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Geometric fitting by two coaxial cylinders

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Abstract. Fitting two coaxial cylinders to data is a standard problem in computational metrology and reverse engineering processes, which also arises in medical imaging. There are many fitting criteria that can be used. One criterion that is widely used in metrology, when the errors in data are thought to be normally distributed, for example, is that of minimizing the sum of squared minimal distance. A similar numerical method is developed to fit two coaxial cylinders in the general position to 3D data, and numerical examples are given.

AMS subject classifications: 65D10, 65C60

Key words: geometric fitting, two coaxial cylinders

1. Introduction

In geometric fitting, also known as best fitting, the error distances are defined as the shortest distances from the given 2D or 3D points to the geometric feature to be fitted. This quality is desirable in many fields of science and engineering, including astronomy, biology, physics, quality control, and metrology [5, 6].

Furthermore, fitting a cylinder that minimizes the sum of the squares of the distance of the points from the cylinder is a recognized problem in, for instance, computational metrology [34](problem C5), computer vision [26], and engineering of a geometric shape [23]. Different approaches have been made to solve this problem resulting in different algorithms that use least squares methods. Fitting an implicit cylinder to given data is considered in [12] and [23, 24]. Fitting a right circular cylinder, a cylinder whose base is perpendicular to its sides. In addition, fitting a special case right circular cylinder is determined in [7], and discussed in [35].

Cylindrical features are common in mechanical designs [25] and reverse engineering processes [10] and arises in medical imaging [9]. Furthermore, finding two coaxial cylinders with minimum difference in their radii that contains all the relevant data points between them is a standard problem in metrology [34] (problem C9). To clarify the problem, assume that the available data, $\mathbf{x}_i = (x_{i1}, x_{i2}, x_{i3})^T,\ i = 1\dots, m$, have errors. Moreover, to find two coaxial cylinders, the suitable criteria proposed are to minimized, where each data point has to be simultaneously associated with one of the cylinders.

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A widely used optimization method in cluster analysis is k-means. This method needs only a data set and a pre-specified number of clusters, k, and minimizing the within cluster square error. Consequently, the algorithm is applicable only if the mean is defined, the k number of clusters has to be estimated. The use of this criterion could be used for division, when the differences in size of geometry of the clusters are big, see [21].

Let W_1 and W_2 be the two subsets of given points, with indexes j and k, associated, respectively, with a small cylinder ς_1 and a large cylinder ς_2. Moreover, $W_1 \cap W_2 = \Phi, W_1 \cup W_2 = \{1, \ldots, m\}$, and $|W_1| = n_1 \geq 8, |W_2| = n_2 \geq 8$. Furthermore, let r_1 and r_2 be the two coaxial cylinders' radii, where $r_1 < r_2$. The common centre will be denoted by $\mathbf{c} = (c_1, c_2, c_3)^T$, and the altitude will be h.

The parametric representations are dependent upon the location of real parameters $t_i, i = 1, \ldots, m$, which are independent of any coordinate system. The parametric representation is important for approximating data that have been measured in an arbitrary coordinate system. In addition, it allows for two closed coaxial cylinders. Further, any explicit surfaces can be specified in the parametric form in an analogous way [35].

Then, a parametric representation of the small right circular coaxial cylinder ς_1 can be written as

$$\mathbf{x}(\mathbf{c}, r_1, \boldsymbol{\theta}, t_j) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + R(\boldsymbol{\theta}) \begin{bmatrix} r_1 \cos(t_j) \\ r_1 \sin(t_j) \\ h_j \end{bmatrix}, \quad 0 < t_j \leq 2\pi. \tag{1}$$

Replacing j by k and r_1 by r_2 in (1) gives the other coaxial cylinder $\varsigma_2, k \in W_2$. $\boldsymbol{\theta} = (\alpha, \beta, \gamma)^T$ where α, β and γ denote, respectively, the common rotation angles in the (x_1, x_2), (x_1, x_3) and (x_2, x_3) plane, known as Euler rotation angles. So, $R(\boldsymbol{\theta})$ will be the unknown rotation matrix.

