GLASNIK MATEMATIČKI
Vol. 51(71)(2016), 1 – 15

SOME FACTORIZATIONS IN THE TWISTED GROUP ALGEBRA OF SYMMETRIC GROUPS

Milena Sošić
University of Rijeka, Croatia

Abstract. In this paper we will give a similar factorization as in [3, 4], where Svrtan and Meljanac examined certain matrix factorizations on Fock-like representation of a multiparametric quon algebra on the free associative algebra of noncommuting polynomials equipped with multiparametric partial derivatives. In order to replace these matrix factorizations (given from the right) by twisted algebra computation, we first consider the natural action of the symmetric group S_n on the polynomial ring R_n in n^2 commuting variables $X_{a\,b}$ and also introduce a twisted group algebra (defined by the action of S_n on R_n) which we denote by $\mathcal{A}(S_n)$. Here we consider some factorizations given from the left because they will be more suitable in calculating the constants (= the elements which are annihilated by all multiparametric partial derivatives) in the free algebra of noncommuting polynomials.

1. Introduction

Following the papers [3, 4] by Meljanac and Svrtan, where an explicit Fock-like representation of a multiparametric quon algebra on the free associative algebra of noncommuting polynomials equipped with multiparametric partial derivatives (see also [2]) is constructed, our task here is to replace the 'nonobvious' matrix level factorizations by 'somewhat' simpler algebraic manipulations in a twisted group algebra $\mathcal{A}(S_n)$.

Similar factorizations in ordinary group algebra were used by Zagier in one parameter case ([8]) and in [3] in the multiparameter case the factorizations were given on the matrix level for the sake of Hilbert space realizability of multiparametric quon algebras. We are motivated by a different problem

2010 *Mathematics Subject Classification.* 05E15.
Key words and phrases. Symmetric group, polynomial ring, group algebra, twisted group algebra.

(computation of constants in multiparametric quon algebras), therefore the factorizations here are algebraically much simpler.

More general factorizations in braid group algebra we can find in [1].

In order to construct $\mathcal{A}(S_n)$ we first consider the natural action of the symmetric group S_n on the polynomial ring R_n in n^2 commuting variables $X_{a\,b}$ and let $\mathcal{A}(S_n) = R_n \rtimes \mathbb{C}[S_n]$ be the associated (twisted) group algebra. Further, we give some factorizations of certain canonical elements in $\mathcal{A}(S_n)$ in terms of simpler elements of $\mathcal{A}(S_n)$. Then by representing $\mathcal{A}(S_n)$ on the free unital associative complex algebra $\mathcal{B}$ (= the algebra of noncommuting polynomials) by using multiparametric partial derivatives, we obtain more easily some matrix factorizations. Similarly, we can apply some factorizations in $\mathcal{A}(S_n)$ in the problem of computing constants (i.e the elements which are annihilated by all multiparametric partial derivatives) in the algebra $\mathcal{B}$. This will be elaborated in the forthcoming paper. The explicit formulas for basic constants in the subspaces of $\mathcal{B}$ up to total degree four are given in [7].

2. The algebra $\mathcal{A}(S_n)$

Let S_n denote the symmetric group on n letters, i.e S_n is the set of all permutations of a set $M = \{1, 2, \ldots, n\}$ equipped with a composition as the binary operation on S_n (where the permutations are regarded as bijections from M to itself). Note that the groups S_n, $n \geq 3$ are not abelian.

Let $X = \{X_{a\,b} \mid 1 \leq a, b \leq n\}$ be a set of n^2 commuting variables $X_{a\,b}$ and let $R_n := \mathbb{C}[X_{a\,b} \mid 1 \leq a, b \leq n]$ denote the polynomial ring, i.e the commutative ring of all polynomials in n^2 variables $X_{a\,b}$ over the set $\mathbb{C}$ (of complex numbers), with $1 \in R_n$ as a unit element of R_n.

First, let S_n act on the set X as follows

$$g.X_{a\,b} = X_{g(a)\,g(b)}\,g. \tag{2.1}$$

This action of S_n on X induces the action of S_n on R_n given by

$$g.p(\ldots, X_{a\,b}, \ldots) = p(\ldots, X_{g(a)\,g(b)}, \ldots)\,g \tag{2.2}$$

for every $g \in S_n$ and any $p \in R_n$.

In what follows we are going to study a kind of twisted group algebra, which we denote by $\mathcal{A}(S_n)$ and call it a twisted group algebra of the symmetric group S_n with the coefficients in the polynomial ring R_n.

Recall that the usual group algebra $\mathbb{C}[S_n] = \left\{\sum_{\sigma\in S_n} c_\sigma \sigma \mid c_\sigma \in \mathbb{C}\right\}$ of the symmetric group S_n is a free vector space (generated with the set S_n), where the multiplication is given by

$$\left(\sum_{\sigma\in S_n} c_\sigma \sigma\right) \cdot \left(\sum_{\tau\in S_n} d_\tau \tau\right) = \sum_{\sigma,\tau\in S_n} (c_\sigma d_\tau)\,\sigma\tau.$$

Here we have used the simplified notation $\sigma\tau$ for the composition $\sigma \circ \tau$, i.e the product of σ and τ in S_n.

Now we define more general group algebra

$$\mathcal{A}(S_n) := R_n \rtimes \mathbb{C}[S_n] \tag{2.3}$$

a twisted group algebra of the symmetric group S_n with coefficients in the polynomial ring R_n.

Here $\rtimes$ denotes the semidirect product. The elements of the set $\mathcal{A}(S_n)$ are the linear combinations

$$\sum_{g_i \in S_n} p_i\, g_i \qquad \text{with} \quad p_i \in R_n$$

and the multiplication in $\mathcal{A}(S_n)$ is given by

$$(p_1 g_1) \cdot (p_2 g_2) := (p_1 \cdot (g_1.p_2))\, g_1 g_2, \tag{2.4}$$

where $g_1.p_2$ is defined by (2.2) and $g_1 g_2$ is the product of g_1 and g_2 in S_n.

It is easy to see that the algebra $\mathcal{A}(S_n)$ is associative but not commutative.

Let

$$I(g) = \{(a,b) \mid 1 \le a < b \le n,\, g(a) > g(b)\}$$

denote the set of inversions of $g \in S_n$.

