UDC 532.542.58:628.12/.14]:519.65
Original scientific paper
Received: 18.06.2015.

An iterative algorithm for initializing the flow in a pipe system with more reservoirs

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SUMMARY

For the analysis of a pressure pipe system with multiple reservoirs, a numerical algorithm for initializing the flow in the system is developed. This algorithm finds real hydraulic head values in junctions of a pressure pipe system by using an iterative method. Also, it determines the orientation and amount of the flow rate for an arbitrary given pressure pipe system and a number of reservoirs. By comparing the presented method with a classical three-reservoir problem, it might be said that the proposed method performs a hydraulic head correction in junctions in an inverse way. The method is implemented in the computer code developed for the analysis of an arbitrary pressure pipe system.

KEY WORDS: *multiple reservoirs, three-reservoirs problem, hydraulic head, pressure pipe system, iterative method.*

1. INTRODUCTION

In hydraulic engineering, a water supply system is the most common type of pressure systems. In order to design one of these, it is necessary to perform the analysis of steady and unsteady flow. This paper focuses on the steady flow analysis, especially on the initialization of such an analysis for the case of water networks with multiple reservoirs. In fact, regardless of the choice of the computational algorithm (pressure-based method or discharge-based method), the steady flow analysis starts by assigning the flow to all pipes, which must necessarily satisfy the continuity equation in all nodes, where the node represents a junction of two or more pipes. In addition, it is necessary to satisfy Kirchhoff's second law [1, 2, 3], which is achieved by using an iterative procedure.

Determination of the reservoir outflow q for water networks with one reservoir is almost trivial. That is because the flow from the reservoir is equal to the sum of all node consumptions q (Figure 1).

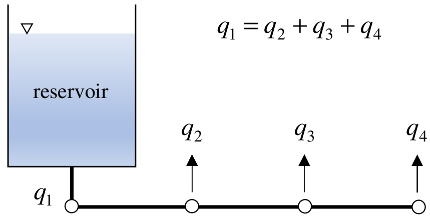

Fig. 1 *Reservoir flow determination for the pipe system with one reservoir*

However, if a pipe system combines multiple reservoirs (Figure 2), the determination of reservoir outflows is more complicated and requires an iterative procedure. That is because there are multiple unknowns in one available equation. This problem is usually solved by an iterative procedure [4] in which, firstly, the amount of pressure is assumed and, secondly, the pressure gradients or achieved flows are computed. The iterative procedure converges to the exact result when the achieved flows satisfy the continuity equation (Kirchhoff's first law).

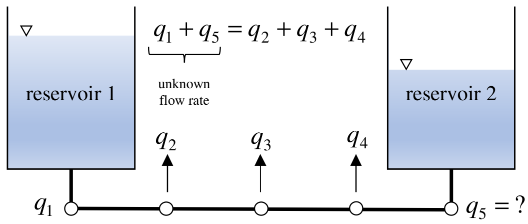

Fig. 2 *Reservoir flow determination for the pipe system with two reservoirs*

In this paper, the given problem is solved in an inverse way so that first the reservoirs' flows are assumed and only then the pressure corrections in nodes are made accordingly. It should be emphasized that this method (unlike the previously mentioned one) can be used in terms of initialization for complex pipe systems with more reservoirs. Also, except of the outflows quantities, the algorithm predicts the orientation of the flow, so it can determine which reservoirs are getting filled with water. At the end of the paper, the numerical algorithm is presented and numerical examples are given.

2. THEORETICAL BACKGROUND

2.1 INDEXATION OF PIPES AND JUNCTIONS

In terms of a computational algorithm, a pipe is an element of a pressure system between two contiguous junctions. Apart from this, a pipe can be considered as an element of a pressure system between a junction and some other functional elements of the system (i.e. pumping station). To identify the position of all geometrical and kinematical values of the flow, they will be labelled with an index j, which will indicate a particular pipe for which that value is mentioned. The index j can take discrete values from one to a maximal value of n_j, which defines the last pipe in the pressure system. Obviously, in order to have a functional purpose, the pipe system layout and pipe numeration should be given in advance.

