GLASNIK MATEMATIČKI
Vol. 51(71)(2016), 17 – 22

ON TWO DIOPHANTINE EQUATIONS OF RAMANUJAN-NAGELL TYPE

Zhongfeng Zhang and Alain Togbé
Zhaoqing University, China and Purdue University North Central, USA

Abstract. In this paper, we prove two conjectures of Ulas ([21]) on two Diophantine equations of Ramanujan-Nagell type. In fact, we show that the following equations

$$x^2 + (2^{m+1}+1)2^n = 2^{4(m+1)} + 2^{3(m+1)} + 2^{2m} + 2^{m+1} + 1,$$

$$x^2 + \frac{1}{3}\left(2^{2m+6}-1\right)2^n = \frac{1}{9}\left(49\cdot 4^{2m+5} - 11\cdot 4^{m+3} + 1\right)$$

have exactly four solutions.

1. Introduction

The Diophantine equation

$$x^2 + 7 = 2^{n+2}. \tag{1.1}$$

is called *the Ramanujan-Nagell equation*. In 1960, Nagell ([16]) proved that the following solutions:

$$(x, n) = (1, 1), (3, 2), (5, 3), (11, 5), (181, 13)$$

are the only solutions to equation (1.1). *A generalized Ramanujan-Nagell equation* is the Diophantine equation

$$x^2 + D = k^n \ \text{ in integers } x \geq 1, \ \ n \geq 1. \tag{1.2}$$

The literature on the generalized Ramanujan-Nagell equation is very rich. One can see for examples [1]–[21]. One aspect of the study of equation (1.2)

2010 *Mathematics Subject Classification.* 11D41.
Key words and phrases. Diophantine equation.
The first author was supported by the Foundation for Distinguished Young Teacher in Higher Education of Guangdong, the Guangdong Provincial Natural Science Foundation (No. 2016A030313013), China and NSF of China (No. 11271142). The second author thanks Purdue University North Central for the support.

is to determine the integer solutions (x, k, n). In 1850, Lebesgue ([13]) proved that the above equation has no solutions when $D = 1$. In 1965, Chao Ko ([11]) proved that the only solution of equation (1.2) with $D = -1$ is $x = 3,\ k = 2$. J.H.E. Cohn ([9]) solved the above equation for several values of the parameter D in the range $1 \leq D \leq 100$. A couple of the remaining values of D in the above range were covered by Mignotte and De Weger in [15], and the remaining ones in the recent paper [7]. Recently, several authors become interested in the case when only the prime factors of D are specified. For example, the case when $D = \prod_i p_i^{a_i}$ with a fixed prime numbers p_i was studied for $p = 2, 3, 5, 7, 13 \ldots$. See [1, 3, 14] for the recent surveys on this type of equation.

Many mathematicians studied also a more generalized Ramanujan-Nagell type of the form

$$x^2 = Ak^n + B, \quad k \in \mathbb{Z}_{\geq 2}, \quad A, B \in \mathbb{Z} \setminus \{0\}. \tag{1.3}$$

In 1996, Stiller ([19]) considered the equation

$$x^2 + 119 = 15 \cdot 2^n$$

and proved that this equation has exactly 6 solutions. This motivated Ulas ([21]) to search for equations of the type (1.3) having five or more solutions. Besides proving many results, he also set many conjectures.

The aim of this paper is to consider [21, Conjectures 4.2 and 4.3] and to prove the following two theorems.

Theorem 1.1. *For each positive integer m, the Diophantine equation*

$$x^2 + (2^{m+1} + 1)2^n = 2^{4(m+1)} + 2^{3(m+1)} + 2^{2m} + 2^{m+1} + 1 \tag{1.4}$$

has exactly four solutions in integers (x, n) with $n = 0, m+2, 2m+3, 3m+3$.

Theorem 1.2. *For each positive integer m, the Diophantine equation*

$$x^2 + \frac{1}{3}(2^{2m+6} - 1)2^n = \frac{1}{9}(49 \cdot 4^{2m+5} - 11 \cdot 4^{m+3} + 1) \tag{1.5}$$

has exactly four solutions in integers (x, n) with $n = 0, 3, 2m+7, 2m+8$.

