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FINITE p-GROUPS ALL OF WHOSE MAXIMAL SUBGROUPS, EXCEPT ONE, HAVE ITS DERIVED SUBGROUP OF ORDER $\leq p$

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Abstract. Let G be a finite p-group which has exactly one maximal subgroup H such that $|H'| > p$. Then we have $\mathrm{d}(G) = 2$, $p = 2$, H' is a four-group, G' is abelian of order 8 and type $(4, 2)$, G is of class 3 and the structure of G is completely determined. This solves the problem Nr. 1800 stated by Y. Berkovich in [3].

We consider here only finite p-groups and our notation is standard (see [1]). If G is a p-group all of whose maximal subgroups have its derived subgroups of order $\leq p$, then such groups G are characterized in [3, §137]. But there is no way to determine completely the structure of such p-groups.

It is quite surprising that we can determine completely (in terms of generators and relations) the title groups, where exactly one maximal subgroup has the commutator subgroup of order $> p$. We shall prove our main theorem (Theorem 8) starting with some partial results about the title groups. However, Propositions 4 and 6 are also of independent interest.

Proposition 1. *Let G be a title p-group. Then we have $\mathrm{d}(G) \leq 3$, $\mathrm{cl}(G) \leq 3$, $p^2 \leq |G'| \leq p^3$ and G' is abelian of exponent $\leq p^2$. Also, G has at most one abelian maximal subgroup.*

Proof. Let H be the unique maximal subgroup of G with $|H'| > p$. This gives $|G'| \geq p^2$. Let $K \neq L$ be maximal subgroups of G which are both distinct from H. We have $|K'| \leq p$, $|L'| \leq p$ and so $K'L' \leq \mathrm{Z}(G)$ and $|K'L'| \leq p^2$. By a result of A. Mann ([1, Exercise 1.69]), we get $|G' : (K'L')| \leq p$. This implies that $|G'| \leq p^3$, G' is abelian and G is of class ≤ 3. Since $K'L'$ is elementary

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abelian, we also get $\exp(G') \le p^2$. If G would have more than one abelian maximal subgroup, then (by the above argument) $|G'| \le p$, a contradiction. Hence G has at most one abelian maximal subgroup.

Note that each nonabelian p-group X has exactly $0, 1$ or $p+1$ abelian maximal subgroups and in the last case $|X'| = p$ (Exercise 1.6(a) in [1]). Suppose that $\mathrm{d}(G) \ge 4$. Then G has at least $1 + p + p^2 + p^3$ distinct maximal subgroups and so the set $\mathcal{S}$ of maximal subgroups of G with the commutator group of order p has at least $p + p^2 + p^3 - 1$ elements. Since G' has at most p^2+p+1 pairwise distinct subgroups of order p (and the maximum is achieved if $G' \cong \mathrm{E}_{p^3}$), it follows that there are $K \ne L \in \mathcal{S}$ such that $K' = L'$. By the above argument (using a result of A. Mann), we get $|G'| = p^2$ and so G' has at most $p+1$ pairwise distinct subgroups of order p (where the maximum is achieved if $G' \cong \mathrm{E}_{p^2}$). If $M \in \mathcal{S}$, then considering G/M', we see that there are at most $p+1$ elements $N \in \mathcal{S}$ such that $N' = M'$. This gives

$$p + p^2 + p^3 - 1 \le (p+1)^2, \text{ and so } p^3 - p \le 2 \quad \text{or} \quad p(p^2-1) \le 2,$$

a contradiction. Our proposition is proved. □

Proposition 2. *Let G be a title p-group. Then the subgroup:*

$$H_0 = \langle M' \mid M \text{ is any maximal subgroup of G with } |M'| \le p \rangle$$

is noncyclic and so H_0 is elementary abelian of order p^2 or p^3 and $H_0 \le \mathrm{Z}(G)$.

