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A full-Newton step feasible interior-point algorithm for $P_*(\kappa)$-LCP based on a new search direction

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Abstract. In this paper, we present a full-Newton step feasible interior-point algorithm for a $P_*(\kappa)$ linear complementarity problem based on a new search direction. We apply a vector-valued function generated by a univariate function on nonlinear equations of the system which defines the central path. Furthermore, we derive the iteration bound for the algorithm, which coincides with the best-known iteration bound for these types of algorithms. Numerical results show that the proposed algorithm is competitive and reliable.

Keywords: interior-point methods, $P_*(\kappa)$-linear complementarity problem, full-Newton step, polynomial complexity

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1. Introduction

Given $M \in R^{n\times n}$ and $q \in R^n$, the standard linear complementarity problem (LCP) is to find a vector pair $(x, s) \in R^n \times R^n$ such that

$$s = Mx + q, \quad xs = 0, \quad (x, s) \geq 0, \tag{1}$$

where xs denotes Hadamard product of vectors x and s, i.e., $xs = [x_1s_1, \dots, x_ns_n]^T$. We shall also use the notation $\frac{x}{s} = [\frac{x_1}{s_1}, \dots, \frac{x_n}{s_n}]^T$, where $s_i \neq 0$ for all $1 \leq i \leq n$. For an arbitrary univariate function f and a vector x, we will use the notation $f(x) = [f(x_1), \dots, f(x_n)]^T$.

If M is a symmetric positive semidefinite matrix, then the LCP is called the monotone LCP, which finds many applications in engineering and economics [6]. In this paper, we consider problem (1) with M being a $P_*(\kappa)$-matrix. The class of $P_*(\kappa)$-matrices was introduced by Kojima et al. [16]. Let κ be a nonnegative number. A matrix $M \in R^{n\times n}$ is called a $P_*(\kappa)$-matrix if and only if

$$(1 + 4\kappa) \sum_{i\in I_+} x_i(Mx)_i + \sum_{i\in I_-} x_i(Mx)_i \geq 0, \quad \forall x \in R^n,$$

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where $I_+ = \{i \in I : x_i(Mx)_i \geq 0\}$ and $I_- = \{i \in I : x_i(Mx)_i < 0\}$ are two index sets. Note that the $P_*(\kappa)$-LCP contains the class of the monotone LCP as a special case ($\kappa = 0$). The theoretical importance of this class of LCPs lays in the fact that this is the largest class for which the polynomial global convergence of IPMs can be proved without additional conditions. There is a number of different interior-point methods (IPMs) to solve $P_*(\kappa)$-LCPs. Kojima et al. [16] first proved the existence of the central path for $P_*(\kappa)$-LCP and generalized the primal-dual interior-point algorithm for linear optimization (LO) [20] to $P_*(\kappa)$-LCP. Their algorithm has polynomial iteration complexity $O((1+\kappa)\sqrt{n}L)$, which is still the best complexity result for solving $P_*(\kappa)$-LCPs. Kojima et al. [17] also introduced a new potential reduction algorithm for solving LCPs. Potra and Sheng [22] defined a predictor-corrector algorithm for the $P_*(\kappa)$-matrix LCPs. Illés et al. [11] presented a polynomial path-following interior-point algorithm for general LCPs. Miao [19] extended the Mizuno-Todd-Ye (MTY) predictor-corrector method to $P_*(\kappa)$-LCP. Lesaja and Roos [18] proposed a unified analysis of the IPM for $P_*(\kappa)$-LCP based on the class of eligible kernel functions which was first introduced by Bai et al. [5] for LO problems.

