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Sturm's theorems for conformable fractional differential equations

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Abstract. In the present paper, we make use of local properties of the recently established definition of a conformable fractional derivative. Sturm's separation and Sturm's comparison theorems are proved for differential equations involving a conformable fractional derivative of order $0 < \alpha \leq 1$.

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1. Introduction

In the last decades, the interest in fractional differential equations is rapidly growing, and classic fractional derivatives such as Caputo, Riemann-Liouville or Hadamard seem to be well-developed (see e.g. [8, 9]). One thing that all these have in common is that they are defined as integrals with different singular kernels, i.e., they have a non-local structure. Due to this fact, none of the classic fractional derivatives satisfies an analog of integer-order product rule: $(fg)' = f'g + fg'$ for C^1-functions f, g. On the other hand, a recently introduced definition of the so-called conformable fractional derivative (see Definition 1 below) involves a limit instead of an integral. This local definition enables us to prove many properties analogous to those of integer-order derivative (cf. [1, 7]). Nowadays, the Cauchy problems involving a conformable fractional derivative [4] and fractional semigroups [2] are also investigated. We note that the notion of a conformable fractional derivative was generalized in [5] to time scales. In this paper, we state and prove Sturm's theorems (see e.g. [6] for classic statements) for differential equations with conformable fractional derivatives. Our results may be used as a foundation for studies of oscillatory properties of conformable fractional differential equations.

For the simplicity, we denote by $|u|$ the Euclidean norm of a vector $u \in \mathbb{R}^n$ without any respect to the dimension $n \in \mathbb{N}$. In the present paper, we always assume $0 < \alpha \leq 1$.

The paper is organized as follows. In the next section, we recall some basic definitions and known results that will be in force when proving the main results.

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Here we also prove a result on the existence and uniqueness of a solution of an initial value problem. Section 3 is devoted to the main results of this paper, and at the end, we present an example comparing solutions of fractional equations with constant coefficients.

2. Preliminary results

Here we recall basic notions, and provide results helpful for the main section. The basic definition is from [7].

Definition 1. *Let $0 < \alpha \leq 1$. The conformable fractional derivative of a function $f\colon [a,\infty) \to \mathbb{R}^n$ is defined as*

$$
\begin{aligned}
{}_a\mathrm{D}_\alpha\, f(t) &= \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon}, \quad t > a,\\
{}_a\mathrm{D}_\alpha\, f(a) &= \lim_{t \to a^+} {}_a\mathrm{D}_\alpha\, f(t).
\end{aligned}
$$

If ${}_a\mathrm{D}_\alpha\, f(t_0)$ exists and is finite, we say that f is α–differentiable at t_0. For $2 \leq n \in \mathbb{N}$ we denote ${}_a\mathrm{D}^n_\alpha\, f(t) = {}_a\mathrm{D}_\alpha\, {}_a\mathrm{D}^{n-1}_\alpha f(t)$.

The conformable fractional integral is defined as

$$
{}_a\mathrm{I}_\alpha\, f(t) = \int_a^t \frac{f(s)}{(s-a)^{1-\alpha}} \mathrm{d}s, \quad t \geq a.
$$

Whenever $a = 0$, we omit the lower index a.

Note that if $f\colon [a,\infty) \to \mathbb{R}^n$ is differentiable at $t_0 \geq a$, then ${}_a\mathrm{D}_\alpha\, f(t_0) = (t_0 - a)^{1-\alpha} f'(t_0)$. We add several lemmas on properties of conformable fractional derivative. The next one is from [7].

Lemma 1. *Let $f\colon [a,\infty) \to \mathbb{R}^n$ be a continuous function, and $0 < \alpha \leq 1$. Then*

$$
{}_a\mathrm{D}_\alpha\, {}_a\mathrm{I}_\alpha\, f(t) = f(t), \quad t > a.
$$

It is elementary to prove the next two lemmas using the continuity of an α–differentiable function [7, Theorem 2.1] in the first case, and the mean value theorem for α–differentiable functions [7, Theorem 2.4] in the second.

