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NORMALIZERS OF IRREDUCIBLE SUBFACTORS

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Abstract

We consider normalizers of an infinite index irreducible inclusion $N \subseteq M$ of II_1 factors. Unlike the finite index setting, an inclusion $uNu^* \subseteq N$ can be strict, forcing us to also investigate the semigroup of one sided normalizers. We relate these one sided normalizers of N in M to projections in the basic construction and show that every trace one projection in the relative commutant $N' \cap \langle M, e_N \rangle$ is of the form $u^* e_N u$ for some unitary $u \in M$ with $uNu^* \subseteq N$ generalizing the finite index situation considered by Pimsner and Popa. We use this to show that each normalizer of a tensor product of irreducible subfactors is a tensor product of normalizers modulo a unitary. We also examine normalizers of infinite index irreducible subfactors arising from subgroup–group inclusions $H \subseteq G$. Here the one sided normalizers arise from appropriate group elements modulo a unitary from $L(H)$. We are also able to identify the finite trace $L(H)$-bimodules in $\ell^2(G)$ as double cosets which are also finite unions of left cosets.

1 Introduction

Dixmier, [3], was the first to recognize the importance of the normalizer $\mathcal{N}(A)$ for a von Neumann subalgebra A of a II_1 factor M. In the case of maximal abelian self-adjoint subalgebras (masas), he classified the masas according to whether $\mathcal{N}(A)''$ was M (regular), was a proper subfactor (semiregular), or was equal to A (singular). He also provided examples of each type by considering inclusions $H \subseteq G$ of suitably chosen group-subgroup pairs. Masas satisfy $A = A' \cap M$ and so their commutants are large. The opposite end of the spectrum is the condition $N' \cap M = \mathbb{C}1$, which defines an irreducible subfactor. Such subfactors will be the focus of our study. The isolated examples of singular subfactors in [20] were the starting point for a systematic examination of this phenomenon for inclusions of the form $M \rtimes_\alpha H \subseteq M \rtimes_\alpha G$ in [25, 26]. The algebra generated by the fixed point subfactor of a finite group action was determined in [8]. Singularity was connected to strong singularity of masas in [22, 24], and one consequence of this was the formula

$$\mathcal{N}(A_1 \mathbin{\overline{\otimes}} A_2)'' = \mathcal{N}(A_1)'' \mathbin{\overline{\otimes}} \mathcal{N}(A_2)'' \tag{1.1}$$

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of [24] for singular masas, which simply says that the tensor product of singular masas is again singular. Subsequently Chifan, [1], proved (1.1) for general masas (see also [16]). These papers collectively have provided strong motivation for the work undertaken here. It also depends heavily on the recent theory of perturbations, developed primarily by Popa, [16, 17, 18, 19, 5], building on the work of Christensen [2] in which an important averaging technique is developed.

A second crucial ingredient is the theory of subfactors [10, 9, 15, 13]. The link between normalizers and subfactors was made by Pimsner and Popa in [13]. Every normalizer $u \in \mathcal{N}(N)$ gives rise to a projection u^*e_Nu in the relative commutant $N' \cap \langle M, e_N\rangle$ for the basic construction $\langle M, e_N\rangle$ (see the next section for explanations of terminology). These projections have canonical trace equal to one. In the finite index case, Proposition 1.7 of [13], shows that every such projection arises from a normalizer in this way. In the infinite index situation, this breaks down and we are forced to work with the more general one sided normalizers, those unitaries $u \in M$ satisfying $uNu^* \subseteq N$. This containment can be strict, so normalizers and their one sided counterparts are distinct in general, as we show in Example 5.4. In the case of a finite index inclusion of factors $N \subseteq M$, each one sided normalizing unitary u induces an equivalence of containments $N \subseteq M$ and $uNu^* \subseteq uMu^* = M$ which then have equal finite indices. This is incompatible with $uNu^* \subsetneq N \subseteq M$ unless the first two algebras are equal, in which case u is a normalizing unitary. A second case where equality occurs is for masas. For a masa $A \subseteq M$, any unitary u which is a one sided normalizer of A has the property that $uAu^* \subseteq uMu^* = M$ is a masa in M contained inside the masa A. The defining property of masas then implies that $uAu^* = A$ and u is also a normalizing unitary.

The contents of the paper are as follows. Section 2 establishes notation and reviews some well-known facts about the basic construction. Section 3 examines the interplay between one sided normalizers and projections in $N' \cap \langle M, e_N\rangle$ when N is irreducible. Here it is shown that every such projection f satisfies $\mathrm{Tr}(f) \geq 1$, and is of the form u^*e_Nu for a one sided normalizer u precisely when $\mathrm{Tr}(f) = 1$. This generalizes Lemma 1.9 of [13], which handles the finite index case, and shows that the consideration of one sided normalizers rather than just normalizers is essential. These results occur in Theorem 3.5 which is the technical basis for section 4, in which we characterize both one sided normalizers and normalizers for tensor products (Theorems 4.1 and 4.2). The last section is devoted to group-subgroup inclusions. When $L(H) \subseteq L(G)$, we characterize the normalizers and one sided normalizers of $L(H)$ in terms of their counterparts at the group level. The ranges of projections in $L(H)' \cap \langle L(G), e_{L(H)}\rangle$ are the $L(H)$-bimodules in $\ell^2(G)$. Those that correspond to projections of finite trace are characterized algebraically in terms of left cosets and double cosets in Theorem 5.2, while the subsequent examples show that the situation is much more complicated for projections of infinite trace.

The following useful analogy between masas and subfactors has been implicit in much of the last two sections. For a masa $A \subseteq M$, the Pukánszky invariant is defined by using the algebra $\mathcal{A}' = (A \cup JAJ)'$, and this can also be viewed as the relative commutant $A' \cap \langle M, e_A\rangle$. It is type I, and the integers (including ∞) which comprise the Pukánszky invariant come from the various summands of type I_n in $e_A^\perp \mathcal{A}'$. For irreducible inclusions of factors $N \subseteq M$, essentially the same algebra $N' \cap \langle M, e_N\rangle$ occurs, where e_N is central just as e_A is central in the masa case. When an abelian subgroup $H \subseteq G$ generates a masa $L(H)$ in $L(G)$, it is often the case that the Pukánszky invariant can be determined from the structure of the double cosets HgH in G [23, 4]. These may be identified with $L(H)$-bimodules in $\ell^2(G)$, and as such they play a significant part in Section 5 where subfactors arising from subgroups are considered. The interplay between these various quantities has been studied extensively in the theory of finite index inclusions of factors [10, 9, 11] but the methods developed there do not seem helpful for the infinite index situation.

2 Notation and preliminaries

The basic object of study in this paper is an inclusion $N \subseteq M$ of II_1 factors, where the unique normalized faithful normal trace on M is denoted by τ. We will always assume that these factors are separable although this is just for notational convenience; the results are valid in general. We always assume that M is in standard form, so that it is represented as left multiplication operators on the Hilbert space $L^2(M,\tau)$, or simply $L^2(M)$. We reserve the letter ξ to denote the image of $1 \in M$ in this Hilbert space, and J will denote the isometric conjugate linear operator on $L^2(M)$ defined by

$$J(x\xi) = x^*\xi, \qquad x \in M, \tag{2.1}$$

and extended by continuity to $L^2(M)$ from the dense subspace $M\xi$. Then $L^2(N)$ is a closed subspace of $L^2(M)$, and e_N denotes the projection of $L^2(M)$ onto $L^2(N)$. The basic construction is the von Neumann algebra generated by M and e_N, and is denoted $\langle M, e_N\rangle$. Since $M' \cap B(L^2(M)) = JMJ$, we also have $\langle M, e_N\rangle' \cap B(L^2(M)) = JNJ$. This shows that $\langle M, e_N\rangle$ is either type II_1 or II_∞, and in both cases there is a unique semifinite normal trace Tr with the property that $\mathrm{Tr}(e_N) = 1$. The Jones index can be described as $\mathrm{Tr}(1)$, although this is not the original definition. These are standard facts in subfactor theory, and can be found in [10, 13, 9]. These sources also contain the following properties of the Jones projection e_N which we now list. We will use them subsequently without comment. The unique trace preserving conditional expectation of M onto N is denoted $\mathbb{E}_N$.

