is the coefficient of $q^{\mu}x^k$ in F_{r}, which is denoted by $\langle q^{\mu}x^k\rangle F_{\mathrm{r}}$ and is equal to

$$\sum_{i_1, i_2 \geq 0} \binom{k-i_1-1}{i_1} \binom{k-2i_1-1}{\mu-k-i_1-i_2} \binom{2k+i_2-2}{2k-2}. \tag{2}$$

(Throughout this paper we adopt the convention: if a binomial coefficient has a negative numerator or denominator, then the value of the coefficient is zero.)

The perimeter generating function for RCCP's is an algebraic function:

$$G_{\mathrm{r}}(x,y) = (1-y)\left[1 - \frac{2\sqrt{2}}{3\sqrt{2} - \sqrt{1+x+\sqrt{(1-x)^2 - 16xy/(1-y)^2}}}\right] \tag{3}$$

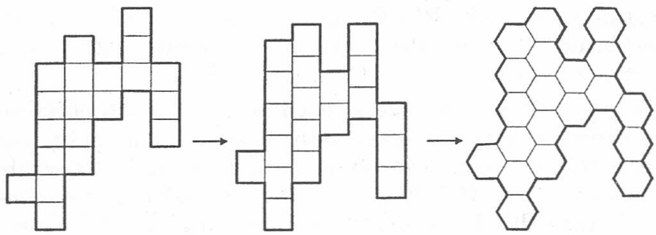


Figure 3. Shifting the columns of a RCCP we obtain a HCCP with no weak upper contacts.

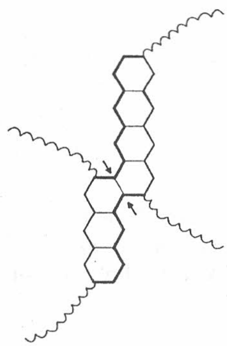


Figure 4. A weak upper contact.

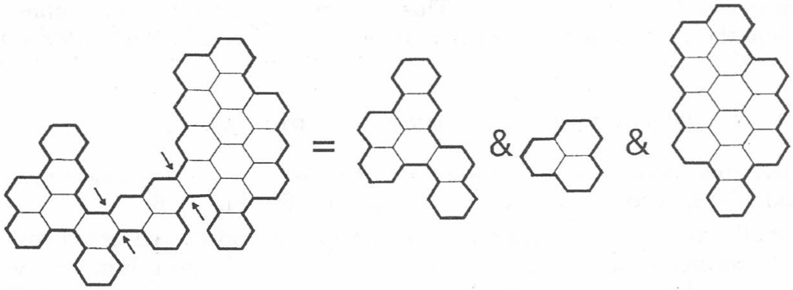


Figure 5.

The number of RCCP's having k columns (*i.e.* $2k$ horizontal edges) and $2v$ vertical edges appears as $\langle x^k y^v \rangle G_r$ and is given by a certain threefold sum of binomial coefficients. Formula (3) was obtained by Feretić and Svrtan[4] (1993). *Cf.* the results of Delest[3] (case x=y, 1988), Brak, Guttmann and Enting[5] (case x=y, 1990) and Lin[6] (1990).

The generating function for RCCP's in three variables q, x and y is also known. It is a non-algebraic one (see Bousquet-Melou,[7] Feretić and Svrtan[4], Brak and Guttmann,[8] Lin and Tzeng[9]).

Honeycomb lattice. When we take a RCCP and shift each of its non-initial columns by half an edge down with respect to the previous column, the resulting figure is in substance a HCCP (column-convex polyomino on the honeycomb lattice). See Figure 3. But it is a special HCCP, since it has no *weak upper contacts* (Figure 4). To obtain an arbitrary HCCP, we sometimes have to glue together several of such special HCCP's (Figure 5).

Thus the area *gf* for HCCP's is given by

$$F_{\mathrm{h}}(q,x) = F_{\mathrm{r}} + F_{\mathrm{r}}^2 + F_{\mathrm{r}}^3 + \ldots = \frac{qx(1-q)^3}{(1-q)^4 - 2qx(1-q)^2 - q^3x^2} \tag{4}$$

and the number of HCCP's with area μ and k columns is

$$\langle q^{\mu} x^{k} \rangle F_{\mathrm{h}} = \sum_{i_1 \geq 0} 2^{k-2i_1-1} \binom{k-i_1-1}{i_1} \binom{\mu+k-i_1-2}{2k-2} . \tag{5}$$

The shifting of the columns procedure may affect the vertical perimeter of a polyomino. For this reason, the perimeter *gf*'s for RCCP's and HCCP's are not so closely related as the area *gf*'s. In fact, the perimeter *gf* for HCCP's is the function $G_{\mathrm{h}}(x, y, z)$ given implicitly by

$$\hat{G} = x \frac{\left[y(\alpha - \hat{G})^2 + (1+y)^2 \alpha \hat{G}\right]\left[z(\alpha - \hat{G})^2 + (1+z)^2 \alpha \hat{G}\right]}{(1-yz)^2 \alpha (\alpha - \hat{G})^2} \tag{6}$$

where $\alpha = 1 - (y + z)\hat{G}$ and $\hat{G} = G_{\mathrm{h}}/(1-yz)$. Notice that Eq. (6) can be rewritten as a quadratic equation in $\alpha\hat{G}/(\alpha - \hat{G})^2$. This remark may be used to calculate G_{h} explicitly, but the final formula is rather unhandy. *Cf.* Lin and Wu[10] (1990), Feretić and Svrtan[4] (1993).