Lemma 2. *Let $0 < \alpha \leq 1$, f be differentiable at $g(t)$, and g α–differentiable at $t > a$. Then*

$$
{}_a\mathrm{D}_\alpha (f \circ g)(t) = f'(g(t))\, {}_a\mathrm{D}_\alpha\, g(t).
$$

Lemma 3. *If $f\colon (a,b) \to \mathbb{R}$ is α–differentiable in (a,b) and ${}_a\mathrm{D}_\alpha\, f$ is positive (negative) on the whole (a,b), then f is increasing (decreasing) on (a,b).*

The interested reader might compare Lemma 2 with the chain rule in [1, Theorem 2.11]. Note that we immediately obtain the statement complementary to Lemma 3: If the function f is α–differentiable on (a,b) and increasing (decreasing) on (a,b), then ${}_a\mathrm{D}_\alpha\, f(c) \geq 0$ (${}_a\mathrm{D}_\alpha\, f(c) \leq 0$) for all $c \in (a,b)$.

In the present paper, when talking about a solution of a fractional differential equation with the conformable fractional derivative we shall always have a *continuous* solution on mind.

Next, we present an integral equation corresponding to the initial value problem for a conformable fractional differential equation

$$\begin{aligned} {}_a\mathrm{D}_\alpha\, x(t) &= f(t, x(t)), \quad t > a, \\ x(a) &= x_a. \end{aligned} \tag{1}$$

Lemma 4. *Let $f \in C([a,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$ be a given function. Then the solution x of the initial value problem* (1) *satisfies*

$$x(t) = x_a + \int_a^t \frac{f(s, x(s))}{(s-a)^{1-\alpha}} \mathrm{d}s, \quad t \geq a. \tag{2}$$

Proof. Since f and x are continuous, ${}_a\mathrm{I}_\alpha\, f(t)$ exists. That means that ${}_a\mathrm{I}_\alpha\, {}_a\mathrm{D}_\alpha\, x(t)$ exists. Then by [1, Corollary 2.7], x is differentiable, and by [1, Lemma 2.8]

$${}_a\mathrm{I}_\alpha\, {}_a\mathrm{D}_\alpha\, x(t) = x(t) - x(a). \tag{3}$$

Therefore, applying the operator ${}_a\mathrm{I}_\alpha$ on equation (1) yields

$$x(t) - x(a) = {}_a\mathrm{I}_\alpha\, f(t)$$

for any $t > a$. That was to be proved. □

Note that the latter lemma holds for f defined on $[a, T] \times B(u, r)$, where $T > a$ and

$$B(u, r) = \left\{ x \in C([a, T], \mathbb{R}^n) \mid \max_{t \in [a,T]} |x(t) - u| \leq r \right\}$$

is a closed ball in $C([a, T], \mathbb{R}^n)$.

The following result gives a sufficient condition for the existence of a unique solution.

Theorem 1. *Let $0 < \alpha \leq 1$ and let $f \colon [a, T] \times B(x_a, r) \to \mathbb{R}^n$ be a given continuous function, where $T > a$. Suppose that $|f(t, x)| \leq M$ for all $t \in [a, T]$, $x \in B(x_a, r)$, for some $M > 0$. Moreover, let $f(t, \cdot)$ be Lipschitz continuous with the constant L for all $t \in [a, T]$. Then there exists a unique solution of the initial value problem* (1) *defined on $[a, T_1]$ with $T_1 = \min\left\{T, a + \left(\frac{\alpha r}{M}\right)^{\frac{1}{\alpha}}\right\}$.*

Proof. Let us introduce the Banach space $Z = C([a, T_1], \mathbb{R}^n)$ equipped with the Bielicki norm $\|x\|_Z = \max_{t \in [a,T_1]} \mathrm{e}^{-L\beta t}|x(t)|$, where

$$\beta := \left(\frac{\alpha + L\mathrm{e}^L}{\alpha} \right)^{\frac{1}{\alpha}}.$$

Define the operator $F \colon Z \to Z$ as

$$(Fx)(t) := x_a + \int_a^t \frac{f(s, x(s))}{(s-a)^{1-\alpha}} \mathrm{d}s.$$

Clearly, F is well defined. Moreover, F maps $B(x_a,r)$ into itself. Indeed, for any $t\in[a,T_1]$ we have

$$|(Fx)(t)-x_a|\le\int_a^t\frac{|f(s,x(s))|}{(s-a)^{1-\alpha}}\mathrm{d}s\le\frac{M(t-a)^\alpha}{\alpha}\le\frac{M(T_1-a)^\alpha}{\alpha}\le r.$$