Roos et al. [23] first analyzed the primal-dual full-Newton step feasible IPM for LO and obtained the currently best known iteration bound for small-update methods, namely, $O(\sqrt{n}\log\frac{n}{\epsilon})$. Wang et al. [26] extended Roos et al.'s full-Newton step primal-dual interior-point algorithm for LO to $P_*(\kappa)$-LCP. Darvay [7] proposed a full-Newton step primal-dual interior-point algorithm for LO that is based on a new class of search directions. The search direction of his algorithm was introduced by using an algebraic equivalent transformation of the nonlinear equations which defines the central path and then applying Newton's method for the new system of equations. Infeasible IPMs for LO based on this technique were proposed in [3, 8]. Achache [2], Asadi and Mansouri [4] and Kheirfam [12] presented numerical results on LCPs based on this technique. Later on, Achache [1], Wang and Bai [27, 28, 29] and Wang et al. [30] extended Darvay's algorithm for LO to convex quadratic optimization (CQO), semidefinite optimization (SDO), second-order cone optimization (SOCO), symmetric cone optimization (SCO) and $P_*(\kappa)$-LCP, respectively. Kheirfam introduced an infeasible IPM for SCO in [13]. Kheirfam and Mahdavi-Amiri [14] and Kheirfam [15] presented a new full-Newton step interior-point algorithm for SCO and the Cartesian $P_*(\kappa)$-LCP over symmetric cones based on modified Newton direction which differs from Darvay's search direction only by a constant multiplier, respectively. Furthermore, Wang proposed a new polynomial interior-point algorithm for the monotone LCPs over symmetric cones with full Nesterov-Todd step [25]. However, Pan et al. [21] devised an infeasible IPM for LO based on a logarithmic equivalent transformation.

Recently, Darvay et al. [9] introduced a new IPM for LO which is based on a new algebraic reformulation of the central path. Later, Darvay and Takács [10] generalized this approach to SCO. In a recent report, Takács and Darvay [24] presented a new full-Newton step infeasible IPM for SCO based on the search direction given by Darvay and Takács in [10].

An interesting question here is whether a new class of search directions can be found where the full-Newton step feasible interior-point algorithm based on the new search directions is well defined. In this paper, we offer a different search direction from the usual Newton directions, modified Newton directions and Darvay's directions, in order to analyze the full-Newton step feasible interior-point algorithm for $P_*(\kappa)$-LCP. These directions are based on a new algebraic equivalent transformation of the nonlinear equations of the system which defines the central path. We develop some new results and prove that the complexity bound of the proposed algorithm is $O((4+7\kappa)\sqrt{n}\log\frac{n}{\epsilon})$. The complexity bound obtained here is the same as small-update methods.

The paper is organized as follows. In Sect. 2, we first propose the new search directions, then present the full-Newton step feasible interior-point algorithm based on the new directions for $P_*(\kappa)$-LCP. In Sect. 3, we analyze the algorithm and derive the currently best known iteration bound for small-update methods. Some numerical results are reported in Sect. 4. Finally, some conclusions and remarks are given in Sect. 5.

2. A new full-Newton step feasible IPM

In this section, we first recall the central path for $P_*(\kappa)$-LCP. Then, we derive the new search directions based on a new equivalent algebraic transformation for $P_*(\kappa)$-LCP. Finally, the generic full-Newton step feasible interior-point algorithm based on the new search directions is provided.

2.1. Central path

The basic idea underlying IPMs is to replace the second equation in (1) by the parameterized equation $xs = \mu e$, with parameter $\mu > 0$ and e denoting the all-one vector $(1, 1, \ldots, 1)^T$. The system (1) becomes:

$$s = Mx + q, \quad xs = \mu e, \quad (x, s) \geq 0. \tag{2}$$

Throughout the paper, we assume that $P_*(\kappa)$-LCP satisfies the interior point condition (IPC), i.e., there exists a pair $(x^0, s^0) > 0$ such that $s^0 = Mx^0 + q$, which implies the existence of a solution for $P_*(\kappa)$-LCP [16]. Since M is a $P_*(\kappa)$-matrix and the IPC holds, the parameterized system (2) has a unique solution $(x(\mu), s(\mu))$ for each $\mu > 0$ (cf. Lemma 4.3 in [16]), which is called the μ-center of $P_*(\kappa)$-LCP. The set of μ-centers (with μ running through all positive real numbers) gives a homotopy path, which is called the central path of $P_*(\kappa)$-LCP. If $\mu \to 0$, then the limit of the central path exists, and since the limit points satisfy the complementarity condition, i.e., $xs = 0$, the limit yields a solution for $P_*(\kappa)$-LCP [16].