2. INTRODUCTION OF THE CHECKERBOARD LATTICE

On the checkerboard lattice, we have two types of cells and also two types of horizontal edges, vertical edges and lattice points. See Figure 6.

We shall partition the column-convex polyominoes on the checkerboard lattice (CCCP's) according to the colour of the bottom cell of their first column: if the colour is black (resp. white), then we speak of a b-polyomino (resp. w-polyomino). Further, let k be a column of some CCCP. If k has a white cell at the top and a black cell at the bottom, then we say that k is a w:b column. The b:b, w:w and b:w columns are defined similarly. For example, the CCCP in Figure 7 is a b-polyomino. Its first column is w:b, the second is b:w, the third is b:b and the fourth is w:w.

Let $G_b(u, x, y, z)$ and $G_w(u, x, y, z)$ be the perimeter *gf*'s for b- and w-polyominoes, with the variables $u, x,\ldots$ marking the u-edges, x- edges,... Obviously, $G_b(x, x, y, y)$ is the perimeter *gf* for RCCP's. Moreover, there is a perimeter preserving bijection between the b- polyominoes with no u-edges and HCCP's (Figure 8). Hence $G_b(0, x, y, z)$ is the perimeter generating function for HCCP's.

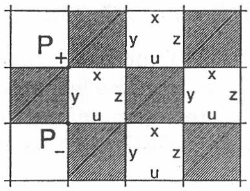


Figure 6. The checkerboard lattice. We say that the lattice points like P_+ are *positive*; those like P_- are *negative*.

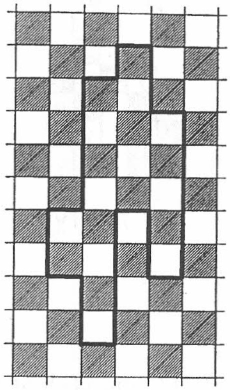


Figure 7.

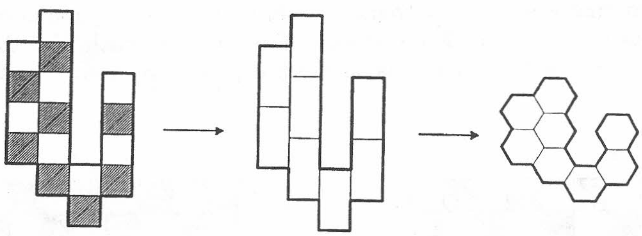


Figure 8.

Having these facts in view, F.Y. Wu suggested the problem of deriving function G_b to Tzeng and Lin. However, putting this suggestion into practice revealed the unpleasant fact that G_b is, unfortunately, a formidably complicated function (see Tzeng and Lin,[11] 1991).

In this paper, we solve the complementary problem of deriving the (relatively simple, as Theorem 1 will show) area *gf*'s for the *b*- and *w*-polyominoes, F_b and F_w. Functions F_b and F_w are in six variables q_b, q_w, $x_{b:b}$, $x_{w:w}$, $x_{b:w}$ and $x_{w:b}$. The exponent of q_b represents the number of black cells, the exponent of $x_{w:b}$ represents the number of *w:b* columns, *etc.*

3. AN INTERMEDIATE OBJECT: THE THIN POLYOMINOES

Let P be a CCCP. Next, let SW(P) and SE(P) be the lower left corner of the first column of P and the lower right corner of the last column of P. The lattice path starting from SW(P) with a horizontal step, going along the lower boundary of P and ending (again with a horizontal step) at SE(P) will be called the *bottom path* of P. See Figure 9 for an example.

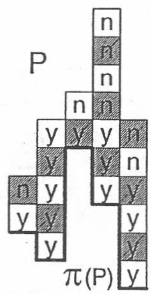


Figure 9. The bottom path of P is thickened.

Let c and π(P) be a cell and the bottom path of some CCCP P. We say that $\pi(P)$ touches c if c and $\pi(P)$ have at least one point in common. In general, the bottom path need not touch all the cells of a polyomino. In Figure 9, the cells touched by $\pi(P)$ are labelled y and the others are labelled n. If the bottom path touches every cell of a CCCP P, we say that P is a *thin* polyomino. Note that a thin polyomino is uniquely determined by its bottom path. For instance, if the bottom path of a thin polyomino P is the one in Figure 10a, then P is surely the polyomino shown in Figure 10b.

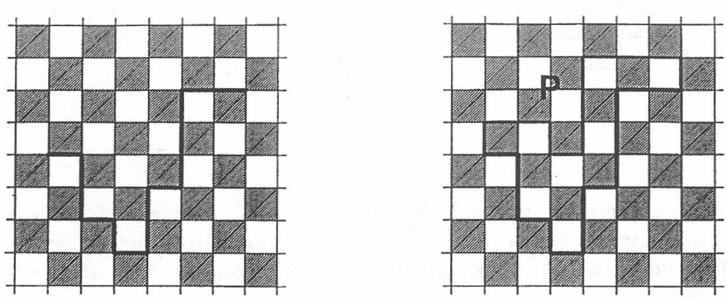


Figure 10. a) b)

In other words, the thin polyominoes are *encoded* by their bottom paths.