Junction indexation is made in a similar way. Therefore, all values which are related to junctions (i.e. hydraulic head H and node consumption q) will have an index i which can take discrete values from one to a maximal value of n_i, which defines the last junction in the pressure system.

In cases where it is necessary to point out a specific junction o and its contiguous pipes, i.e. pipes connected with a junction o, pipes are labeled with an index oi, where o represents the observed junction and i the ordinal number of one of the contiguous junctions.

2.2 KIRCHHOFF'S LAWS

The steady flow model is based on Kirchhoff's laws. Firstly, at any junction in the pipe system the volumetric flow rate into that junction is equal to the volumetric flow rate out of that junction. Secondly, for each pipe the Darcy–Weisbach equation [5] should be satisfied.

The first Kirchhoff's law can be interpreted as principle of flow Q conservation. Mathematical notation of the first rule is:

$$\sum_{i=1}^{n} Q_{oi} = q_o \tag{1}$$

in which Q_{oi} represents volumetric flow rate in a pipe oi and q consumption in a junction o. The index i represents the pipe connection counter which goes from 1 to n, where n represents the total number of contiguous junctions.

The second Kirchhoff's law is based on the principle of conservation of energy inside a close section of pipes i.e. inside a pipe loop. The law defines that the sum of all energy loss ΔE inside each loop is equal to zero, which can mathematically be written as [6]:

$$\sum_{i=1}^{n} \Delta E_{oi} = 0 \tag{2}$$

in which n is the total number of pipes that form a loop. This law applies to all loops in a pipe system. Also, it should be noted that this law is changed if there is an element in the observed loop, which adds or subtracts energy from the flow in the loop. Therefore, if there is a pump station in the loop, then Eq. (2) becomes:

$$\sum_{i=1}^{n} \Delta E_{oi} = H_{man} \tag{3}$$

in which H_{man} labels the manometric head of the pump station. On the other hand, if there is a turbine in the loop, Eq. (2) becomes:

$$\sum_{i=1}^{n} \Delta E_{oi} = -H_{turb} \tag{4}$$

in which H_{turb} labels a pressure drop on a turbine.

2.3 BERNOULLI'S PRINCIPLE

The principle of energy conservation can be applied for the purpose of defining Bernoulli's equation for a steady flow condition, which can be written as:

$$z_1 + \frac{p_1}{\rho g} + \frac{v_1^2}{2g} = z_2 + \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + \Delta E \tag{5}$$

in which z labels geodetic height of the pipe centerline, ρ fluid density, g gravitational acceleration, v average flow velocity and ΔE energy loss caused by overcoming hydrodynamic resistance between section 1 and 2. It is necessary to note that the velocity head defined by the third term on the LHS and RHS will be neglected in the developed numerical procedure due to the relatively small contribution to the total energy. For water system distribution this is most frequently justified.

2.4 DARCY–WEISBACH EQUATION

Energy loss ΔE can be expressed as a function of relevant geometrical and kinematical quantities of the flow. By combining the analytical flow analysis with the Vaschy-Buckingham Π theorem [7], a Darcy–Weisbach equation can be derived:

$$\Delta E = \lambda \frac{L}{D} \frac{v^2}{2g} \tag{6}$$

in which λ denotes the Darcy friction factor, L pipe length and D pipe diameter. The Darcy friction factor λ represents a function of a Reynolds number Re and a relative roughness ε/D [8].

3. THREE RESERVOIR PROBLEM

As described in the introduction, determination of reservoir flow is very complex for networks with multiple reservoirs. In order to clarify the problem, let us consider a network with three reservoirs denoted as A, B and C connected with three pipes. The three pipes are connected in the junction J, in which the continuity equation (1) must be satisfied (Figure 3).