Next we show that F is a contraction. Let $t\in[a,T_1]$ be arbitrary and fixed, and $x,y\in B(x_a,r)$. Then

$$\begin{aligned}|(Fx)(t)-(Fy)(t)|&\le\int_a^t\frac{|f(s,x(s))-f(s,y(s))|}{(s-a)^{1-\alpha}}\mathrm{d}s\\&\le L\int_a^t\frac{\mathrm{e}^{L\beta s}}{(s-a)^{1-\alpha}}\mathrm{d}s\|x-y\|_Z.\end{aligned}$$

Now, we split $\int_a^t=\int_a^{a+\varepsilon}+\int_{a+\varepsilon}^t$ for $\varepsilon:=\beta^{-1}$. If $t<a+\varepsilon$, omit the second integral in the following inequalities for the estimations to hold.

$$\begin{aligned}|(Fx)(t)-(Fy)(t)|&\le L\left(\mathrm{e}^{L\beta(a+\varepsilon)}\int_a^{a+\varepsilon}\frac{\mathrm{d}s}{(s-a)^{1-\alpha}}+\frac{1}{\varepsilon^{1-\alpha}}\int_{a+\varepsilon}^t\mathrm{e}^{L\beta s}\mathrm{d}s\right)\|x-y\|_Z\\&=L\left(\mathrm{e}^{L\beta(a+\varepsilon)}\frac{\varepsilon^\alpha}{\alpha}+\frac{\mathrm{e}^{L\beta t}-\mathrm{e}^{L\beta(a+\varepsilon)}}{L\beta\varepsilon^{1-\alpha}}\right)\|x-y\|_Z.\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{e}^{-L\beta t}|(Fx)(t)-(Fy)(t)|&\le\varepsilon^\alpha\left(\frac{L\mathrm{e}^{L\beta(a-t+\varepsilon)}}{\alpha}+\frac{1-\mathrm{e}^{L\beta(a-t+\varepsilon)}}{\beta\varepsilon}\right)\|x-y\|_Z\\&\le\varepsilon^\alpha\left(\frac{L\mathrm{e}^L}{\alpha}+1-\mathrm{e}^L\right)\|x-y\|_Z=(1-\varepsilon^\alpha\mathrm{e}^L)\|x-y\|_Z.\end{aligned}$$

Taking the maximum over all $t\in[a,T_1]$, one obtains $\|Fx-Fy\|_Z\le(1-\varepsilon^\alpha\mathrm{e}^L)\|x-y\|_Z$. The statement follows by the Banach fixed point theorem and Lemma 4. $\square$

The following definition of α–Wronskian is from [3], and will be in force in the next section.

Definition 2. *Let x, y be given functions α–differentiable on $[a,b]$, $\alpha\in(0,1]$. We set*

$${}_aW_\alpha[x,y](t):=\begin{vmatrix}x(t)&y(t)\\ {}_a\mathrm{D}_\alpha\,x(t)&{}_a\mathrm{D}_\alpha\,y(t)\end{vmatrix}.$$

3. Sturm's theorems

In this section, we consider the scalar fractional differential equation of second order of the form

$${}_a\mathrm{D}_\alpha^2\,x(t)+({}_a\mathrm{D}_\alpha\,x(t))p(t)+x(t)q(t)=0,\quad t>a \tag{4}$$

with continuous functions p, q. Classically [6], two functions x, y continuous on $[a,b]$ will be called linearly dependent if there exist $c_1,c_2\in\mathbb{R}$ such that $|c_1|+|c_2|>0$ and

$c_1x(t)+c_2y(t)\equiv 0$ for all $t\in[a,b]$. In the other case, they are linearly independent. Clearly, if one of two functions is identically equal to zero, they cannot be linearly independent. Using the formula

$$ {}_aW_\alpha[x,y](t)=\mathrm{e}^{-\int_{a_1}^{t}\frac{p(s)}{(s-a)^{1-\alpha}}\mathrm{d}s}\,{}_aW_\alpha[x,y](a_1),\quad t\in(a,b) $$

for two solutions, x and y, of (4) and some $a_1\in(a,b)$, which follows from [3, Theorem 2.2], we immediately obtain the next equivalent condition.

Lemma 5. *Two solutions x, y of equation* (4) *defined on* (a,b) *for some $a<b$ are linearly independent if and only if ${}_aW_\alpha[x,y](t)\neq 0$ for all $t\in(a,b)$.*

One of the main results of this paper follows. It is Sturm's separation theorem for fractional differential equations with the conformable derivative.