The next two sections will be devoted to the proofs of the above theorems. After giving the four solutions of each of equations (1.4), (1.5), we will use an elementary method to prove that these equations have no other solutions. The technique consists in considering different intervals and showing that there is no value of x in those intervals verifying equations (1.4), (1.5). For the sake of completeness, we will give all details for solving each of these equations.

2. Proof of Theorem 1.1

Let p be a prime and n an integer, we denote by $v_p(n)$ the p-adic valuation of n. Assume $x \geq 0$. Then, it is easy to check that

$(x, n) = (2^{2m+2} + 2^m, 0),\ (2^{2m+2} + 2^m - 1, m+2),\ (2^{2m+2} - 2^m - 1, 2m+3),$

$$(2^m+1, 3m+3)$$

are solutions of equation (1.4). We will prove that these solutions are the only solutions of equation (1.4), with $x \geq 0$.

(i) $n \geq 3m+4$: one has

$$x^2+(2^{m+1}+1)2^n \geq (2^{m+1}+1)2^{3m+4} > 2^{4(m+1)}+2^{3(m+1)}+2^{2m}+2^{m+1}+1,$$

which is impossible.

(ii) $1 \leq n \leq m+1$: we get $2^{2m+2}+2^m-1 < x < 2^{2m+2}+2^m$. We deduce also that this is impossible.

(iii) $m+3 \leq n \leq 2m+2$: let $x = 2^{2m+2}-1+\alpha$, $n = m+2+\delta$. Then, we see that $-2^m < \alpha < 2^m$, $1 \leq \delta \leq m$. One has

$$\begin{aligned}(2^{m+1}+1)2^{m+2}(2^\delta-1) &= (2^{2m+2}-1+2^m)^2-(2^{2m+2}-1+\alpha)^2\\ &= (2^{2m+3}-2+2^m+\alpha)(2^m-\alpha).\end{aligned}$$

If $\alpha = 0$, then $2^{2m+3}-2+2^m = (2^{m+3}+4)(2^\delta-1)$ and we obtain $4|2^m-2$. We deduce that $m=1$. We come to a contradiction. Thus, we assume $\alpha \neq 0$ and let $\alpha = 2^t a, 2 \nmid a, 0 \leq t < m$. Then,

$$(2^{m+1}+1)2^{m+2}(2^\delta-1) = (2^{2m+3}-2+2^m+2^t a)(2^m-2^t a). \tag{2.1}$$

Obviously, $t \geq 1$ and $m \geq 2$ since $m > t \geq 1$. Equation (2.1) becomes

$$(2^{m+1}+1)2^{m+1-t}(2^\delta-1) = (2^{2m+2}-1+2^{m-1}+2^{t-1}a)(2^{m-t}-a).$$

One has $v_2(2^{2m+2}-1+2^{m-1}+2^{t-1}a) = m+1-t \geq 2$ and then $2|2^{t-1}a-1$. This implies that $t=1$. We obtain $v_2(2^{2m+2}-1+2^{m-1}+a) = m \geq 2$, i.e. $v_2(2^{m-1}+a-1) = m$. This yields $v_2(a-1) = m-1$. On the other hand, $-2^m < \alpha = 2a < 2^m$, so $-2^{m-1}-1 < a-1 < 2^{m-1}-1$. Thus, we get $a = -2^{m-1}+1$. Therefore, $v_2(2^{2m+2}-1+2^{m-1}+a) = v_2(2^{2m+2}) = 2m+2$. This gives a contradiction.

(iv) $2m+4 \leq n \leq 3m+2, m \geq 2$: let $x = 2^{2m+1}-2^m-1+\alpha$, $n = 2m+3+\delta$. Then, we have $-2^{2m+1}+2^{m+1}+2 < \alpha < 2^{2m+1}$, $1 \leq \delta < m$. One can see that

$$\begin{aligned}(2^{m+1}+1)2^{2m+3}(2^\delta-1) &= (2^{2m+2}-2^m-1)^2-(2^{2m+1}-2^m-1+\alpha)^2\\ &= (3\cdot 2^{2m+1}-2^{m+1}-2+\alpha)(2^{2m+1}-\alpha).\end{aligned}$$