Then to every $g \in S_n$ we associate a monomial in the ring R_n defined by

$$X_g := \prod_{(a,b) \in I(g^{-1})} X_{a\,b} \left(= \prod_{a<b,\, g^{-1}(a) > g^{-1}(b)} X_{a\,b} \right), \tag{2.5}$$

which encodes all inversions of g^{-1} (and of g too).

More generally, for any subset $A \subseteq \{1, 2, \ldots, n\}$ we will use the notation

$$X_A := \prod_{(a,b) \in A \times A,\, a<b} X_{a\,b} \cdot X_{b\,a} = \prod_{(a,b) \in A \times A,\, a<b} X_{\{a,\,b\}}, \tag{2.6}$$

where

$$X_{\{a,\,b\}} := X_{a\,b} \cdot X_{b\,a}. \tag{2.7}$$

Definition 2.1. *To each $g \in S_n$ we assign a unique element $g^* \in \mathcal{A}(S_n)$ defined by*

$$g^* := X_g\, g \tag{2.8}$$

with X_g defined by (2.5).

In what follows we will use the elements $g^* \in \mathcal{A}(S_n)$ defined by (2.8).

Theorem 2.2. *For every $g_1^*, g_2^* \in \mathcal{A}(S_n)$ we have*

$$g_1^* \cdot g_2^* = X(g_1, g_2)\, (g_1 g_2)^*, \tag{2.9}$$

where the multiplication factor is given by

(2.10)

$$X(g_1,g_2)=\prod_{(a,b)\in I(g_1^{-1})\setminus I((g_1g_2)^{-1})} X_{\{a,\,b\}}\left(=\prod_{(a,b)\in I(g_1)\cap I(g_2^{-1})} X_{\{g_1(a),\,g_1(b)\}}\right).$$

PROOF. By using the notations (2.8) and abbreviating $g_1g_2=g$ we have

$$\begin{aligned}
g_1^*\cdot g_2^* &= (X_{g_1}\,g_1)\cdot(X_{g_2}\,g_2)=(X_{g_1}\cdot g_1.X_{g_2})\,g\\
&=\left(X_{g_1}\cdot\prod_{(c,d)\in I(g_2^{-1})} X_{g_1(c)\,g_1(d)}\right)g\\
&=\left(X_{g_1}\cdot\prod_{(g_1^{-1}(a),g_1^{-1}(b))\in I(g_2^{-1})} X_{a\,b}\right)g\\
&=\left(X_{g_1}\cdot\prod_{(a,b)\in I(g^{-1})\setminus I(g_1^{-1})} X_{a\,b}\cdot\prod_{(b,a)\in I(g_1^{-1})\setminus I(g^{-1})} X_{a\,b}\right)g\\
&=\left(\prod_{(a,b)\in I(g_1^{-1})} X_{a\,b}\cdot\prod_{(a,b)\in I(g^{-1})\cap I(g_1^{-1})} X_{a\,b}^{-1}\cdot\prod_{(a,b)\in I(g^{-1})} X_{a\,b}\right.\\
&\qquad\left.\cdot\prod_{(a,b)\in I(g_1^{-1})\setminus I(g^{-1})} X_{b\,a}\right)g\\
&=\left(\prod_{(a,b)\in I(g_1^{-1})\setminus I(g^{-1})} X_{a\,b}\cdot\prod_{(a,b)\in I(g_1^{-1})\setminus I(g^{-1})} X_{b\,a}\cdot\prod_{(a,b)\in I(g^{-1})} X_{a\,b}\right)g\\
&=\prod_{(a,b)\in I(g_1^{-1})\setminus I(g^{-1})} X_{\{a,\,b\}}\cdot\left(\prod_{(a,b)\in I(g^{-1})} X_{a\,b}\,g\right)=X(g_1,g_2)\,g^*.
\end{aligned}$$

Here we have used the following properties

$$\prod_{(a,b)\in I((g_1g_2)^{-1})} X_{a\,b}=\prod_{(a,b)\in I((g_1g_2)^{-1})\cap I(g_1^{-1})} X_{a\,b}\cdot\prod_{(a,b)\in I((g_1g_2)^{-1})\setminus I(g_1^{-1})} X_{a\,b},$$

$$\prod_{(a,b)\in I(g_1^{-1})} X_{a\,b}=\prod_{(a,b)\in I(g_1^{-1})\setminus I((g_1g_2)^{-1})} X_{a\,b}\cdot\prod_{(a,b)\in I(g_1^{-1})\cap I((g_1g_2)^{-1})} X_{a\,b}$$

and the proof is finished. □

COROLLARY 2.3.

(2.11) $$g_1^*\cdot g_2^*=(g_1g_2)^*\quad if\quad l(g_1g_2)=l(g_1)+l(g_2)$$

where $l(g) := Card\, I(g)$ *is the length of* $g \in S_n$.

PROOF. It is easy to see that in the case $l(g_1) + l(g_2) = l(g_1 g_2)$ we have $X(g_1, g_2) = 1$, so (2.9) implies (2.11). ∎

The factor $X(g_1, g_2)$ takes care of the reduced number of inversions in the group product of $g_1, g_2 \in S_n$.

EXAMPLE 2.4. Let $g_1 = 132, g_2 = 312 \in S_3$. Then $g_1 g_2 = 213, l(g_1) = 1$, $l(g_2) = 2$, $l(g_1 g_2) = 1$. Note that $g_1^{-1} = 132$, $g_2^{-1} = 231$, so

$$g_1^* \cdot g_2^* = (X_{2\,3}\, g_1) \cdot (X_{1\,3} X_{2\,3}\, g_2) = X_{2\,3} X_{1\,2} X_{3\,2}\, g_1 g_2 = X_{\{2,\,3\}} X_{1\,2}\, g_1 g_2.$$

On the other hand we have: $(g_1 g_2)^* = X_{1\,2}\, g_1 g_2$, since $(g_1 g_2)^{-1} = 213$. Thus we get $g_1^* \cdot g_2^* = X_{\{2,\,3\}}\, (g_1 g_2)^*$ and $X(g_1, g_2) = X_{\{2,\,3\}}$.

