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Comparison of accurate solutions of nonlinear Hammerstein fuzzy integral equations

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Abstract. In this paper, efficient numerical techniques have been proposed to solve nonlinear Hammerstein fuzzy integral equations. The proposed methods are based on Bernstein polynomials and Legendre wavelets approximation. Usually, nonlinear fuzzy integral equations are very difficult to solve both analytically and numerically. The present methods applied to the integral equations is reduced to solve the system of nonlinear algebraic equations. Again, this system has been solved by Newton's method. The numerical results obtained by present methods have been compared with those of the homotopy analysis method. Illustrative examples have been discussed to demonstrate the validity and applicability of the presented methods.

AMS subject classifications: 45G10, 65R20

Key words: Bernstein polynomial, Legendre wavelets, Hammerstein integral equation, fuzzy integral equation, fuzzy calculus

1. Introduction

The application of fuzzy integral equations has been developed in recent years. The concept of fuzzy sets was originally introduced by Zadeh [32], led to the definition of fuzzy numbers and their implementations in fuzzy control [11] and approximate reasoning problems [33]. The study of fuzzy integral equations was invented by investigations of Kaleva [18] and Seikkala [27] for fuzzy Volterra integral equations. In the past few years, many works were written by several authors in the theory of fuzzy integral equations. The approximate analytical methods like the Adomian decomposition method [4], the homotopy analysis method [20], and the homotopy perturbation method [3] have been used to solve fuzzy integral equations. Numerical techniques are available in the literature to solve fuzzy integral equations. Fuzzy Fredholm integral equations have been solved by the Nystrom method [1], the sinc function [19], the residual minimization method [17], the fuzzy transforms method [14] and others.

Previously, many numerical methods have been used to solve nonlinear Hammerstein integral equations. Nonlinear Hammerstein fuzzy Fredholm integral equations and Hammerstein-Volterra delay integral equations have been solved by the learned

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researchers like Bica et al. [8, 9]. Recently, nonlinear Hammerstein-Fredholm integral equations have been solved by the B-spline wavelet method and the variational iteration method [25], and fuzzy integro-differential equations have been solved by the two-dimensional Legendre wavelet method [26]. Bernstein polynomial approximation has been applied by many researchers to solve integral equations. Fuzzy Volterra integral equations [21] and fuzzy Fredholm integral equations [15] have been solved by Bernstein polynomials. Bernstein polynomials have been applied to solve systems of nonlinear Hammerstein-Fredholm integral equations in [24].

In this paper, nonlinear Hammerstein fuzzy Fredholm integral equations have been solved by the Bernstein polynomial collocation method (BPCM) and the Legendre wavelet method (LWM). These proposed methods are used to solve a Hammerstein integral equation by reducing to a system of nonlinear algebraic equations. Again, this system has been solved by Newton's method. Also, the results obtained by the presented methods have been compared with the results obtained by the homotopy analysis method (HAM). Some illustrative examples are introduced to justify the accuracy and applicability of the present methods. The comparison between the present methods establishes that the Bernstein polynomial collocation method provides more accurate solutions than Legendre wavelet method solutions.

This paper is organized as follows: in Section 2, we present some preliminaries and notations of nonlinear Hammerstein fuzzy Fredholm integral equations. In Section 3, we discuss the properties of Bernstein polynomial and function approximation. In Section 4, we discuss the properties of Legendre wavelets and function approximation. In Section 5, we establish function approximation of a nonlinear Hammerstein fuzzy Fredholm integral equation by two-dimensional Bernstein polynomial and two dimensional Legendre wavelets. In Section 6, we discuss the convergence analysis of the present method. Section 7 deals with illustrative examples which show the efficiency and accuracy of these presented methods with regard to the HAM.

2. Preliminaries of a fuzzy integral equation

In this section, the most basic notations used in fuzzy calculus are introduced. We start with defining a fuzzy number.

Definition 1 (see [16]). *A fuzzy number u is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$; $0 \leq r \leq 1$ which satisfy the following properties:*

(1) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function over $[0, 1]$,

(2) $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function over $[0, 1]$,

(3) $\underline{u}(r) \leq \overline{u}(r), \quad 0 \leq r \leq 1$.

For arbitrary $u(r) = (\underline{u}(r), \overline{u}(r))$, $v(r) = (\underline{v}(r), \overline{v}(r))$ and $k > 0$, we define addition $(u + v)$ and scalar multiplication by k as

(i) $(\underline{u+v})(r) = \underline{u}(r) + \underline{v}(r)$,

(ii) $(\overline{u+v})(r) = \overline{u}(r) + \overline{v}(r)$,

(iii) $\left(\underline{ku}\right)(r) = k\underline{u}(r)$, $\left(\overline{ku}\right)(r) = k\overline{u}(r)$.

Definition 2 (see [30])**.** *For arbitrary fuzzy numbers $u, v \in E$ (the space E is the set of all fuzzy numbers), we use the distance*

$$D(u,v) = \sup_{0 \le r \le 1} \left[\max\left\{|\overline{u}(r) - \overline{v}(r)|, |\underline{u}(r) - \underline{v}(r)|\right\}\right]$$

and it is shown that (E, D) is a complete metric space.