If $\alpha = 0$, then $v_2((3\cdot 2^{2m+1}-2^{m+1}-2+\alpha)(2^{2m+1}-\alpha)) = 2m+2$. This is impossible. Thus, we assume $\alpha \neq 0$ and let $\alpha = 2^t a, 2 \nmid a, 0 < t < 2m+1$ as α is even. Then, we obtain

$$(2^{m+1}+1)2^{2m+3}(2^\delta-1) = (3\cdot 2^{2m+1}-2^{m+1}-2+2^t a)(2^{2m+1}-2^t a). \tag{2.2}$$

Then, $v_2(3\cdot 2^{2m}-2^m-1+2^{t-1}a) = 2m+2-t \geq 2$. We deduce that $v_2(2^{t-1}a-1) \geq 1$, which is a contradiction. This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

Assume $x \geq 0$. Then, it is easy to check that

$$(x, n) = (\frac{1}{3}(14 \cdot 4^{m+2} - 2), 0),\ (\frac{1}{3}(14 \cdot 4^{m+2} - 5), 3),$$
$$(\frac{1}{3}(10 \cdot 4^{m+2} - 1), 2m+7),\ (\frac{1}{3}(2 \cdot 4^{m+2} + 1), 2m+8)$$

are solutions of equation (1.5). We will prove that these solutions are the only solutions of equation (1.5), with $x \geq 0$.

(i) $n \geq 2m+9$: one has

$$x^2 + \frac{1}{3}(2^{2m+6} - 1)2^n \geq \frac{1}{3}(2^{2m+6} - 1)2^{2m+9} > \frac{1}{9}(49 \cdot 4^{2m+5} - 11 \cdot 4^{m+3} + 1).$$

We deduce a contradiction.

(ii) $1 \leq n \leq 2$: in this case, we obtain $\frac{1}{3}(14 \cdot 4^{m+2} - 5) < x < \frac{1}{3}(14 \cdot 4^{m+2} - 2)$, which is impossible.

(iii) $4 \leq n \leq 2m+6$: let $x = \frac{1}{3}(12 \cdot 4^{m+2} - 3 + \alpha)$, with $3|\alpha$. Then, we get $-2 \cdot 4^{m+2} + 2 < \alpha < 2 \cdot 4^{m+2} - 2$. Therefore, we have

$$\begin{aligned}\frac{1}{3}(2^{2m+6} - 1)2^n(2^{2m+7-n} - 1) &= \frac{1}{9}(12 \cdot 4^{m+2} - 3 + \alpha)^2 - \frac{1}{9}(10 \cdot 4^{m+2} - 1)^2 \\ &= \frac{1}{9}(22 \cdot 4^{m+2} - 4 + \alpha)(2 \cdot 4^{m+2} - 2 + \alpha).\end{aligned}$$

If $\alpha = 0$, then $v_2((22 \cdot 4^{m+2} - 4 + \alpha)(2 \cdot 4^{m+2} - 2 + \alpha)) = 3 = n$. This contradicts the condition $4 \leq n \leq 2m+6$. Thus, we assume $\alpha \neq 0$ and put $\alpha = 2^t a, 2 \nmid a, 1 \leq t \leq 2m+4$. We see that

$$\frac{1}{3}(2^{2m+6} - 1)2^n(2^{2m+7-n} - 1) = \frac{1}{9}(22 \cdot 4^{m+2} - 4 + 2^t a)(2 \cdot 4^{m+2} - 2 + 2^t a). \tag{3.1}$$

We will study equation (3.1) according to the values of t.

If $t \geq 3$, then $v_2((22 \cdot 4^{m+2} - 4 + 2^t a)(2 \cdot 4^{m+2} - 2 + 2^t a)) = 3 < n$, which is impossible.

If $t = 1$, then $v_2(22 \cdot 4^{m+2} - 4 + 2^t a) = 1$. So we have $v_2(2 \cdot 4^{m+2} - 2 + 2^t a) = n-1$. Therefore, we deduce that $v_2(2^{2m+4} - 1 + a) = n-2$. On the other hand, $-2^{2m+4} < a - 1 < 2^{2m+4} - 2$. Since $3|a$, one gets $n - 2 < 2m+4, a - 1 \neq 0$. Therefore, we have $v_2(a-1) = n-2$. Put $a = 2^{n-2}u + 1, 2 \nmid u, 4 \leq n \leq 2m+5$. Then, we obtain $\alpha = 2^{n-1}u + 2, 2 \nmid u, 4 \leq n \leq 2m+5$.