2.2. New search directions

Following [7], we replace the parameterized equation, i.e., $xs = \mu e$, in (2) by an equivalent algebraic transformation $\psi\left(\frac{xs}{\mu}\right) = \psi(e)$, where $\psi(t)$ is a real valued function on $[0, \infty)$ such that $\psi(0) = 0$ and differentiable on $(0, \infty)$ such that $\psi^{'}(t) > 0$ for all $t > 0$; i.e., $\psi(t)$ is strictly increasing, thus, one-to-one. Under this transformation, the original perturbed system (2) is transformed into the following equivalent one:

$$\begin{aligned} -Mx + s = q,\ \ x, s \geq 0, \\ \psi(\tfrac{xs}{\mu}) = \psi(e). \end{aligned} \tag{3}$$

Since system (2) has a unique solution, we conclude that system (3) has a unique solution as well. A promising way to obtain the search directions for $P_*(\kappa)$-LCP is to apply Newton's method to system (3). For any strictly feasible point x and s, we find displacements Δx and Δs such that

$$\begin{aligned} &-M(x+\Delta x) + (s + \Delta s) = q, \\ &\psi\big(\tfrac{xs}{\mu} + \tfrac{x\Delta s + s\Delta x + \Delta x \Delta s}{\mu}\big) = \psi(e). \end{aligned} \tag{4}$$

The second equation of the system (4) is equivalent to

$$\psi\Big(\frac{xs}{\mu} + \frac{x\Delta s + s\Delta x + \Delta x\Delta s}{\mu}\Big) = \psi(e).$$

Neglecting the quadratic term $\Delta x \Delta s$ in the above equation and using Taylor's theorem we get

$$\psi\Big(\frac{xs}{\mu}\Big) + \psi^{'}\Big(\frac{xs}{\mu}\Big)\Big(\frac{x\Delta s + s\Delta x}{\mu}\Big) = \psi(e),$$

which is equivalent to the equation

$$s\Delta x + x\Delta s = \mu\Big(\psi^{'}\Big(\frac{xs}{\mu}\Big)\Big)^{-1}\Big(\psi(e) - \psi\Big(\frac{xs}{\mu}\Big)\Big).$$

Thus, we can rewrite the system (4) as follows

$$\begin{aligned} -M\Delta x + \Delta s = 0, \\ s\Delta x + x\Delta s = \mu\big(\psi^{'}(\tfrac{xs}{\mu})\big)^{-1}\big(\psi(e) - \psi(\tfrac{xs}{\mu})\big). \end{aligned} \tag{5}$$

Introducing the variance vector

$$v := \sqrt{\frac{xs}{\mu}}, \tag{6}$$

and the scaled search directions

$$d_x := \frac{v\Delta x}{x}, \quad d_s := \frac{v\Delta s}{s}, \tag{7}$$

the system (5) is further simplified

$$\begin{aligned} -\overline{M}d_x + d_s = 0, \\ d_x + d_s = p_v, \end{aligned} \tag{8}$$

where $\overline{M} := DMD$ with $D := X^{\frac{1}{2}}S^{-\frac{1}{2}}, X := \mathrm{diag}(x), S := \mathrm{diag}(s)$ and

$$p_v := \frac{\psi(e) - \psi(v^2)}{v\psi'(v^2)}.$$

Since M is a $P_*(\kappa)$-matrix, it follows from

$$\begin{aligned}(1+4\kappa)&\sum_{i\in I_+} x_i(\overline{M}x)_i + \sum_{i\in I_-} x_i(\overline{M}x)_i \\ &= (1+4\kappa)\sum_{i\in I_+} x_i\big(DMDx\big)_i + \sum_{i\in I_-} x_i\big(DMDx\big)_i \\ &= (1+4\kappa)\sum_{i\in I_+} (Dx)_i(MDx)_i + \sum_{i\in I_-} (Dx)_i(MDx)_i \geq 0\end{aligned}$$

that $\overline{M}$ also is a $P_*(\kappa)$-matrix. Thus, the system (8) has a unique solution (see [16, Lemma 4.1]). By choosing function $\psi(t)$ appropriately, the system (8) can be used to define a class of search directions. For example:

- $\psi(t) = t$ yields $p_v = v^{-1} - v$ which gives the classical search directions [23].

- $\psi(t) = \sqrt{t}$ yields $p_v = 2(e - v)$ which gives the search directions introduced by Darvay [7].