In addition, the matrix $R(\boldsymbol{\theta}) \in R^{3\times 3}$ is orthogonal, so $\det(R(\boldsymbol{\theta})) = 1$. Moreover, R can be written as a product of three elementary rotations

$$\begin{aligned} R(\boldsymbol{\theta}) &= R_1(\alpha) R_2(\beta) R_3(\gamma) \\ &= \begin{bmatrix} C_\alpha & S_\alpha & 0 \\ -S_\alpha & C_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_\beta & 0 & S_\beta \\ 0 & 1 & 0 \\ -S_\beta & 0 & C_\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\gamma & S_\gamma \\ 0 & -S_\gamma & C_\gamma \end{bmatrix} \\ &= \begin{bmatrix} C_\beta C_\alpha & -S_\gamma S_\beta C_\alpha + C_\gamma S_\alpha & C_\alpha S_\beta C_\gamma + S_\alpha S_\gamma \\ -C_\beta S_\alpha & S_\gamma S_\beta S_\alpha + C_\gamma C_\alpha & -S_\alpha S_\beta C_\gamma + C_\alpha S_\gamma \\ -S_\beta & -S_\gamma C_\beta & C_\beta C_\gamma \end{bmatrix}, \end{aligned}$$

where the notations C_α and S_α are used for simplicity to denote $\cos(\alpha)$ and $\sin(\alpha)$, respectively, and the other rotation angles are notated similarly.

Let $\mathbf{a} = (\mathbf{c}, r_1, r_2, \boldsymbol{\theta})^T \in \mathbf{R}^8$, and define the orthogonal distance vector

$$\mathbf{v}_i(\mathbf{a}, t_i) = \begin{cases} \mathbf{x}_j - \mathbf{x}(\mathbf{c}, r_1, \boldsymbol{\theta}, t_j), & j \in W_1; \\ \mathbf{x}_k - \mathbf{x}(\mathbf{c}, r_2, \boldsymbol{\theta}, t_k), & k \in W_2. \end{cases}$$

Let $t_i(\mathbf{a}), i = 1, \ldots, m$ be such that for any $\mathbf{a}$, $\|\mathbf{v}_i\|_2^2$ is minimized with respect to t_i. Then, the problem of fitting two coaxial cylinders to data using orthogonal distance regression is the minimization of the objective function

$$\hat{G}(\mathbf{a}, t_i(\mathbf{a})) = \sum_{j}^{n_1} \|\mathbf{x}_j - \mathbf{x}(\mathbf{a}, t_j(\mathbf{a}))\|_2^2 + \sum_{k}^{n_2} \|\mathbf{x}_k - \mathbf{x}(\mathbf{a}, t_k(\mathbf{a}))\|_2^2. \tag{2}$$

A Gauss-Newton type method is popular for solving orthogonal distances problems, see [1, 2, 4, 7, 8, 13, 14, 18, 19, 35, 36, 38, 39, 40]. This method requires a nonsingular Jacobian matrix. To be more precise, The Jacobian matrix must be a full rank, which is in this case 8. The condition is not satisfied for fitting a parametric right circular cylinder in general position and orthogonal distance regression. Furthermore, this condition is not satisfied for fitting two coaxial cylinders, as a result of

$$\hat{G}(\mathbf{c}, r_1, r_2, \alpha, \beta, \gamma) = \hat{G}(\mathbf{c}, r_1, r_2, \alpha - \pi, \pi - \beta, \gamma - \pi). \tag{3}$$

In fact, a direct analog of the trust region Levenberg-Marquardt algorithm can be used for solving the orthogonal distance regression problem, (2), and a local convergence for the solution can get it when Jacobian is rank deficient, in which Gauss-Newton type method is not effective [11, 20]. This algorithm will be subjected in future work.

This paper proposes geometric fitting by a general position of two parametric coaxial cylinders. This type of algorithm has been proposed for the usual orthogonal distance regression problem [27, 30, 31, 32, 33]. The problem will be described in the next section. A corresponding algorithm and starting values are developed in Section 3. Numerical examples are given in Section 4.