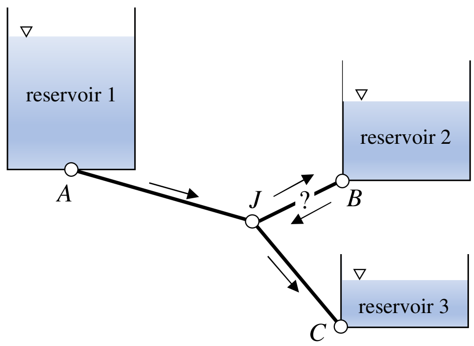

Fig. 3 *Classical example of the three-reservoir problem*

Each of the reservoirs represents one point where there can be only one value of the head. The same applies to the junction. Also, it should be noted that for each pipe the Darcy–Weisbach equation (6) must be satisfied.

If the piezometric head of the reservoirs is known, then it is a classical problem known as the *three-reservoir problem* [9]. In the available literature, the problem is solved iteratively by adjusting the piezometric head at junction J to satisfy Kirchhoff's law. In other words, the piezometric head at the junction J should satisfy the head loss equation for each pipe i.e. the Darcy-Weisbach equation (6). This equation may be used to define the pipe parameter S as:

$$S_{oi} = \lambda_{oi} \frac{L_{oi}}{D_{oi}\, 2gA_{oi}^2} \tag{7}$$

which enables us to rewrite Eq. (6) in the form:

$$\Delta E_{oi} = S_{oi} Q_{oi}^2 \tag{8}$$

The former Eq. can be applied to a pipe network illustrated in Figure 3 to obtain the following expressions (9):

$$\begin{aligned} \Delta E_{JA} &= | H_J - H_A | = S_{JA} Q_{JA}^2 \\ \Delta E_{JB} &= | H_J - H_B | = S_{JB} Q_{JB}^2 \\ \Delta E_{JC} &= | H_J - H_C | = S_{JC} Q_{JC}^2 \end{aligned} \tag{9}$$

It should be noted that the positive value Q_{JA} means that the flow goes from J to A, so it will be negative if the actual flow goes from A to J.

The second condition that should be satisfied at the junction J is the previously mentioned continuity equation (1). For the considered pipe system it can be written as:

$$Q_{JA} + Q_{JB} + Q_{JC} = 0 \tag{10}$$

The procedure of retrieving a solution to the posed problem is the following. In the first step, the piezometric head H_J at the junction J should be guessed. After the head differences are known, the flow rates, as well as head losses in all pipes can be computed from Eq. (9). The calculated flow rates can be checked by Eq. (10). Therefore, if the calculated flow in junction J is too high, then the larger head value H_J is required. Likewise, if the calculated flow is too low, then the lower head value H_J is required. The new value H_J is used as an initial guess in the next iteration step and the whole procedure is repeated until Eq. (10) is satisfied.

It is important to note that the number of unknown values (H_J, Q_{JA}, Q_{JB}, Q_{JC}) is greater than the available head loss equations (9). However, due to the additional equation (10), it is possible to solve this problem directly [9]. Nevertheless, the available literature shows deficiency in direct methods for larger systems.

Also, it is interesting to notice in Figure 3 that, due to the position of the reservoir B, it is impossible to accurately predict the flow direction in the pipe JB. For the pipes JA and JC, the flow direction is logically predicted due to the fact that the reservoir A has the highest hydraulic head point while the reservoir B has the lowest.

Contrary to methods usually used for solving the three-reservoir problem, we focus on developing a method for an arbitrary number of reservoirs, as well as larger and more complex pipe systems. Namely, to initialize the computational procedure for the steady flow condition in a given pipe system with more reservoirs, the flow rate from the reservoirs should be determined in advance. For this purpose, a numerical procedure is developed and illustrated in the following chapters.

4. THE ITERATIVE PROCEDURE

As discussed previously, a computational procedure needed to determine the flow in a relatively simple pipe system with just one junction and few reservoirs (Figure 3) is not trivial. Hereafter an iterative method developed for solving complex pipe systems with multiple reservoirs is described.

The first condition that should be satisfied is given by the equality between the sum of flows from reservoirs and the total consumption in junctions (Figure 4). The second condition is defined by the hydraulic heads in the junctions [1, 2].