3.1. *A »relaxed« coding for the bottom paths*

The bottom paths of CCCP's are self-avoiding checkerboard lattice paths over the step-set {(1, 0), (0, 1), (0, -1)}, with the first and the last step being (1, 0) steps.

Now let us describe our coding for bottom paths. The codes are words over the *alphabet* $\{e_{+-}, e_{-+}, U^{+}_{+}, U^{-}_{-}, U^{-}_{+}, U^{+}_{-}, D^{+}_{+}, D^{-}_{-}, D^{+}_{-}, D^{-}_{+}, n_{+}, n_{-}\}$. The twelve letters serve to keep a partial record of the horizontal edges and vertical *segments* that we come upon in the course of traversing a bottom path from left to right "(a segment = a maximal union of consecutive collinear edges)". Precisely, we write

e_{+-} when we meet a horizontal edge starting at a positive point and ending at a negative point;

U^{-}_{+} when we meet an upward *segment* starting at a positive point and ending at a negative point;

D^{+}_{-} when we meet a downward segment starting at a positive point and ending at a negative point;

D^{+}_{+} when we meet a nonempty downward segment starting at a positive point and ending at a positive point;

n_{+} when we meet an empty vertical segment »starting« and »ending« at a positive point.

We omit to declare the roles of the remaining letters, but this will certainly make no difficulties to the reader.

An example: the bottom path in Figure 10a is encoded by the word $e_{+-}\, D^{-}_{-}\, e_{-+}\, D^{+}_{-}\, e_{-+}\, U^{+}_{+}\, e_{+-}\, U^{+}_{-}\, e_{+-}\, n_{-}\, e_{-+}$.

Of course, our coding for bottom paths is not injective (Figure 11). Thus it does not satisfy the requirements of the Dyck- Schützenberger-Viennot[12, 13] (DSV) methodology. Anyway, this coding proves to be useful in the calculation of the area *gf* for thin polyominoes.

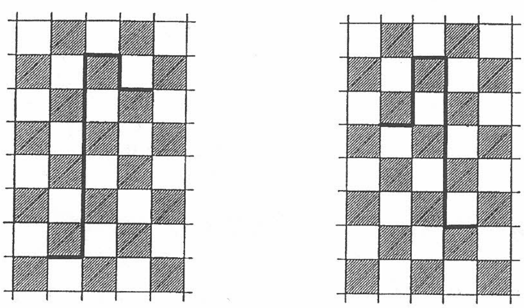


Figure 11. Both of these bottom paths are encoded by the word $e_{+-}\, U^{-}_{-}\, e_{-+}\, D^{+}_{-}\, e_{-+}$.

We shall say that a bottom path is positive (resp. negative) if it starts at a positive (resp. negative) point. Let $\mathscr{B}_{+}$ and $\mathscr{B}_{-}$ be the *languages* formed by the words encoding positive paths and negative paths, respectively. Examining how the positive and negative paths may begin, we find that the languages $\mathscr{B}_{+}$ and $\mathscr{B}_{-}$ have the following *grammar*:

$$\mathscr{B}_+ = \mathrm{e}_{+-}\left[1 + (\mathrm{U}^+_- + \mathrm{D}^-_+)\,\mathscr{B}_+ + (\mathrm{U}^-_- + \mathrm{D}^-_- + \mathrm{n}_-)\,\mathscr{B}_-\right], \tag{7}$$

$$\mathscr{B}_- = \mathrm{e}_{-+}\left[1 + (\mathrm{U}^+_+ + \mathrm{D}^+_+ + \mathrm{n}_+)\,\mathscr{B}_+ + (\mathrm{U}^-_+ + \mathrm{D}^+_-)\,\mathscr{B}_-\right]. \tag{8}$$

In view of our next move, it is convenient to rearrange the right sides of Eqs. (7) and (8) writing

$$\mathrm{D}^-_+\,\mathscr{B}_+ \quad \text{as} \quad \mathrm{D}^-_+\,\mathrm{e}_{+-}\,(U^+_-\,\mathscr{B}_+ + U^-_-\,\mathscr{B}_-) + D^-_+\left[\mathscr{B}_+ - e_{+-}\,(U^+_-\mathscr{B}_+ + U^-_-\,\mathscr{B}_-)\right],$$

$$\mathrm{D}^-_-\,\mathscr{B}_- \quad \text{as} \quad \mathrm{D}^-_-\,\mathrm{e}_{-+}\,(U^+_+\,\mathscr{B}_+ + U^-_+\,\mathscr{B}_-) + D^-_-\left[\mathscr{B}_- - e_{-+}\,(U^+_+\mathscr{B}_+ + U^-_+\,\mathscr{B}_-)\right]$$

and »developing« $\mathrm{D}^+_+\,\mathscr{B}_+$ and $\mathrm{D}^+_-\,\mathscr{B}_-$ in a likewise manner.