2. The problem

For simplicity, let R contain the row vectors $\boldsymbol{\lambda}_p, p = 1, \ldots, 3$ and the column vectors $\boldsymbol{\mu}_p, p = 1, \ldots, 3$. For instance, $\boldsymbol{\lambda}_1 = [C_\beta C_\alpha \quad -S_\gamma S_\beta C_\alpha + C_\gamma S_\alpha \quad C_\alpha S_\beta C_\gamma + S_\alpha S_\gamma]$, and $\boldsymbol{\mu}_1 = [C_\beta C_\alpha \quad -C_\beta S_\alpha \quad -S_\beta]^T$. Further, define

$$\boldsymbol{\nu}_j = \begin{bmatrix} r_1 \cos(t_j) \\ r_1 \sin(t_j) \\ h_j \end{bmatrix}, j \in W_1, \quad \text{and} \quad \boldsymbol{\nu}_k = \begin{bmatrix} r_2 \cos(t_k) \\ r_2 \sin(t_k) \\ h_k \end{bmatrix}, k \in W_2.$$

Minimizing the sum of the squares of the distance between a data point and its closest point on the coaxial cylinder in general position can be defined to minimize

$$\begin{aligned}\xi(W_1, W_2, \mathbf{a}, t_i(\mathbf{a})) = \sum_j &\{(x_{j1} - c_1 - \boldsymbol{\lambda}_1\boldsymbol{\nu}_j)^2 + (x_{j2} - c_2 - \boldsymbol{\lambda}_2\boldsymbol{\nu}_j)^2 \\ &+(x_{j3} - c_3 - \boldsymbol{\lambda}_3\boldsymbol{\nu}_j)^2\} + \sum_k \{(x_{k1} - c_1 - \boldsymbol{\lambda}_1\boldsymbol{\nu}_k)^2 \\ &+(x_{k2} - c_2 - \boldsymbol{\lambda}_2\boldsymbol{\nu}_k)^2 + (x_{k3} - c_3 - \boldsymbol{\lambda}_3\boldsymbol{\nu}_k)^2\}\end{aligned} \tag{4}$$

To calculate the value of t_i, that corresponds to the orthogonal point on the two coaxial cylinders, at each iteration for given $\mathbf{a}$, we must solve the following necessary condition

$$\frac{\partial \xi}{\partial t_i} = 0,\ i = 1, \dots, m, \quad \text{i.e.,} \quad \frac{\partial \xi}{\partial \mathrm{t_j}} = 0 \quad \text{and} \quad \frac{\partial \xi}{\partial \mathrm{t_k}} = 0. \tag{5}$$

In a nutshell, expanding $\mathbf{v}_i(\mathbf{a}, t_i(\mathbf{a}))^T \nabla_{t_i(\mathbf{a})} \mathbf{v}_i(\mathbf{a}, t_i(\mathbf{a}))$, where $\nabla_{t_i(\mathbf{a})}$ denotes the partial derivative with respect to t, results in

$$\nabla_{t_i(\mathbf{a})} \mathbf{v}_i(\mathbf{a}, t_i(\mathbf{a})) = \begin{cases} -r_1 \left[\, \boldsymbol{\mu}_1\ \boldsymbol{\mu}_2 \,\right] \begin{bmatrix} -S_{t_j} \\ C_{t_j} \end{bmatrix}, \\ -r_2 \left[\, \boldsymbol{\mu}_1\ \boldsymbol{\mu}_2 \,\right] \begin{bmatrix} -S_{t_k} \\ C_{t_k} \end{bmatrix} \end{cases}$$

where the notations S_{t_j} and C_{t_j} denote $\sin(t_j)$ and $\cos(t_j)$, respectively, and similarly for S_{t_k} and C_{t_k}. Furthermore, using the trigonometric identities

$$\sin(t_i(\mathbf{a})) = \frac{2\omega_i}{1+\omega_i^2},\ \cos(t_i(\mathbf{a})) = \frac{1-\omega_i^2}{1+\omega_i^2}, \quad \text{and} \quad \tan(\frac{\mathrm{t_i}(\mathbf{a})}{2}) = \omega_\mathrm{i},$$

it follows that the necessary condition (5) requires the solution, for each i, of the polynomial