Remark 1. *If the fuzzy function $f(t)$ is continuous in the metric D (see [4]), its definite integral exists. Also,*

$$\left(\underline{\int_a^b f(t;r)dt}\right) = \int_a^b \underline{f}(t;r)dt,$$
$$\left(\overline{\int_a^b f(t;r)dt}\right) = \int_a^b \overline{f}(t;r)dt.$$

We have followed [16] and defined the integral of a fuzzy function using the Riemann integral concept.

Definition 3 (see [6])**.** *If $f : R \longrightarrow E$ is a fuzzy function (where E is a subset of a Banach space) and $t_0 \in R$, then the derivative $f'(t_0)$ of f at a point t_0 is defined by*

$$f'(t_0) = \lim_{h \to 0^+} \frac{f(t_0 + h) - f(t_0)}{h}, \tag{1}$$

provided that this limit taken with respect to the metric D exists.

In this paper, we consider a nonlinear Hammerstein fuzzy Fredholm integral equation of the form

$$u(t) = g(t) + \int_0^1 H(t,s)F(u(s))ds, \quad H(t,s) \in C([0,1] \times [0,1]), \quad t \in [0,1], \tag{2}$$

where u, g and F are fuzzy functions, and $H(t,s)$ is positive in $[0,1]$. Let

$$u(t,r) = \left(\underline{u}(t,r), \overline{u}(t,r)\right),$$
$$g(t,r) = \left(\underline{g}(t,r), \overline{g}(t,r)\right),$$
$$F(u(t,r)) = \left(\underline{F}(u(t,r)), \overline{F}(u(t,r))\right).$$

Equation (2), in crisp sense, converted into a system as

$$\underline{u}(t,r) = \underline{g}(t,r) + \int_0^1 H(t,s)\underline{F}\left(u(s,r)\right)ds, \tag{3}$$

$$\overline{u}(t,r) = \overline{g}(t,r) + \int_0^1 H(t,s)\overline{F}\left(u(s,r)\right)ds. \tag{4}$$

3. Bernstein polynomials and function approximation

The general form of the n-th degree Bernstein polynomials on the interval $[0,1]$ defined in [24, 7] as

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0, 1, \ldots, n, \tag{5}$$

where $\binom{n}{i} = \dfrac{n!}{i!(n-i)!}$.

Explicitly, if f is a continuous function on $[0,1]$, then $B_n(f)$ is called the n-th Bernstein polynomial for f defined as (see [22])

$$B_n(f)(x) = \sum_{i=0}^{n} f(\frac{i}{n}) B_{i,n}(x).$$

Note that each of these $n+1$ polynomials having degree n satisfies the following properties:

$$\begin{aligned} B_{i,n}(x) &= 0, \quad if \quad i < 0 \quad or \quad i > n, \\ B_{i,n}(0) &= B_{i,n}(1) = 0, \quad for \quad 1 \le i \le n-1, \\ \sum_{i=0}^{n} B_{i,n}(x) &= 1. \end{aligned}$$

Bernstein polynomials defined above form a complete basis [24, 7] over the interval $[0,1]$. A function $u(x)$ defined over $[0,1]$ can be approximated by Bernstein polynomial basis functions of degree n as

$$u(x) \approx \sum_{i=0}^{n} c_i B_{i,n}(x). \tag{6}$$

Similarly, in a two-dimensional case, any function $u(x,t) \in [0,1] \times [0,1]$ can be approximated as

$$u(x,t) \approx \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} c_{i,j} B_{i,n_1}(x) B_{j,n_2}(t). \tag{7}$$

We suppose $\|.\|$ to be the maximum norm on $[0,1]$; then the error bound

$$|(B_n f)(x) - f(x)| \le \frac{1}{2n} x(1-x) \|f''\|. \tag{8}$$

given in Chapter 10 of [13] shows that the rate of convergence is at least $\frac{1}{n}$ for $f \in C^2[0,1]$. Namely, for each function $f : [0,1] \to R$ and a point of continuity x of f we have

$$\lim_{n \to \infty} (B_n f)(x) = f(x), \tag{9}$$

and the convergence is uniform if f is continuous. Voronovskaya [28] derived that if the second derivative $f''(x)$ of the function f exists, then

$$\lim_{n\to\infty} n((B_n f)(x) - f(x)) = \frac{1}{2}x(1-x)f''(x). \tag{10}$$

Therefore, $(B_n f)(x) = f(x) + O(1/n)$.

4. Legendre wavelets and function approximation

Wavelets constitute a family of functions constructed from dilation and translation of a single function called mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets as

$$\Psi_{a,b}(t) = |a|^{-\frac{1}{2}}\Psi\left(\frac{t-b}{a}\right),\ a, b \in R,\ a \neq 0. \tag{11}$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, $a_0 > 1$, $b_0 > 0$ and n, and k are positive integers, we have the following family of discrete wavelets:

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}}\psi(a_0^k t - nb_0),$$

where $\psi_{k,n}(t)$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(t)$ form an orthonormal basis.