(i) $e_N(x\xi) = \mathbb{E}_N(x)\xi$, $x \in M$.

(ii) $e_N x e_N = \mathbb{E}_N(x)e_N = e_N\mathbb{E}_N(x)$, $x \in M$.

(iii) $x \mapsto e_N x$ and $x \mapsto x e_N$ are injective maps for $x \in M$.

(iv) $\{x e_N y \colon\ x, y \in M\}$ generates a strongly dense subalgebra of $\langle M, e_N\rangle$.

(v) $\mathrm{Tr}(x e_N y) = \tau(xy)$ for $x, y \in M$.

(vi) Me_N is $*$-strongly dense in $\langle M, e_N\rangle e_N$.

(vii) $e_N\langle M, e_N\rangle e_N = Ne_N = e_N N$.

(viii) $M \cap \{e_N\}' = N$.

(ix) Let $N_i \subseteq M_i$, $i = 1, 2$, be inclusions of II_1 factors and let Tr_i be the canonical trace on $\langle M_i, e_{N_i}\rangle$, $i = 1, 2$. Then

$$\langle M_1, e_{N_1}\rangle \mathbin{\overline{\otimes}} \langle M_2, e_{N_2}\rangle \cong \langle M_1 \mathbin{\overline{\otimes}} M_2, e_{N_1 \overline{\otimes} N_2}\rangle, \tag{2.2}$$

and $\mathrm{Tr}_1 \otimes \mathrm{Tr}_2$ is the canonical trace on the tensor product.

The last two sections are concerned with inclusions $H \subseteq G$ of groups. The canonical basis for $\ell^2(G)$ is denoted $\{\delta_g\colon\ g \in G\}$, and we assume that G is represented on this Hilbert space by the left regular representation λ, so that $\lambda_s\delta_t = \delta_{st}$ for $s, t \in G$. The right regular representation ρ satisfies $\rho(s) = J\lambda_s J$. As is standard, $L(G)$ is used for the von Neumann algebra generated by the left regular representation.

We will require the following lemma in Theorem 5.2. A similar result can be found in [25], but this is not quite in the form that we need, so we offer a slightly more general version here.

Lemma 2.1. *Let $N \subseteq M$ be an inclusion of II_1 factors on $L^2(M)$ such that $N' \cap M = \mathbb{C}1$. Let $\{\phi_1, \dots, \phi_n\}$ be a set of automorphisms of M with the property that the restriction of each $\phi_j^{-1}\phi_i$ to N is not implemented by a unitary in M whenever $i \neq j$. Let $X \subseteq N$ and $Y \subseteq JMJ$ be self-adjoint subsets which generate their respective containing factors and assume that $1 \in X$. Then the von Neumann subalgebra of $\mathbb{M}_n(B(L^2(M)))$ generated by*

$$\left\{ \begin{pmatrix} \phi_1(x)y & & \\ & \ddots & \\ & & \phi_n(x)y \end{pmatrix} :\ x \in X, y \in Y \right\}$$

is

$$\left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} :\ t_i \in B(L^2(M)) \right\}.$$

Proof. By the double commutant theorem, it suffices to show that the commutant of the first set of operators is the set of diagonal scalar matrices. Commutation with

$$\left\{ \begin{pmatrix} y & & \\ & \ddots & \\ & & y \end{pmatrix} :\ y \in Y \right\}$$

allows us to consider a matrix $(m_{ij}) \in \mathbb{M}_n(M)$. The conditions for this to commute with

$$\left\{ \begin{pmatrix} \phi_1(x) & & \\ & \ddots & \\ & & \phi_n(x) \end{pmatrix} :\ x \in X \right\}$$

are

$$\phi_i(x)m_{ij} = m_{ij}\phi_j(x), \qquad x \in X, \quad 1 \leq i, j \leq n, \tag{2.3}$$

which then must hold for all $x \in N$. Since $\phi_i(N)$ has trivial relative commutant in M, while (2.3) shows that $m_{ij}m_{ij}^* \in \phi_i(N)' \cap M$, we conclude that each m_{ij} is a scalar multiple of a unitary. The case $i = j$ in (2.3) places $m_{ii} \in \phi_i(N)' \cap M = \mathbb{C}1$ so the diagonal entries are scalars.

Suppose that some $m_{ij} \neq 0$ for a pair of integers $i \neq j$. By scaling we may replace m_{ij} in (2.3) by a unitary $u \in M$. If we apply ϕ_j^{-1}, then we obtain

$$\phi_j^{-1}(\phi_i(x))\phi_j^{-1}(u) = \phi_j^{-1}(u)x, \qquad x \in N, \tag{2.4}$$

which is contrary to the hypothesis that $\phi_j^{-1}\phi_i$ is not unitarily implemented on N. This shows that $m_{ij} = 0$ for $i \neq j$, completing the proof. □

3 One sided normalizers and the basic construction

Suppose that $N \subset M$ is an inclusion of II_1 factors. When $[M : N] < \infty$, [13] shows that $e_N\langle M, e_N\rangle = e_N M$ and there is a bijective correspondence between $\mathcal{U}(M)/\mathcal{U}(N)$ and projections with trace 1 in $\langle M, e_N\rangle$. In the infinite index situation these properties no longer hold. An example in [6] shows that the first of these conditions can fail in the infinite index situation.

Proposition 3.1. *Let $N \subset M$ be an inclusion of II_1 factors. Then the following conditions are equivalent:*

1. $[M : N] < \infty$.
2. $e_N\langle M, e_N\rangle = e_N M$.
3. *Every projection in $\langle M, e_N\rangle$ with trace 1 is of the form u^*e_Nu for some unitary $u \in M$.*

Proof. The equivalence between the first two conditions can be found in the remarks preceding [21, Corollary 4.2.5] and $1 \Rightarrow 3$ is the first part of Proposition 1.7 of [13]. It remains to show that $3 \Rightarrow 2$. This follows as given any unitary $V \in \langle M, e_N\rangle$, the projection V^*e_NV has trace one and so by hypothesis is of the form u^*e_Nu, for some unitary $u \in M$. Then $e_NV = e_NVu^*e_Nu \in e_NM$, as $e_NMe_N = e_NN$. Condition 2 follows as the unitaries V span $\langle M, e_N\rangle$. □

Throughout the remainder of this section $N \subseteq M$ will denote an irreducible inclusion of II_1 factors. The unitary group of M is written as $\mathcal{U}(M)$ and we use the notation

$$\mathcal{N}(N) = \{u \in \mathcal{U}(M) \colon\ uNu^* = N\}, \quad \mathcal{ON}(N) = \{u \in \mathcal{U}(M) \colon\ uNu^* \subseteq N\}$$

to denote respectively the group of unitary normalizers and the semigroup of one sided unitary normalizers of N. Unlike the first half of Proposition 1.7 of [13], the second half, characterizing projections of trace one in the relative commutant $N' \cap \langle M, e_N\rangle$, does generalize to the infinite index situation. In Theorem 3.5, we show that the trace of a projection in the relative commutant $N' \cap \langle M, e_N\rangle$ must be greater than or equal to one, and those projections of trace one are central and of the form u^*e_Nu for some $u \in \mathcal{ON}(N)$. The techniques of [13] are intrinsically finite index in nature, so we are forced to take a more circuitous approach. We begin by showing that such projections are central.