EXAMPLE 2.5. For $g_1 = 132$, $g_2 = 231$ we have $g_1 g_2 = 321$, $l(g_1) = 1$, $l(g_2) = 2$, $l(g_1 g_2) = 3$. Further $g_1^{-1} = 132$, $g_2^{-1} = 312$ and $(g_1 g_2)^{-1} = 321$, so we get:

$$g_1^* \cdot g_2^* = (X_{2\,3}\, g_1) \cdot (X_{1\,2} X_{1\,3}\, g_2) = X_{2\,3} X_{1\,3} X_{1\,2}\, g_1 g_2,$$

$(g_1 g_2)^* = X_{1\,2} X_{1\,3} X_{2\,3}\, g_1 g_2$. Thus $g_1^* \cdot g_2^* = (g_1 g_2)^*$ and $X(g_1, g_2) = 1$.

We denote by $t_{a,b}$, $1 \le a \le b \le n$ the following cyclic permutation in S_n

$$t_{a,b}(k) := \begin{cases} k & 1 \le k \le a-1 \quad \text{or} \quad b+1 \le k \le n \\ b & k = a \\ k-1 & a+1 \le k \le b \end{cases} \tag{2.12}$$

which maps b to $b-1$ to $b-2 \cdots$ to a to b and fixes all $1 \le k \le a-1$ and $b+1 \le k \le n$ (compare with notation of $t_{a,b}$ in [3]).

Let $t_{b,a}$ denote the inverse of $t_{a,b}$. Then

$$t_{b,a}(k) := \begin{cases} k & 1 \le k \le a-1 \quad \text{or} \quad b+1 \le k \le n \\ k+1 & a \le k \le b-1 \\ a & k = b. \end{cases} \tag{2.13}$$

Then the sets of inversions are given by

$$I(t_{a,b}) = \{(a, j) \mid a+1 \le j \le b\},$$

$$I(t_{b,a}) = \{(i, b) \mid a \le i \le b-1\},$$

so the corresponding elements in $\mathcal{A}(S_n)$ have the form

$$t^*_{a,b} = \left(\prod_{a \le i \le b-1} X_{i\,b} \right) t_{a,b} \tag{2.14}$$

$$t^*_{b,a} = \left(\prod_{a+1 \le j \le b} X_{a\,j} \right) t_{b,a}. \tag{2.15}$$

Remark 2.6. Observe that if $b = a$ then $t^*_{a,a} = id$ (where $I(t_{a,a}) = \emptyset$). In the case $b = a+1$ we have $t_{a,a+1} = t_{a+1,a}$ and we also denote it by $t_a (= t_{a,a+1})$, $1 \le a \le n-1$ (the transposition of adjacent letters a and $a+1$). Now it is easy to see that $t^*_a = X_{a\,a+1}\, t_a$, with $I(t_a) = \{(a, a+1)\}$.

Theorem 2.2 implies the following more specific properties that will be presented in the following four corollaries.

Corollary 2.7. *For each* $1 \le a \le n-1$ *we have*

$$(t^*_a)^2 = X_{\{a,\,a+1\}}\, id. \tag{2.16}$$

Here we have used that $t_a t_a = id$ and $X_{\{a,\,a+1\}} = X_{a\,a+1} \cdot X_{a+1\,a}$.

Corollary 2.8 (Braid relations). *We have*

(*i*) $t^*_a \cdot t^*_{a+1} \cdot t^*_a = t^*_{a+1} \cdot t^*_a \cdot t^*_{a+1}$ *for each* $1 \le a \le n-2$,

(*ii*) $t^*_a \cdot t^*_b = t^*_b \cdot t^*_a$ *for each* $1 \le a, b \le n-1$ *with* $|a-b| \ge 2$.

Corollary 2.9. *For each* $g \in S_n$, $1 \le a < b \le n$ *we have*

$$g^* \cdot t^*_{b,a} = \left(\prod_{a<j\le b,\, g(a)>g(j)} X_{\{g(a),\, g(j)\}} \right) (g t_{b,a})^*.$$

In the case $g \in S_j \times S_{n-j}$, $1 \le j \le k \le n$ *we have*

$$g^* \cdot t^*_{k,j} = (g t_{k,j})^*. \tag{2.17}$$

Compare (2.17) with Corollary 2.3.

Corollary 2.10 (Commutation rules). *We have*

(*i*) $t^*_{m,k} \cdot t^*_{p,k} = (t^*_k)^2 \cdot t^*_{p,k+1} \cdot t^*_{m-1,k}$ *if* $1 \le k \le m < p \le n$.

(*ii*) *Let* $w_n (= n\, n-1 \cdots 2\, 1)$ *be the longest permutation in* S_n. *Then for every* $g \in S_n$ *we have*

$$(g w_n)^* \cdot w^*_n = w^*_n \cdot (w_n g)^* \left(= \prod_{a<b,\, g^{-1}(a)<g^{-1}(b)} X_{\{a,\,b\}} \right) g^*.$$

3. Decompositions of certain canonical elements in $\mathcal{A}(S_n)$

Here we will decompose any permutation g in S_n into cycles.

Observe first that any permutation $g \in S_n$ can be represented uniquely as $g = g_1 t_{k_1,1}$ with $g_1 \in S_1 \times S_{n-1}$ and $1 \le k_1 \le n$. Then $g(k_1) = g_1(t_{k_1,1}(k_1)) = g_1(1) = 1$, so k_1 should be $g^{-1}(1)$.

Subsequently, the permutation $g_1 \in S_1 \times S_{n-1}$ can be represented uniquely as $g_1 = g_2 t_{k_2,2}$ with $g_2 \in S_1 \times S_1 \times S_{n-2}$ and $2 \le k_2 \le n$. Then $g_1(k_2) = g_2(t_{k_2,2}(k_2)) = g_2(2) = 2$ implies $k_2 = g_1^{-1}(2)$.