$$A_{i2}(\omega_i^2 - 1) - A_{i1}\omega_i = 0,$$

with respect to ω_i, where

$$A_{i2} = \begin{cases} r_1(\mathbf{x}_i - \mathbf{c})^T \boldsymbol{\mu}_2, & \text{if } i \in W_1, \\ r_2(\mathbf{x}_i - \mathbf{c})^T \boldsymbol{\mu}_2, & \text{if } i \in W_2 \end{cases}$$

and

$$A_{i1} = \begin{cases} 2r_1(\mathbf{x}_i - \mathbf{c})^T \boldsymbol{\mu}_1, & \text{if } i \in W_1, \\ 2r_2(\mathbf{x}_i - \mathbf{c})^T \boldsymbol{\mu}_1, & \text{if } i \in W_2 \end{cases}.$$

Thus,

$$t_i(\mathbf{a}) = 2\tan^{-1}(\omega_i), \tag{6}$$

and the parameter $t_i(\mathbf{a})$ should be chosen to minimize the ith term of the objective function $\xi(\mathbf{a}, t(\mathbf{a}))$. More details can be found in [7].

Likewise, for each i, the value of h_i must satisfy the necessary condition

$$\frac{\partial \xi}{\partial h_i} = 0,\ i = 1, \dots, m. \ \text{i.e.,} \ \frac{\partial \xi}{\partial h_j} = 0 \text{ and } \frac{\partial \xi}{\partial h_k} = 0.$$

Then,

$$h_i = \begin{cases} -(\mathbf{x}_j - \mathbf{c})^T \boldsymbol{\mu}_3, \\ -(\mathbf{x}_k - \mathbf{c})^T \boldsymbol{\mu}_3. \end{cases} \tag{7}$$

The necessary conditions to minimize (4) with respect to the centre and radii are

$$\frac{\partial \xi}{\partial c_1} = \frac{\partial \xi}{\partial c_2} = \frac{\partial \xi}{\partial c_3} = \frac{\partial \xi}{\partial r_1} = \frac{\partial \xi}{\partial r_2} = 0,$$

which give the following linear system

$$\begin{bmatrix} m & 0 & 0 & & \\ 0 & m & 0 & \mathbf{p}_1 & \mathbf{p}_2 \\ 0 & 0 & m & & \\ & \mathbf{p}_1^T & & n_1 & 0 \\ & \mathbf{p}_2^T & & 0 & n_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} \sum_j (x_{j1} - \mu_{13} h_j) + \sum_k (x_{k1} - \mu_{13} h_k) \\ \sum_j (x_{j2} - \mu_{23} h_j) + \sum_k (x_{k2} - \mu_{23} h_k) \\ \sum_j (x_{j3} - \mu_{33} h_j) + \sum_k (x_{k3} - \mu_{33} h_k) \\ \sum_j \left(\mathbf{x}_j^T \begin{bmatrix} \boldsymbol{\mu}_1 & \boldsymbol{\mu}_2 \end{bmatrix} \begin{bmatrix} \cos(t_j) \\ \sin(t_j) \end{bmatrix} \right) \\ \sum_k \left(\mathbf{x}_k^T \begin{bmatrix} \boldsymbol{\mu}_1 & \boldsymbol{\mu}_2 \end{bmatrix} \begin{bmatrix} \cos(t_k) \\ \sin(t_k) \end{bmatrix} \right) \end{bmatrix}, \tag{8}$$

where

$$\mathbf{p}_1 = \begin{bmatrix} \boldsymbol{\mu}_1 & \boldsymbol{\mu}_2 \end{bmatrix} \begin{bmatrix} \sum_j \cos(t_j) \\ \sum_j \sin(t_j) \end{bmatrix} \quad \text{and} \quad \mathbf{p}_2 = \begin{bmatrix} \boldsymbol{\mu}_1 & \boldsymbol{\mu}_2 \end{bmatrix} \begin{bmatrix} \sum_k \cos(t_k) \\ \sum_k \sin(t_k) \end{bmatrix}.$$