Lemma 3.2. *Let $u \in \mathcal{ON}(N)$, and let $\phi\colon\ N \to N$ be the $*$-homomorphism defined by $\phi(x) = uxu^*$, $x \in N$. Then e_N is a central projection in $\phi(N)' \cap \langle M, e_N\rangle$. In particular, this projection is central in $N' \cap \langle M, e_N\rangle$.*

Proof. Let v be a fixed but arbitrary unitary in $\phi(N)' \cap \langle M, e_N\rangle$. We begin by establishing that $v\xi = \lambda\xi$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

Let $\eta = v\xi \in L^2(M)$. By $\|\cdot\|_2$-density of $M\xi$ in $L^2(M)$, we may find a sequence $\{x_n\}_{n=1}^{\infty}$ in M such that

$$\lim_{n\to\infty} \|\eta - x_n\xi\|_2 = 0. \tag{3.1}$$

Noting that v commutes with $\phi(N)$ and with JNJ, we obtain

$$\begin{aligned} J\phi(w)J\phi(w)\eta &= J\phi(w)J\phi(w)v\xi = vJ\phi(w)J\phi(w)\xi \\ &= v\xi = \eta, \qquad w \in \mathcal{U}(N), \end{aligned} \tag{3.2}$$

so η is an invariant vector for $J\phi(w)J\phi(w)$. Then

$$\|J\phi(w)J\phi(w)x_n\xi - \eta\|_2 = \|J\phi(w)J\phi(w)(x_n\xi - \eta)\|_2 \le \|x_n\xi - \eta\|_2. \tag{3.3}$$

For each $n \in \mathbb{N}$, let $y_n \in M$ be such that $y_n\xi$ is the unique element of minimal $\|\cdot\|_2$-norm in

$$\overline{\text{conv}}^w\{\phi(w)x_n\phi(w)^*\xi\colon\ w \in \mathcal{U}(N)\},$$

by [21, Section 8.2]. Taking convex combinations and norm limits in (3.3) shows that

$$\|y_n\xi - \eta\|_2 \le \|x_n\xi - \eta\|_2, \tag{3.4}$$

so (3.1) implies that

$$\lim_{n\to\infty} \|y_n\xi - \eta\|_2 = 0. \tag{3.5}$$

Moreover, uniqueness of $y_n\xi$ shows that

$$\phi(w)y_n\phi(w)^* = y_n, \qquad w \in \mathcal{U}(N), \tag{3.6}$$

and so

$$\phi(w)y_n = y_n\phi(w), \qquad w \in \mathcal{U}(N). \tag{3.7}$$

Thus $y_n \in \phi(N)' \cap M = \mathbb{C}1$ since N is irreducible. From (3.1) and (3.5), we conclude that $\eta = \lambda\xi$ for some $\lambda \in \mathbb{C}$. Since $\|\eta\|_2 = \|v\xi\|_2 = 1$, it follows that $|\lambda| = 1$.

For an arbitrary $x \in M$,

$$ve_Nx\xi = v\mathbb{E}_N(x)\xi = vJ\mathbb{E}_N(x^*)J\xi. \tag{3.8}$$

Since v commutes with $JNJ = \langle M, e_N\rangle'$, (3.8) shows that

$$\begin{aligned} ve_Nx\xi &= J\mathbb{E}_N(x^*)Jv\xi = J\mathbb{E}_N(x^*)J\lambda\xi \\ &= \lambda\mathbb{E}_N(x)\xi = \lambda e_Nx\xi, \qquad x \in M. \end{aligned} \tag{3.9}$$

Thus $ve_N = \lambda e_N$, so

$$ve_Nv^* = (ve_N)(e_Nv^*) = \lambda e_N\bar{\lambda}e_N = e_N, \tag{3.10}$$

since $|\lambda|^2 = 1$, showing that v commutes with e_N. Since $v \in \phi(N)' \cap \langle M, e_N\rangle$ was an arbitrary unitary, we conclude that e_N is central in this algebra.

The second statement of the lemma is an immediate consequence of taking u to be 1, whereupon $\phi(N) = N$. $\square$

Lemma 3.3. *Let $u \in \mathcal{ON}(N)$ be a fixed but arbitrary unitary. Then u^*e_Nu is a minimal projection in $N' \cap \langle M, e_N\rangle$ and is also central in this algebra.*

Proof. As in the proof of Lemma 3.2, let $\phi\colon\ N \to N$ be the $*$-homomorphism defined by $\phi(x) = uxu^*$ for $x \in N$. If $y \in N' \cap \langle M, e_N\rangle$ and $x \in N$,

$$(u^*yu)x = u^*y\phi(x)u = u^*\phi(x)yu = x(u^*yu). \tag{3.11}$$

Then (3.11) shows that $u^*yu \in N' \cap \langle M, e_N\rangle$ whenever $y \in N' \cap \langle M, e_N\rangle$. In particular, $u^*e_Nu \in N' \cap \langle M, e_N\rangle$ by letting y be e_N.

To establish minimality of u^*e_Nu, consider a projection $q \in N' \cap \langle M, e_N\rangle$ satisfying $q \leq u^*e_Nu$. Then $uqu^* \leq e_N$, so there is a projection $p \in N$ such that $uqu^* = pe_N$, as implied by the relation $e_N\langle M, e_N\rangle e_N = Ne_N$ from Section 2. For each $x \in N$,

$$\begin{aligned}\phi(x)pe_N &= uxu^*uqu^* = uxqu^* = uqxu^*\\ &= uqu^*\phi(x) = pe_N\phi(x) = p\phi(x)e_N,\end{aligned} \tag{3.12}$$

and so $\phi(x)p = p\phi(x)$ for $x \in N$. Thus $p \in \phi(N)' \cap N \subseteq \phi(N)' \cap M = \mathbb{C}1$, showing that $p = 0$ or $p = 1$. It follows that $q = 0$ or $q = u^*e_Nu$, proving minimality of u^*e_Nu in $N' \cap \langle M, e_N\rangle$.

We now show centrality of u^*e_Nu. For $z \in N$ and $y \in N' \cap \langle M, e_N\rangle$,

$$uyu^*\phi(z) = \phi(z)uyu^* \tag{3.13}$$

by taking $x = \phi(z)$ in (3.11). So $uyu^* \in \phi(N)' \cap \langle M, e_N\rangle$. By Lemma 3.2, e_N is central in the latter algebra from which we obtain

$$e_Nuyu^* = uyu^*e_N. \tag{3.14}$$

In (3.14), multiply on the left by u^* and on the right by u. The result is that u^*e_Nu commutes with y for $y \in N' \cap \langle M, e_N\rangle$, showing centrality of u^*e_Nu. $\square$

Recall the following proposition from [13, Proposition 1.7]. No adjustments are needed for the infinite index case.

Proposition 3.4. *Let $u, v \in \mathcal{U}(M)$. Then $u^*e_Nu = v^*e_Nv$ if, and only if, there exists $w \in \mathcal{U}(N)$ such that $v = wu$.*

The following theorem is the main technical result which links one sided normalizers to certain projections in $N' \cap \langle M, e_N\rangle$. The construction of the $*$-homomorphism ϕ in the proof comes from [16].