Legendre wavelets $\psi_{n,m}(t) = \psi(k, n, m, t)$ have four arguments. $n = 1, 2, \ldots, 2^{k-1}$, $k \in Z^+$, m is the order of Legendre polynomials and t is normalized time. They are defined on the interval $[0, 1)$ as [29, 31]

$$\psi_{n,m}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}2^{\frac{k}{2}}P_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}} \\ 0, & \text{otherwise} \end{cases}, \tag{12}$$

where $m = 0, 1, \ldots, M-1$ and $n = 1, 2, \ldots, 2^{k-1}$.

The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, the dilation parameter is $a = 2^{-k}$ and the translation parameter is $b = (2n-1)2^{-k}$.

Here $P_m(t)$ are the well-known Legendre polynomials of order m, which are defined on the interval $[-1, 1]$, and can be determined with the aid of the following recurrence formulae:

$$\begin{aligned} P_0(t) &= 1, \\ P_1(t) &= t, \\ P_{m+1}(t) &= \left(\frac{2m+1}{m+1}\right)tP_m(t) - \left(\frac{m}{m+1}\right)P_{m-1}(t), \quad m = 1, 2, 3, \ldots \end{aligned}$$

The two-dimensional Legendre wavelets are defined as [5, 26]

$$\psi_{n_1,m_1,n_2,m_2}(x,t) = \begin{cases} AP_{m_1}(2^{k_1}x - 2n_1 + 1)P_{m_2}(2^{k_2}t - 2n_2 + 1), & \frac{n_1-1}{2^{k_1-1}} \le x < \frac{n_1}{2^{k_1-1}}, \ \frac{n_2-1}{2^{k_2-1}} \le t < \frac{n_2}{2^{k_2-1}}, \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

where $A = \sqrt{\left(m_1 + \frac{1}{2}\right)\left(m_2 + \frac{1}{2}\right)} 2^{\frac{k_1+k_2}{2}}$, and $n_1 = 1, 2, \ldots, 2^{k_1-1}$, $n_2 = 1, 2, \ldots, 2^{k_2-1}$, $k_1, k_2 \in Z^+$, and m_1, m_2 are the order of Legendre polynomials. $\psi_{n_1,m_1,n_2,m_2}(x,t)$ forms a basis for $L^2([0,1) \times [0,1))$.

A function $f(x,t)$ defined over $[0,1)\times[0,1)$ can be expanded in terms of Legendre wavelets as [5, 26]

$$f(x,t) = \sum_{n_1=1}^{\infty}\sum_{m_1=0}^{\infty}\sum_{n_2=1}^{\infty}\sum_{m_2=0}^{\infty} c_{n_1,m_1,n_2,m_2}\psi_{n_1,m_1,n_2,m_2}(x,t), \tag{14}$$

If the infinite series in equation (14) is truncated, then it can be written as

$$f(x,t) = \sum_{n_1=1}^{2^{k_1-1}}\sum_{m_1=0}^{M_1-1}\sum_{n_2=1}^{2^{k_2-1}}\sum_{m_2=0}^{M_2-1} c_{n_1,m_1,n_2,m_2}\psi_{n_1,m_1,n_2,m_2}(x,t) = C^T\Psi(x,t) \tag{15}$$

where C and $\Psi(x,t)$ are $\left(2^{k_1-1}2^{k_2-1}M_1M_2 \times 1\right)$ vectors given by

$$\begin{aligned} C =& [c_{1,0,1,0}, \ldots, c_{1,0,1,M_2-1}, \ldots, c_{1,0,2^{k_2-1},M_2-1}, \ldots, \\ & c_{2^{k_1-1},M_1-1,2^{k_2-1},M_2-1}]^T \end{aligned} \tag{16}$$

$$\begin{aligned} \Psi(x,t) =& [\psi_{1,0,1,0}(x,t), \ldots, \psi_{1,0,1,M_2-1}(x,t), \ldots, \\ & \psi_{1,0,2^{k_2-1},M_2-1}(x,t), \ldots, \psi_{2^{k_1-1},M_1-1,2^{k_2-1},M_2-1}(x,t)]^T. \end{aligned} \tag{17}$$

5. Analysis of present methods

5.1. Method I: Bernstein polynomial collocation method

Consider equation (3) for solving by the Bernstein polynomial collocation method; first, approximate the unknown function $\underline{u}(t,r)$ by using equation (7) as

$$\underline{u}(t,r) \approx \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} c_{i,j}B_{i,n_1}(t)B_{j,n_2}(r). \tag{18}$$

Equation (3) can be reduced as

$$\begin{aligned} &\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} c_{i,j}B_{i,n_1}(t)B_{j,n_2}(r) \\ &\qquad = \underline{g}(t,r) + \int_0^1 H(t,s)\underline{F}\left(\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} c_{i,j}B_{i,n_1}(s)B_{j,n_2}(r)\right) ds. \end{aligned} \tag{19}$$

Utilizing (19) with the collocation points t_l and r_m defined as

$$t_l = t_0 + lh_1, \qquad t_0 = 0, \qquad h_1 = \frac{1}{n_1}, \qquad l = 0, 1, \ldots, n_1,$$

$$r_m = r_0 + mh_2, \qquad r_0 = 0, \qquad h_2 = \frac{1}{n_2}, \qquad m = 0, 1, \ldots, n_2,$$

(19) reduces to a system of $(n_1+1)(n_2+1)$ number of nonlinear algebraic equations with the same number of unknowns as $c_{i,j}$, $i = 0, 1, \ldots, n_1$, $j = 0, 1, \ldots, n_2$. This algebraic system has been solved by Newton's method to obtain the unknowns $c_{i,j}$, $i = 0, 1, \ldots, n_1$, $j = 0, 1, \ldots, n_2$. Hence we get the solution $\underline{u}(t, r)$ from (18) and the same algorithm can be applied to obtain the approximate solution of $\overline{u}(t, r)$.