Proof. Suppose that H_0 is cyclic. Then we have $|H_0| = p$ and so $|G'| = p^2$ because (by [1, Exercise 1.69]) $|G' : H_0| \le p$ and Proposition 1 implies that $|G'| \ge p^2$. This gives that $H' = G'$, where H is the unique maximal subgroup of G with $|H'| > p$. Consider the nonabelian factor group G/H_0. In this case G/H_0 has exactly one nonabelian maximal subgroup H/H_0. Since $\mathrm{d}(G/H_0) = 2$ or 3, the last statement would imply that the nonabelian p-group G/H_0 would have exactly p or $p + p^2$ abelian maximal subgroups, a contradiction (by [1, Exercise 1.6(a)]). □

Proposition 3. *Let G be a title p-group. Then we have* $\mathrm{d}(G) = 2$.

Proof. Assume that $\mathrm{d}(G) = 3$ and we use the notation from Proposition 2.

First suppose that $H_0 = G'$ so that G is of class 2 with an elementary abelian commutator subgroup. For any $x, y \in G$, we get $[x^p, y] = [x, y]^p = 1$ and this implies that $\mho_1(G) \le \mathrm{Z}(G)$. It follows $\Phi(G) = \mho_1(G)G' \le \mathrm{Z}(G)$ and $G/\Phi(G) \cong \mathrm{E}_{p^3}$. Let X be any maximal subgroup of G so that $X/\Phi(G) \cong \mathrm{E}_{p^2}$ and all $p+1$ maximal subgroups of X which contain $\Phi(G)$ are abelian. This implies $|X'| \le p$. But then each maximal subgroup of G has its derived subgroup of order $\le p$, contrary to our assumption.

Now assume $H_0 \ne G'$. In this case $H_0 \cong \mathrm{E}_{p^2}$, $H_0 \le \mathrm{Z}(G)$ and $|G'| = p^3$. There are exactly $p + p^2$ maximal subgroups M_i of G such that $|M_i'| \le p$,

$i = 1, 2, ..., p + p^2$. Since H_0 has exactly $p + 1$ subgroups of order p, it follows that there exist the indices $i \neq j \in \{1, 2, ..., p + p^2\}$ such that $M_i' = M_j'$ is of order p. Again by [1, Exercise 1.69] we have $|G' : (M_i'M_j')| \leq p$ and this gives $|G'| \leq p^2$, a contradiction. Our proposition is proved. □

Proposition 4. *Let G be a two-generator p-group, $p > 2$, with $G' \cong \mathrm{C}_{p^2}$. Then each maximal subgroup of G is nonabelian.*

Proof. Assume that G has an abelian maximal subgroup M so that $|M/\Phi(G)| = p$. Take an element $a \in M \setminus \Phi(G)$ and an element $b \in G \setminus M$ so that we have $G = \langle a, b\rangle$ and $G' = \langle [a, b]\rangle$. Since G' is cyclic, [1, Theorem 7.1(c)] implies that G is regular. We have $b^p \in \Phi(G) < M$ and so $[a, b^p] = 1$. Hence

$$(a^{-1}b^{-p}a)b^p = ((b^{-1})^a)^p b^p = 1 \text{ and so } (b^a)^p = b^p.$$

By [1, Theorem 7.2(a)] (about regular p-groups), the last relation gives $((b^{-1})^a b)^p = 1$ or equivalently $[a, b]^p = 1$, a contradiction. □

Remark 5. The assumption $p > 2$ in Proposition 4 is essential. This shows a 2-group of maximal class and order 16.

Proposition 6. *Let G be a two-generator p-group, $p > 2$, with $G' \cong \mathrm{E}_{p^2}$. Then G has an abelian maximal subgroup.*

Proof. By [3, Proposition 137.4], each proper subgroup of G has its derived subgroup of order at most p. Then we may apply [3, Proposition 137.5] and so for each $x, y \in G$, we get $[x^p, y] = [x, y]^p = 1$. This gives that $\mho_1(G) \leq \mathrm{Z}(G)$ and therefore we obtain that $\Phi(G) = \mho_1(G)G'$ is abelian. Let M be a maximal subgroup of G which centralizes G'. We have $|M : \Phi(G)| = p$ and M centralizes $\mho_1(G)$ and G' so that $\Phi(G) \leq \mathrm{Z}(M)$. This implies that M is abelian and we are done. □

Remark 7. The assumption $p > 2$ in Proposition 5 is essential. Let G ba a faithful and splitting extension of an elementary abelian group of order 8 by a cyclic group of order 4. Then we have $\mathrm{d}(G) = 2$ and $G' \cong \mathrm{E}_4$ but G has no abelian maximal subgroup.