Theorem 2. *Let x, y be linearly independent solutions of* (4) *defined on* (a,b) *(b is allowed to be $+\infty$), p and q given continuous functions, and* $0<\alpha\leq 1$. *Then x has a zero between any two successive zeros of y. Thus the zeros of x and y occur alternately.*

Proof. Linear independence yields

$$ {}_aW_\alpha[x,y](t)=x(t)\,{}_a\mathrm{D}_\alpha\,y(t)-({}_a\mathrm{D}_\alpha\,x(t))y(t)\neq 0,\quad t\in(a,b). \tag{5} $$

So ${}_aW_\alpha[x,y]$ does not change the sign over (a,b). Suppose that $t_1,t_2\in(a,b)$ are two successive zeros of y. Note that ${}_a\mathrm{D}_\alpha\,y(t_i)\neq 0$ for each $i=1,2$. Otherwise, Theorem 1 gives $y\equiv 0$ — a contradiction with linear independence. This also means that the zeros of y (and also of x) are isolated.

Hence, from (5),

$$ x(t_1)({}_a\mathrm{D}_\alpha\,y(t_1))x(t_2)\,{}_a\mathrm{D}_\alpha\,y(t_2)>0. \tag{6} $$

Let us assume without any loss of generality that ${}_a\mathrm{D}_\alpha\,y(t_1)>0$. By Lemma 3, y is increasing at t_1. Since y is continuous, and t_2 is a zero of y next to t_1, y is decreasing at t_2. By the corollary of Lemma 3, ${}_a\mathrm{D}_\alpha\,y(t_2)<0$. Similarly, the case ${}_a\mathrm{D}_\alpha\,y(t_1)<0$ gives ${}_a\mathrm{D}_\alpha\,y(t_2)>0$. Therefore, $({}_a\mathrm{D}_\alpha\,y(t_1))\,{}_a\mathrm{D}_\alpha\,y(t_2)<0$, i.e., $x(t_1)x(t_2)<0$ by (6). The continuity of x yields the existence of $t_3\in(t_1,t_2)$ such that $x(t_3)=0$. Note that x has only one zero in (t_1,t_2). Indeed, if there were $t_4\neq t_3$ in (t_1,t_2) such that $x(t_4)=0$, then applying the above arguments would result in the existence of $t_5\in(t_3,t_4)$ or $t_5\in(t_4,t_3)$ such that $y(t_5)=0$, i.e. there is another zero of y between t_1 and t_2, what is a contradiction. □

The next result is Sturm's comparison theorem for conformable fractional differential equations.

Theorem 3. *Let x and y be nontrivial solutions of the equations*

$$ {}_a\mathrm{D}_\alpha^2\,x(t)+x(t)r(t)=0,\quad t>a, $$
$$ {}_a\mathrm{D}_\alpha^2\,y(t)+y(t)r_1(t)=0,\quad t>a, $$

respectively, where $r(t)\geq r_1(t)$ for $t>a$ are given continuous functions. Then exactly one of the following conditions holds:

(1) x has at least one zero between any two zeros of y,

(2) $r(t) = r_1(t)$ for all $t > a$, and x is a constant multiple of y.

Proof. Let us suppose that (1) does not hold. Let $a < t_1 < t_2$ be two consecutive zeros of y. Thus ${}_aW_\alpha[x,y](t_i) = x(t_i)\,{}_a\mathrm{D}_\alpha\, y(t_i)$ for $i = 1, 2$. Without any loss of generality, we assume that $x(t), y(t) > 0$ on (t_1, t_2) (otherwise, take $-x$ or $-y$). Similarly to the proof of Theorem 2,

$$ {}_a\mathrm{D}_\alpha\, y(t_1) > 0 > {}_a\mathrm{D}_\alpha\, y(t_2). $$

Therefore,

$$ \begin{aligned} {}_aW_\alpha[x,y](t_1) &= x(t_1)\,{}_a\mathrm{D}_\alpha\, y(t_1) \geq 0, \\ {}_aW_\alpha[x,y](t_2) &= x(t_2)\,{}_a\mathrm{D}_\alpha\, y(t_2) \leq 0. \end{aligned} \tag{7} $$

For the derivative, we have

$$ {}_a\mathrm{D}_\alpha\, {}_aW_\alpha[x,y](t) = x(t)y(t)(r(t) - r_1(t)) \geq 0, \quad t \in (t_1, t_2). $$