5.2. Method II: Legendre wavelet method

Consider equation (3) for solving by the Legendre wavelet method; first, approximate the unknown function $\underline{u}(t, r)$ by using (15) as

$$\underline{u}(t, r) = C^T \Psi(t, r). \tag{20}$$

Now, (3) can be reduced as

$$C^T \Psi(t, r) = \underline{g}(t, r) + \int_0^1 H(t, s) \underline{F}\left(C^T \Psi(s, r)\right) ds. \tag{21}$$

In order to use the Gauss-Legendre integration formula for (21), we transfer the interval $[0, 1]$ to $[-1, 1]$ by means of the transformation $\tau = 2s - 1$.

Therefore, (21) can be written as

$$C^T \Psi(t, r) = \underline{g}(t, r) + \frac{1}{2} \int_{-1}^1 H\left(t, \frac{\tau+1}{2}\right) \underline{F}\left(C^T \Psi\left(\frac{\tau+1}{2}, r\right)\right) d\tau. \tag{22}$$

By using the Gauss-Legendre integration formula, we get

$$C^T \Psi(t, r) = \underline{g}(t, r) + \frac{1}{2} \sum_{j=1}^{M} w_j H\left(t, \frac{\tau_j+1}{2}\right) \underline{F}\left(C^T \Psi\left(\frac{\tau_j+1}{2}, r\right)\right), \tag{23}$$

where τ_j are m zeros of Legendre polynomials P_{m+1} and w_j are the corresponding weights. Now we collocate (23) at $t_p = \frac{2p-1}{2^{k_1} M_1}$, $p = 1, 2, \ldots, 2^{k_1-1} M_1$ and $r_q = \frac{2q-1}{2^{k_2} M_2}$, $q = 1, 2, \ldots, 2^{k_2-1} M_2$, we have

$$C^T \Psi(t_p, r_q) = \underline{g}(t_p, r_q) + \frac{1}{2} \sum_{j=1}^{M} w_j H\left(t_p, \frac{\tau_j+1}{2}\right) \underline{F}\left(C^T \Psi\left(\frac{\tau_j+1}{2}, r_q\right)\right) \tag{24}$$

Equation (24) gives a system of $2^{k_1-1} M_1 \times 2^{k_2-1} M_2$ number of algebraic equations with the same number of unknowns for C^T. Again, solving this system numerically by Newton's method, we can find the value for unknowns for C^T and hence obtain the approximate solution for $\underline{u}(t, r)$. The same algorithm can be applied to obtain the approximate solution of $\overline{u}(t, r)$.

6. Convergence analysis

In this section, we have discussed the convergence of Bernstein polynomial collocation methods.

Theorem 1. *Let $u(t) \in H^k(0,1)$ (Sobolov space) and $u_n(t) = \sum_{j=0}^{n} b_j \tilde{U}_j(t)$ be the best approximation polynomial of $u(t)$ in L_w^2-norm. Then the truncation error is*

$$\|u(t) - u_n(t)\|_{L_w^2[0,1]} \leq C_0 n^{-k} \|u(t)\|_{H^k(0,1)},$$

where C_0 is independent of n and $u(t)$.

Proof. See [10]. □

Theorem 2. *Let $u(t) \in H^{(k)}(0,1)$ (Sobolov space) and $u_n(t) = \sum_{i=0}^{n} c_i B_{i,n}(t)$ be the approximate solution of $u(t)$, where, $B_{i,n}(t)$ are Bernstein polynomials, $i = 0, 1, \ldots, n$. Then the error term $\|e_n(t)\|$ is bounded and converges to zero.*

Proof. The Bernstein polynomials are not orthogonal. These polynomials can be expressed in terms of second kind shifted Chebyshev polynomials which are orthogonal with respect to the weight function $\tilde{w}(t) = w(2t-1)$ and $w(t) = \sqrt{1-t^2}$.

$$B_{i,n}(t) = \frac{1}{2^n}\binom{n}{i} \sum_{s=0}^{n} d_s^{i,n} \frac{1}{2^s} \sum_{m=0}^{[s/2]} \{\binom{s}{m} - \binom{s}{m+1}\} \tilde{U}_{s-2m}(t),$$

where $\tilde{U}(t) = U(2t-1)$ are shifted Chebyshev polynomials and $d_s^{i,n} = \sum_k (-1)^{s-k} \binom{i}{k}\binom{n-i}{s-k}$. The summation over s is chosen as follows: For $i < n-i$,

(1) $k = 0$ to s, for $s \leq i$,

(2) $k = 0$ to i, for $i < s \leq n-i$,

(3) $k = s-(n-i)$ to $n-i$, for $n-i < s \leq n$.