By repeating the above procedure for every $1 \le j \le n$ we can deduce that the permutation $g_{j-1} \in S_1^{j-1} \times S_{n-j+1}$ can be represented uniquely as $g_{j-1} =$

$g_j t_{k_j,j}$ with $g_j \in S_1^j \times S_{n-j}$ and $j \le k_j \le n$, where $g_{j-1}(k_j) = g_j(t_{k_j,j}(k_j)) = g_j(j) = j$ implies $k_j = g_{j-1}^{-1}(j)$. Thus we get the decomposition:

$$g = t_{k_n,n} \cdot t_{k_{n-1},n-1} \cdots t_{k_j,j} \cdots t_{k_2,2} \cdot t_{k_1,1} \left(= \overleftarrow{\prod_{1 \le j \le n}} t_{k_j,j} \right). \tag{3.1}$$

Example 3.1. By applying the decomposition (3.1) on all permutations in $S_3 = \{123, 132, 312, 321, 231, 213\}$ we obtain

$$123 = t_{3,3}t_{2,2}t_{1,1}, \quad 132 = t_{3,3}t_{3,2}t_{1,1}, \quad 312 = t_{3,3}t_{3,2}t_{2,1},$$
$$321 = t_{3,3}t_{3,2}t_{3,1}, \quad 231 = t_{3,3}t_{2,2}t_{3,1}, \quad 213 = t_{3,3}t_{2,2}t_{2,1},$$

so the corresponding elements in the algebra $\mathcal{A}(S_3)$ are given by

$$123^* = t^*_{3,3} \cdot t^*_{2,2} \cdot t^*_{1,1}, \quad 132^* = t^*_{3,3} \cdot t^*_{3,2} \cdot t^*_{1,1}, \quad 312^* = t^*_{3,3} \cdot t^*_{3,2} \cdot t^*_{2,1},$$
$$321^* = t^*_{3,3} \cdot t^*_{3,2} \cdot t^*_{3,1}, \quad 231^* = t^*_{3,3} \cdot t^*_{2,2} \cdot t^*_{3,1}, \quad 213^* = t^*_{3,3} \cdot t^*_{2,2} \cdot t^*_{2,1}.$$

The following calculation shows the general situation, which will be used later in many calculations. Assume that $\alpha_3^* = \sum_{g \in S_3} g^*$. Then we get

$$\begin{aligned}
\alpha_3^* = \sum_{g \in S_3} g^* &= t^*_{3,3} \cdot t^*_{2,2} \cdot t^*_{1,1} + t^*_{3,3} \cdot t^*_{3,2} \cdot t^*_{1,1} + t^*_{3,3} \cdot t^*_{3,2} \cdot t^*_{2,1} \\
&\quad + t^*_{3,3} \cdot t^*_{3,2} \cdot t^*_{3,1} + t^*_{3,3} \cdot t^*_{2,2} \cdot t^*_{3,1} + t^*_{3,3} \cdot t^*_{2,2} \cdot t^*_{2,1} \\
&= \left(t^*_{3,3}\right) \cdot \left(t^*_{3,2} \cdot \left(t^*_{3,1} + t^*_{2,1} + t^*_{1,1}\right) + t^*_{2,2} \cdot \left(t^*_{3,1} + t^*_{2,1} + t^*_{1,1}\right)\right) \\
&= \left(t^*_{3,3}\right) \cdot \left(t^*_{3,2} + t^*_{2,2}\right) \cdot \left(t^*_{3,1} + t^*_{2,1} + t^*_{1,1}\right)
\end{aligned}$$

i.e

$$\alpha_3^* = \beta_1^* \cdot \beta_2^* \cdot \beta_3^*, \tag{3.2}$$

where we have used the notations

$$\begin{aligned}
\beta_1^* &= t^*_{3,3} \ (= id)\,; \\
\beta_2^* &= t^*_{3,2} + t^*_{2,2} \ \left(= t^*_{3,2} + id\right); \\
\beta_3^* &= t^*_{3,1} + t^*_{2,1} + t^*_{1,1} \ \left(= t^*_{3,1} + t^*_{2,1} + id\right).
\end{aligned}$$

Therefore, we can conclude that the element $\alpha_3^* \in \mathcal{A}(S_3)$ given by $\alpha_3^* = \sum_{g \in S_3} g^*$ can be written in the product form (3.2).

In the next theorem we will prove that the element $\alpha_n^* \in \mathcal{A}(S_n)$ given by $\alpha_n^* = \sum_{g \in S_n} g^*$, $n \ge 1$ can be decomposed into the product of simpler elements of the algebra $\mathcal{A}(S_n)$ which we denote by β^*_{n-k+1} for each $1 \le k \le n$.

Definition 3.2. *For every $1 \le k \le n$ we define*

$$\beta^*_{n-k+1} := t^*_{n,k} + t^*_{n-1,k} + \cdots + t^*_{k+1,k} + t^*_{k,k} \left(= \overleftarrow{\sum_{k \le m \le n}} t^*_{m,k} \right). \tag{3.3}$$

REMARK 3.3. Now it is easy to see that

$$\begin{aligned}
\beta_n^* &:= t_{n,1}^* + t_{n-1,1}^* + \cdots + t_{2,1}^* + t_{1,1}^* \quad (\text{if} \quad k=1),\\
\beta_{n-1}^* &:= t_{n,2}^* + t_{n-1,2}^* + \cdots + t_{3,2}^* + t_{2,2}^* \quad (\text{if} \quad k=2),\\
&\vdots\\
\beta_3^* &:= t_{n,n-2}^* + t_{n-1,n-2}^* + t_{n-2,n-2}^* \quad (\text{if} \quad k=n-2),\\
\beta_2^* &:= t_{n,n-1}^* + t_{n-1,n-1}^* \quad (\text{if} \quad k=n-1),\\
\beta_1^* &:= t_{n,n}^* (= id) \quad (\text{if} \quad k=n).
\end{aligned}$$

THEOREM 3.4. *Let α_n^* be the following canonical element in $\mathcal{A}(S_n)$:*

$$\alpha_n^* = \sum_{g \in S_n} g^*. \tag{3.4}$$

Then α_n^* has the following factorization

$$\alpha_n^* = \beta_1^* \cdot \beta_2^* \cdots \beta_n^* \left(= \overleftarrow{\prod_{1 \le k \le n}} \beta_{n-k+1}^* \right). \tag{3.5}$$