If there is some $t_0 \in (t_1, t_2)$ such that ${}_a\mathrm{D}_\alpha\, {}_aW_\alpha[x,y](t_0) > 0$, then from

$$ \begin{aligned} f(t_1) &= f(a) + {}_a\mathrm{I}_\alpha\, {}_a\mathrm{D}_\alpha\, f(t_1), \\ f(t_2) &= f(a) + {}_a\mathrm{I}_\alpha\, {}_a\mathrm{D}_\alpha\, f(t_2) \end{aligned} $$

for a continuous function f, and using (7), we obtain

$$ 0 \geq {}_aW_\alpha[x,y](t_2) = {}_aW_\alpha[x,y](t_1) + \int_{t_1}^{t_2} \frac{{}_a\mathrm{D}_\alpha\, {}_aW_\alpha[x,y](s)}{(s-a)^{1-\alpha}} \mathrm{d}s > 0, $$

what is a contradiction. Hence, $r(t) = r_1(t)$ for all $t \in (t_1, t_2)$. Note that now x and y solve the same equation. So, Theorem 2 yields that x and y are linearly dependent.

Obviously, if (2) is valid, then x cannot have a zero between two successive zeros of y, i.e., (1) does not hold. The proof is complete. $\square$

For the final result of this section, we shall need the fractional version of the Picone identity. For the simplicity, we omit the argument t.

Lemma 6. *Let u, v be nontrivial solutions of the equations*

$$ \begin{aligned} {}_a\mathrm{D}_\alpha(p\, {}_a\mathrm{D}_\alpha\, u) + qu = 0 \quad &\textit{on } (a, \infty), \\ {}_a\mathrm{D}_\alpha(p_1\, {}_a\mathrm{D}_\alpha\, v) + q_1 v = 0 \quad &\textit{on } (a, \infty), \end{aligned} \tag{8} $$

respectively, where q and q_1 are given continuous functions, p and p_1 are α–differentiable. Then

$$ \begin{aligned} &{}_a\mathrm{D}_\alpha \left(\frac{u}{v} \left(p({}_a\mathrm{D}_\alpha\, u)v - p_1 u({}_a\mathrm{D}_\alpha\, v) \right) \right) \\ &\qquad = (q_1 - q)u^2 + (p - p_1)({}_a\mathrm{D}_\alpha\, u)^2 + p_1 \left({}_a\mathrm{D}_\alpha\, u - ({}_a\mathrm{D}_\alpha\, v)\frac{u}{v} \right)^2. \end{aligned} \tag{9} $$

Proof. The statement is obtained by direct differentiation of the left-hand side, applying the rules for the derivative of a product and fraction of two functions [7]

$$ {}_a\mathrm{D}_\alpha(fg) = ({}_a\mathrm{D}_\alpha\, f)g + f({}_a\mathrm{D}_\alpha\, g), \quad {}_a\mathrm{D}_\alpha \left(\frac{f}{g}\right) = \frac{({}_a\mathrm{D}_\alpha\, f)g - f({}_a\mathrm{D}_\alpha\, g)}{g^2}, $$

and using (8). □

The following result is a generalization of Theorem 3.

Theorem 4. *Let u, v be nontrivial solutions of equations* (8)*, where $0 < p_1(t) \leq p(t)$, $q(t) \leq q_1(t)$ for $t > a$ are given continuous functions, and p and p_1 are α–differentiable. Then between any two zeros $t_1, t_2 > a$ of u, there exists at least one $t_0 \in [t_1, t_2]$ such that $v(t_0) = 0$.*

Proof. Let $a < t_1 < t_2$ be two consecutive zeros of u. Let us assume that $u(t) > 0$ for all $t \in (t_1, t_2)$, and conversely, that $v(t) > 0$ for all $t \in [t_1, t_2]$ (take $-u$ or $-v$ if needed). Then applying the operator ${}_a\mathrm{I}_\alpha \cdot (t_2) - {}_a\mathrm{I}_\alpha \cdot (t_1)$ to the Picone identity (9), we get