For $i = n-i$ (n is an even integer),

(1) $k = 0$ to s, for $s \leq i$,

(2) $k = s-i$ to i, for $i < s \leq n$.

For $i > n-i$, i and $n-i$ above are interchanged.

We can approximate $u(t) \approx \sum_{i=0}^{n} c_i B_{i,n}(t)$ which can be expressed as $u(t) \approx \sum_{j=0}^{n} b_j \tilde{U}_j(t)$, where b_j's are the scalar multiplication of c_j's.

If $u(t)$ is approximated by $u_n(t) = \sum_{j=0}^{n} b_j \tilde{U}_j(t)$, then we find $\overline{b}_j$ as approximation of b_j and $\overline{u}_n(t) = \sum_{j=0}^{n} \overline{b}_j \tilde{U}_j(t)$. We have

$$\begin{aligned}\|u(t) - \overline{u}_n(t)\|_2 &= \|u(t) - u_n(t) + u_n(t) - \overline{u}_n(t)\|_2 \\ &\leq \|u(t) - u_n(t)\|_2 + \|u_n(t) - \overline{u}_n(t)\|_2.\end{aligned}$$

Now,

$$\|u_n(t)-\overline{u}_n(t)\|_2 = \left[\int_0^1 (u_n(t)-\overline{u}_n(t))^2\tilde{w}(t)dt\right]^{1/2}$$
$$= \left[\int_0^1 \left(\sum_{j=0}^{n}(b_j-\overline{b}_j)\tilde{U}_j(t)\right)^2 \tilde{w}(t)dt\right]^{1/2}$$
$$= \left(\sum_{j=0}^{n}\frac{\pi}{4}(b_j-\overline{b}_j)^2\right)^{1/2}.$$

Using $(n+1)$-point Gauss-Chebyshev rule, we have (see [12])

$$|b_j-\overline{b}_j| \le C_1 n^{-k+1}$$

and

$$\|u_n-\overline{u}_n\|_2 \le C_2(n+1)^{1/2}n^{-k+1}.$$

Thus,

$$\|u(t)-\overline{u}_n(t)\|_2 \le C_0 n^{-k}\|u(t)\| + C_2(n+1)^{1/2}n^{-k+1}.$$

Hence the error term $\|e_n(t)\| = \|u(t)-\overline{u}_n(t)\| \longrightarrow 0$ as $n,k\longrightarrow\infty$. □

7. Illustrative examples

Example 1. *Consider the following nonlinear Hammerstein fuzzy integral equation*

$$u(t) = g(t) + \int_0^1 H(t,s)e^{-s}(u(s))^2 ds, \quad t\in[0,1] \tag{25}$$

with

$$H(t,s) = \begin{cases} \frac{1}{6}s^2(1-t)^2(3t-s-2ts), & 0\le s\le t\le 1\\ \frac{1}{6}t^2(1-s)^2(3s-t-2ts), & 0\le t\le s\le 1\end{cases}$$

and

$$g(t,r) = g_1(t,r)+g_2(t,r), \quad t,r\in[0,1],$$

where

$$g_1(t,r) = \left[(1-t)^2(3t+1)\left(1-\frac{1-r}{10}\right),(1-t)^2(3t+1)\left(1+\frac{1-r}{10}\right)\right],$$
$$g_2(t,r) = \left[t^2(2-t)\left(e-\frac{1-r}{10}\right),t^2(2-t)\left(e+\frac{1-r}{10}\right)\right].$$

Equation (25)*, in crisp sense, converted into a system as*

$$\underline{u}(t,r) = \underline{g}(t,r) + \int_0^1 H(t,s)e^{-s}\left(\underline{u}(s,r)\right)^2 ds, \tag{26}$$
$$\overline{u}(t,r) = \overline{g}(t,r) + \int_0^1 H(t,s)e^{-s}\left(\overline{u}(s,r)\right)^2 ds, \tag{27}$$

where

$$g(t,r) = (\underline{g}(t,r), \overline{g}(t,r)) = (\underline{g_1}(t,r) + \underline{g_2}(t,r), \overline{g_1}(t,r) + \overline{g_2}(t,r)).$$

The exact solution of this problem is not known. This problem has been solved by the Bernstein polynomial collocation method and the Legendre wavelet method, and compared with the approximate analytical method like the homotopy analysis method [23, 2]. Since the exact solution of this problem is unknown, the solution in the HAM has been considered as a standard solution.

- ***Comparison with the HAM solution***

In homotopy analysis method [23, 2], the m^{th} order deformation equation approximating $\underline{u}(t,r)$ is given by

$$\begin{aligned} L\left[\underline{u}_m(t,r) - \chi_m \underline{u}_{m-1}(t,r)\right] =& \hbar \Re_m\left(\underline{u}_0, \underline{u}_1, \ldots, \underline{u}_{m-1}\right) \\ =& \hbar \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} N(\varphi(t,r;q))|_{q=0}, \end{aligned}$$

where

$$N(\varphi(t,r;q)) = \varphi(t,r;q) - \underline{g}(t,r) - \int_0^1 H(t,s) e^{-s} (\varphi(s,r;q))^2 ds$$

and $\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}.$

Here auxiliary parameter [2] $\hbar = -1$ belongs to the convergence region of the HAM series solution.