The iterative procedure for initializing the flow in a pipe system with more junctions and reservoirs is summarized in the following description. Choose any flow values that satisfy the first Kirchhoff's law (1). With those values compute initial pipe flows Q, initial pressure losses ΔH and initial hydraulic heads H. Input those initial values in the Newton-Raphson iterative algorithm [10, 11]. However, in the Newton-Raphson iterative algorithm a hydraulic head H for one reservoir is held constant while others are progressively corrected. The choice of a reservoir which has a constant hydraulic head is arbitrary. In this way hydraulic heads in the pipe system are obtained, which satisfies assumed outgoing reservoir flows, but not the required hydraulic heads in certain water reservoirs. Based on the differences of obtained and given hydraulic heads for each of the reservoirs, the correction of reservoir flows is performed. With new reservoir flows, new initial values and hydraulic heads are computed. This procedure is repeated until the adjusted reservoir flows satisfy the required hydraulic heads for each of the reservoirs.

To explain the computational procedure in detail, it is necessary to clarify the idea behind it. If a computed hydraulic head H in an observed reservoir is lower than required, it is necessary to increase reservoir flow q in the next iterative step. Herein, it should be stressed that the flow may be directed out of or to the reservoir. If it runs out of the reservoir, it has a positive sign (+) and (-) in the opposite direction. The main idea is to add a correction factor to those reservoirs for which required hydraulic heads are not reached. As mentioned earlier, hydraulic head for one reservoir is held constant inside a Newton-Raphson iterative loop. For this reservoir, the flow q is corrected depending on the rest of the corrected flows, so that that the continuity equation is satisfied (1). In other words, this reservoir flow is corrected last.

For each of the reservoirs (beside the one for which hydraulic head H is held constant) the correction of the flow q is performed according to the equation:

$$q^{(k)} = q^{(k-1)} \pm \Delta q \tag{11}$$

in which $q^{(k)}$ denotes the reservoir flow at k iteration, $q^{(k-1)}$ reservoir flow at k-1 iteration and Δq *the* correction of the value for the flow at the present iteration. The reservoir flow for reservoir 1 (for which the hydraulic head H is held constant) is corrected depending on the reservoir flow 2, as described earlier (Figure 4).

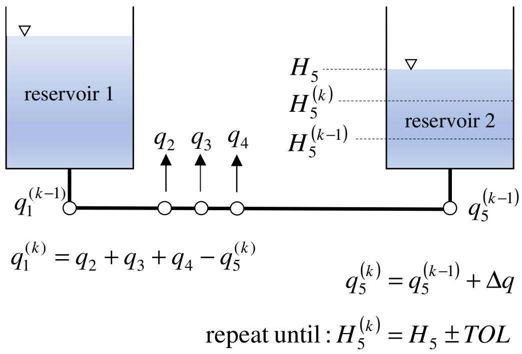

Fig. 4 *Determining the flow from multiple reservoirs by the proposed iterative procedure*

Elaborating this idea, it was concluded that the best approach of assuming the next iteration would be by using linear interpolation or extrapolation (Figure 5). By doing so, Eq. (11) gets a little bit more complex due to the increasing number of points used for interpolation (or extrapolation). Therefore, instead of just having previous iteration that contributes to the present one, there is also one preceding it. Accordingly, Eq. (11) takes the form:

$$q^{(k)} = q^{(k-2)} + \Delta q\,, \tag{12}$$

in which $q^{(k\text{-}2)}$ labels the reservoir flow q at k-2 or the antepenultimate iteration. The penultimate iteration k-1 is included within the flow correction value Δq, which can be obtained by geometrical equality (Figure 5) given in the form:

$$\frac{H^{(k-1)} - H^{(k-2)}}{q^{(k-1)} - q^{(k-2)}} = \frac{H_R - H^{(k-2)}}{q^{(k)} - q^{(k-2)}}\,, \tag{13}$$

which can be further used to rewrite Eq. (12) in the form:

$$q^{(k)} = q^{(k-2)} + \frac{(H_R - H^{(k-2)})(q^{(k-1)} - q^{(k-2)})}{H^{(k-1)} - H^{(k-2)}}\,, \tag{14}$$

where H_R denotes given or requested reservoir hydraulic head H, $H^{(k\text{-}2)}$ hydraulic head for antepenultimate iteration and $H^{(k\text{-}1)}$ for penultimate one.