PROOF. By considering decomposition (3.1) of $g \in S_n$ and the property (2.17) we can write:

$$\begin{aligned}
\alpha_n^* &= \sum_{g \in S_n} g^* = \sum_{\substack{g_1 \in S_1 \times S_{n-1}\\ 1 \le k_1 \le n}} (g_1 t_{k_1,1})^* = \sum_{\substack{g_1 \in S_1 \times S_{n-1}\\ 1 \le k_1 \le n}} g_1^* t_{k_1,1}^*\\
&= \left(\sum_{g_1 \in S_1 \times S_{n-1}} g_1^* \right) \cdot \left(\sum_{1 \le k_1 \le n} t_{k_1,1} \right)\\
&= \left(\sum_{\substack{g_2 \in S_1^2 \times S_{n-2}\\ 2 \le k_2 \le n}} (g_2 t_{k_2,2})^* \right) \cdot \left(\sum_{1 \le k_1 \le n} t_{k_1,1}^* \right)\\
&= \left(\sum_{g_2 \in S_1^2 \times S_{n-2}} g_2^* \right) \cdot \left(\sum_{2 \le k_2 \le n} t_{k_2,2}^* \right) \cdot \left(\sum_{1 \le k_1 \le n} t_{k_1,1}^* \right) = \cdots\\
&= \left(t_{k_n,n}^* \right) \cdot \left(\sum_{n-1 \le k_{n-1} \le n} t_{k_{n-1},n-1}^* \right) \cdots \left(\sum_{2 \le k_2 \le n} t_{k_2,2}^* \right) \cdot \left(\sum_{1 \le k_1 \le n} t_{k_1,1}^* \right)\\
&= \overleftarrow{\prod_{1 \le k \le n}} \beta_{n-k+1}^*
\end{aligned}$$

and the proof is finished. □

Let us introduce some new elements in the algebra $\mathcal{A}(S_n)$ by which we will reduce β^*_{n-k+1}, $1 \le k \le n-1$. The motivation is to show that the element $\alpha^*_n \in \mathcal{A}(S_n)$ can be expressed in turn as products of yet simpler elements of the algebra $\mathcal{A}(S_n)$.

Definition 3.5. *For every $1 \le k \le n-1$ we define the following elements in the algebra $\mathcal{A}(S_n)$*

$$\gamma^*_{n-k+1} := \left(id - t^*_{n,k}\right) \cdot \left(id - t^*_{n-1,k}\right) \cdots \left(id - t^*_{k+1,k}\right) = \overleftarrow{\prod_{k+1 \le m \le n}} \left(id - t^*_{m,k}\right),$$

$$\begin{aligned}\delta^*_{n-k+1} &:= \left(id - (t^*_k)^2\, t^*_{n,k+1}\right) \cdot \left(id - (t^*_k)^2\, t^*_{n-1,k+1}\right) \cdots \left(id - (t^*_k)^2\, t^*_{k+1,k+1}\right) \\ &= \overleftarrow{\prod_{k+1 \le m \le n}} \left(id - (t^*_k)^2\, t^*_{m,k+1}\right)\end{aligned}$$

where $(t^*_k)^2$ is given by (2.16) *and $t^*_{k+1,k+1} = id$.*

Proposition 3.6. *For every $1 \le k \le n-1$ we have the following factorization*

$$\beta^*_{n-k+1} = \delta^*_{n-k+1} \cdot \left(\gamma^*_{n-k+1}\right)^{-1}.$$

Proof. Let $\beta^*_{n-k+1,p} := \overleftarrow{\sum}_{k \le m \le p} t^*_{m,k}$ for every $k \le p \le n$. Then we obtain:

$$\begin{aligned}\beta^*_{n-k+1,p} \cdot \left(id - t^*_{p,k}\right) &= \overleftarrow{\sum_{k \le m \le p}} t^*_{m,k} - \overleftarrow{\sum_{k \le m \le p}} t^*_{m,k}\, t^*_{p,k} \\ &= t^*_{p,k} + \overleftarrow{\sum_{k \le m \le p-1}} t^*_{m,k} - \overleftarrow{\sum_{k+1 \le m \le p}} t^*_{m,k}\, t^*_{p,k} - t^*_{k,k}\, t^*_{p,k} \\ &= \overleftarrow{\sum_{k \le m \le p-1}} t^*_{m,k} - \overleftarrow{\sum_{k+1 \le m \le p}} t^*_{m,k}\, t^*_{p,k} \\ &= \overleftarrow{\sum_{k \le m \le p-1}} t^*_{m,k} - \overleftarrow{\sum_{k+1 \le m \le p}} (t^*_k)^2\, t^*_{p,k+1}\, t^*_{m-1,k} \\ &= \overleftarrow{\sum_{k \le m \le p-1}} t^*_{m,k} - \overleftarrow{\sum_{k \le m \le p-1}} (t^*_k)^2\, t^*_{p,k+1}\, t^*_{m,k} \\ &= \overleftarrow{\sum_{k \le m \le p-1}} \left(id - (t^*_k)^2\, t^*_{p,k+1}\right) \cdot t^*_{m,k} \\ &= \left(id - (t^*_k)^2\, t^*_{p,k+1}\right) \cdot \beta^*_{n-k+1,p-1}\end{aligned}$$

i.e

$$\beta^*_{n-k+1,p} \cdot \left(id - t^*_{p,k}\right) = \left(id - (t^*_k)^2\, t^*_{p,k+1}\right) \cdot \beta^*_{n-k+1,p-1}. \tag{3.6}$$

for every $k \le p \le n$. Note that $\beta^*_{n-k+1,k} = id$ and $\beta^*_{n-k+1,n} = \beta^*_{n-k+1}$, so for $p = n$ the identity (3.6) is given by

$$\beta^*_{n-k+1} \cdot \big(id - t^*_{n,k}\big) = \big(id - (t^*_k)^2\, t^*_{n,k+1}\big) \cdot \beta^*_{n-k+1,n-1}. \tag{3.7}$$

By multiplying (3.7) from right to left with $\Big(id - t^*_{n-1,k}\Big) \cdots \Big(id - t^*_{k+2,k}\Big) \cdot \Big(id - t^*_{k+1,k}\Big)$ and by using above identities (3.6) for all $k \le p \le n-1$ it is easy to check that

$$\begin{aligned}\beta^*_{n-k+1} &\cdot \big(id - t^*_{n,k}\big) \cdot \big(id - t^*_{n-1,k}\big) \cdots \big(id - t^*_{k+2,k}\big) \cdot \big(id - t^*_{k+1,k}\big)\\ &= \big(id - (t^*_k)^2\, t^*_{n,k+1}\big) \cdots \big(id - (t^*_k)^2\, t^*_{k+2,k+1}\big) \cdot \big(id - (t^*_k)^2\big)\end{aligned}$$

i.e

$$\beta^*_{n-k+1} \cdot \gamma^*_{n-k+1} = \delta^*_{n-k+1}$$

for every $1 \le k \le n-1$ whence arises the identity of the Proposition 3.6. ∎

Example 3.7. By applying (3.5) and Proposition 3.6 we will illustrate the factorization of $\alpha^*_n \in \mathcal{A}(S_n)$ in cases $n = 2, 3, 4$ (recall that $\beta^*_1 = id$).