Now, using the m^{th} order deformation equation of $\underline{u}(t,r)$, we recursively obtain

$$\begin{aligned} &\underline{u}_0(t,r) = 0, \\ &\underline{u}_1(t,r) = \frac{1}{10}(9 + r + 9t + rt - 47t^2 + 20et^2 - 3rt^2 + 28t^3 - 10et^3 + 2rt^3), \\ &\underline{u}_2(t,r) = \frac{1}{25} e^{-t} r^2 t^6 - \frac{2}{5} e^{1-t} r t^6 + \ldots, \end{aligned}$$

and so on.

Thus the HAM solution is

$$\underline{U}_m(t,r) = \sum_{i=0}^{m} \underline{u}_i(t,r).$$

Similarly, using the m^{th} order deformation equation of $\overline{u}(t,r)$, we recursively obtain

$$\begin{aligned} &\overline{u}_0(t,r) = 0, \\ &\overline{u}_1(t,r) = \frac{1}{10}(11 - r + 11t - rt - 53t^2 + 20et^2 + 3rt^2 + 32t^3 - 10et^3 - 2rt^3), \\ &\overline{u}_2(t,r) = \frac{1}{100} e^{-1-t}\left(4er^2t^6 + 40e^2rt^6 - 128ert^6 + \ldots\right), \end{aligned}$$

and so on.

r	t	$\underline{u}(t,r)$			$\overline{u}(t,r)$		
		Standard solution (HAM)	BPCM	LWM	Standard solution (HAM)	BPCM	LWM
0.25	0.2	1.13904	1.13903	1.13926	1.30376	1.30375	1.30403
	0.4	1.41286	1.41289	1.41283	1.57078	1.57081	1.57075
	0.6	1.75035	1.75035	1.75035	1.89387	1.89387	1.89386
	0.8	2.15758	2.15761	2.15785	2.29349	2.29352	2.29381
0.5	0.2	1.16649	1.16648	1.16672	1.27631	1.2763	1.27657
	0.4	1.43917	1.4392	1.43914	1.54445	1.54449	1.54443
	0.6	1.77427	1.77427	1.77426	1.86994	1.86994	1.86993
	0.8	2.18023	2.18026	2.18051	2.27084	2.27087	2.27115
0.75	0.2	1.19394	1.19393	1.19418	1.24885	1.24884	1.24910
	0.4	1.46549	1.46552	1.46546	1.51813	1.51816	1.51810
	0.6	1.79818	1.79818	1.79817	1.84602	1.84602	1.84601
	0.8	2.20288	2.20291	2.20317	2.24819	2.24821	2.24849

Table 1: *Numerical solutions for Example 1*

Error	$\underline{u}(t,r)$		$\overline{u}(t,r)$	
$0 \leq t, r \leq 1$	BPCM	LWM	BPCM	LWM
L_∞	0.3E-4	0.29E-3	0.4E-4	0.32E-3
L_2	0.754983E-4	0.600999E-3	0.768115E-4	0.690145E-3

Table 2: *Error analysis for Example 1 with regard to the HAM*

Thus the HAM solution is

$$\overline{U}_m(t,r) = \sum_{i=0}^{m} \overline{u}_i(t,r).$$

The numerical results obtained by the Bernstein polynomial collocation method (BPCM) for $n_1 = n_2 = 4$ *and the Legendre wavelet method (LWM) for* $M_1 = M_2 = 4, k_1 = k_2 = 1$ *have been compared with the results obtained by the* 2^{nd} *order homotopy analysis method (HAM) of* $u(t,r) = (\underline{u}(t,r), \overline{u}(t,r))$ *for* $r = 0.25,\ 0.5,\ 0.75$. *These results are shown in Tables 1 and 2. From Table 2, it may be easily observed that* L_∞ *and* L_2 *errors for the Bernstein polynomial collocation method are better than other method solutions.*

Example 2. *Consider the following nonlinear Hammerstein fuzzy integral equation*

$$u(t) = g(t) + \int_0^1 \frac{ts}{3}(u(s))^2 ds, \quad t \in [0,1], \tag{28}$$

where

$$g(t,r) = \left[\frac{11}{12}t - \frac{1}{6}(1-r), \quad \frac{11}{12}t + \frac{1}{6}(1-r)\right], \quad t, r \in [0,1].$$

r	t	$\underline{u}(t,r)$			$\overline{u}(t,r)$		
		Standard solution (HAM)	BPCM	LWM	Standard solution (HAM)	BPCM	LWM
0.25	0.2	0.06780	0.06917	0.06917	0.32811	0.33258	0.33257
	0.4	0.26061	0.26334	0.26334	0.53122	0.54015	0.54015
	0.6	0.45341	0.45751	0.45751	0.73434	0.74772	0.74772
	0.8	0.64621	0.65168	0.65168	0.93745	0.95530	0.95529
0.5	0.2	0.11089	0.1126	0.11260	0.28443	0.28817	0.28818
	0.4	0.30511	0.30853	0.30853	0.48552	0.49301	0.49302
	0.6	0.49934	0.50446	0.50446	0.68662	0.69785	0.69786
	0.8	0.69356	0.70039	0.70040	0.88771	0.90269	0.90270
0.75	0.2	0.15410	0.15621	0.15621	0.24086	0.24399	0.24399
	0.4	0.34986	0.35408	0.35408	0.44006	0.44631	0.44631
	0.6	0.54562	0.55195	0.55195	0.63926	0.64864	0.64863
	0.8	0.74138	0.74982	0.74983	0.83845	0.85096	0.85095