Now we replace the syllables $\mathrm{D}^+_+\,\mathrm{e}_{+-}\,\mathrm{U}^-_-$, $\mathrm{D}^-_-\,\mathrm{e}_{-+}\,\mathrm{U}^+_+$, $\mathrm{D}^+_+\,\mathrm{e}_{+-}\,\mathrm{U}^+_-$, $\mathrm{D}^-_-\,\mathrm{e}_{-+}\,\mathrm{U}^-_+$, $\mathrm{D}^+_-\,\mathrm{e}_{-+}\,\mathrm{U}^+_+$, $\mathrm{D}^-_+\,\mathrm{e}_{+-}\,\mathrm{U}^-_-$, $\mathrm{D}^+_-\,\mathrm{e}_{-+}\,\mathrm{U}^-_+$ and $\mathrm{D}^-_+\,\mathrm{e}_{+-}\,\mathrm{U}^+_-$ by ${}^+_+\mathrm{V}^-_-\,\mathrm{e}_{+-}$, ${}^-_-\mathrm{V}^+_+\,\mathrm{e}_{-+}$, ${}^+_+\mathrm{V}^+_-\,\mathrm{e}_{+-}$, ${}^-_-\mathrm{V}^-_+\,\mathrm{e}_{-+}$, ${}^+_-\mathrm{V}^+_+\,\mathrm{e}_{-+}$, ${}^-_+\mathrm{V}^-_-\,\mathrm{e}_{+-}$, ${}^+_-\mathrm{V}^-_+\,\mathrm{e}_{-+}$ and ${}^-_+\mathrm{V}^+_-\,\mathrm{e}_{+-}$ respectively.* Immediately afterwards, we let the letters of the thus modified languages $\mathscr{B}_+$ and $\mathscr{B}_-$ commute (making the letters commute is a part of the DSV-methodology). In this way, these two languages turn into generating functions, which will be called B_+ and B_-.

In fact, the *gf*'s B_+ and B_- refer to a coding for bottom paths which is somewhat different from the one described at the beginning of this section. This modified coding uses the D-letters only for those downward segments which are not immediately followed by an upward segment. Similarly, it uses the U-letters only for those upward segments which are not immediately preceded by a downward segment. To register the event of a downward segment immediately followed by an upward segment, an appropriate V-letter is used.

From the rearrangements of Eqs. (7) and (8) we can easily read off the following linear equations for B_+ and B_-:

$$B_+ = a_{11}\,\mathrm{e}_{+-}\,B_+ + a_{12}\,\mathrm{e}_{+-}\,B_- + \mathrm{e}_{+-}\,, \tag{9}$$

$$B_- = a_{21}\,\mathrm{e}_{-+}\,B_+ + a_{22}\,\mathrm{e}_{-+}\,B_- + \mathrm{e}_{-+}\,, \tag{10}$$

where

$$a_{11} = \mathrm{U}^+_- + \mathrm{D}^-_+ + \mathrm{e}_{+-}({}^-_+\mathrm{V}^+_- - \mathrm{D}^-_+\,\mathrm{U}^+_-) + \mathrm{e}_{-+}({}^-_-\mathrm{V}^+_+ - \mathrm{D}^-_-\,\mathrm{U}^+_+)\,,$$

$$a_{12} = \mathrm{U}^-_- + \mathrm{D}^-_- + \mathrm{n}_- + \mathrm{e}_{+-}({}^-_+\mathrm{V}^-_- - \mathrm{D}^-_+\,\mathrm{U}^-_-) + \mathrm{e}_{-+}({}^-_-\mathrm{V}^-_+ - \mathrm{D}^-_-\,\mathrm{U}^-_+)\,,$$

$$a_{21} = \mathrm{U}^+_+ + \mathrm{D}^+_+ + \mathrm{n}_+ + \mathrm{e}_{+-}({}^+_+\mathrm{V}^+_- - \mathrm{D}^+_+\,\mathrm{U}^+_-) + \mathrm{e}_{-+}({}^+_-\mathrm{V}^+_+ - \mathrm{D}^+_-\,\mathrm{U}^+_+)\,,$$

$$a_{22} = \mathrm{U}^-_+ + \mathrm{D}^+_- + \mathrm{e}_{+-}({}^+_+\mathrm{V}^-_- - \mathrm{D}^+_+\,\mathrm{U}^-_-) + \mathrm{e}_{-+}({}^+_-\mathrm{V}^-_+ - \mathrm{D}^+_-\,\mathrm{U}^-_+)\,.$$

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* The notation »V« is intended to suggest the word »valley«.

Therefore $$B_+ = \frac{e_{+-}\left[1 + (a_{12} - a_{22})\, e_{-+}\right]}{(1 - a_{11}\, e_{+-})\,(1 - a_{22}\, e_{-+}) - a_{12}\, a_{21}\, e_{+-}\, e_{-+}} \tag{11}$$

and $$B_- = \frac{e_{-+}\left[1 + (a_{21} - a_{11})\, e_{+-}\right]}{(1 - a_{11}\, e_{+-})\,(1 - a_{22}\, e_{-+}) - a_{12}\, a_{21}\, e_{+-}\, e_{-+}} \tag{12}$$

3.2. How to recover the area gf for thin polyominoes?

Now that we know the generating function for the codes of the bottom paths, it is not too hard to obtain the area *gf* for the corresponding family of polyominoes, *i.e.* for the thin polyominoes. In fact, all we have to do is make an appropriate change of variables in functions B_+ and B_-.