In this paper, we restrict our analysis to the case where $\psi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$, this yields

$$p_v = e - v^2.$$

The new search directions d_x and d_s are obtained by solving the system (8) with $p_v = e - v^2$ so that Δx and Δs are computed via (7). The new iterate is obtained by taking a full-Newton step according to

$$x^+ := x + \Delta x, \quad s^+ := s + \Delta s. \tag{9}$$

For the analysis of the algorithm, we define a norm-based proximity measure as follows:

$$\delta(v) := \delta(x, s; \mu) := \|p_v\| = \|e - v^2\|. \tag{10}$$

Note that

$$\delta(v) = 0 \Leftrightarrow v = e \Leftrightarrow xs = \mu e.$$

Therefore, the value of $\delta(v)$ can be considered a measure of the distance between the given pair (x, s) and the corresponding μ-center $(x(\mu), s(\mu))$.

2.3. Generic full-Newton step feasible interior-point algorithm

Here, the generic full-Newton step feasible interior-point algorithm is presented.

Generic feasible IPM for $\mathbf{P}_*(\kappa) - \mathbf{LCP}$

Input : Accuracy parameter $\epsilon > 0$;
barrier update parameter θ, $0 < \theta < 1$;
threshold parameter $0 < \tau < 1$;
a strictly feasible solution x^0, s^0 and $\mu^0 > 0$
such that $\delta(x^0, s^0; \mu^0) \leq \tau$.
begin :
$x := x^0, s := s^0, \mu := \mu^0$;
while $n\mu > \epsilon$ **do**
begin
solve (8) to obtain (d_x, d_s) and then use (7)
to obtain $(\Delta x, \Delta s)$;
$x := x + \Delta x$;
$s := s + \Delta s$;
$\mu := (1 - \theta)\mu$.
end
end

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3. Analysis of the algorithm

Let us define

$$q_v := d_x - d_s.$$

Then, using the above equation and the second equation of (8) we have

$$d_x = \frac{p_v + q_v}{2}, \quad d_s = \frac{p_v - q_v}{2},$$

which implies

$$d_x d_s = \frac{p_v^2 - q_v^2}{4}. \tag{11}$$

Lemma 1 (see [26, Lemma 3.1])**.** *Let $\delta := \delta(x, s; \mu)$. Then*

$$-\kappa\delta^2 \leq d_x^T d_s.$$

From Lemma 1 and (10), we can conclude that

$$\|q_v\|^2 = \|p_v\|^2 - 4d_x^T d_s \leq \delta^2 + 4\kappa\delta^2 = (1 + 4\kappa)\delta^2. \tag{12}$$

The following lemma shows the strict feasibility of the full-Newton step.

Lemma 2. *Let $\delta := \delta(x, s; \mu) < \frac{2}{1+\sqrt{1+4\kappa}}$. Then the full-Newton step is strictly feasible.*

Proof. Let $0 \leq \alpha \leq 1$. We define

$$x(\alpha) = x + \alpha\Delta x, \quad s(\alpha) = s + \alpha\Delta s.$$

From (6), (7) and the second equation of (8), we get

$$\begin{aligned} x(\alpha)s(\alpha) &= \frac{xs}{v^2}(v+\alpha d_x)(v+\alpha d_s) \\ &= \mu\big(v^2+\alpha v(d_x+d_s)+\alpha^2 d_x d_s\big) \\ &= \mu\Big(v^2+\alpha v p_v+\alpha^2\frac{p_v^2-q_v^2}{4}\Big) \\ &= \mu\Big((1-\alpha)v^2+\alpha\big(v^2+vp_v+\alpha\frac{p_v^2-q_v^2}{4}\big)\Big) \\ &= \mu\Big((1-\alpha)v^2+\alpha\big((v+\frac{p_v}{2})^2-(1-\alpha)\frac{p_v^2}{4}-\alpha\frac{q_v^2}{4}\big)\Big). \end{aligned} \tag{13}$$