Theorem 3.5. (i) *Each non–zero projection $f \in N' \cap \langle M, e_N\rangle$ satisfying* $\mathrm{Tr}(f) \leq 1$ *has the form u^*e_Nu for some $u \in \mathcal{ON}(N)$.*

(ii) *Each non–zero projection $f \in N' \cap \langle M, e_N\rangle$ satisfies* $\mathrm{Tr}(f) \geq 1$.

Proof. By Lemma 3.3, each projection u^*e_Nu, where $u \in \mathcal{ON}(N)$, lies in $N' \cap \langle M, e_N\rangle$. Conversely, let f be a non–zero projection in $N' \cap \langle M, e_N\rangle$ satisfying $\mathrm{Tr}(f) \le 1$, and choose a projection $p \in N$ such that $\tau(p) = \mathrm{Tr}(f)$. Then $\mathrm{Tr}(pe_N) = \tau(p)$, so f and pe_N are equivalent projections in the factor $\langle M, e_N\rangle$. Let $V \in \langle M, e_N\rangle$ be a partial isometry with $VV^* = pe_N$ and $V^*V = f$. For $x, y \in N$,

$$VxV^*VyV^* = VxfyV^* = VfxyV^* = VxyV^*, \tag{3.15}$$

and this shows that $x \mapsto VxV^*$ defines a $*$-homomorphism $\psi\colon\ N \to \langle M, e_N\rangle$. Since $e_NV = V$, the range of ψ is contained in $e_N\langle M, e_N\rangle e_N = Ne_N$, and so there is a $*$-homomorphism $\phi\colon\ N \to N$ such that

$$VxV^* = \phi(x)e_N = e_N\phi(x), \qquad x \in N. \tag{3.16}$$

If we multiply (3.16) on the left by V^* and use $f \in N' \cap \langle M, e_N\rangle$, $fV^* = V^*$ and $V^*e_N = V^*$, we obtain

$$xV^* = V^*\phi(x), \qquad x \in N. \tag{3.17}$$

As in [16] (see also the discussion in [21, Section 9.4]) there exists a non–zero partial isometry $w \in M$ such that

$$xw^* = w^*\phi(x), \qquad x \in N. \tag{3.18}$$

Multiplication by w on the right in (3.18) shows that $w^*w \in N' \cap M = \mathbb{C}1$, and so w is a unitary. Since $\phi(1)e_N = VV^* = pe_N$ from (3.16), while $\phi(1) = 1$ from (3.18), it follows that $p = 1$ and $\mathrm{Tr}(f) = 1$. Thus $\mathrm{Tr}(f) < 1$ is impossible, establishing (ii).

From (3.18), $wNw^* = \phi(N) \subseteq N$, and so $w \in \mathcal{ON}(N)$. Now consider $W = e_Nw \in \langle M, e_N\rangle$. This is a partial isometry because $WW^* = e_N$. Moreover, $W^*W = w^*e_Nw \in N' \cap \langle M, e_N\rangle$ and is central by Lemma 3.3. For $x \in N$,

$$\begin{aligned} W^*Vx &= W^*\phi(x)V = w^*e_N\phi(x)V \\ &= w^*\phi(x)e_NV = xw^*e_NV = xW^*V, \end{aligned} \tag{3.19}$$

using (3.17) and (3.18). Thus the operator W^*V lies in $N' \cap \langle M, e_N\rangle$. Now

$$(W^*V)(W^*V)^* = W^*VV^*W = w^*e_Nw, \tag{3.20}$$

so W^*V is a partial isometry. Also

$$(W^*V)^*(W^*V) = V^*WW^*V = fV^*WW^*Vf \tag{3.21}$$

and so is a projection q in $N' \cap \langle M, e_N\rangle$ below f. From (3.20), q is equivalent in $N' \cap \langle M, e_N\rangle$ to the central projection w^*e_Nw, so equality must hold. Thus $w^*e_Nw = q \le f$, and faithfulness of the trace Tr gives $w^*e_Nw = f$ since both projections have unit trace. This completes the proof of (i), and (ii) has already been proved. □

4 Tensor products

Throughout this section, we will consider two irreducible inclusions $N_i \subseteq M_i$, $i = 1, 2$, of II_1 factors. Our objective is to relate the normalizer of the tensor product to the normalizers of the individual algebras. In the context of masas $A_i \subseteq M_i$, $i = 1, 2$, Chifan, [1], has shown that $\mathcal{N}(A_1 \overline{\otimes} A_2)'' = \mathcal{N}(A_1)'' \overline{\otimes} \mathcal{N}(A_2)''$, and we will obtain a similar relationship for the N_i's below. We will also be able to identify explicitly the normalizing unitaries for the tensor product.

We let $M = M_1 \overline{\otimes} M_2$ and $N = N_1 \overline{\otimes} N_2$. Tomita's commutant theorem ensures that $N \subseteq M$ is an irreducible inclusion. The basic construction behaves well with respect to tensor products, and there is a natural isomorphism

$$\langle M, e_N \rangle \cong \langle M_1, e_{N_1} \rangle \overline{\otimes} \langle M_2, e_{N_2} \rangle \tag{4.1}$$

where $e_N = e_{N_1} \otimes e_{N_2}$. Let Tr_i, $i = 1, 2$, denote the canonical traces on $\langle M_i, e_{N_i} \rangle$ so that the canonical trace Tr on $\langle M, e_N \rangle$ is given by $\mathrm{Tr} = \mathrm{Tr}_1 \otimes \mathrm{Tr}_2$.

We begin by using the results of Section 3 to determine the one sided normalizers of the tensor product.

Theorem 4.1. *Each unitary $v \in \mathcal{ON}(N)$ has the form $w(u_1 \otimes u_2)$ where $w \in \mathcal{U}(N_1 \overline{\otimes} N_2)$ and $u_i \in \mathcal{ON}(N_i)$, $i = 1, 2$.*

Proof. It is clear that any unitary of the stated form is a one sided normalizer of $N_1 \overline{\otimes} N_2$. Conversely, consider a one sided unitary normalizer v of $N_1 \overline{\otimes} N_2$. Then $v^* e_N v$ is both a minimal and central projection in $N' \cap \langle M, e_N \rangle$, by Lemma 3.3. Two applications of Tomita's commutant theorem show that

$$N' \cap \langle M, e_N \rangle = (N_1' \cap \langle M_1, e_{N_1} \rangle) \overline{\otimes} (N_2' \cap \langle M_2, e_{N_2} \rangle) \tag{4.2}$$

and

$$Z(N' \cap \langle M, e_N \rangle) = Z(N_1' \cap \langle M_1, e_{N_1} \rangle) \overline{\otimes} Z(N_2' \cap \langle M_2, e_{N_2} \rangle), \tag{4.3}$$

where $Z(\cdot)$ denotes the center of an algebra. If these centers are decomposed as direct sums of their atomic and diffuse parts, then minimality for $v^* e_N v$ implies that $v^* e_N v = p_1 \otimes p_2$ for minimal projections $p_i \in Z(N_i' \cap \langle M_i, e_{N_i} \rangle)$, $i = 1, 2$. By Theorem 3.5, $\mathrm{Tr}_i(p_i) \geq 1$, forcing equality since $\mathrm{Tr}(v^* e_N v) = 1$. A second application of Theorem 3.5 gives the existence of unitaries $u_i \in \mathcal{ON}(N_i)$ such that $p_i = u_i^* e_{N_i} u_i$ for $i = 1, 2$. Thus

$$v^* e_N v = u_1^* e_{N_1} u_1 \otimes u_2^* e_{N_2} u_2 = (u_1 \otimes u_2)^* e_N (u_1 \otimes u_2). \tag{4.4}$$

By Lemma 3.4, there exists a unitary $w \in \mathcal{U}(N_1 \overline{\otimes} N_2)$ such that $v = w(u_1 \otimes u_2)$. $\square$

The case of one sided normalizers above easily leads to a similar result for unitary normalizers.