The determinant g of the coefficient matrix in (8) is

$$\begin{aligned} g = m^3 n_1 n_2 + m[&\left(\sum_j \sin(t_j)\right)^2 \left(\sum_k \cos(t_k)\right)^2 \\ &+ \left(\sum_k \sin(t_k)\right)^2 \left(\sum_j \cos(t_j)\right)^2] - m^2 n_1 [\left(\sum_k \sin(t_k)\right)^2 \left(\sum_k \cos(t_k)\right)^2] \\ &- m^2 n_2 [\left(\sum_j \sin(t_j)\right)^2 \left(\sum_j \cos(t_j)\right)^2] \end{aligned} \tag{9}$$

In fact, $g \geq 0$, however, $g = 0$ only in uninteresting cases like $t_i = 0, i = 1, \ldots, m$. Thus, normally (8) has a unique solution [27, 30, 31, 32, 33].

Now, the remaining necessary conditions are

$$\frac{\partial \xi}{\partial \alpha} = \frac{\partial \xi}{\partial \beta} = \frac{\partial \xi}{\partial \gamma} = 0. \tag{10}$$

These three equations for the unknowns α, β and γ are highly nonlinear. Nevertheless, the rotation angles can be found using direct minimization of ξ, see [28, 29]. The first condition, with respect to α, in (10) gives

$$\begin{aligned} &\sum_j (\mathbf{x}_j - \mathbf{c})^T \frac{dR_1}{d\alpha} R_2 R_3 \boldsymbol{\nu}_j + \sum_k (\mathbf{x}_k - \mathbf{c})^T \frac{dR_1}{d\alpha} R_2 R_3 \boldsymbol{\nu}_k \\ &\quad = \sum_j \boldsymbol{\nu}_j^T R_3^T R_2^T R_1^T \frac{dR_1}{d\alpha} R_2 R_3 \boldsymbol{\nu}_j + \sum_k \boldsymbol{\nu}_k^T R_3^T R_2^T R_1^T \frac{dR_1}{d\alpha} R_2 R_3 \boldsymbol{\nu}_k. \end{aligned} \tag{11}$$

Because

$$R_1^T \frac{dR_1}{d\alpha} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

if we define $\mathbf{v}_j = R_2R_3\boldsymbol{\nu}_j$ and $\mathbf{v}_k = R_2R_3\boldsymbol{\nu}_k$, then

$$\sum_j^m \mathbf{v}_j^T R_1^T \frac{dR_1}{d\alpha}\mathbf{v}_j = \sum_k^m \mathbf{v}_k^T R_1^T \frac{dR_1}{d\alpha}\mathbf{v}_k = 0.$$

Consequently, the necessary condition (11) will be

$$\sum_j (\mathbf{x}_j - \mathbf{c})^T \frac{dR_1}{d\alpha} R_2R_3\boldsymbol{\nu}_j + \sum_k (\mathbf{x}_k - \mathbf{c})^T \frac{dR_1}{d\alpha} R_2R_3\boldsymbol{\nu}_k = 0. \tag{12}$$

Define $\hat{\mathbf{v}}_j = (\mathbf{x}_j - \mathbf{c})^T$ and $\hat{\mathbf{v}}_k = (\mathbf{x}_k - \mathbf{c})^T$. Thus the solution for α can be determined as

$$\tan(\alpha(\beta,\gamma)) = \frac{\sum_j(\hat{v}_{j1}v_{j2} - \hat{v}_{j2}v_{j1}) + \sum_k(\hat{v}_{k1}v_{k2} - \hat{v}_{k2}v_{k1})}{\sum_j(\hat{v}_{j1}v_{j1} + \hat{v}_{j2}v_{j2}) + \sum_k(\hat{v}_{k1}v_{k1} + \hat{v}_{k2}v_{k2})}. \tag{13}$$

In the same way, the second and the third condition in (10) give

$$\sum_j (\mathbf{x}_j - \mathbf{c})^T R_1 \frac{dR_2}{d\beta} R_3\boldsymbol{\nu}_j + \sum_k (\mathbf{x}_k - \mathbf{c})^T R_1 \frac{dR_2}{d\beta} R_3\boldsymbol{\nu}_k = 0,$$

$$\sum_j (\mathbf{x}_j - \mathbf{c})^T R_1R_2 \frac{dR_3}{d\gamma} \boldsymbol{\nu}_j + \sum_k (\mathbf{x}_k - \mathbf{c})^T R_1R_2 \frac{dR_3}{d\gamma} \boldsymbol{\nu}_k = 0.$$