Recall that the thin polyomino associated to a given bottom path consists of all cells above the path such that the cell and the path have at least one point in common. Now, a horizontal edge marked by e_{+-} is the lower side of a black cell and a horizontal edge marked by e_{-+} is the lower side of a white cell. Thus, to begin changing the variables of B_+ and B_-, we substitute

$$e_{+-} = q_b\, x_{b:b} \quad , \quad e_{-+} = q_w\, x_{w:w}\ . \tag{13}$$

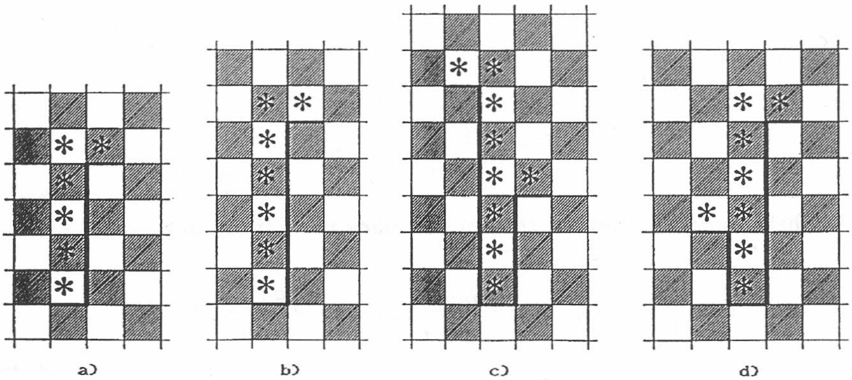

Figure 12. The cells of P touched by the vertical segment or the relevant valley are marked by asterisks.

Let π be a bottom path, let P be the thin polyomino associated to π and let s be a vertical segment of π.

First, suppose that s is encoded by U_+^+. As s starts and ends at positive points, its length is an even number, say $2k$. A glance at Figure 12a makes it clear that s touches exactly $2k+2$ cells of P. But two of those cells lie immediately above the horizontal edges of π. Therefore, they have already been taken into account by the substitutions Eq. (13). On the other hand, since s is encoded by U_+^+, it does not come immediately after some downward segment. Hence for the remaining $2k$ touched cells, the only part of π that touches them is s. In order to take into account these new touched cells, of which k are black and k are white, we shall make the substitution

$$\mathrm{U}_{+}^{+} = \sum_{k \geq 1} (q_b\, q_w)^k = \frac{q_b\, q_w}{1 - q_b\, q_w}\,. \tag{14}$$

For similar reasons, we also let

$$\mathrm{U}_{-}^{-} = \frac{q_b\, q_w}{1 - q_b\, q_w}\,. \tag{15}$$

Now suppose that the vertical segment s is encoded by U_{+}^{-}. Then, the length of s is an odd number, say $2k - 1$ ($k \in N$). s touches $2k + 1$ cells of P (Figure 12b) and our forthcoming substitution should take into account $2k$-1 of them. Also note that by Eq. (13) the column to the left of s is supposed to be a $w{:}w$ one. However, the $2k - 1$ new touched cells, of which k are black and $k - 1$ are white, turn that column into a $b{:}w$ one. These facts led us to make the substitution:

$$\mathrm{U}_{+}^{-} = \left(\sum_{k \geq 1} q_b^k\, q_w^{k-1}\right) \cdot x_{w:w}^{-1} \cdot x_{b:w} = \frac{q_b}{1 - q_b\, q_w} \cdot x_{w:w}^{-1} \cdot x_{b:w}\,. \tag{16}$$

Similarly, we let

$$\mathrm{U}_{-}^{+} = \frac{q_w}{1 - q_b\, q_w} \cdot x_{b:b}^{-1} \cdot x_{w:b}\,. \tag{17}$$

For the D- and n-variables, the substitutions to be made are

$$\mathrm{D}_{+}^{+} = \frac{q_b\, q_w}{1 - q_b\, q_w}\,, \quad \mathrm{D}_{-}^{-} = \frac{q_b\, q_w}{1 - q_b\, q_w}\,, \quad \mathrm{D}_{-}^{+} = \frac{q_b}{1 - q_b\, q_w}\, x_{w:w}^{-1}\, x_{b:w}\,,$$

$$\mathrm{D}_{+}^{-} = \frac{q_w}{1 - q_b\, q_w}\, x_{b:b}^{-1}\, x_{w:b}\,, \quad n_{+} = 1\,, \quad n_{-} = 1\,. \tag{18)–(23}$$

Next, let v be a »valley« of π encoded by ${}^{+}_{+}V^{+}_{-}$. There are two cases to be considered:

i) the left »hill-side« of v is longer than the right »hill-side« (Figure 12c). Let the length of the left hill-side be $2k$. Then the length of the right hill-side is one among the k values 1, 3,..., $2k - 1$. Further, the column of P springing up from the valley v consists of $2k + 1$ cells. The lowermost cell has already been taken into account by the substitutions Eq. (13) while the remaining $2k$ cells (k black and k white ones) should be taken into account now.

ii) the left hill-side of v is shorter than the right hill-side (Figure 12d). Suppose that the length of the right hill-side is $2k + 1$. Then, there are k possibilities for the length of the left hill-side: 2, 4, ..., $2k$. In the column of P springing up from the valley v, there are $2k+2$ cells and $2k+1$ of them (k black and $k+1$ white ones) have not yet been taken into account. Further, the column in question is a $w{:}b$ column, whereas Eq. (13) anticipated that it would be a $b{:}b$ one.