(i) In the case $n = 2$ we have $\alpha^*_2 = \beta^*_2$ and

$$\alpha^*_2 = \big(id - (t^*_1)^2\big) \cdot \big(id - t^*_{2,1}\big)^{-1}.$$

(ii) For $n = 3$ we have $\alpha^*_3 = \beta^*_2 \cdot \beta^*_3$, where

$$\begin{aligned}\beta^*_2 &= \big(id - (t^*_2)^2\big) \cdot \big(id - t^*_{3,2}\big)^{-1},\\ \beta^*_3 &= \big(id - (t^*_1)^2 \cdot t^*_{3,2}\big) \cdot \big(id - (t^*_1)^2\big) \cdot \big(id - t^*_{2,1}\big)^{-1} \cdot \big(id - t^*_{3,1}\big)^{-1}.\end{aligned}$$

(iii) For $n = 4$ we have $\alpha^*_4 = \beta^*_2 \cdot \beta^*_3 \cdot \beta^*_4$, where

$$\begin{aligned}\beta^*_2 &= \big(id - (t^*_3)^2\big) \cdot \big(id - t^*_{4,3}\big)^{-1},\\ \beta_3 &= \big(id - (t^*_2)^2 \cdot t^*_{4,3}\big) \cdot \big(id - (t^*_2)^2\big) \cdot \big(id - t^*_{3,2}\big)^{-1} \cdot \big(id - t^*_{4,2}\big)^{-1},\\ \beta^*_4 &= \big(id - (t^*_1)^2 \cdot t^*_{4,2}\big) \cdot \big(id - (t^*_1)^2 \cdot t^*_{3,2}\big) \cdot \big(id - (t^*_1)^2\big) \cdot \big(id - t^*_{2,1}\big)^{-1}\\ &\quad \cdot \big(id - t^*_{3,1}\big)^{-1} \cdot \big(id - t^*_{4,1}\big)^{-1}.\end{aligned}$$

Lemma 3.8. *We have*

(i) $t^*_{b,a} = t_{1,n} \cdot t^*_{b+1,a+1} \cdot t_{n,1}, \qquad 1 \le a \le b \le n,$

(ii) $X_{\{a,\,a+1\}}\, id = t_{1,n} \cdot X_{\{a+1,\,a+2\}} \cdot t_{n,1}, \qquad 1 \le a \le n-1.$

PROOF. (i) By (2.12), (2.13) and (2.15) we get

$$t_{1,n} \cdot t^*_{b+1,a+1} \cdot t_{n,1} = t_{1,n} \cdot \left(\prod_{a+2 \le j \le b+1} X_{a+1\,j} \right) t_{b+1,a+1} \cdot t_{n,1}$$
$$= \left(\prod_{a+1 \le j \le b} X_{a\,j} \right) t_{1,n} \cdot t_{b+1,a+1} \cdot t_{n,1} = t^*_{b,a}.$$

Here we have used $t_{b,a} = t_{1,n} t_{b+1,a+1} t_{n,1}$ (recall that $t_{1,n} = t^{-1}_{n,1}$).

(ii) Directly from the definition of $t_{1,n}$:

$$t_{1,n} \cdot X_{\{a+1,\, a+2\}} \cdot t_{n,1} = X_{\{a,\, a+1\}} \cdot t_{1,n} \cdot t_{n,1} = X_{\{a,\, a+1\}}\, id$$

(this is equivalent to $(t^*_a)^2 = t_{1,n} \cdot (t^*_{a+1})^2 \cdot t_{n,1}$). ∎

REMARK 3.9. The elements $\delta^*_{n-k+1} \in \mathcal{A}(S_n)$, $1 \le k \le n-1$ from Definition 3.5 can be rewritten as:

$$\delta^*_{n-k+1} = \big(id - X_{\{k,\,k+1\}}\, t^*_{n,k+1}\big) \cdot \big(id - X_{\{k,\,k+1\}}\, t^*_{n-1,k+1}\big)$$
$$\cdots \big(id - X_{\{k,\,k+1\}}\, t^*_{k+2,k+1}\big) \cdot \big(id - X_{\{k,\,k+1\}}\, t^*_{k+1,k+1}\big)$$

or shorter

$$\delta^*_{n-k+1} = \overleftarrow{\prod_{k+1 \le m \le n}} \big(id - X_{\{k,\,k+1\}}\, t^*_{m,k+1}\big)\,. \tag{3.8}$$

Our next goal is to give a formula for the inverse of α^*_n. In order to do this we first need to determine the inverse of δ^*_{n-k+1} for all $1 \le k \le n-1$, because

$$(\alpha^*_n)^{-1} = \gamma^*_n \cdot (\delta^*_n)^{-1} \cdot \gamma^*_{n-1} \cdot \big(\delta^*_{n-1}\big)^{-1} \cdots \gamma^*_2 \cdot (\delta^*_2)^{-1}\,.$$

Let us introduce a more accurate label

$$\delta^*_{n-k+1,n} := \delta^*_{n-k+1} \tag{3.9}$$

where δ^*_{n-k+1} is given by (3.8).

Let us denote by

$$Des(\sigma) := \{1 \le i \le n-1 \mid \sigma(i) > \sigma(i+1)\}$$

the descent set of a permutation $\sigma \in S_n$.
Let $des(\sigma) = Card(Des(\sigma))$ be the number of descents of σ.
Note that for $g \in S^k_1 \times S_{n-k}$

$$Des(g) = \{k+1 \le i \le n-1 \mid g(i) > g(i+1)\}.$$

PROPOSITION 3.10. *The inverse of* $\delta^*_{n-k+1,n}$, $1 \le k \le n-1$ *is given by the formula*

$$\big(\delta^*_{n-k+1,n}\big)^{-1} = (\Delta_{n-k+1,n})^{-1} \cdot \big(\varepsilon^*_{n-k+1,n}\big) \tag{3.10}$$

where

$$\Delta_{n-k+1,n} := \big(id - X_{\{k,\,k+1\}}\big)\cdot\big(id - X_{\{k,\,k+1\,k+2\}}\big)\cdots\big(id - X_{\{k,\,k+1,\,\ldots,\,n\}}\big)\,,$$

$$\varepsilon^*_{n-k+1,n} := \sum_{g\in S_1^k\times S_{n-k}} \omega_{n-k+1,n}(g)\,g^*$$

and

$$\omega_{n-k+1,n}(g) := \prod_{i\in Des(g^{-1})} X_{\{k,\,k+1,\,\ldots,\,i\}}.$$