Given

$$R_2^T\frac{dR_2}{d\beta} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad R_3^T\frac{dR_3}{d\gamma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

and the orthogonality of D, define further $\hat{\mathbf{v}}_j = (\mathbf{x}_j - \mathbf{c})^T R_1$, $\hat{\mathbf{v}}_k = (\mathbf{x}_k - \mathbf{c})^T R_1$, $\mathbf{v}_j = R_3\boldsymbol{\nu}_j$, and $\mathbf{v}_k = R_3\boldsymbol{\nu}_j$. Then,

$$\tan(\beta(\alpha,\gamma)) = \frac{\sum_j(\hat{v}_{j1}v_{j3} - \hat{v}_{j3}v_{j1}) + \sum_k(\hat{v}_{k1}v_{k3} - \hat{v}_{k3}v_{k1})}{\sum_j(\hat{v}_{j1}v_{j1} + \hat{v}_{j3}v_{j3}) + \sum_k(\hat{v}_{k1}v_{k1} + \hat{v}_{k3}v_{k3})}. \tag{14}$$

Now define $\hat{\mathbf{v}}_j = (\mathbf{x}_j - \mathbf{c})^T R_1R_2$ and $\hat{\mathbf{v}}_k = (\mathbf{x}_k - \mathbf{c})^T R_1R_2$. Then

$$\tan(\gamma(\alpha,\beta)) = \frac{\sum_j(\hat{v}_{j2}u_{j3} - \hat{v}_{j3}u_{j2}) + \sum_k(\hat{v}_{k2}u_{k3} - \hat{v}_{k3}u_{k2})}{\sum_j(\hat{v}_{j2}u_{j2} + \hat{v}_{j3}u_{j3}) + \sum_k(\hat{v}_{k2}u_{k2} + \hat{v}_{k3}u_{k3})}. \tag{15}$$

If one of the denominators ($d_j, j = 1,\ldots,3$, for example) of the angles α, β and γ in (13), (14) or (15), respectively, becomes zero, then $\frac{\pi}{2}$ is the value of the corresponding jth rotation angle. Moreover, $\theta_j, j = 1,\ldots,3$ will be replaced by $\theta_j = \theta_j + \pi$ if $n_j\cos(\theta_j) + d_j\sin(\theta_j) < 0, j = 1,\ldots,3$, where n_j denotes the nominator of the jth angle of $\boldsymbol{\theta}$ [29]. Due to equality (3), minima or global minima are not unique, but because ξ is continuous and bounded below, this is not a problem [28]. In addition, the rotation angles $\boldsymbol{\theta} = (\alpha,\beta,\gamma)^T$ can be easily done by using NAG subroutine FMIN [28] or by using the MATLAB command FMINSEARCH, with starting intervals $[0, 2\pi]$. For the algorithm, see [22].

Naturally, the subsets W_1 and W_2 can be determined by calculating $\|\mathbf{x}_j - \mathbf{x}(\mathbf{c}, r_1, \boldsymbol{\theta}, t_j)\|^2$ and $\|\mathbf{x}_k - \mathbf{x}(\mathbf{c}, r_k, \boldsymbol{\theta}, t_k)\|^2$, for each i. If $\|\mathbf{x}_j - \mathbf{x}(\mathbf{c}, r_1, \boldsymbol{\theta}, t_j)\|^2 < \|\mathbf{x}_k - \mathbf{x}(\mathbf{c}, r_2, \boldsymbol{\theta}, t_k)\|^2$, then $i \in W_1$, otherwise i is in W_2.

3. An algorithm and starting values

Fitting two coaxial parametric cylinders to given data $\mathbf{x}_i, i = 1, \ldots, m$ can be summarized in the following steps.

STEP 0. Input: $\mathbf{a}^{(0)} = (\mathbf{c}^{(0)}, r_1^{(0)}, r_2^{(0)}, \boldsymbol{\theta}^{(0)})^T$, and a tolerance (**Tol**). Set $\xi^{(0)} = \infty$, $l = 1$.