These considerations lead us to let

$$ {}^{+}_{+}V^{+}_{-} = \sum_{k \geq 1} k(q_b\, q_w)^k + \left(\sum_{k \geq 1} k\, q_b^k\, q_w^{k+1}\right) \cdot x_{b:b}^{-1}\, x_{w:b} = \frac{q_b\, q_w}{(1 - q_b\, q_w)^2}\,(1 + q_w\, x_{b:b}^{-1}\, x_{w:b}) \tag{24} $$

Likewise, we infer that for the remaining seven V-variables the substitutions to be made are

$$ {}^{-}_{-}V^{-}_{+} = \frac{q_b\, q_w}{(1 - q_b\, q_w)^2}\,(1 + q_b\, x_{w:w}^{-1}\, x_{b:w})\,, \quad {}^{+}_{+}V^{-}_{-} = \frac{q_b\, q_w}{(1 - q_b\, q_w)^2}\,(1 + q_b\, q_w)\,, $$

$$ {}^{-}_{-}V^{+}_{+} = \frac{q_b\, q_w}{(1 - q_b\, q_w)^2}\,(1 + q_b\, q_w)\,, \quad {}^{+}_{-}V^{+}_{+} = \frac{q_b\, q_w}{(1 - q_b\, q_w)^2}\,(1 + q_b\, x_{w:w}^{-1}\, x_{b:w})\,, $$

(25)–(31)

$$ {}^{-}_{+}V^{-}_{-} = \frac{q_b\, q_w}{(1 - q_b\, q_w)^2}\,(1 + q_w\, x_{b:b}^{-1}\, x_{w:b})\,, \quad {}^{+}_{-}V^{-}_{+} = \frac{q_b}{(1 - q_b\, q_w)^2}\,(1 + q_b\, q_w) x_{w:w}^{-1}\, x_{b:w}\,, $$

$$ {}^{-}_{+}V^{+}_{-} = \frac{q_w}{(1 - q_b\, q_w)^2}\,(1 + q_b\, q_w) x_{b:b}^{-1}\, x_{w:b}\,. $$

The bottom paths of the (thin or not) b-polyominoes are positive and the bottom paths of the w-polyominoes are negative. Thus, if we put Eqs. (13)-(31) into Eq. (11), we obtain the area gf for thin b- polyominoes $f_b(q_b, q_w, x_{b:b}, x_{w:w}, x_{b:w}, x_{w:b})$. The formula for f_b is

$$ f_b = \frac{n}{d} \tag{32} $$

where

$$ n = q_b\, x_{b:b}(1 - q_b\, q_w)^2 \{(1 - q_b\, q_w)^2 + [(1 + q_b\, q_w)\, x_{w:w} - 2q_b\, x_{b:w}]\,(1 - q_b\, q_w)\, q_w + $$
$$ + [q_b\, x_{b:w}^2 + q_w\, x_{w:w}^2 - (1 + q_b\, q_w)\, x_{w:w}\, x_{b:w}]\, q_b\, q_w^2\} $$

and

$$ d = \{(1 - q_b\, q_w)^2 - 2(1 - q_b\, q_w) q_b\, q_w\, x_{w:b} - [(1 + q_b\, q_w) x_{b:b}\, x_{w:b} + (x_{b:b}\, x_{w:w} - x_{w:b}^2) q_w] q_b^2\, q_w\} \cdot $$
$$ \cdot \{(1 - q_b\, q_w)^2 - 2(1 - q_b\, q_w) q_b\, q_w\, x_{b:w} - [(1 + q_b\, q_w) x_{w:w}\, x_{b:w} + (x_{b:b}\, x_{w:w} - x_{b:w}^2) q_b] q_b\, q_w^2\} - $$
$$ - [(1 - q_b\, q_w)(1 + q_b\, q_w) + (q_b\, x_{b:b} + q_w\, x_{w:w}) q_b\, q_w]^2 q_b\, q_w\, x_{b:b}\, x_{w:w}\,. $$

Once that the function f_b is known, the area *gf* for the thin w-polyominoes f_w can be found immediately. The (say) downward translation by one lattice unit is a bijection between the thin w- polyominoes counted by $\langle q_b^i\, q_w^j\, x_{b:b}^k\, x_{w:w}^l\, x_{b:w}^m\, x_{w:b}^n\rangle\, f_w$ and the thin b-polyominoes counted by $\langle q_b^j\, q_w^i\, x_{b:b}^l\, x_{w:w}^k\, x_{b:w}^n\, x_{w:b}^m\rangle\, f_b$. This relationship between the coefficients of f_w and f_b implies that

$$f_w(q_b, q_w, x_{b:b}, x_{w:w}, x_{b:w}, x_{w:b}) = f_b(q_w, q_b, x_{w:w}, x_{b:b}, x_{w:b}, x_{b:w}) . \tag{33}$$

4. THE AREA *gf* FOR ALL CCCP's

Defining a general CCCP amounts to the same thing as choosing a thin polyomino and deciding how many cells we wish to add at the top of each of the thin polyomino's columns. See Figure 13. Note that if we have, for example, a b:b column and we add an even number of cells at the top of it, we will obtain again a *b:b* column. But, if we add an odd number of cells, we will obtain a *w:b* column.