Proposition 8. *Let G be a title p-group and $\Gamma_1 = \{H_1, H_2, ..., H_p, H\}$ be the set of all maximal subgroups of G, where $|H'| > p$. Then G' is abelian of order p^3, $H' \cong \mathrm{E}_{p^2}$, $H' \leq \mathrm{Z}(G)$ and $H_1', H_2', ..., H_p'$ are pairwise distinct subgroups of order p contained in H'. If $G = \langle x, y\rangle$ for some $x, y \in G$, then $[x, y] \in G' \setminus H'$ and $[x, y] \notin \mathrm{Z}(G)$ so that G is of class 3. Finally, G/H' is nonmetacyclic minimal nonabelian and so if $a \in G \setminus G'$ is such that $a^p \in G'$, then $a^p \in H'$.*

Proof. Let H_0 be the subgroup of G' as defined in Proposition 2. Then $H_0 \leq \mathrm{Z}(G)$ and H_0 is elementary abelian of order p^2 or p^3. Suppose for a moment that $H_0 = G'$. We have $G = \langle x, y\rangle$ for some $x, y \in G$ and $[x, y] \in H_0$

so that $G/\langle[x,y]\rangle$ is abelian and $G' = \langle[x,y]\rangle$ is of order p, a contradiction. It follows that $H_0 \neq G'$ which gives that $H_0 \cong \mathrm{E}_{p^2}$, $|G' : H_0| = p$ and G' is abelian of order p^3. Since $\mathrm{d}(G/H_0) = 2$ and $|G'/H_0| = p$, it follows that G/H_0 is minimal nonabelian (see [2, Lemma 65.2(a)]). In particular, we have $H' \leq H_0$ which together with $|H'| > p$ implies $H' = H_0 \cong \mathrm{E}_{p^2}$. If G/H' is metacyclic, then a result of N. Blackburn (see [1, Lemma 44.1] and [1, Corollary 44.6]) gives that G is also metacyclic. This is a contradiction because G' is noncyclic. Hence G/H' is nonmetacyclic minimal nonabelian so that [2, Lemma 65.1] gives that G'/H' is a maximal cyclic subgroup of G/H'. Thus for each element $a \in G \setminus G'$ such that $a^p \in G'$, we get $a^p \in H'$.

We have $G = \langle x, y\rangle$ for some $x, y \in G$. It is clear that $\langle[x,y]\rangle$ is not normal in G. Indeed, if $\langle[x,y]\rangle \trianglelefteq G$, then $G/\langle[x,y]\rangle$ is abelian and so $\langle[x,y]\rangle = G'$ is of order $\leq p^2$ (noting that $\exp(G') \leq p^2$), a contradiction. We have proved that $\langle[x,y]\rangle$ is not normal in G. In particular, $[x,y] \notin \mathrm{Z}(G)$ and so $[x,y] \in G' \setminus H'$ and G is of class 3.

If $\Gamma_1 = \{H_1, H_2, ..., H_p, H\}$ is the set of all maximal subgroups of G, then we have $H_i' \leq H_0 = H'$ for all $i = 1, 2, ..., p$. We claim that $H_1', H_2', ..., H_p'$ are pairwise distinct subgroups of order p. Indeed, if $|H_i'H_j'| \leq p$ for some $i \neq j$, $i, j \in \{1, 2, ..., p\}$, then a result of A. Mann (see [1, Exercise 1.69]) implies $|G' : (H_i'H_j')| \leq p$ and so $|G'| \leq p^2$, a contradiction. Our proposition is proved. ∎

Remark 9. If X is a two-generator p-group of class 2, then it is well known that X' is cyclic. Hence if G is any two-generator p-group, then $G'/\mathrm{K}_3(G)$ is cyclic, where $\mathrm{K}_3(G) = [G', G]$.