STEP 1. Determine the following:

- $t_j^{(l)}$ and $t_k^{(l)}$.
- $h_j^{(l)}$ and $h_k^{(l)}$.
- $\mathbf{c}^{(l)}$, $r_1^{(l)}$ and $r_2^{(l)}$ by solving the linear system (8).
- The objective function $\xi^{(l)}$ and (4) .

STEP 2. Determine: $\boldsymbol{\theta}^{(k)}$ using (13),(14),(15).

STEP 3. If $|\xi^{(l)} - \xi^{(l-1)}| < Tol$, then STOP.

STEP 4. Determine: $W_1^{(k)}$ and $W_2^{(k)}$.

STEP 5. Set $l = l + 1$, then go to **STEP 1**.

The idea of the proposed algorithm is to fix all variables expect one or some groups, to globally minimize problem (4) with respect to the rest, then to fix these variables and globally minimize (4) with respect to some other variables and so on.

It is necessary to provide starting points for any iterative algorithm. The algebraic fitting can be used to find initial Euler rotation angles $\boldsymbol{\theta}^{(0)}$ and an initial centre $\mathbf{c}^{(0)}$. The cylinder is represented by

$$\mathbf{x}^T A \mathbf{x} + \mathbf{b}^T \mathbf{x} + e = 0, \tag{16}$$

where A is the symmetric positive definite matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

and e is a scalar. Equation (16) contains ten linear coefficients

$$\mathbf{v} = (a_{11}, a_{22}, a_{33}, a_{12}, a_{13}, a_{23}, b_1, b_2, b_3, e)^T,$$

that can be found by minimizing

$$\|Z\mathbf{v}\|_2 \ \text{ subject to } \ \|\mathbf{v}\|_2 = 1,$$

where

$$Z = \begin{bmatrix} x_{i1}^2 & x_{i2}^2 & x_{i3}^2 & 2x_{i1}x_{i2} & 2x_{i1}x_{i3} & 2x_{i2}x_{i3} & x_{i1} & x_{i2} & x_{i3} & 1 \end{bmatrix},$$

$i = 1, \dots, m$. This problem is equivalent to finding the right singular vector associated with the smallest singular value of Z [4, 7, 15, 35, 37].

Using the relations between parametric and implicit forms and the Hanson and Norris procedure for finding Euler rotation angles [17], let $A = Q\Lambda Q^T$ be the eigendecomposition of A [16] and suppose the eigenvalues matrix Λ has been approximated as $\Lambda = \overline{\lambda} I$, where I is the identity matrix of order 3 and $\overline{\lambda}$ is the average of the eigenvalues. Explicit expressions for the angles can be written as follows:

$$\begin{aligned}
\tan(\gamma) &= \frac{Q(2,2)}{Q(3,2)}, \\
\tan(\beta) &= \frac{C_\gamma Q(2,2)}{Q(1,2) + Q(2,2)}, \\
\tan(\alpha) &= \frac{S_\gamma Q(3,1) + C_\gamma Q(2,1)}{C_\beta Q(1,1) - S_\gamma S_\beta Q(2,1) + S_\beta C_\gamma Q(3,1)}.
\end{aligned}$$

and the centre can be determined as

$$\mathbf{c} = -\frac{\mathbf{b}}{2\overline{\lambda}}.$$

The data mean can be used as an initial centre. Further, the starting radius $r_1^{(0)}$ can be set to the minimum distance between the initial centre and input data, and $r_2^{(0)}$ is set to the maximum. More details can be found in [3, 7].

4. Numerical experiments

This section presents two examples to illustrate the application of least squares fitting of two coaxial parametric cylinders to data. The first subset W_1 of size $|W_1|$ data points is generated by selecting a particular cylinder. The second subset W_2 is generated in the same way, with $r_2 > r_1$, where $|W_1| = |W_2| = m/2$ to make the calculation much easier. Then, random perturbations are introduced for these data on the interval $[0.0, 1.0]$, and the MATLAB command "rand" is used. The initial subsets $W_1^{(0)}$ and $W_2^{(0)}$ are determined by taking a random permutation of the integers from 1 to m using the MATLAB command "randperm". The algorithm terminates when the tolerance is reduced to less than 10^{-6}.