Proof. By (3.8) and (3.9) we have

$$\delta^*_{n-k+1,n} = \big(id - X_{\{k,\,k+1\}}\,t^*_{n,k+1}\big)\cdot \overleftarrow{\prod_{k+1\le m\le n-1}} \big(id - X_{\{k,\,k+1\}}\,t^*_{m,k+1}\big)$$

or shortly

$$\delta^*_{n-k+1,n} = \big(id - X_{\{k,\,k+1\}}\,t_{n,k+1}\big)\cdot\delta^*_{n-k+1,n-1} \tag{3.11}$$

where

$$\begin{aligned}\delta^*_{n-k+1,n-1} &= \overleftarrow{\prod_{k+1\le m\le n-1}} \big(id - X_{\{k,\,k+1\}}\,t^*_{m,k+1}\big)\\ &= t_{1,n}\cdot\left(\overleftarrow{\prod_{k+1\le m\le n-1}} \big(id - X_{\{k+1,\,k+2\}}\,t^*_{m+1,k+2}\big)\right)\cdot t_{n,1}\\ &= t_{1,n}\cdot\left(\overleftarrow{\prod_{k+2\le m\le n}} \big(id - X_{\{k+1,\,k+2\}}\,t^*_{m,k+2}\big)\right)\cdot t_{n,1} = t_{1,n}\cdot\delta^*_{n-k,n}\cdot t_{n,1}.\end{aligned}$$

Here we have used property (ii) of the Lemma 3.8. Thus we obtain

$$\delta^*_{n-k+1,n} = \big(id - X_{\{k,\,k+1\}}\,t^*_{n,k+1}\big)\cdot t_{1,n}\cdot\delta^*_{n-k,n}\cdot t_{n,1}$$

i.e the identity

$$\big(\delta^*_{n-k+1,n}\big)^{-1}\cdot\big(id - X_{\{k,\,k+1\}}\,t^*_{n,k+1}\big) = t_{1,n}\cdot\big(\delta^*_{n-k,n}\big)^{-1}\cdot t_{n,1}$$

which takes the form:

$$(\Delta_{n-k+1,n})^{-1}\cdot\varepsilon^*_{n-k+1,n}\cdot\big(id - X_{\{k,\,k+1\}}\,t^*_{n,k+1}\big) = t_{1,n}\cdot(\Delta_{n-k,n})^{-1}\cdot\varepsilon^*_{n-k,n}\cdot t_{n,1}$$

or

$$\varepsilon^*_{n-k+1,n}\cdot\big(id - X_{\{k,\,k+1\}}\,t^*_{n,k+1}\big) = \big(id - X_{\{k,\,k+1,\,\ldots,\,n\}}\big)\cdot\varepsilon^*_{n-k+1,n-1} \tag{3.12}$$

where

$$\begin{aligned}\varepsilon^*_{n-k+1,n-1} &= t_{1,n}\cdot\varepsilon^*_{n-k,n}\cdot t_{n,1},\\ id - X_{\{k,\,k+1,\,\ldots,\,n\}} &= \Delta_{n-k+1,n}\cdot t_{1,n}\cdot(\Delta_{n-k,n})^{-1}\cdot t_{n,1}.\end{aligned}$$

To prove the formula (3.10) (by induction), it suffices to prove the identity (3.12). Notice that (3.12) is equivalent to

$$\varepsilon^*_{n-k+1,n} = \big(id - X_{\{k,\,k+1,\,\ldots,\,n\}}\big)\cdot \varepsilon^*_{n-k+1,n-1}\cdot \big(id - X_{\{k,\,k+1\}}\,t^*_{n,k+1}\big)^{-1}.$$

We first calculate

$$\begin{aligned}
\varepsilon^*_{n-k+1,n}\cdot X_{\{k,\,k+1\}}\,t^*_{n,k+1} &= \sum_{\sigma\in S_1^k\times S_{n-k}} \omega_{n-k+1,n}(\sigma)\,\sigma^*\cdot X_{\{k,\,k+1\}}\,t^*_{n,k+1}\\
&= \sum_{\sigma\in S_1^k\times S_{n-k}} \omega_{n-k+1,n}(\sigma)\cdot X_{\{k,\,\sigma(k+1)\}}\,\sigma^*\cdot t^*_{n,k+1}\\
&= \sum_{\sigma\in S_1^k\times S_{n-k}} \omega_{n-k+1,n}(\sigma)\cdot X_{\{k,\,\sigma(k+1)\}}\\
&\quad\cdot \prod_{k+1\le j<\sigma(k+1)} X_{\{j,\,\sigma(k+1)\}}\,(\sigma\,t_{n,k+1})^*\\
&= \sum_{\sigma\in S_1^k\times S_{n-k}} \omega_{n-k+1,n}(\sigma)\cdot \prod_{k\le j<\sigma(k+1)} X_{\{j,\,\sigma(k+1)\}}\,(\sigma\,t_{n,k+1})^*\\
&= \sum_{g\in S_1^k\times S_{n-k}} \omega_{n-k+1,n}(g\,t^{-1}_{n,k+1})\cdot \prod_{k\le j<g(n)} X_{\{j,\,g(n)\}}\,g^*
\end{aligned}$$

where we used that $g=\sigma\,t_{n,k+1}$ implies $\sigma = g\,t^{-1}_{n,k+1}$, so $\sigma(k+1)=g(n)$. On the other hand, by the formula

$$\text{(3.13)}\quad Des(t_{n,k+1}\,g^{-1}) = \big(Des(g^{-1})\setminus\{g(n)\}\big)\cup\{g(n)-1\} \qquad \text{if}\quad g(n)\le n$$

where

$$Des(g^{-1})\setminus\{g(n)\} = Des(g^{-1}) \qquad \text{when}\quad g(n)=n,\quad g(n)\notin Des(g^{-1})$$