The inequality $x(\alpha)s(\alpha) > 0$ holds if

$$\min\Big((v+\frac{p_v}{2})^2-(1-\alpha)\frac{p_v^2}{4}-\alpha\frac{q_v^2}{4}\Big) > 0,$$

and this relation is satisfied if

$$\begin{aligned} \|(1-\alpha)\frac{p_v^2}{4}+\alpha\frac{q_v^2}{4}\| \le \min_i\big(v+\frac{p_v}{2}\big)_i^2 &= \min_i\Big(v+\frac{e-v^2}{2}\Big)_i^2 \\ &= \min_i\frac{(-v_i^2+2v_i+1)^2}{4}. \end{aligned} \tag{14}$$

On the other hand, from (10) we have

$$\delta = \|e-v^2\| \ge |1-v_i^2|,\ i=1,2,\ldots,n,$$

which implies

$$\sqrt{1-\delta} \le v_i \le \sqrt{1+\delta},\ i=1,2,\ldots,n. \tag{15}$$

One can easily verify that $f(v_i) = -v_i^2+2v_i+1$ for $\sqrt{1-\delta} \le v_i \le \sqrt{1+\delta}$ is concave. Thus, for any $i=1,2,\ldots,n$, we have

$$f(v_i) \ge \min\{f(\sqrt{1+\delta}), f(\sqrt{1-\delta})\} = f(\sqrt{1-\delta}) = \delta+2\sqrt{1-\delta} \ge \delta+2(1-\delta) = 2-\delta.$$

Therefore

$$\frac{(2-\delta)^2}{4} \le \min_i\frac{f(v_i)^2}{4}. \tag{16}$$

From the triangle inequality, (10) and (12), it follows that

$$\begin{aligned} \|(1-\alpha)\frac{p_v^2}{4}+\alpha\frac{q_v^2}{4}\| &\le (1-\alpha)\frac{\|p_v\|^2}{4}+\alpha\frac{\|q_v\|^2}{4} \\ &\le (1-\alpha)\frac{\delta^2}{4}+\alpha\frac{(1+4\kappa)\delta^2}{4} \\ &= \frac{(1+4\alpha\kappa)\delta^2}{4} \le \frac{(1+4\kappa)\delta^2}{4}. \end{aligned} \tag{17}$$

Due to (16) and (17) the inequality (14) holds if

$$(1+4\kappa)\delta^2 \leq (2-\delta)^2,$$

which implies that

$$\delta < \frac{2}{1+\sqrt{1+4\kappa}}.$$

Therefore, the inequality (14) holds, for $\delta < \frac{2}{1+\sqrt{1+4\kappa}}$, i.e., $x(\alpha)s(\alpha) > 0$. Since $x(\alpha)$ and $s(\alpha)$ are linear functions of α and $x(0) = x > 0$ and $s(0) = s > 0$, it follows that $x(1) = x^+ > 0$ and $s(1) = s^+ > 0$. This completes the proof. □

The next lemma investigates the effect of a full-Newton step on the proximity measure.

Lemma 3. *Let* $\delta := \delta(x, s; \mu) < \frac{2}{1+\sqrt{1+4\kappa}}$. *Then*

$$\delta(x^+, s^+; \mu) \leq \frac{(1+2\kappa)\delta^2}{1+\sqrt{1-\delta}}.$$

Thus $\delta(x^+, s^+; \mu) \leq (1+2\kappa)\delta^2$, *which shows the quadratic convergence of the Newton step.*

Proof. Let $v^+ := \sqrt{\frac{x^+s^+}{\mu}}$. Then from (13) with $\alpha = 1$ and $v^2 = e - p_v$ we have

$$\begin{aligned}(v^+)^2 = v^2 + vp_v + \frac{p_v^2}{4} - \frac{q_v^2}{4} &= e - (e-v)p_v + \frac{p_v^2}{4} - \frac{q_v^2}{4} \\ &= e - \frac{p_v^2}{e+v} + \frac{p_v^2}{4} - \frac{q_v^2}{4} \\ &= e - \frac{3e-v}{e+v} \cdot \frac{p_v^2}{4} - \frac{q_v^2}{4}. \qquad (18)\end{aligned}$$