Example 1. *As shown in the last section, the starting points*

$$\mathbf{a}^{(0)} = (2.0710, -3.0982, -0.2680, 1.9714, 4.0816, -1.0613, 0.2023, 0.6275)^T$$

are determined, to fit the data $\mathbf{x}_i, i = 1, \dots, 24$, *starting with* $W_1^{(0)} = \{1, 3, 5, 7, 9 - 12, 15, 20, 23, 24\}$ *and* $W_2^{(0)} = \{2, 4, 6, 8, 13,$
$14, 16 - 19, 21, 22\}$. *The objective value is reduced from* $\xi^{(0)} = 33.706235$ *to* $\xi^{(38)} = 0.953025$. *As expected, the method gives* $W_1 = \{1, \dots, 12\}$ *and* $W_2 = \{13, \dots, 24\}$ *in the third iteration. The result is shown in Figure* 1, *with the final solution*

$$\mathbf{a}^{(38)} = (2.0762, -3.1958, -0.2177, 2.0192, 3.2700, -1.4567, 0.5818, 0.0307)^T.$$

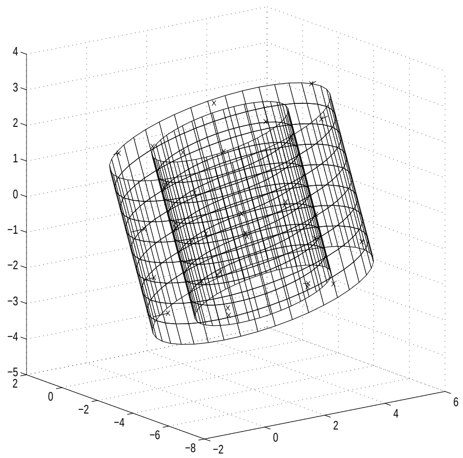

Figure 1: *Fitting two coaxial cylinders to 24 data points*

Example 2. *The starting points*

$$\mathbf{a}^{(0)} = (-1.7221, 1.5614, -0.6577, 3.6485, 7.0902, -0.0926, 0.1675, -1.8094)^T$$

are determined to fit $m = 100$ *data points, starting with* $|W_1^{(0)}| = |W_2^{(0)}| = 50$*, where* W_1 *and* W_2 *are randomly generated permutations of integers from 1 to 100 . The objective value is reduced from* $\xi^{(0)} = 381.179881$ *to* $\xi^{(28)} = 11.435341$*. As expected, the method gives* $W_1 = \{1, \ldots, 50\}$ *and* $W_2 = \{51, \ldots, 100\}$ *in the third iteration. The result is shown in Figure* 2*, with the final solution*

$$\mathbf{a}^{(38)} = (-1.5012, 1.5810, 0.6902, 4.1812, 6.1308, -0.0201, 0.1723, 4.7079)^T.$$

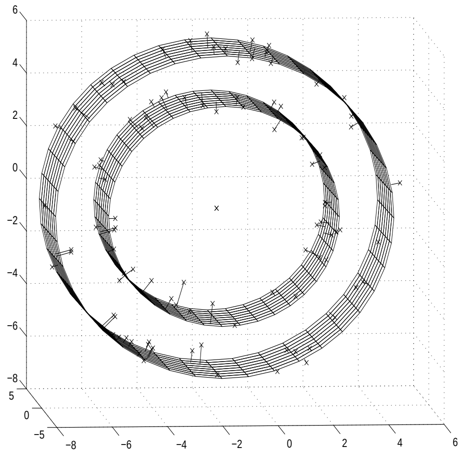

Figure 2: *Fitting two coaxial cylinders to 100 data points*

Geometric fitting by two coaxial cylinders to the measured data in 3-space requires an iterative solution to linear and non-linear subproblems and does not need a derivative for Jacobian or Hessian matrices. Nevertheless, the algorithm is known to be slow [5, 27, 28, 29, 30, 31, 32, 33]. Geometric fitting, like orthogonal distance regression, is also sensitive to the effects of outliers.

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