Substituting $t = 1, a = 2^{n-2}u + 1, 2 \nmid u$ into equation (3.1) and simplifying one has

$$13 \cdot 2^{4m+10} = (6 \cdot 2^{n-1}u + 6 \cdot 2^{n-1} + 5) \cdot 2^{2m+6} + (2^{n-1}u)^2 - 2 \cdot 2^{n-1}u - 3 \cdot 2^n. \tag{3.2}$$

Since $(2^{n-1}u)^2 - 2 \cdot 2^{n-1}u - 3 \cdot 2^n < (2^{2m+5})^2 + 2 \cdot 2^{2m+5} = 2^{4m+10} + 2^{2m+6}$, one gets

$$(6 \cdot 2^{n-1}u + 6 \cdot 2^{n-1} + 6) \cdot 2^{2m+6} > 12 \cdot 2^{4m+10},$$

i.e.

$$2^{n-1}u + 2^{n-1} + 1 > 2^{2m+5}.$$

So $u \geq 2^{2m+6-n} - 1$. On the other hand, from $\alpha = 2^{n-1}u + 2 < 2^{2m+5} - 2$, we have $u < 2^{2m+6-n}$. Therefore, $u = 2^{2m+6-n} - 1$. We replace this into (3.2) and simplify to obtain

$$2^{2m+4} + 1 = 2^{2m+7-n} + 2^{n-3}.$$

This is impossible since $2m + 7 - n > 0, n - 3 > 0$.

If $t = 2$, then $v_2(2 \cdot 4^{m+2} - 2 + 2^t a) = 1$. One can see that $v_2(22 \cdot 4^{m+2} - 4 + 2^t a) = n - 1$. Therefore, we get $v_2(11 \cdot 2^{2m+3} - 1 + a) = n - 3$. On the other hand, $-2^{2m+3} - 1 < a - 1 < 2^{2m+3} - 1$. Since $3|a$, one has $n - 3 < 2m + 3, a - 1 \neq 0, -2^{2m+3}$. Therefore, we have $v_2(a - 1) = n - 3$. Put $a = 2^{n-3}u + 1, 2 \nmid u, 4 \leq n \leq 2m + 5$. Then, we get $\alpha = 2^{n-1}u + 4, 2 \nmid u, 4 \leq n \leq 2m + 5$.

Replacing $t = 2, a = 2^{n-3}u + 1, 2 \nmid u$ into equation (3.1) and simplifying, we obtain

$$13 \cdot 2^{4m+10} = (6 \cdot 2^{n-1}u + 6 \cdot 2^{n-1} + 17) \cdot 2^{2m+6} + (2^{n-1}u)^2 + 2 \cdot 2^{n-1}u - 3 \cdot 2^n. \tag{3.3}$$

Since $(2^{n-1}u)^2 + 2 \cdot 2^{n-1}u - 3 \cdot 2^n \leq (2^{2m+5} + 1)^2 + 2 \cdot (2^{2m+5} - 5) - 3 \cdot 2^n < 2^{4m+10} + 2 \cdot 2^{2m+6}$, one has

$$(6 \cdot 2^{n-1}u + 6 \cdot 2^{n-1} + 19) \cdot 2^{2m+6} > 12 \cdot 2^{4m+10},$$

i.e.

$$2^{n-1}u + 2^{n-1} + 4 > 2^{2m+5}.$$

As $n \geq 4$ we deduce that $u \geq 2^{2m+6-n} - 1$. On the other hand, from $\alpha = 2^{n-1}u + 4 < 2^{2m+5} - 2$, one has $u < 2^{2m+6-n}$. Therefore, $u = 2^{2m+6-n} - 1$. Thus, equation (3.3) implies

$$9 \cdot 2^{2m+5-n} + 2^{n-4} = 1 + 2^{2m+3}. \tag{3.4}$$

So, we get $n = 4$ or $n = 2m + 5$. If $n = 4$, then (3.4) becomes $9 \cdot 2^{2m+1} + 1 = 1 + 2^{2m+3}$, which is impossible. If $n = 2m + 5$, then equation (3.4) gives $9 + 2^{2m+1} = 1 + 2^{2m+3}$, i. e. $1 + 2^{2m-2} = 2^{2m}$, which is also impossible. This completes the proof of Theorem 1.2. □

Acknowledgements.