Theorem 4.2. *Each unitary $v \in \mathcal{N}(N)$ has the form $w(u_1 \otimes u_2)$ where $w \in \mathcal{U}(N_1 \overline{\otimes} N_2)$ and $u_i \in \mathcal{N}(N_i)$, $i = 1, 2$.*

Proof. Clearly each unitary of the stated form is a unitary normalizer of N. Conversely, let $v \in \mathcal{N}(N)$. Viewing v as a one sided normalizer, Theorem 4.1 implies that v has the form $v = w(u_1 \otimes u_2)$ where $w \in \mathcal{U}(N_1 \overline{\otimes} N_2)$ and $u_i \in \mathcal{ON}(N_i)$ for $i = 1, 2$. Then $w^* v \in \mathcal{N}(N_1 \overline{\otimes} N_2)$, showing that $u_i \in \mathcal{N}(N_i)$, otherwise $(u_1 \otimes u_2)^*(N_1 \overline{\otimes} N_2)(u_1 \otimes u_2)$ would be strictly contained in $N_1 \overline{\otimes} N_2$. $\square$

Corollary 4.3. *Let $N_i \subseteq M_i$, $i = 1, 2$, be irreducible inclusions of* II_1 *factors. Then*

$$\mathcal{ON}(N_1 \mathbin{\overline{\otimes}} N_2)'' = \mathcal{ON}(N_1)'' \mathbin{\overline{\otimes}} \mathcal{ON}(N_2)'' \tag{4.5}$$

and

$$\mathcal{N}(N_1 \mathbin{\overline{\otimes}} N_2)'' = \mathcal{N}(N_1)'' \mathbin{\overline{\otimes}} \mathcal{N}(N_2)''. \tag{4.6}$$

Proof. This is immediate from the characterizations of one sided and two-sided normalizers in Theorems 4.1 and 4.2. □

Corollary 4.4. *Let $N_i \subseteq M_i$, $i = 1, 2$ be inclusions of* II_1 *factors with N_i singular in M_i. Then $N_1 \mathbin{\overline{\otimes}} N_2$ is singular in $M_1 \mathbin{\overline{\otimes}} M_2$.*

Proof. This is immediate from Corollary 4.3 as a singular subfactor is automatically irreducible. □

Remark 4.5. The result of Theorem 4.2 can be false in more general situations. Consider two regular masas $A_i \subseteq M_i$, $i = 1, 2$. Then $A_1 \mathbin{\overline{\otimes}} A_2$ is a regular masa in $M_1 \mathbin{\overline{\otimes}} M_2$ since

$$\{u_1 \otimes u_2 \colon\ u_i \in \mathcal{N}(A_i)\} \subseteq \mathcal{N}(A_1 \mathbin{\overline{\otimes}} A_2).$$

Choose projections $p_i \in A_i$ of equal trace $1/2$, so that $p_1 \otimes 1$ and $1 \otimes p_2$ have equal trace in $A_1 \mathbin{\overline{\otimes}} A_2$. From [14], there exists a unitary $u \in \mathcal{N}(A_1 \mathbin{\overline{\otimes}} A_2)$ such that $u(p_1 \otimes 1)u^* = 1 \otimes p_2$. Then u cannot have the form $(u_1 \otimes u_2)w$ for $w \in \mathcal{U}(A_1 \mathbin{\overline{\otimes}} A_2)$ and $u_i \in \mathcal{N}(A_i)$, since this would imply that

$$\begin{aligned} 1 \otimes p_2 = u(p_1 \otimes 1)u^* &= (u_1 \otimes u_2)(p_1 \otimes 1)(u_1 \otimes u_2)^* \\ &= (u_1 p_1 u_1^* \otimes 1), \end{aligned} \tag{4.7}$$

which is impossible.

5 Group factors

This section is concerned with irreducible inclusions of II_1 factors which arise from infinite index inclusions of countable discrete groups. We examine the normalizers of $L(H) \subseteq L(G)$ when $H \subseteq G$, and relate these to the algebraic normalizers of H as a subgroup of G. One sided normalizers will again play a role, so apart from the standard notation

$$\mathcal{N}_G(H) = \{g \in G\colon\ gHg^{-1} = H\}$$

for the normalizer, we also introduce the semigroup of one sided normalizers

$$\mathcal{ON}_G(H) = \{g \in G\colon\ gHg^{-1} \subseteq H\}.$$

Subsequently we will exhibit situations where these two normalizers are distinct. To examine these normalizers, we look at the $L(H)$-bimodules in $\ell^2(G)$. Each projection $f \in L(H)' \cap \langle L(G), e_{L(H)}\rangle$ has a range which is invariant under left and right multiplications by elements of $L(H)$. Conversely, any norm closed $L(H)$-bimodule in $\ell^2(G)$ is the range of such a projection

f. The connection between bimodules and one-sided normalizers is then given by Theorem 3.5.

In the case of finite index inclusions of factors, bimodules have been extensively studied, [9, 11, 13], but new phenomena occur in the infinite index situation. In this section we investigate the structure of $L(H)' \cap \langle L(G), e_{L(H)} \rangle$. In Theorem 5.2 we characterize the bimodules for projections of finite trace, although the structure can be much more complicated when projections of infinite trace are considered. We recall from Theorem 3.5 (ii) that any non–zero projection $f \in L(H)' \cap \langle L(G), e_{L(H)} \rangle$ satisfies $\mathrm{Tr}(f) \geq 1$.

We will need the simple lemma below, which characterizes algebraically group-subgroup inclusions which give rise to an irreducible inclusion of subfactors. The proof is standard and can be constructed by following the argument that a countable discrete group G gives rise to a factor $L(G)$ if, and only if, G is I.C.C. from [12].

Lemma 5.1. *Let $K \subseteq H \subseteq G$ be an inclusion of countable discrete groups.*

(i) *$L(H)$ is irreducible in $L(G)$ if and only if each $g \in G \backslash \{e\}$ has infinitely many H-conjugates;*

(ii) *If G is I.C.C., $L(H)$ is irreducible in $L(G)$ and K has finite index in H, then $L(K)$ is irreducible in $L(G)$.*

In order to motivate the next theorem, consider an inclusion $H \subseteq G$ of countable discrete groups such that $L(H)' \cap L(G) = \mathbb{C}1$. Then, for $g \in G$, the operator $\lambda_g e_{L(H)} \lambda_g^*$ has unit trace and is the projection onto $\lambda_g \ell^2(H)$, the closed subspace generated by a left coset gH of H. If we have a closed subspace generated by a double coset HgH with associated projection f, then $f \in L(H)'$. Moreover, if HgH is written as the disjoint union of left cosets $\{g_i H\}_{i \in I}$, then

$$f = \sum_{i \in I} \lambda_{g_i} e_{L(H)} \lambda_{g_i}^* \in L(H)' \cap \langle L(G), e_{L(H)} \rangle,$$

and so $\mathrm{Tr}(f)$ is the cardinality of the set I, which is either ∞ or a finite integer. These remarks also apply to finite unions of double cosets. The following result gives the converse of this discussion.