We may write

$$\begin{aligned}\delta(x^+, s^+; \mu) = \left\|e - (v^+)^2\right\| &= \left\|\frac{3e-v}{e+v} \cdot \frac{p_v^2}{4} + \frac{q_v^2}{4}\right\| \\ &\leq \left\|\frac{3e-v}{e+v}\right\|_\infty \left\|\frac{p_v^2}{4}\right\| + \left\|\frac{q_v^2}{4}\right\| \\ &\leq \left\|\frac{3e-v}{e+v}\right\|_\infty \left\|\frac{p_v}{2}\right\|^2 + \left\|\frac{q_v}{2}\right\|^2. \qquad (19)\end{aligned}$$

From $\delta < \frac{2}{1+\sqrt{1+4\kappa}} \leq 1$ and (15) it follows that

$$\left|\frac{3-v_i}{1+v_i}\right| = \frac{3-v_i}{1+v_i},$$

and

$$\left\|\frac{3e-v}{e+v}\right\|_\infty = \max_i \left|\frac{3-v_i}{1+v_i}\right| = \max_i \frac{3-v_i}{1+v_i} \leq \frac{3-\sqrt{1-\delta}}{1+\sqrt{1-\delta}}. \qquad (20)$$

Substituting (10), (12) and (20) into (19) it follows that

$$\begin{aligned}\delta(x^+, s^+; \mu) &\leq \frac{3-\sqrt{1-\delta}}{1+\sqrt{1-\delta}} \cdot \frac{\delta^2}{4} + \frac{(1+4\kappa)\delta^2}{4} \\ &= \frac{(1+\kappa+\kappa\sqrt{1-\delta})\delta^2}{1+\sqrt{1-\delta}} \\ &\leq \frac{(1+2\kappa)\delta^2}{1+\sqrt{1-\delta}}.\end{aligned}$$

Thus, the proof is complete. □

The next lemma gives the effect of a full-Newton step on duality gap.

Lemma 4. *After a full-Newton step it holds*

$$(x^+)^T s^+ \leq n\mu.$$

Proof. Using (18), we get

$$\begin{aligned}(x^+)^T s^+ = e^T(x^+ s^+) &= \mu e^T({v^+}^2) \\ &= \mu e^T\Big(e - \frac{3e-v}{e+v} \cdot \frac{p_v^2}{4} - \frac{q_v^2}{4}\Big) \\ &= n\mu - \mu \Big\| \sqrt{\frac{3e-v}{e+v}} \cdot \frac{p_v}{2} \Big\|^2 - \frac{\mu}{4}\|q_v\|^2 \leq n\mu.\end{aligned}$$

This completes the proof of the lemma. □

The following lemma investigates the effect on the proximity measure after a main iteration of the algorithm.

Lemma 5. *Let* $(x, s) > 0$ *such that* $\delta := \delta(x, s; \mu) < \frac{2}{1+\sqrt{1+4\kappa}}$ *and* $\mu^+ = (1-\theta)\mu$ *where* $\theta \in (0, 1)$. *Then*

$$\delta(x^+, s^+; \mu^+) \leq \frac{1}{1-\theta}\big(\theta\sqrt{n} + \delta(x^+, s^+; \mu)\big).$$

Proof. After updating $\mu^+ = (1-\theta)\mu$, the vector v^+ is divided by the factor $\sqrt{1-\theta}$. Using (10) and the triangle inequality, we obtain

$$\begin{aligned}\delta(x^+, s^+; \mu^+) = \Big\| e - \frac{(v^+)^2}{1-\theta} \Big\| &= \frac{1}{1-\theta}\big\|(1-\theta)e - (v^+)^2\big\| \\ &\leq \frac{1}{1-\theta}\big(\theta\sqrt{n} + \|e - (v^+)^2\|\big) \\ &= \frac{1}{1-\theta}\big(\theta\sqrt{n} + \delta(x^+, s^+; \mu)\big).\end{aligned}$$