$$ \begin{aligned} 0 &= \left[\frac{u(t)}{v(t)}(p(t)({}_a\mathrm{D}_\alpha\, u(t))v(t) - p_1(t)u(t)({}_a\mathrm{D}_\alpha\, v(t)))\right]_{t=t_1}^{t_2} \\ &= \int_{t_1}^{t_2} \frac{(q_1(t) - q(t))u^2(t)}{(t-a)^{1-\alpha}} \mathrm{d}t + \int_{t_1}^{t_2} \frac{(p(t) - p_1(t))({}_a\mathrm{D}_\alpha\, u(t))^2}{(t-a)^{1-\alpha}} \mathrm{d}t \\ &\quad + \int_{t_1}^{t_2} \frac{p_1(t)}{(t-a)^{1-\alpha}} \left({}_a\mathrm{D}_\alpha\, u(t) - ({}_a\mathrm{D}_\alpha\, v(t))\frac{u(t)}{v(t)}\right)^2 \mathrm{d}t \\ &\geq \int_{t_1}^{t_2} \frac{p_1(t)}{(t-a)^{1-\alpha}} \left({}_a\mathrm{D}_\alpha\, u(t) - ({}_a\mathrm{D}_\alpha\, v(t))\frac{u(t)}{v(t)}\right)^2 \mathrm{d}t \\ &= \int_{t_1}^{t_2} \frac{p_1(t)v^2(t)}{(t-a)^{1-\alpha}} \left({}_a\mathrm{D}_\alpha \left(\frac{u(t)}{v(t)}\right)\right)^2 \mathrm{d}t \geq 0 \end{aligned} $$

with the aid of (3). Now, the right-hand side of the above inequality is zero if and only if

$$ {}_a\mathrm{D}_\alpha \left(\frac{u(t)}{v(t)}\right) = 0 $$

for all $t \in (t_1, t_2)$, i.e., u/v is constant on (t_1, t_2). Then continuity of u yields that $u(t) = 0$ for all $t \in [t_1, t_2]$, which is in contradiction with the positivity of u on (t_1, t_2). □

Finally, we present an example of equations with constant coefficients, illustrating the application of the latter theorem.

Example 1. *Let us consider the couple of equations*

$$ {}_a\mathrm{D}_{\frac{1}{2}}(4\, {}_a\mathrm{D}_{\frac{1}{2}} u) + u = 0 \quad \text{on } (a, \infty), \tag{10} $$

$$ {}_a\mathrm{D}_{\frac{1}{2}}(4\, {}_a\mathrm{D}_{\frac{1}{2}} v) + q_1 v = 0 \quad \text{on } (a, \infty) \tag{11} $$

for $a \in \mathbb{R}$ and parameter $q_1 \in \mathbb{R}$.

It can be easily verified that

$$u(t) = \sin\left(\frac{\pi}{4} + \sqrt{t-a}\right) \tag{12}$$

solves (10) along with $u(a) = \frac{\sqrt{2}}{2}$, ${}_a\mathrm{D}_{\frac{1}{2}}u(a) = \frac{\sqrt{2}}{4}$. Similarly, equation (11) has a solution

$$v(t) = \sin\left(c_1 + \sqrt{q_1(t-a)}\right) \tag{13}$$

satisfying $v(a) = \sin(c_1)$, ${}_a\mathrm{D}_{\frac{1}{2}}v(a) = \frac{\sqrt{q_1}}{2}\cos(c_1)$. Therefore, if $q_1 < 1$, function v oscillates more slowly than u, and eventually one of its zeros will not lie between two successive zeros of u. On the other hand, v oscillates faster than u whenever $q_1 > 1$. This coincides with the statement of Theorem 4, and is depicted in Figure 1.

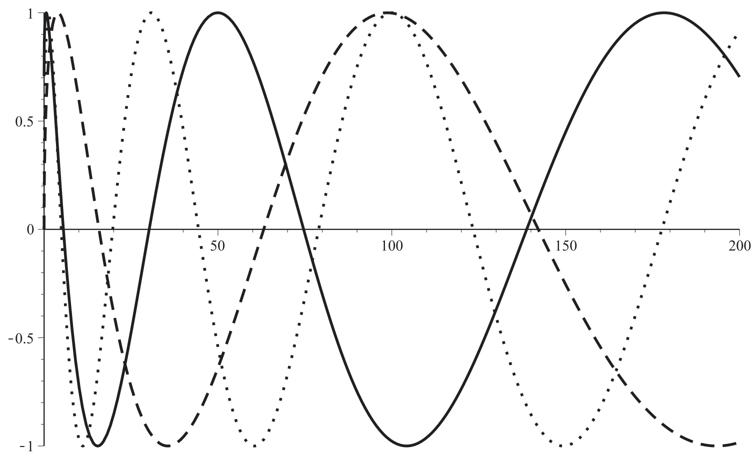

Figure 1: *For $a = 0$, function u of* (12) *(solid), v of* (13) *with $c_1 = 0$ and $q_1 = \frac{5}{8}$ (dashed), $q_1 = 2$ (dotted)*

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