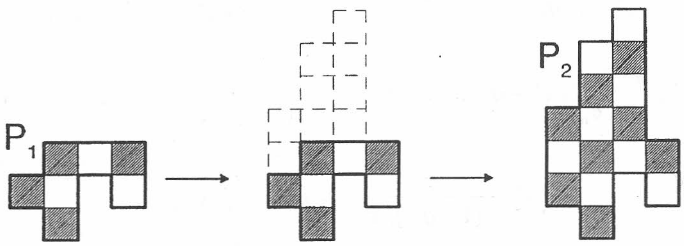

Figure 13. To obtain the polyomino P_2, we start with the thin polyomino P_1 and add 2, 3, 4, 0 cells at the top of its 1st, 2nd, 3rd, 4th columns, respectively.

From these remarks we infer that there is a change of variables by which the area *gf*'s for thin polyominoes f_b and f_w can be transformed into F_b and F_w, the area *gf*'s for general CCCP's. Actually, the substitutions that we have to make are

$$x_{b:b} = [1 + q_b\, q_w + (q_b\, q_w)^2 + \dots]\, x_{b:b} + [1 + q_b\, q_w + (q_b\, q_w)^2 + \dots]\, q_w\, x_{w:b} =$$

$$= \frac{1}{1 - q_b\, q_w}\,(x_{b:b} + q_w\, x_{w:b}) ,$$

(34)–(37)

$$x_{w:w} = \frac{1}{1 - q_b\, q_w}\,(x_{w:w} + q_b\, x_{b:w}) , \qquad x_{b:w} = \frac{1}{1 - q_b\, q_w}\,(x_{b:w} + q_w\, x_{w:w}) ,$$

$$x_{w:b} = \frac{1}{1 - q_b\, q_w}\,(x_{w:b} + q_b\, x_{b:b}) .$$

Putting Eqs. (34)–(37) into Eq. (32), after a good deal of rear-ranging we obtain our main result:

Theorem 1. We have

$$F_b = \frac{q_b A^3 BCD}{A^2 BCB'C' - [(1+q_b q_w)A^2 + q_b q_w(q_b D + q_w D')](q_b BCD + q_w B'C'D')}, \tag{38}$$

where

$$A = 1 - q_b q_w ,$$

$$B = 1 - q_b q_w + q_w x_{w:w} , \quad B' = 1 - q_b q_w + q_b x_{b:b} ,$$

$$C = 1 - q_b q_w - q_b q_w x_{b:w} , \quad C' = 1 - q_b q_w - q_b q_w x_{w:b} ,$$

$$D = x_{b:b} + q_w x_{w:b} , \quad D' = x_{w:w} + q_b x_{b:w} .$$

Of course,

$$F_w(q_b, q_w, x_{b:b}, x_{w:w}, x_{b:w}, x_{w:b}) = F_b(q_w, q_b, x_{w:w}, x_{b:b}, x_{w:b}, x_{b:w}) . \tag{39}$$

Let P be an arbitrary CCCP and let k be a column of P. Further, let $|k|_b$ and $|k|_w$ be the numbers of black cells and white cells contained in k. If k is respectively a $b{:}b$, $w{:}w$, $b{:}w$, $w{:}b$ column, then $|k|_w = |k|_b - 1$, $|k|_w = |k|_b + 1$, $|k|_w = |k|_b$, $|k|_w = |k|_b$. Hence the polyomino P satisfies

$$|P|_w = |P|_b - |P|_{b:b} + |P|_{w:w} . \tag{40}$$

Here, $|P|_w$ denotes the number of white cells of P, $|P|_{b:b}$ denotes the number of $b{:}b$ columns of P and so on.

Thus, the coefficient $\langle q_b^{\mu_b} q_w^{\mu_w} x_{b:b}^{bb} x_{w:w}^{ww} x_{b:w}^{bw} x_{w:b}^{wb}\rangle F_b + F_w$ may be different from zero only in the case $\mu_w = \mu_b - bb + ww$. In that case, the coefficient in question is given by the sevenfold sum

$$\sum_{i_1,\dots,i_7 \ge 0} (-1)^{bb+ww+i_5+i_6} \binom{i_1+i_2}{i_1}\binom{i_3+i_4}{i_3}\binom{i_1+i_3}{i_5}\binom{i_2+i_4}{i_6}\binom{i_1+i_2-1}{i_3+i_4} \cdot$$

$$\cdot \binom{bb+i_1-i_5-1}{i_1-1}_{\leftarrow}\binom{ww+i_2-i_6-1}{i_2-1}_{\leftarrow}\binom{wb-i_3+i_5-1}{i_1-1}_{\leftarrow}\binom{bw-i_4+i_6-1}{i_2-1}_{\leftarrow} \cdot \tag{41}$$

$$\cdot \binom{bb+ww+wb+bw+i_1+i_2+i_3+i_4+i_7-2}{i_7}\binom{i_1+i_2-i_3-i_4-1}{bb+wb+bw-\mu_b+i_1+i_2+i_7-1} .$$

Some of the binomial coefficients (*bc*'s) appearing in Eq. (41) are indicated by an $\leftarrow$ arrow. Those *bc*'s are much the same as the usual *bc*'s, the only difference being

the agreement $\begin{bmatrix} -1 \\ -1 \end{bmatrix}_{\leftarrow} = 1$.