This completes the proof. □

It is clear that

$$1 + 4\kappa < (3 + 8\kappa)^2 = \big(4(1 + 2\kappa) - 1\big)^2.$$

Taking the square root of both sides, we may write

$$\sqrt{1 + 4\kappa} < 4(1 + 2\kappa) - 1,$$

and this relation implies that

$$\frac{1}{2(1 + 2\kappa)} < \frac{2}{1 + \sqrt{1 + 4\kappa}}. \tag{21}$$

Corollary 1. *Let* $\delta = \delta(x, s; \mu) \leq \frac{1}{2(1+2\kappa)}$, $\theta = \frac{1}{(4+7\kappa)\sqrt{n}}$ *and* $n \geq 4$. *Then*

$$\delta(x^+, s^+; \mu^+) \leq \frac{1}{2(1 + 2\kappa)}.$$

Proof. Since $n \geq 4$ and $\kappa \geq 0$, we have

$$1 - \theta = 1 - \frac{1}{(4 + 7\kappa)\sqrt{n}} \geq 1 - \frac{1}{2(4 + 7\kappa)} = \frac{7 + 14\kappa}{2(4 + 7\kappa)}.$$

Moreover, from $\delta \leq \frac{1}{2(1+2\kappa)}$ it follows that

$$1 + \sqrt{1 - \delta} \geq 1 + \sqrt{1 - \frac{1}{2(1 + 2\kappa)}} = 1 + \sqrt{\frac{1 + 4\kappa}{2(1 + 2\kappa)}} \geq 1 + \sqrt{\frac{1}{2}} > \frac{17}{10}.$$

Using (21) and Lemma 3 we obtain

$$\delta(x^+, s^+; \mu) \leq \frac{(1 + 2\kappa)\delta^2}{1 + \sqrt{1 - \delta}} \leq \frac{\frac{1}{4(1+2\kappa)}}{\frac{17}{10}} = \frac{5}{34(1 + 2\kappa)}.$$

Finally, from (21) and Lemma 5 it follows that

$$\begin{aligned}
\delta(x^+, s^+; \mu^+) &\leq \frac{2(4 + 7\kappa)}{7 + 14\kappa}\Big(\frac{1}{4 + 7\kappa} + \frac{5}{34(1 + 2\kappa)}\Big) \\
&= \frac{2}{7 + 14\kappa}\Big(1 + \frac{5(4 + 7\kappa)}{34(1 + 2\kappa)}\Big) \leq \frac{2}{7 + 14\kappa}\big(1 + \frac{20}{34}\big) \\
&= \frac{54}{119(1 + 2\kappa)} < \frac{1}{2(1 + 2\kappa)}.
\end{aligned}$$

This completes the proof. □

The algorithm starts from a strictly feasible point (x^0, s^0) such that $\delta(x^0, s^0; \mu^0) \leq \frac{1}{2(1+2\kappa)}$. The algorithm stops if $n\mu \leq \epsilon$. Otherwise, we compute the search directions d_x and d_s from (8) at the current iterate, then we apply (6), (7) and (9) to get the new iterate (x^+, s^+). It follows from Lemma 2 and Lemma 4 that (x^+, s^+)

is strictly feasible, and $(x^+)^T s^+ \le \mu n$. After the update μ to $\mu^+ = (1-\theta)\mu$ with $\theta = \frac{1}{(4+7\kappa)\sqrt{n}}$, by Corollary 1, we have

$$\delta(x^+, s^+; \mu^+) \le \frac{1}{2(1+2\kappa)}.$$

This implies that the algorithm is well defined.

Lemma 6. *Assume that* (x^0, s^0) *is a strictly feasible solution of* (2), $\mu^0 = \frac{(x^0)^T s^0}{n}$ *and* $\delta(x^0, s^0; \mu^0) \le \frac{1}{2(1+2\kappa)}$. *Moreover, let* (x^k, s^k) *be the point obtained after k iterations. Then the inequality* $(x^k)^T s^k \le \epsilon$ *is satisfied for*

$$k \ge \frac{1}{\theta} \log \frac{(x^0)^T s^0}{\epsilon}.$$

Proof. After k iterations, Lemma 4 implies that

$$(x^k)^T s^k \le n\mu^k = n(1-\theta)^k \mu^0 = (1-\theta)^k (x^0)^T s^0,$$

hence $(x^k)^T s^k \le \epsilon$ holds if

$$(1-\theta)^k (x^0)^T s^0 \le \epsilon.$$

Taking logarithms, we obtain

$$k \log(1-\theta) + \log((x^0)^T s^0) \le \log \epsilon.$$

Using $-\log(1-\theta) \ge \theta$, we conclude that the above inequality holds if

$$-k\theta + \log((x^0)^T s^0) \le \log \epsilon.$$

Thus the result is obtained. □

The following theorem gives an upper bound for the total number of iterations produced by the algorithm.