we obtain

$$\begin{aligned}
&\omega_{n-k+1,n}(g\,t^{-1}_{n,k+1})\cdot \prod_{k\le j<g(n)} X_{\{j,\,g(n)\}}\\
&= \prod_{i\in Des(t_{n,k+1}\,g^{-1})} X_{\{k,\,k+1,\,\ldots,\,i\}}\cdot \prod_{k\le j<g(n)} X_{\{j,\,g(n)\}}\\
&= \sum_{\substack{i\in Des(t_{n,k+1}\,g^{-1})\\ i\neq g(n)-1}} X_{\{k,\,k+1,\,\ldots,\,i\}}\cdot \underbrace{X_{\{k,\,k+1,\,\ldots,\,g(n)-1\}}}_{if\quad i=g(n)-1}\cdot \prod_{k\le j<g(n)} X_{\{j,\,g(n)\}}\\
&= \sum_{\substack{i\in Des(t_{n,k+1}\,g^{-1})\\ i\neq g(n)-1}} X_{\{k,\,k+1,\,\ldots,\,i\}}\cdot X_{\{k,\,k+1,\,\ldots,\,g(n)\}}\\
&= \begin{cases} \omega_{n-k+1,n}(g) & \text{if}\quad g(n)<n\\ X_{\{k,\,k+1,\,\ldots,\,n\}}\cdot\omega_{n-k+1,n}(g) & \text{if}\quad g(n)=n. \end{cases}
\end{aligned}$$

Therefore

$$\varepsilon^*_{n-k+1,n}\cdot X_{\{k,\,k+1\}}\,t^*_{n,k+1} = \begin{cases} \sum_{g\in S_1^k\times S_{n-k}} \omega_{n-k+1,n}(g)\,g^* & \text{if} \quad g(n)<n \\ \sum_{g\in S_1^k\times S_{n-k}} X_{\{k,\,k+1,\,\dots,\,n\}}\cdot\omega_{n-k+1,n}(g)\,g^* & \text{if} \quad g(n)=n \end{cases}$$
$$= \begin{cases} \varepsilon^*_{n-k+1,n} & \text{if} \quad g(n)<n \\ X_{\{k,\,k+1,\,\dots,\,n\}}\cdot\varepsilon^*_{n-k+1,n} & \text{if} \quad g(n)=n. \end{cases}$$

Finally, we get

$$\varepsilon^*_{n-k+1,n}\cdot\big(id - X_{\{k,\,k+1\}}\,t^*_{n,k+1}\big) = \varepsilon^*_{n-k+1,n} - \varepsilon^*_{n-k+1,n}\cdot X_{\{k,\,k+1\}}\,t^*_{n,k+1}$$
$$= \sum_{\substack{g\in S_1^k\times S_{n-k}\\ g(n)<n}} \omega_{n-k+1,n}(g)\,g^* + \sum_{\substack{g\in S_1^k\times S_{n-k}\\ g(n)=n}} \omega_{n-k+1,n}(g)\,g^*$$
$$- \sum_{\substack{g\in S_1^k\times S_{n-k}\\ g(n)<n}} \omega_{n-k+1,n}(g)\,g^* - \sum_{\substack{g\in S_1^k\times S_{n-k}\\ g(n)=n}} X_{\{k,\,k+1,\,\dots,\,n\}}\cdot\omega_{n-k+1,n}(g)\,g^*$$
$$= \big(id - X_{\{k,\,k+1,\,\dots,\,n\}}\big)\cdot \sum_{\substack{g\in S_1^k\times S_{n-k}\\ g(n)=n}} \omega_{n-k+1,n}(g)\,g^*$$
$$= \big(id - X_{\{k,\,k+1,\,\dots,\,n\}}\big)\cdot\varepsilon^*_{n-k+1,n-1}$$

where we have used that

$$\sum_{\substack{g'\in S_1^k\times S_{n-k}\\ g'(n)=n}} \omega_{n-k+1,n}(g')\,(g')^*$$
$$= \sum_{\substack{g'\in S_1^k\times S_{n-k}\\ g'(n)=n}} t_{1,n}\cdot\big(t_{n,1}\cdot\omega_{n-k+1,n}(g')\,(g')^*\cdot t_{1,n}\big)\cdot t_{n,1}$$
$$= t_{1,n}\cdot\left(\sum_{g\in S_1^{k+1}\times S_{n-k-1}} \omega_{n-k,n}(g)\,g^*\right)\cdot t_{n,1}$$
$$= t_{1,n}\cdot\varepsilon^*_{n-k,n}\cdot t_{n,1} = \varepsilon^*_{n-k+1,n-1}.$$

This proves (3.12) and the proof of the Proposition 3.10 is now completed. ∎

The matrix factorizations from the right given in [3, 4] one can replace by twisted algebra factorizations (from the right). But here we have presented the factorizations from the left because they are more suitable for computing constants in multiparametric algebra of noncommuting polynomials (this will be treated in a forthcoming paper).

References

[1] G. Duchamp, A. Klyachko, D. Krob and J.-Y. Thibon, *Noncommutative symmetric functions III: Deformations of Cauchy and convolution algebras,* Discrete Math. Theor. Comput. Sci. **1** (1997), 159–216.

[2] S. Meljanac, A. Perica and D. Svrtan, *The energy operator for a model with a multiparametric infinite statistics,* J. Phys. A **36 2**3 (2003), 6337–6349.

[3] S. Meljanac and D. Svrtan, *Determinants and inversion of Gram matrices in Fock representation of q_{kl}-canonical commutation relations and applications to hyperplane arrangements and quantum groups. Proof of an extension of Zagier's conjecture,* preprint, arXiv:math-ph/0304040vl.

[4] S. Meljanac and D. Svrtan, *Study of Gram matrices in Fock representation of multiparametric canonical commutation relations, extended Zagier's conjecture, hyperplane arrangements and quantum groups,* Math. Commun. **1** (1996), 1–24.

[5] J.J. Rotman, An introduction to the theory of groups, Fourth edition, Springer-Verlag, New York, 1995.

[6] J.J. Rotman, Advanced modern algebra, Prentice Hall, New Jersey, 2002.

[7] M. Sošić, *Computing constants in some weight subspaces of free associative complex algebra,* Int. J. Pure Appl. Math. **81** (2012), 165–190.

[8] D. Zagier, *Realizability of a model in infinite statistics,* Commun. Math. Phys. **147** (1992), 199–210.

M. Sošić
Department of Mathematics
University of Rijeka
51 000 Rijeka
Croatia
E-mail: msosic@math.uniri.hr

Received: 12.2.2015.