Table 3: *Numerical solutions for Example 2*

Error $0 \le t, r \le 1$	$\underline{u}(t,r)$ BPCM	$\underline{u}(t,r)$ LWM	$\overline{u}(t,r)$ BPCM	$\overline{u}(t,r)$ LWM
L_∞	0.844E-2	0.845E-2	0.1785E-1	0.1784E-1
L_2	0.166465E-1	0.166393E-1	0.362121E-1	0.362082E-1

Table 4: *Error analysis for Example 2 with regard to the HAM*

Equation (28)*, in crisp sense, converted into a system as*

$$\underline{u}(t,r) = \frac{11}{12}t - \frac{1}{6}(1-r) + \int_0^1 \frac{ts}{3}\left(\underline{u}(s,r)\right)^2 ds \tag{29}$$

$$\overline{u}(t,r) = \frac{11}{12}t + \frac{1}{6}(1-r) + \int_0^1 \frac{ts}{3}\left(\overline{u}(s,r)\right)^2 ds \tag{30}$$

The exact solution of this problem is not known. This problem has been solved by the Bernstein polynomial collocation method and the Legendre wavelet method, and compared with the approximate analytical method like the homotopy analysis method [23, 2]. Since, the exact solution of this problem is unknown, the solution in the HAM has been considered as a standard solution.

- ***Comparison with the HAM solution***

In the homotopy analysis method [23, 2], the m^{th} *order deformation equation approximating* $\underline{u}(t,r)$ *is given by*

$$\begin{aligned} L\left[\underline{u}_m(t,r) - \chi_m \underline{u}_{m-1}(t,r)\right] =& \hbar \Re_m\left(\underline{u}_0, \underline{u}_1, \ldots, \underline{u}_{m-1}\right) \\ =& \hbar \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}} N(\varphi(t,r;q))|_{q=0}, \end{aligned}$$

where

$$N(\varphi(t,r;q)) = \varphi(t,r;q) - \frac{11}{12}t + \frac{1}{6}(1-r) - \int_0^1 \frac{ts}{3}(\varphi(s,r;q))^2 ds$$

and $\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}$.

Here, an auxiliary parameter [2] $\hbar = -1$ *belongs to the convergence region of the HAM series solution.*

Now, using the m^{th} *order deformation equation of* $\underline{u}(t,r)$*, we recursively obtain*

$$\begin{aligned}
&\underline{u}_0(t,r) = 0,\\
&\underline{u}_1(t,r) = -\frac{1}{6} + \frac{r}{6} + \frac{11t}{12},\\
&\underline{u}_2(t,r) = \frac{211t}{5184} + \frac{2rt}{81} + \frac{r^2t}{216},\\
&\textit{and so on.}
\end{aligned}$$

Thus the HAM solution is

$$\underline{U}_m(t,r) = \sum_{i=0}^{m} \underline{u}_i(t,r).$$

Similarly, using the m^{th} *order deformation equation of* $\overline{u}(t,r)$*, we recursively obtain*

$$\begin{aligned}
&\overline{u}_0(t,r) = 0,\\
&\overline{u}_1(t,r) = \frac{1}{6} - \frac{r}{6} + \frac{11t}{12},\\
&\overline{u}_2(t,r) = \frac{563t}{5184} - \frac{7rt}{162} + \frac{r^2t}{216},\\
&\textit{and so on.}
\end{aligned}$$

Thus the HAM solution is

$$\overline{U}_m(t,r) = \sum_{i=0}^{m} \overline{u}_i(t,r).$$

The numerical results obtained by the Bernstein polynomial collocation method (BPCM) for $n_1 = n_2 = 2$ *and the Legendre wavelet method (LWM) for* $M_1 = M_2 = 4, k_1 = k_2 = 1$ *have been compared with the results obtained by the* 3^{rd} *order homotopy analysis method (HAM) of* $u(t,r) = (\underline{u}(t,r), \overline{u}(t,r))$ *for* $r = 0.25,\ 0.5,\ 0.75$*. These results are shown in Tables 3 and 4. From Table 4, it may be easily observed that* L_∞ *and* L_2 *errors for the Bernstein polynomial collocation method are better than other method solutions.*

Example 3. *Consider the following nonlinear Hammerstein fuzzy integral equation*

$$u(t) = g(t) + \int_0^1 ts(u(s))^3 ds, \quad t \in [0,1], \tag{31}$$

where

$$g(t,r) = g_1(t,r) + g_2(t,r), \quad t, r \in [0,1],$$

where

$$g_1(t,r) = [2+r, 3-r],$$
$$g_2(t,r) = \left[-t\left(\frac{87}{10} + \frac{47}{4}r + 4r^2 + \frac{1}{2}r^3\right), -t\left(\frac{49}{10} - \frac{9}{2}r + \frac{5}{2}r^2 - \frac{1}{2}r^3\right)\right].$$