Theorem 5.2. *Let $H \subseteq G$ be an inclusion of countable discrete groups such that $L(H)' \cap L(G) = \mathbb{C}1$, and let $f \in L(H)' \cap \langle L(G), e_{L(H)} \rangle$ be a non–zero projection such that $\mathrm{Tr}(f) < \infty$. Then $\mathrm{Tr}(f)$ is an integer, and there exist $g_1, \ldots, g_n \in G$ such that the range of f is the direct sum $\bigoplus_{i=1}^{n} \lambda_{g_i} \ell^2(H)$. In particular, the range of f is a finite sum of $L(H)$-bimodules generated by double cosets HgH each of which is a finite sum of right $L(H)$-modules generated by left cosets gH.*

Proof. We consider a non–zero projection $f \in L(H)' \cap \langle L(G), e_{L(H)} \rangle$ with $\mathrm{Tr}(f) < \infty$, and we write $\mathrm{Tr}(f) = (n-1) + \mu$ where $n \in \mathbb{N}$ and $\mu \in (0, 1]$. In the course of the proof it will be shown that $\mu = 1$.

Choose a projection $p \in L(H)$ with $\tau(p) = \mu$. Following the approach of [13], the diagonal projections

$$P_1 = \begin{pmatrix} f & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} e_{L(H)} & & & & \\ & e_{L(H)} & & & \\ & & \ddots & & \\ & & & e_{L(H)} & \\ & & & & pe_{L(H)} \end{pmatrix} \tag{5.1}$$

in $\mathbb{M}_n(\langle L(G), e_{L(H)}\rangle)$ have equal finite traces and so are equivalent in this factor. Thus there exists a column matrix $V = (v_1, \ldots, v_n)^T$ with entries $v_i \in \langle L(G), e_{L(H)}\rangle$ such that $V^*V = f$ and $VV^* = P_2$. In particular, $v_i^* e_{L(H)} = v_i^*$, $1 \le i \le n$. As in [16] (see also [21, Section 8.4]), the map $x \mapsto VxV^*$ defines a homomorphism $\psi\colon\ L(H) \to \mathbb{M}_n\langle L(G), e_{L(H)}\rangle$ whose range lies under P_2, and so there is a homomorphism $\phi\colon\ L(H) \to \mathbb{M}_n(L(H))$ such that

$$\psi(x) = \phi(x)P_2, \qquad \phi(1) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & p \end{pmatrix}, \tag{5.2}$$

for $x \in L(H)$. Then

$$Vx = Vfx = Vxf = VxV^*V = \phi(x)P_2V = \phi(x)V, \qquad x \in L(H), \tag{5.3}$$

so

$$xV^* = V^*\phi(x), \qquad x \in L(H), \tag{5.4}$$

by taking adjoints in (5.3). Note that (5.4) is an equality of $1 \times n$ row operators with entries from $\langle L(G), e_{L(H)}\rangle$. Let $\eta_j = v_j^*\xi \in \ell^2(G)$, $1 \le j \le n$. For a fixed j, the Kaplansky density theorem allows us to choose a uniformly bounded net $(w_\alpha e_{L(H)})$ converging $*$-strongly to $v_j^* e_{L(H)}$, where $w_\alpha \in L(G)$. For each $y \in L(H)$,

$$v_j^* y\xi = \lim_\alpha w_\alpha y\xi = \lim_\alpha Jy^*Jw_\alpha\xi = \eta_j y, \tag{5.5}$$

and this equality then holds for each j, $1 \le j \le n$. Now apply (5.4) to column vectors whose only non-zero entries are ξ in the j^{th} component, $1 \le j \le n$, and use (5.5) to conclude that

$$x(\eta_1, \eta_2, \ldots, \eta_n) = (\eta_1, \eta_2, \ldots, \eta_n)\phi(x), \qquad x \in L(H), \tag{5.6}$$

where the right action of $L(H)$ on $\ell^2(G)$ is used to define the multiplication on the right-hand side of this equation. By putting $x = 1$ in (5.6), we see that $\eta_n = \eta_n p$, so (5.6) can also be written as

$$u(\eta_1, \ldots, \eta_n)\phi(u^*) = (\eta_1, \ldots, \eta_n), \qquad u \in \mathcal{U}(L(H)). \tag{5.7}$$

Choose a sequence $(y_{1,m}\xi, \ldots, y_{n,m}\xi)$, $m \ge 1$, converging to $(\eta_1, \ldots, \eta_n)$ in $\|\cdot\|_2$-norm where $y_{i,m} \in L(G)$. The convex sets

$$K_m = \overline{\text{conv}}^w\{u(y_{1,m}, \ldots, y_{n,m})\phi(u^*)\colon\ u \in \mathcal{U}(L(H))\}$$

are weakly compact in $L(G) \times \cdots \times L(G)$, so the image in $\ell^2(G) \oplus \cdots \oplus \ell^2(G)$ is also weakly compact and weakly closed. Since K_m is invariant for the action $u \cdot \phi(u^*)$, the unique element $(w_{1,m}\xi, \ldots, w_{n,m}\xi) \in K_m$ of minimal $\|\cdot\|_2$-norm, with $w_{i,m} \in L(G)$, satisfies

$$x(w_{1,m}, \ldots, w_{n,m}) = (w_{1,m}, \ldots, w_{n,m})\phi(x), \qquad x \in L(H). \tag{5.8}$$

Moreover, $\lim\limits_{m\to\infty} \|\eta_i - w_{i,m}\xi\|_2 = 0$ for $1 \le i \le n$. It follows from (5.8) that $\sum\limits_{i=1}^{n} w_{i,m}w_{i,m}^* \in L(H)' \cap L(G) = \mathbb{C}1$ for each $m \ge 1$. Thus the $\|\cdot\|_2$-norm and operator norm agree for $(w_{1,m}, \ldots, w_{n,m})$, $m \ge 1$. Since these converge to $(\eta_1, \ldots, \eta_n)$, they are bounded in $\|\cdot\|_2$-norm and hence in operator norm. By dropping to a subnet, we may further assume that they converge weakly to a row operator $(w_1, \ldots, w_n) \in L(G) \times \cdots \times L(G)$, whereupon $\eta_i = w_i\xi$ for $1 \le i \le n$. From (5.5), we conclude that

$$v_j^* y\xi = \eta_j y = w_j \xi y = w_j y \xi, \qquad y \in L(H), \tag{5.9}$$

and so $v_j^* e_{L(H)} = w_j e_{L(H)}$ for $1 \le j \le n$. Moreover, (5.6) becomes

$$x(w_1, \ldots, w_n) = (w_1, \ldots, w_n)\phi(x), \qquad x \in L(H), \tag{5.10}$$

and $w_n p = w_n$, by putting $x = 1$.