Let (sp1) and (sp2) be the following two symmetry properties:

$$\text{(sp1)} \quad \varphi(q_b, q_w, x_{b:b}, x_{w:w}, x_{b:w}, x_{w:b}) = \varphi(q_b, q_w, x_{b:b}, x_{w:w}, x_{w:b}, x_{b:w}) \; ,$$

$$\text{(sp2)} \quad \varphi(q_b, q_w, x_{b:b}, x_{w:w}, x_{b:w}, x_{w:b}) = \varphi(q_w, q_b, x_{w:w}, x_{b:b}, x_{w:b}, x_{b:w}) \; .$$

Now, none of the functions f_b, f_w, F_b and F_w has either of the above properties. Function $f_b + f_w$ possesses only the property (sp2), while $F_b + F_w$ possesses both (sp1) and (sp2). Basically, to give a combinatorial proof that $F_b + F_w$ possesses (sp1), we just have to reflect the CCCP's in a horizontal line crossing the vertical lattice edges at their midpoints. This proof does not apply to $f_b + f_w$ because the reflection in a horizontal line transforms some of the thin polyominoes into non-thin polyominoes.

5. TWO SPECIAL CASES

Rectangular lattice. Our first check of the formula (38) will be to see if the appropriate change of variables does really transform F_b into F_r, the area *gf* for CCP's on the rectangular lattice. Thus, we put

$$q_b = q \; , \quad q_w = q \; , \quad x_{b:b} = x \; , \quad x_{w:w} = x \; , \quad x_{b:w} = x \; , \quad x_{w:b} = x \; . \tag{42}$$

This turns the numerator of F_b into

$$qx(1-q)^3\,(1+q)^4\,(1-q^2+qx)\,(1-q^2-q^2x) \tag{43}$$

and the denominator of F_b into

$$\left[(1-q)^4 - qx(1-q)^2\,(1+q) - q^3x^2\right](1+q)^4\,(1-q^2+qx)\,(1-q^2-q^2x) \; . \tag{44}$$

When Eq. (43) is divided by Eq. (44), the common factors cancel, and what results is the known formula (1) for F_r.

Honeycomb lattice. In section 2, we said that there is a perimeter preserving 1-1 correspondence between the b-polyominoes with no u-edges and the honeycomb lattice CCP's. Now, note that a b-polyomino has no u-edges if all of its columns are $w{:}b$ columns. In fact, the correspondence of section 2 maps the b-polyominoes with only $w{:}b$ columns having μ black cells, μ white cells and k columns bijectively onto the HCCP's having area μ and k columns. Hence the change of variables

$$q_b = q^{1/2} \; , \quad q_w = q^{1/2} \; , \quad x_{b:b} = 0 \; , \quad x_{w:w} = 0 \; , \quad x_{b:w} = 0 \; , \quad x_{w:b} = x \tag{45}$$

should turn F_b into F_h, the area generating function for HCCP's.

Indeed, the substitution Eq. (45) changes the numerator of Eq. (38) into

$$qx(1-q)^5 \tag{46}$$

and the denominator into

$$\left[(1-q)^4 - 2qx(1-q)^2 - q^3x^2\right](1-q)^2 . \tag{47}$$

Evidently, the formula for F_h obtained by dividing Eq. (46) by Eq. (47) is exactly the same one as that derived in section 1.

Remark. So far our results Eqs. (38) and (39) have resisted to two checks. Nevertheless, to be completely sure that Eqs. (38) and (39) are correct, we subjected them to one more check. In addition to the coefficients of $F_b + F_w$, which are given by Eq. (41), we also calculated the coefficients of F_b alone. Evaluating these coefficients we found how the 38997082 b-polyominoes and the twice as many CCCP's having the area 20 and 6 columns are distributed according to six parameters: black cells, white cells, *b:b, w:w, b:w* and *w:b* columns. Then, we rederived these two six-parameter distributions by a different method: the Temperley[1] recurrences. To the satisfaction, all the numerical values obtained by the two different methods fully agreed.

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SAŽETAK

Prebrojavanje stupčano konveksnih poliominoe na šahovskoj mreži prema njihovoj površini

Svjetlan Feretić

Izračunana je F_b, funkcija izvodnica za broj stupčano konveksnih poliominoe na šahovskoj mreži koji imaju zadanu površinu. Zanimljivo je da se površinske funkcije izvodnice za stupčano konveksne poliominoe na pravokutnoj i šesterokutnoj mreži mogu dobiti kao specijalni slučajevi funkcije F_b. Tematski je ovaj rad srodan radu Tzenga i Lina, koji su, na Wuov poticaj, stupčano konveksne poliominoena šahovskoj mreži prebrojili prema opsegu.