Proposition 10. *If G is a title p-group, then $p = 2$.*

Proof. Assume that $p > 2$ and we use Proposition 6 together with the notation introduced there.

First suppose that G' is not elementary abelian. Then we have $\mathrm{o}([x,y]) = p^2$ and $\langle[x,y]^p\rangle$ is a subgroup of order p contained in H'. Let H_i', $i \in \{1, 2, ..., p\}$, be such that $H_i' \neq \langle[x,y]^p\rangle$ which gives $G' = H_i' \times \langle[x,y]\rangle$. We consider the factor group $\bar{G} = G/H_i'$. Since $\mathrm{d}(\bar{G}) = 2$, $p > 2$, and $\bar{G}' \cong \mathrm{C}_{p^2}$, we may use Proposition 4 saying that each maximal subgroup of $\bar{G}$ is nonabelian. But $\bar{H}_i = H_i/H_i'$ is an abelian maximal subgroup of $\bar{G}$, a contradiction.

We have proved that G' is elementary abelian of order p^3. Let $\{H_1', H_2', ..., H_p', K\}$ be the set of all $p+1$ subgroups of order p in H' and consider the factor group G/K. All $p+1$ maximal subgroups of G/K are nonabelian, $\mathrm{d}(G/K) = 2$, $p > 2$, and $(G/K)' = G'/K \cong \mathrm{E}_{p^2}$. By Proposition 5, G/K possesses an abelian maximal subgroup, a contradiction. We have proved that we must have $p = 2$. ∎

Theorem 11. *Let G be a p-group with exactly one maximal subgroup H such that $|H'| > p$. Then we have* $\mathrm{d}(G) = 2$, *$p = 2$ and G' is abelian of order* 8 *and type* $(4,2)$. *Also,* $[G',G] = \Omega_1(G') \le \mathrm{Z}(G)$, $\Phi(G) = \mathrm{C}_G(G')$ *is abelian and* $\mho_2(G) \le \mathrm{Z}(G)$. *Let* $\{H_1, H_2, H\}$ *be the set of maximal subgroups of* G. *Then* $H_1' = \langle z_1 \rangle$ *and* $H_2' = \langle z_2 \rangle$ *are both of order* 2, $\langle z_1, z_2 \rangle = \Omega_1(G') = H' \cong \mathrm{E}_4$, $\mathrm{d}(H) = 3$ *and* $\mho_1(G') = \langle z_1 z_2 \rangle$. *Finally,* H *is the unique maximal subgroup of* G *which contains an element acting invertingly on* G'. *We have the following two possibilities:*

(i) $\mathrm{d}(H_1) = \mathrm{d}(H_2) = 2$ *in which case* H_1 *and* H_2 *are minimal nonabelian. In this case either* H_1 *and* H_2 *are both metacyclic and* G *is isomorphic to one of the groups of Theorem 100.3(a) and (b) in [3] or* H_1 *and* H_2 *are both nonmetacyclic and* G *is isomorphic to one of the groups of Theorem 100.3(c) in [3].*

(ii) $\mathrm{d}(H_1) = \mathrm{d}(H_2) = 3$ *and the group* G *is given with:*

$$G = \langle a, b \mid [a,b] = v,\ v^4 = 1,\ [v,a] = z_1,\ [v,b] = z_1^{\epsilon} z_2,\ z_1^2 = z_2^2 = 1,\ v^2 = z_1 z_2,$$
$$[z_1, a] = [z_1, b] = [z_2, a] = [z_2, b] = 1,\ a^{2^m} = z_1^{\alpha} z_2^{\beta},\ b^{2^n} = z_1^{\gamma} z_2^{\delta} \rangle,$$