Let $\{g_i H\colon\ i \ge 1\}$ be a listing of the left H-cosets in G. Then there exist row operators $(z_{1,j}, \ldots, z_{n,j})$, $j \ge 1$, with $z_{i,j} \in L(H)$ such that

$$(w_1, \ldots, w_n) = \sum_{j=1}^{\infty} \lambda_{g_j}(z_{1,j}, \ldots, z_{n,j}), \tag{5.11}$$

where the sum, which could be finite, converges in $\|\cdot\|_2$-norm, and $z_{n,j}p = z_{n,j}$. For each $h \in H$, (5.10) gives

$$\lambda_h \sum_{j=1}^{\infty} \lambda_{g_j}(z_{1,j}, \ldots, z_{n,j}) = \sum_{j=1}^{\infty} \lambda_{g_j}(z_{1,j}, \ldots, z_{n,j})\phi(\lambda_h). \tag{5.12}$$

For convenience, write $Z_j = (z_{1,j}, \ldots, z_{n,j})$ and suppose that the numbering has been chosen so that $\|Z_j\|_2 \ge \|Z_{j+1}\|_2$, for $j \ge 1$, possible because $\|Z_j\|_2 \to 0$ as $j \to \infty$. If $\|Z_j\|_2 \ne 0$, then $S_j = \{i\colon\ \|Z_i\|_2 = \|Z_j\|_2\}$ is a finite set. Each $h \in H$ defines a permutation of the left H-cosets by $g_i H \mapsto h g_i H$, and so there is a permutation π_h of $\{1, 2, \ldots\}$ such that $h g_i H = g_{\pi_h(i)} H$. The map $h \mapsto \pi_h$ is then a homomorphism of H into the group of permutations of $\mathbb{N}$. Moreover, there are maps $\alpha_i\colon\ H \to H$ such that $h g_i = g_{\pi_h(i)}\alpha_i(h)$, $h \in H$, and (5.12) becomes

$$\sum_{j=1}^{\infty} \lambda_{g_{\pi_h(j)}} \lambda_{\alpha_i(h)}(z_{1,j}, \ldots, z_{n,j}) = \sum_{j=1}^{\infty} \lambda_{g_j}(z_{1,j}, \ldots, z_{n,j})\phi(\lambda_h). \tag{5.13}$$

It follows that

$$\lambda_{g_{\pi_h(j)}} \lambda_{\alpha_i(h)} Z_j = \lambda_{g_{\pi_h(j)}} Z_{\pi_h(j)} \phi(\lambda_h) \tag{5.14}$$

for each j. Taking 2-norms, we obtain $\|Z_j\|_2 = \|Z_{\pi_h(j)}\|_2$, so each $h \in H$ permutes the cosets $\{g_iH\colon\ i \in S_j\}$. We will now show that the number of non-zero Z_j's must be at least n.

The range of f is the range of $V^* = V^*e_{L(H)}$ and this operator is $(w_1e_{L(H)}, \ldots, w_ne_{L(H)})$. Thus the range of f is contained in the closure of

$$\left\{\sum_{i=1}^{n} w_i\zeta_i\colon\ \zeta_i \in \ell^2(H)\right\}.$$

Indeed, equality must hold since the projection onto this subspace is

$$\sum_{i=1}^{n} w_ie_{L(H)}w_i^* = V^*V = f.$$

If Z_j, $1 \le j \le r < n$, are the only non-zero Z_j's, then (5.11) shows that the range of f is contained in

$$\left\{\sum_{i=1}^{r} \lambda_{g_i}\zeta_i\colon\ \zeta_i \in \ell^2(H)\right\}$$

and the projection onto this space is $\sum\limits_{i=1}^{r} \lambda_{g_i}e_{L(H)}\lambda_{g_i}^*$ which has trace $r \le n-1$, contradicting $\mathrm{Tr}(f) > n-1$.

Thus we may pick an integer $N \ge n$ such that $\|Z_N\|_2 > \|Z_{N+1}\|_2$. Each $h \in H$ permutes the left cosets $\{g_iH\colon\ 1 \le i \le N\}$, so the restriction of π_h to $\{g_iH\colon\ 1 \le i \le N\}$ gives a homomorphism of H into the finite group of permutations of $\{g_iH\colon\ 1 \le i \le N\}$, and so the kernel K has finite index in H. For each $1 \le i \le N$, $kg_iH = g_iH$ for $k \in K$, so $g_i^{-1} \cdot g_i$ induces a homomorphism $\phi_i\colon\ K \to H$ for $1 \le i \le N$. Since each $Z_j \ne 0$ for $1 \le j \le N$, from (5.11) we can find vectors $(\zeta_{1,j}, \ldots, \zeta_{n,j})$, $1 \le j \le N$, $\zeta_{i,j} \in \ell^2(H)$, such that $\sum\limits_{i=1}^{n} w_i\zeta_{i,j}$ has a non-zero λ_{g_j}-coefficient. A suitable linear combination then gives a vector $\sum\limits_{j=1}^{\infty} \lambda_{g_j}\zeta_j \in \mathrm{Ran}\ f$, where $\zeta_j \in \ell^2(H)$, $j \ge 1$, and are non-zero for $1 \le j \le N$. Pre-multiplication by K and post-multiplication by H allow us to find vectors in the range of f whose first N components are

$$\sum_{i=1}^{N} \lambda_{g_i}\phi_i(\lambda_k)J\lambda_h^*J\zeta_i,$$

which we write in matrix form as

$$(\lambda_{g_1}, \ldots, \lambda_{g_N}) \begin{pmatrix} \phi_1(\lambda_k)J\lambda_h^*J & & \\ & \ddots & \\ & & \phi_N(\lambda_k)J\lambda_h^*J \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_N \end{pmatrix}. \tag{5.15}$$

For $i \ne j$, $\phi_i\phi_j^{-1}(x) = \lambda_{g_i^{-1}g_j}x\lambda_{g_j^{-1}g_i}$ for $x \in L(G)$. If there is a unitary $u \in L(H)$ with $\phi_i\phi_j^{-1}(y) = uyu^*$ for all $y \in L(K)$, then $u^*\lambda_{g_i^{-1}g_j} \in L(K)' \cap L(G) = \mathbb{C}1$ by Lemma 5.1. Hence

$g_i^{-1}g_j \in H$, which is a contradiction as $g_iH \neq g_jH$. We can now apply Lemma 2.1 to deduce that the diagonal matrices in (5.15) generate the von Neumann algebra

$$\left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_N \end{pmatrix} : \ t_i \in B(\ell^2(H)) \right\}.$$

Since $\zeta_i \neq 0$ for $1 \leq i \leq N$, we see that

$$\operatorname{span}\left\{ \begin{pmatrix} \phi_1(\lambda_k)J\lambda_h^*J & & \\ & \ddots & \\ & & \phi_N(\lambda_k)J\lambda_h^*J \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_N \end{pmatrix} : \ k \in K, \quad h \in H \right\}$$

is dense in the direct sum of N copies of $\ell^2(H)$. If $\tilde{f} = \sum_{i=1}^{N} \lambda_{g_i} e_{L(H)} \lambda_{g_i}^*$, then $\tilde{f}$ is a projection of trace N. The range projection of $\tilde{f}f$ is $\tilde{f}$ while the range projection of $f\tilde{f}$ lies under f. Since these range projections are equivalent in $\langle L(G), e_{L(H)} \rangle$, we conclude that $\mathrm{Tr}(\tilde{f}) \leq \mathrm{Tr}(f)$. Thus

$$n \leq N = \mathrm{Tr}(\tilde{f}) \leq \mathrm{Tr}(f) = n - 1 + \mu, \tag{5.16}$$

forcing $\mu = 1$, and $N = n$. In particular, no choice of $N > n$ was possible. Thus only λ_{g_j} terms for $j \leq n$ appear in (5.11) and so $f \leq \tilde{f}$. Equality of the traces then gives $f = \tilde{f}$, and the result follows. □

We can now deduce that, modulo a unitary from the smaller algebra, normalizers of irreducible subfactors coming from group-subgroup inclusions are given by normalizing elements of the group.