The exact solution of this problem is $u(t,r) = [2+t+r, \quad 3-2t-r]$. *Equation* (31), *in crisp sense, converted into a system as*

$$\underline{u}(t,r) = \underline{g}(t,r) + \int_0^1 ts\left(\underline{u}(s,r)\right)^3 ds, \tag{32}$$

$$\overline{u}(t,r) = \overline{g}(t,r) + \int_0^1 ts\left(\overline{u}(s,r)\right)^3 ds, \tag{33}$$

where

$$g(t,r) = (\underline{g}(t,r), \overline{g}(t,r)) = (\underline{g_1}(t,r) + \underline{g_2}(t,r), \overline{g_1}(t,r) + \overline{g_2}(t,r)).$$

Solving (32) *by the Bernstein polynomial collocation method for* $n_1 = n_2 = 1$, *we obtain the unknowns as*

$$c_{0,0} = 2, \quad c_{0,1} = 3, \quad c_{1,0} = 3, \quad c_{1,1} = 4$$

and the solution can be obtained as $\underline{u}(t,r) = \sum_{i=0}^{1}\sum_{j=0}^{1} c_{i,j}B_{i,1}(t)B_{j,1}(r) = 2+t+r$. *Similarly, solving* (33) *by the* 1^{st} *order Bernstein polynomial collocation method, we obtain the unknowns as*

$$c_{0,0} = 3, \quad c_{0,1} = 2, \quad c_{1,0} = 1, \quad c_{1,1} = 0$$

and the solution can be obtained as $\overline{u}(t,r) = \sum_{i=0}^{1}\sum_{j=0}^{1} c_{i,j}B_{i,1}(t)B_{j,1}(r) = 3-2t-r$.

Solving (32) *by the Legendre wavelet method for* $M_1 = M_2 = 2, k_1 = k_2 = 1$, *we obtain the unknowns as*

$$C^T = [3.0003, 0.28862, 0.28885, -0.0000332746]$$

and the solution can be obtained as $\underline{u}(t,r) = C^T\Psi(t,r) = 2+r+t-0.000399295rt$. *Similarly, solving* (33) *by the Legendre wavelet method for* $M_1 = M_2 = 2, k_1 = k_2 = 1$, *we obtain the unknowns as*

$$C^T = [1.52366, -0.216966, -0.563689, 0.0414012]$$

r	t		$\underline{u}(t,r)$			$\overline{u}(t,r)$		
		Exact	HAM	BPCM	LWM	HAM	BPCM	LWM
0.25	0.2	2.45	2.45	2.45	2.44998	2.35	2.35	2.33462
	0.4	2.65	2.65	2.65	2.64996	1.95	1.95	1.91925
	0.6	2.85	2.85	2.85	2.84994	1.55	1.55	1.50387
	0.8	3.05	3.05	3.05	3.04992	1.15	1.15	1.0885
0.5	0.2	2.7	2.72	2.7	2.69996	2.1	2.1	2.10947
	0.4	2.9	2.94	2.9	2.89992	1.7	1.7	1.71893
	0.6	3.1	3.15	3.1	3.09988	1.3	1.3	1.3284
	0.8	3.3	3.37	3.3	3.29984	0.89	0.9	0.93786
0.75	0.2	2.95	2.98	2.95	2.94994	1.85	1.85	1.88431
	0.4	3.15	3.21	3.15	3.14988	1.45	1.45	1.51861
	0.6	3.35	3.45	3.35	3.34982	1.05	1.05	1.15292
	0.8	3.55	3.67	3.55	3.54976	0.649	0.65	0.787223

Table 5: *Numerical solutions for Example 3*

and the solution can be obtained as $\overline{u}(t,r) = C^T\Psi(t,r) = 3-r-2.20108t+0.496815rt$.

The numerical results obtained by the Bernstein polynomial collocation method (BPCM) for $n_1 = n_2 = 1$ *and the Legendre wavelet method (LWM) for* $M_1 = M_2 = 2, k_1 = k_2 = 1$ *have been compared with the results obtained by the* 4^{th} *order homotopy analysis method (HAM) of* $u(t,r) = (\underline{u}(t,r), \overline{u}(t,r))$ *for* $r = 0.25,\ 0.5,\ 0.75$. *These results are shown in Table 5. It is justified that there is a good agreement between HAM results and exact results, and, with other numerical results as well.*

8. Conclusion

In this paper, the Bernstein polynomial collocation method and the Legendre wavelet method have been applied to the nonlinear Hammerstein fuzzy Fredholm integral equations and the obtained results then compared with the results obtained by homotopy analysis method. The presented methods reduce the Hammerstein fuzzy Fredholm integral equation to asystem of nonlinear algebraic equations and this system has been solved by Newton's method. Since the homotopy analysis method is an analytical method, then from the tables, it is justified that the results obtained by presented methods are very accurate with regard to HAM results and also it is cleared that the Bernstein polynomial collocation method is more accurate than theLegendre wavelet method. The illustrative examples have been included to demonstrate the validity and applicability of the proposed techniques. These examples also exhibit the accuracy and efficiency of the presented methods. However, the Bernstein polynomial collocation method provides more accurate solutions in comparison to other methods discussed.

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