where $m \ge 2$, $n \ge 2$, *and* $\alpha, \beta, \gamma, \delta, \epsilon \in \{0,1\}$. *We have here* $|G| = 2^{m+n+3} \ge 2^7$, $G' = \langle v, z_1 \rangle \cong \mathrm{C}_4 \times \mathrm{C}_2$, $[G', G] = \langle z_1, z_2 \rangle = \Omega_1(G') \le \mathrm{Z}(G)$ *and the Frattini subgroup* $\Phi(G) = \langle G', a^2, b^2 \rangle$ *is abelian. Finally, if* $\epsilon = 0$, *then* $H = \Phi(G)\langle ab \rangle$ *and if* $\epsilon = 1$, *we have* $H = \Phi(G)\langle b \rangle$.

Conversely, all groups stated in parts (i) and (ii) of this theorem are p*-groups all of whose maximal subgroups, except one, have its derived subgroup of order* $\le p$.

Proof. We use Proposition 6 together with the notation introduced there. By Proposition 7, we have in addition $p = 2$.

Let X be a maximal subgroup of G. By Schreier's inequality ([2, Theorem A.25.1]), we have

$$\mathrm{d}(X) \le 1 + |G : X|(\mathrm{d}(G) - 1),$$

and so $\mathrm{d}(X) \le 3$. Since $H' \cong \mathrm{E}_4$ and $H' \le \mathrm{Z}(H)$, the maximal subgroup H cannot be two-generator (see Remark 3). It follows that we have $\mathrm{d}(H) = 3$. Since G is a nonmetacyclic two-generator 2-group, we may use [3, Theorem 107.1] saying that such a group has an even number of two-generator maximal subgroups. It follows that we have either $\mathrm{d}(H_1) = \mathrm{d}(H_2) = 2$ or $\mathrm{d}(H_1) = \mathrm{d}(H_2) = 3$.

Set $H_1' = \langle z_1 \rangle$, $H_2' = \langle z_2 \rangle$ so that we have $H' = \langle z_1 \rangle \times \langle z_2 \rangle \cong \mathrm{E}_4$ and $\Phi(G) = H_1 \cap H_2$. Since $(\Phi(G))' \le \langle z_1 \rangle \cap \langle z_2 \rangle = \{1\}$, it follows that $\Phi(G)$ is abelian and so $\Phi(G)$ is a maximal normal abelian subgroup of G (containing G'). Take elements $h_1 \in H_1 \setminus \Phi(G)$ and $h_2 \in H_2 \setminus \Phi(G)$ so that we have $G = \langle h_1, h_2 \rangle$, $[h_1, h_2] = v \in G' \setminus H'$ and $\mathrm{o}(v) \le 4$. If v commutes with both h_1 and h_2, then we get $v \in \mathrm{Z}(G)$, a contradiction. Without loss of generality we may assume that $[v, h_1] \ne 1$ and so we get $[v, h_1] = z_1$.

Assume for a moment that $G' \cong \mathrm{E}_8$ so that v is an involution. We compute

$$[h_1^2, h_2] = [h_1, h_2]^{h_1}[h_1, h_2] = v^{h_1}v = (vz_1)v = v^2 z_1 = z_1.$$

This is a contradiction since $h_1^2 \in \Phi(G)$ and $\langle h_1^2, h_2\rangle \le H_2$, where $H_2' = \langle z_2\rangle$. We have proved that G' is abelian of type $(4, 2)$ and so $\mathrm{o}(v) = 4$ and $1 \ne v^2 \in H'$.