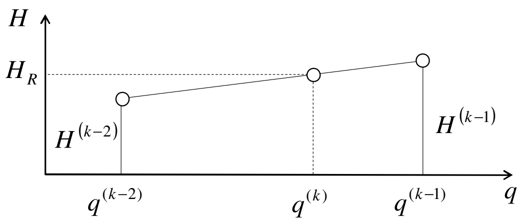

Fig. 5 *Estimation of flow q from reservoir by linear interpolation*

As can be noted, the latest iteration is assumed by using the previous two. It should be emphasized that, if there is a linear dependence between the reservoir flow q and reservoir hydraulic head H, only three iterations are needed for finding the exact value of the reservoir flow q. However, because there is more than just one reservoir in system, the reservoir flow q and reservoir hydraulic head H_R dependence is not strictly linear. This is because reservoirs interact among each other. Hence, a certain number of iteration will be required for finding the exact value of the reservoir flow q. The example of finding the correct value q for nonlinear relationship between the reservoir flow q and reservoir hydraulic head H_R is shown on

Figure 6. If memory consumption is taken into account, note that only the last two iterations are used for finding the next one. All iterations before it are not stored in the memory.

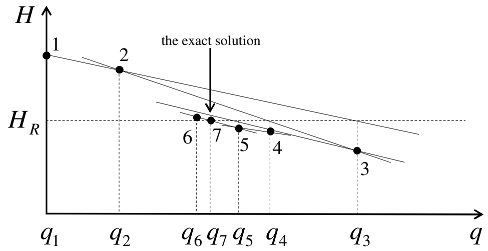

Fig. 6 *Iterative solution for nonlinear relationship between reservoir flow and reservoir hydraulic head*

The exact number of iterations primarily depends on the number of reservoirs in the pipe system. Hence, a greater number of reservoirs interacting among each other will require a greater number of iterations. The proposed linear method is algorithmically simple, relatively fast and precise. It converges to a given accuracy, which can be arbitrarily defined, and ensures the convergence. Although some other, higher-order polynomial functions would theoretically improve the convergence by decreasing the computational time, reducing the number of iteration, an appropriate polynomial order is hard to define uniquely and depends on the number of reservoirs in a pipe system. Also, note that a polynomial function of higher order has more solutions. However, just one of those solutions is right in terms of finding the exact solution for reservoir flow value q. Therefore, for the sake of simplicity and reliability, the linear method is found to be a more appropriate method of achieving that objective.

5. NUMERICAL EXAMPLE

A numerical example was calculated with the presented numerical algorithm implemented in MathCAD 15 [12] for a pipe system with geometrical and topological data illustrated in Figure 7. Note that the system contains 15 junctions and reservoirs are defined on three of them. Apart from that, the consumption in junctions, hydraulic heads of reservoirs and mechanical properties of fluids (density and viscosity) should be defined in advance.

Geometrical features of the pipe system are associated with diameters of sections D and absolute roughness of pipe ε. The geometrical characteristics of the pipe system are defined by the coordinate matrix **XYZ** of the junctions given in the form:

$$\boldsymbol{XYZ} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ \vdots & \vdots & \vdots \\ x_{ni} & y_{ni} & z_{ni} \end{pmatrix}, \tag{15}$$

where x_i, y_i and z_i are the coordinates of the junction denoted by the number i. On the other hand, the topological features are related to a number of junctions ni and number of sections nj. The topological matrix **TOP** defines the junction connections and for the first four pipes in the given pipe system (Figure 7) can be defined as:

$$\boldsymbol{TOP} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \\ \vdots & \vdots \end{pmatrix} \tag{16}$$

and defines connections of sections and junctions. For the considered numerical example (Figure 7), the section's length is given as: Δ *=1000 m*, Δy*=1000 m*, where Δx and Δy defines the distance between junctions. The density of fluid is $\rho=1000\ kg/m^3$ and dynamic viscosity $\mu=0.001307\ Pa{\cdot}s$ (for a temperature of 10ºC). Consumption in all junctions is $q=60\ l/s$. The only exceptions are junctions 1, 8 and 15 that represent reservoirs. The section's diameters vary and are: $D_1=400$ *mm*, $D_2=600$ *mm*, $D_3=550$ *mm*, $D_4=500$ *mm*, $D_5=900$ *mm*, $D_6=750$ *mm*, $D_7=950$ *mm*, $D_8=300$ *mm*, $D_9=200$ *mm*, $D_{10}=700$ *mm*, $D_{11}=150$ *mm*, $D_{12}=100$ *mm*, $D_{13}=100$ *mm*, $D_{14}=400$ *mm*, $D_{15}=200$ *mm*, $D_{16}=250$ *mm*, $D_{17}=100$ *mm*, $D_{18}=100$ *mm*, $D_{19}=100$ *mm*, $D_{20}=150$ *mm*, $D_{21}=150$ *mm*, $D_{22}=150$ *mm*. The absolute pipe roughness is $\varepsilon = 0.01$ mm for all sections. The hydraulic head on junction 1 is $H_1 = 50\ m$ and on junctions *8* and *15* they are *40 m*.

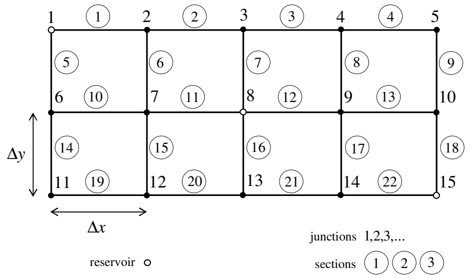

Fig. 7 *Pipe system used for the numerical example*

As a result of numerical computation, pipe flows Q, velocity in sections v, hydraulic heads in junctions H and reservoir flows q are obtained. The most interesting results are those of the distribution of hydraulic heads on the basis of which it is easier to understand the calculated values of reservoir flows q. Despite the fact that reservoirs on junctions 8 and 15 are of the same value, they have different reservoir flows' features. Actually, the reservoir on junction 8 receives the flow in the amount of *291.5 l/s* and is initially acting as a consumer, while the reservoir on junction 15 delivers the flow in the amount of *37.3 l/s*. This is due to hydraulic heads' differences between reservoirs and neighboring junctions. The hydraulic head distribution is illustrated in Figure 8.

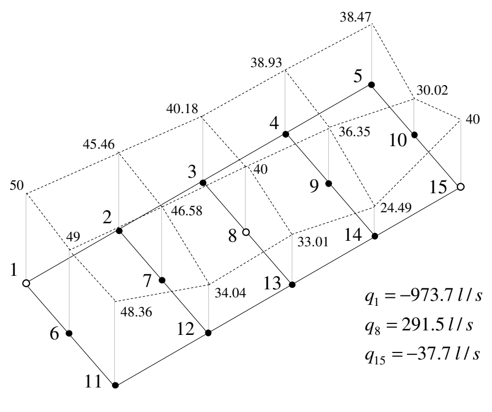

Fig. 8 *Computed hydraulic heads and reservoir flows values*

Furthermore, the influence of the number of iteration on a computation time was examined and for all the numerical experiments the almost linear dependence was evidenced. The same conclusion applies to the relation between the number of reservoirs in the pipe system and number of required iterations to retrieve the solution, as much as the number of reservoirs and computation time. To illustrate this, it was examined using the same topology showed in Figure 7. Computation with 1, 2, 3 ..., 8 reservoirs was executed 10 times, each time varying hydraulic heads and placements of reservoirs in the pipe system. The results are illustrated in Figures 9, 10 and 11.