The authors are grateful to the referees for their useful comments to improve the first version of this paper.

References

[1] F. S. Abu Muriefah and Y. Bugeaud, *The Diophantine equation $x^2 + c = y^n$: a brief overview*, Rev. Colombiana Mat. **40** (2006), 31–37.

[2] R. Apéry, *Sur une équation diophantienne*, C. R. Acad. Sci. Paris, Sér. A **251** (1960), 1451–1452.

[3] S. A. Arif and F. S. Abu Muriefah, *The Diophantine equation $x^2 + q^{2k} = y^n$*, Arab. J. Sci. Eng. Sect. A Sci. **26** (2001), 53–62.

[4] M. Bauer, M. Bennett, *Applications of the hypergeometric method to the generalized Ramanujan-Nagell equation.* Ramanujan J. **6** (2002), 209–270.
[5] F. Beukers, *On the generalized Ramanujan-Nagell equation I*, Acta Arith. **38** (1980/1981), 389–410.
[6] F. Beukers, *On the generalized Ramanujan-Nagell equation II*, Acta Arith. **39** (1981), 113–123.
[7] Y. Bugeaud, M. Mignotte and S. Siksek, *Classical and modular approaches to exponential Diophantine equations. II. The Lebesgue-Nagell equation*, Compos. Math. **142** (2006), 31–62.
[8] Y. Bugeaud and T. N. Shorey, *On the number of solutions of the generalized Ramanujan-Nagell equation*, J. Reine Angew. math. **539** (2001), 55–74.
[9] J. H. E. Cohn, *The Diophantine Equation* $x^2 + C = y^n$, Acta. Arith. **65** (1993), 367–381.
[10] A. Y. Khinchin, Continued fractions, P. Noordhoff Ltd., Groningen, 1963, 3rd edition.
[11] C. Ko, *On the Diophantine equation* $x^2 = y^n + 1$, $xy \neq 0$, Sci. Sinica **14** (1965), 457–460.
[12] M. Le, *A note on the number of solutions of the generalized Ramanujan-Nagell equation* $x^2 - D = k^n$, Acta Arith., **78** (1996), 11–18.
[13] V. A. Lebesgue, *Sur l'impossibilité en nombres entiers de l'équation* $x^m = y^2 + 1$ Nouv. Annal. des Math. **9** (1850), 178–181.
[14] E. Goins, F. Luca, A. Togbé *On the Diophantine equation* $x^2+2^\alpha\cdot 5^\beta\cdot 13^\gamma = y^n$, ANTS VIII Proceedings: A.J. van der Poorten and A. Stein (eds.), ANTS VIII, Lecture Notes in Computer Science 5011 (2008), 430–442.
[15] M. Mignotte and B. M. M. de Weger, *On the Diophantine equations* $x^2 + 74 = y^5$ *and* $x^2 + 86 = y^5$, Glasgow Math. J. **38** (1996), 77–85.
[16] T. Nagell, *The Diophantine Equation* $x^2 + 7 = 2^n$, Ark. Math. **4** (1961), 185–187.
[17] N. Tzanakis and J. Wolfskill, *On the diophantine equation* $y^2 = 4q^n + 4q + 1$, J. Number Theory **23** (1986), 219–237.
[18] N. Saradha and A. Srinivasan, *Generalized Lebesgue-Ramanujan-Nagell equations*, in: Diophantine equations, 207-223, Tata Inst. Fund. Res., Mumbai, 2008.
[19] J. Stiller, *The Diophantine equation* $x^2+119 = 15\cdot 2^n$ *has exactly six solutions*, Rocky Mountain J. Math. **26** (1996), 295–298.
[20] N. Tzanakis, *On the Diophantine equation* $y^2 - D = 2^k$, J. Number Theory **17** (1983), 144–164.
[21] M. Ulas, *Some experiments with Ramanujan-Nagell type Diophantine equations*, Glas. Mat. Ser. III **49** (2014), 287–302.

Z. Zhang
School of Mathematics and Statistics
Zhaoqing University
Zhaoqing 526061
China

A. Togbé
Department of Mathematics
Purdue University North Central
1401 S. U.S. 421 Westville IN 46391
USA
E-mail: `atogbe@pnc.edu`

Received: 29.3.2015.
Revised: 1.8.2015.