Theorem 1. *Let* $\tau = \frac{1}{2(1+2\kappa)}$ *and* $\theta = \frac{1}{(4+7\kappa)\sqrt{n}}$. *Then the algorithm requires at most*

$$O\Big((4+7\kappa)\sqrt{n} \log \frac{(x_0)^T s_0}{\epsilon}\Big)$$

iterations. The output gives an ϵ*-approximate solution for* $P_*(\kappa)$*-LCP.*

4. Numerical results

In this section, we compare the proposed algorithm in this paper with the given algorithm in [30]. We consider the $P_*(\kappa)$-LCP as follows:

$$M = \begin{bmatrix} Q_2 & & & & \\ & Q_3 & & & \\ & & \ddots & & \\ & & & Q_2 & \\ & & & & Q_3 \end{bmatrix}, \text{where } Q_2 = \begin{bmatrix} 0 & 1+4\kappa_1 \\ 1 & 0 \end{bmatrix}, Q_3 = \begin{bmatrix} 0 & 1+4\kappa_2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with a starting point $x^0 = s^0 = e$. Here, we take $M \in R^{50\times 50}$ and $\kappa = \kappa_1 = \kappa_2 \in \{1, 2, 3, 10, 100, 1000\}$. Numerical results were obtained by using MATLAB R2009a (version 7.8.0.347) on Windows XP Enterprise 32-bit operating system. We list the number of iterations (Iter.) and the duality gap (gap) when the algorithms terminate and the CPU time (time) in seconds. In Table 1, we used $\theta = \frac{1}{(4+7\kappa)\sqrt{n}}$ and $\theta = \frac{1}{2(1+4\kappa)\sqrt{n}}$ as required by the algorithms in order to guarantee convergence, respectively. In all experiments, the algorithms terminate after the duality gap satisfies $x^T s \leq 10^{-4}$. We observe that in Table 1, our algorithm is better than

κ	The proposed algorithm			algorithm in [30]		
	Iter.	gap	time	Iter	gap	time
1	1016	9.8841e-005	0.380060	923	9.9024e-005	0.396940
2	1665	9.9709e-005	0.405280	1665	9.9711e-005	0.420972
3	2315	9.9524e-005	0.725418	2407	9.9972e-005	0.853700
10	6861	9.9968e-005	1.961780	7604	9.9852e-005	2.158221
100	65318	1.0000e-004	16.893976	74412	9.9988e-005	20.362354
1000	649890	9.9999e-005	212.079718	742493	1.0000e-004	315.493099

Table 1:

Algorithm in [30]. Although, in theory, the convergence is not guaranteed for bigger θ values, we performed a MATLAB experiment for $\theta = 0.05$. Results are given in Table 2. It can be seen from Table 2 that for bigger θ values, the two algorithms

κ	The proposed algorithm			algorithm in [30]		
	Iter.	gap	time	Iter	gap	time
1	257	9.9016e-005	0.221336	257	9.9080e-005	0.068680
10	257	9.9016e-005	0.134520	257	9.9080e-005	0.067193
100	257	9.9016e-005	0.136514	257	9.9080e-005	0.110972
1000	257	9.9016e-005	0.232406	257	9.9080e-005	0.214892

Table 2:

are almost the same. Therefore, the numerical results show that our algorithm is competitive and reliable.

5. Conclusions and remarks

In this paper, we have proposed a new full-Newton step feasible interior-point algorithm for $P_*(\kappa)$-LCP. The algorithm is based on a new class of search directions obtained by an algebraic equivalent form of the nonlinear equations of the central path. The currently best known iteration bound for $P_*(\kappa)$-LCP is derived. Moreover, our numerical experiments also show that our new algorithm may perform well in practice.

An interesting topic is the generalization of the analysis of the full-Newton step feasible interior-point algorithm to other algebraic equivalent transformations of the central path. Another topic for further research may be the development of the algorithm for the Cartesian $P_*(\kappa)$-LCP over symmetric cones, symmetric cone optimization and semidefinite optimization.

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