We have $\mathrm{K}_3(G) = [G', G] \ge \langle z_1\rangle$. Since $\mathrm{d}(G) = 2$, it follows by Remark 1 that $G'/\mathrm{K}_3(G)$ is cyclic. Suppose that $[v, h_2] = 1$ so that in this case we have $\mathrm{K}_3(G) = \langle z_1\rangle$. We compute

$$[h_1, h_2^2] = [h_1, h_2][h_1, h_2]^{h_2} = vv^{h_2} = v^2 \ne 1.$$

We have $\langle h_1, h_2^2\rangle \le H_1$ and so $v^2 = z_1$. But then we have $G'/\mathrm{K}_3(G) = G'/\langle z_1\rangle \cong \mathrm{E}_4$, a contradiction. We have proved that $[v, h_2] \ne 1$ and so $[v, h_2] = z_2$. This gives

$$\mathrm{K}_3(G) = \langle z_1, z_2\rangle = H' \le \mathrm{Z}(G)$$

and G is of class 3.

We get

$$[h_1^2, h_2] = [h_1, h_2]^{h_1}[h_1, h_2] = v^{h_1}v = (vz_1)v = v^2 z_1,$$

and since $\langle h_1^2, h_2\rangle \le H_2$, it follows that $v^2 z_1 \in \langle z_2\rangle$ and so $v^2 \in \{z_1, z_1z_2\}$. Similarly, we get

$$[h_1, h_2^2] = [h_1, h_2][h_1, h_2]^{h_2} = vv^{h_2} = v(vz_2) = v^2 z_2,$$

and since $\langle h_1, h_2^2\rangle \le H_1$, it follows that $v^2 z_2 \in \langle z_1\rangle$ and so $v^2 \in \{z_2, z_1z_2\}$. As a result, we get $v^2 = z_1z_2$ and so $\mho_1(G') = \langle z_1z_2\rangle$. Note that $H = \Phi(G)\langle h_1h_2\rangle$ and

$$v^{h_1h_2} = (vz_1)^{h_2} = v(z_1z_2) = vv^2 = v^3 = v^{-1}$$

and so h_1h_2 acts invertingly on G'. It follows that $\Phi(G) = \mathrm{C}_G(G')$ and H is the unique maximal subgroup of G which contains an element acting invertingly on G'.

Let $x, y \in G$. Then $\langle x^2, y\rangle$ is contained in one of the maximal subgroups X_i of G, where X_i' is elementary abelian of order ≤ 4 and $\mathrm{cl}(X_i) = 2$ ($i = 1, 2, 3$). It follows

$$[x^4, y] = [(x^2)^2, y] = [x^2, y]^2 = 1,$$

and so we get $\mho_2(G) \le \mathrm{Z}(G)$.

Now suppose that $\mathrm{d}(H_1) = \mathrm{d}(H_2) = 2$. In this case both H_1 and H_2 are minimal nonabelian (see [2, Lemma 65.2(a)]) and H is neither abelian nor minimal nonabelian. Since $\mathrm{d}(G) = 2$ and $H_1' \ne H_2'$ such 2-groups are completely determined in [3, Theorem 100.3] which gives the groups quoted in part (i) of our theorem.

It remains to consider the case $\mathrm{d}(H_1) = \mathrm{d}(H_2) = 3$. By [3, Theorem 107.2(a)], a nonmetacyclic two-generator 2-group G has the property that every maximal subgroup of G is not generated by two elements if and only if G/G' has no cyclic subgroup of index 2. Thus G/G' is abelian of type $(2^m, 2^n)$, where $m \geq 2$, $n \geq 2$ and so $|G| = |G'|2^{m+n} = 2^{m+n+3} \geq 2^7$. There are normal subgroups A and B of G such that $G = AB$, $A \cap B = G'$, $A/G' \cong \mathrm{C}_{2^m}$, $B/G' \cong \mathrm{C}_{2^n}$, $m \geq 2$, $n \geq 2$. Let $a \in A \setminus G'$, $b \in B \setminus G'$ be such that $\langle a \rangle$ covers A/G' and $\langle b \rangle$ covers B/G'. Since G/H' is nonmetacyclic minimal nonabelian, we know that (see [2, Lemma 65.1]) G'/H' is a maximal cyclic subgroup of G/H' and so we have $a^{2^m} \in H'$ and $b^{2^n} \in H'$. We have $G = \langle a, b \rangle$ and so $[a, b] = v$ is an element of order 4 contained in $G' \setminus H'$.