Corollary 5.3. *Let $H \subseteq G$ be an inclusion of countable discrete groups, where G is I.C.C. and $L(H)$ is irreducible in $L(G)$.*

(i) *Each $u \in \mathcal{ON}(L(H))$ has the form $w\lambda_g$ for $w \in \mathcal{U}(L(H))$ and $g \in \mathcal{ON}_G(H)$;*

(ii) *each $u \in \mathcal{N}(L(H))$ has the form $w\lambda_g$ for $w \in \mathcal{U}(L(H))$ and $g \in \mathcal{N}_G(H)$.*

Proof. Given a one-sided normalizer u of $L(H)$ in $L(G)$, the projection $u^*e_{L(H)}u$ lies in $L(H)' \cap \langle L(G), e_{L(H)} \rangle$ so by Theorem 5.2, there exists $g \in G$ with $u^*e_{L(H)}u = \lambda_g^*e_{L(H)}\lambda_g$. Proposition 3.4 gives a unitary $w \in L(H)$ with $u = w\lambda_g$. Since w^*u is a one-sided normalizer of $L(H)$, it follows that $\lambda_g \in \mathcal{ON}(L(H))$ and so $g \in \mathcal{ON}_G(H)$. For (ii), suppose additionally that u is a normalizer of $L(H)$. Then $\lambda_g = w^*u$ normalizes $L(H)$ so that $g \in \mathcal{N}_G(H)$. □

Example 5.4. An immediate consequence of Corollary 5.3 (ii) is that $L(H)$ is singular in $L(G)$ precisely when $\mathcal{N}_G(H) = H$. Here we give examples of singular inclusions $L(H) \subseteq L(G)$ which nevertheless have non-trivial one sided normalizers.

Consider the free group $\mathbb{F}_\infty$, where the generators are written $\{g_i : \ i \in \mathbb{Z}\}$, and for each $n \in \mathbb{Z}$, let H_n be the subgroup generated by $\{g_i : \ i \geq n\}$. The shift $i \mapsto i+1$ on $\mathbb{Z}$ induces an automorphism ϕ of $\mathbb{F}_\infty$ defined on generators by $\phi(g_i) = g_{i+1}$, $i \in \mathbb{Z}$, and ϕ maps H_n into

$H_{n+1} \subseteq H_n$. Then $n \mapsto \phi^n$ gives a homomorphism $\alpha\colon \mathbb{Z} \to \mathrm{Aut}(\mathbb{F}_\infty)$, and we let G be the semidirect product $\mathbb{F}_\infty \rtimes_\alpha \mathbb{Z}$. We abuse notation and write the elements of this group as $w\phi^n$ where $w \in \mathbb{F}_\infty$. The multiplication is

$$(v\phi^n)(w\phi^m) = (v\phi^n(w))\phi^{n+m}, \qquad v, w \in \mathbb{F}_\infty, \quad m, n \in \mathbb{Z}. \tag{5.17}$$

We then consider the inclusion $H_n \subseteq \mathbb{F}_\infty \rtimes_\alpha \mathbb{Z}$. By construction $\phi H_n \phi^{-1} = H_{n+1}$ so ϕ is a one sided normalizer of H_n for each $n \in \mathbb{Z}$. We now show that the only normalizers of H_n lie in H_n.

Suppose that $v\phi^k$ has the property that $v\phi^k H_n \phi^{-k} v^{-1} = H_n$. If $v \in H_n$ then $H_{n+k} = H_n$, forcing $k = 0$, and we see that $v\phi^k \in H_n$. Thus we may assume that $v \notin H_n$. Let j be the minimal integer such that g_j appears in v. Then $j < n$ otherwise $v \in H_n$. Then $H_{n+k} = v^{-1} H_n v \subseteq H_j$, so $n + k \geq j$. Take $r > n$ such that the letter g_r does not appear in the reduced word v. Then there is no cancellation in $v^{-1} g_r v$. In particular, the letter g_j cannot cancel from $v^{-1} g_r v \in v^{-1} H_n v = H_{n+k}$, and so $n + k \leq j$, showing that $v^{-1} H_n v = H_j$. There is also no cancellation in $v g_r v^{-1}$, so $v g_r v^{-1} \in v H_j v^{-1}$ is not contained in H_n, a contradiction. Thus there are no non-trivial normalizers of H_n, so $L(H_n)$ is singular in $L(G)$ although it does have non-trivial one sided normalizers. Further algebraic calculations along the same lines show that

$$\mathcal{ON}_G(H_n) = \{v\phi^r\colon\ v \in H_n, \quad r \geq 0\}, \qquad n \in \mathbb{Z}. \tag{5.18}$$

We omit the easy details.

It is worth noting that the disparity between normalizers and one sided normalizers in this example is extreme; the former generate $L(H_n)$ while the latter generate $L(G)$.

Remark 5.5. Just as in Remark 4.5, the analogous statement to Corollary 5.3 is false in the abelian situation. Let H be an abelian subgroup of an I.C.C. group G such that every element of $G \setminus H$ has infinitely many H-conjugates — this is Dixmier's condition, [3], which is equivalent to $L(H)$ being a masa in $L(G)$. Normalizers of $L(H)$ are not necessarily of the form $u\lambda_g$ for some $g \in \mathcal{N}_G(H)$ and a unitary $u \in L(H)$. This leads to a question to which we do not know the answer. Suppose that $\mathcal{N}_G(H) = H$. Must $L(H)$ be singular in $L(G)$? The methods used to prove singularity of masas coming from subgroups, [22], [20, Lemma 2.1], require additional algebraic conditions on $H \subseteq G$. □

Example 3.5 of [6] demonstrates (for the inclusion of free groups $\mathbb{F}_2 \subseteq \mathbb{F}_3$) that $L(H)' \cap \langle L(G), e_H \rangle$ can contain a II_∞ factor. Our final example shows that, even in the singular infinite index case, this algebra can also be atomic, abelian and generated by its minimal projections of finite trace. Furthermore, the traces of these minimal projections can be uniformly bounded. We note that any countable discrete group G can act on $\mathbb{F}_{|G|}$ by outer automorphisms. Index the generators of $\mathbb{F}_{|G|}$ by $\{g_t\colon\ t \in G\}$ and let $\beta_s \in \mathrm{Aut}(\mathbb{F}_{|G|})$ be defined on generators by $g_t \mapsto g_{st}$, $s, t \in G$. The semidirect product $\mathbb{F}_{|G|} \rtimes_\beta G$ is a countable I.C.C. group.

Example 5.6. Let $\mathbb{Z}_2$ act on $\mathbb{Z}$ by

$$\alpha_m(n) = (-1)^m n, \qquad n \in \mathbb{Z}, \quad m \in \mathbb{Z}_2, \tag{5.19}$$

and then let $\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2$ act on $\mathbb{F}_\infty$ by an action β as described above. Set $G = \mathbb{F}_\infty \rtimes_\beta (\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2)$ and let H be the subgroup generated by $\mathbb{F}_\infty$ and $\mathbb{Z}_2$. Each $g \in G$ has infinitely many H-conjugates and so $L(H)$ is irreducible in $L(G)$. Any $g \in G \backslash H$ contains a non-zero group element $n \in \mathbb{Z}$, and then properties of the semidirect product show that the double coset HgH is a union of two left cosets generated by $\pm n \in \mathbb{Z}$. By Theorem 5.2, we see that each of these double cosets corresponds to a minimal projection in $L(H)' \cap \langle L(G), e_{L(H)} \rangle$ of trace 2, so this algebra is abelian and any projection in it under $1 - e_{L(H)}$ is an orthogonal sum of projections of trace 2.

Many variations on this theme are possible. Replace $\mathbb{Z}_2$ by a group of order n and replace $\mathbb{Z}$ by an infinite group on which it acts. The minimal projections will then all have integer trace bounded by n.

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