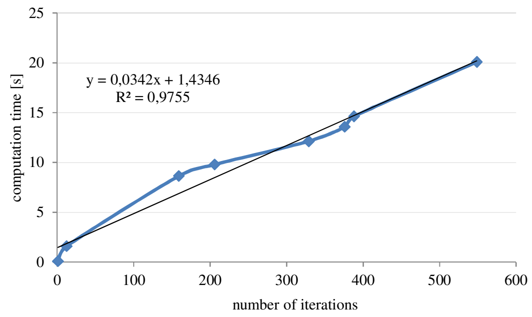

Fig. 9 *Relation between the number of iteration and computational time required to retrieve the solution*

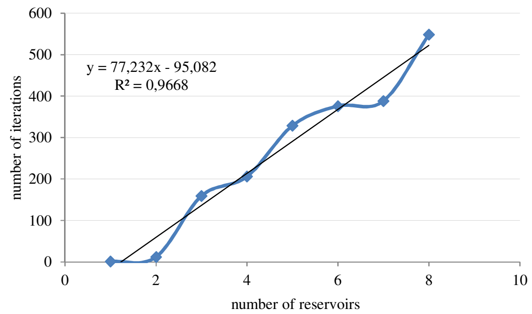

Fig. 10 *Relation between the number of reservoirs and number of iteration required to retrieve the solution*

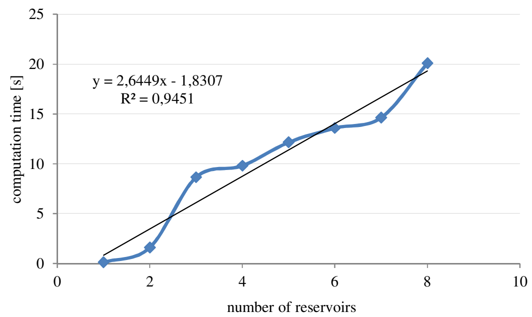

Fig. 11 *Relation between the number of reservoirs and computational time required to retrieve the solution*

6. CONCLUSIONS

The iterative numerical procedure proposed in this paper can be used to calculate the rate of flow from reservoirs in a given pipe system. The computational implementation of the procedure is relatively easy and the procedure can be used to determine the amount and direction of the flow from reservoirs where hydraulic heads are specified in advance. Computation time depends on the complexity of a network topology and number of reservoirs. In this paper, focus was turned to the relationship between a number of reservoirs and computation time, while the network topology was held constant. The relationship showed an approximate linear dependence. Although a linear interpolation method was used in the developed numerical procedure, the numerical procedure can be further improved by introducing a nonlinear interpolation (e.g. a quadratic one) for the purpose of finding the amount and direction of reservoir flow. These methods promise a faster obtainment of the exact solution, but they also present a great challenge in defining the exact solution (e.g. the quadratic equation has two solutions).

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ITERATIVNI ALGORITAM ZA INICIJALIZACIJU TOKA U TLAČNIM SUSTAVIMA S VIŠE VODOSPREMNIKA

Za analizu tlačnih sustava s više rezervoara, razvijen je numerički algoritam za inicijalizaciju toka. Koristeći iterativnu proceduru, razvijeni numerički algoritam se koristi u svrhu definiranja tlaka u tlačnim sustavima. Osim navedenoga, algoritam se koristi u svrhu definiranja orijentacije i iznosa volumetrijskog protoka za proizvoljan tlačni sustav s više vodosprema. Uspoređujući ovaj algoritam s klasičnom metodom koja se koristi za problem triju rezervoara, može se reći da predloženi algoritam korigira iznos tlaka u spojevima tlačne mreže na inverzan način. Numerički algoritam je implementiran u računalni kod razvijen za analizu proizvoljnih tlačnih sustava.

KLJUČNE RIJEČI: *tlačni sustavi s više rezervoara, problem triju rezervoara, hidraulički tlak, tlačni sustav, iterativna metoda.*