Maximal subgroups of G are $M_1 = A\langle b^2 \rangle$, $M_2 = B\langle a^2 \rangle$ and $M_3 = \Phi(G)\langle ab \rangle$, where $\Phi(G) = G'\langle a^2 \rangle\langle b^2 \rangle$ is abelian. Since $\Phi(G) = \mathrm{C}_G(G')$ and $\Omega_1(G') = H' \leq \mathrm{Z}(G)$, we see that $G/\Phi(G) \cong \mathrm{E}_4$ acts faithfully on G' stabilizing the chain $G' > H' > \{1\}$. Interchanging A and B (if necessary), we may assume that $|M_1'| = 2$ and so we may set $v^a = vz_1$ which gives that $[v, a] = z_1$ and $M_1' = \langle z_1 \rangle$, where $z_1 \in H' \setminus \langle v^2 \rangle$. Set $z_2 = z_1 v^2$ so that we have $v^2 = z_1 z_2$. Then we have two possibilities.

(1) We assume $v^b = vz_1z_2 = v^{-1}$ or equivalently $[v, b] = z_1z_2$ so that the element b inverts each element in G'. Since the maximal subgroup H is the unique maximal subgroup of G which contains an element acting invertingly on G', we have in this case $M_2 = B\langle a^2 \rangle = H$, where we should have $H' = \langle z_1, z_2 \rangle$. Indeed, we have

$$[a^2, b] = [a, b]^a[a, b] = v^a v = (vz_1)v = v^2 z_1 = (z_1z_2)z_1 = z_2,$$

and so we get $H' = \langle z_1, z_2 \rangle$. In this case $M_3 = \Phi(G)\langle ab \rangle$ has the property $M_3' = \langle z_2 \rangle$. Indeed, here we have

$$\begin{aligned} &[a^2, ab] = [a^2, b] = z_2, \\ &[ab, b^2] = [a, b^2]^b = ([a, b][a, b]^b)^b = (vv^b)^b = (vv^{-1})^b = 1, \end{aligned}$$

and

$$v^{ab} = (vz_1)^b = (vz_1z_2)z_1 = vz_2 \text{ and so } [v, ab] = z_2.$$

(2) Now we suppose $v^b = vz_2$ or equivalently $[v, b] = z_2$. In this case we get $M_2' = \langle z_2 \rangle$ since

$$[a^2, b] = [a, b]^a[a, b] = v^a v = (vz_1)v = v^2 z_1 = (z_1z_2)z_1 = z_2.$$

Also, we have here $M_3 = H$ because

$$v^{ab} = (vz_1)^b = (vz_2)z_1 = v(z_1z_2) = vv^2 = v^3 = v^{-1}$$

and so ab acts invertingly on G'. We have $[v, ab] = z_1z_2$ and

$$[a^2, ab] = [a^2, b] = [a, b]^a[a, b] = v^a v = (vz_1)v = v^2 z_1 = (z_1z_2)z_1 = z_2,$$

and so we have here $M_3' = \langle z_1, z_2 \rangle$.

In both cases (1) and (2), we may set $v^b = z_1^{\epsilon} z_2$, where in case (1) we have $\epsilon = 1$ and in case (2) we have $\epsilon = 0$. Thus, if $\epsilon = 0$, then $H = \Phi(G)\langle ab\rangle$ and if $\epsilon = 1$, we have $H = \Phi(G)\langle b\rangle$.

Also, we may set

$$a^{2^m} = z_1^{\alpha} z_2^{\beta},\ b^{2^n} = z_1^{\gamma} z_2^{\delta},$$

where $\alpha, \beta, \gamma, \delta \in \{0, 1\}$ since we know that a^{2^m}, $b^{2^n} \in H' = \langle z_1, z_2\rangle$.

Conversely, by inspection of groups given in parts (i) and (ii) of our theorem, we see that all these groups have the title property. Our theorem is proved. ∎

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