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ON THE STRUCTURE OF THE AUTOMORPHISM GROUP OF A MINIMAL NONABELIAN p-GROUP (METACYCLIC CASE)

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Abstract. In this paper we find the complete structure for the automorphism groups of metacyclic minimal nonabelian 2-groups. This, together with [6, 7], gives the complete answer to the Question 15 from [5] (respectively Question 20 from [4]) in the case of metacyclic groups. We also correct some inaccuracies and extend the results from [13].

1. Introduction

All groups considered here are finite and the notation used is standard.

Finite p-groups are an important group class of finite groups. Since the classification of finite simple groups is finally completed, the study of finite p-groups becomes more and more active. Many leading group theorists, for example, Berkovich, Glauberman, Janko etc., turn their attention to the study of finite p-groups, see [1–4, 9, 10, 12]. Since a finite p-group has "too many" normal subgroups and, consequently, there is an extremely large number of nonisomorphic p-groups of a given fixed order, the classification of finite p-groups in the classical sense is impossible. In [1–3] Berkovich and Janko have developed some techniques for working with minimal non-abelian subgroups of finite p-groups. Roughly speaking, they show that some control over the lattice of subgroups in p-groups can be gained by considering maximal abelian subgroups together with minimal non-abelian subgroups. In [12] Janko points out that in studying the structure of non-abelian p-groups G, the minimal non-abelian subgroups of G play an important role since they generate the group G. More precisely, if A is a maximal normal abelian subgroup of G, then

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minimal non-abelian subgroups of G cover the set $G \setminus A$ (see Proposition 1.6 in [12]). A p-group G is said to be *minimal nonabelian* (for brevity, $\mathcal{A}_1$-group), if G is nonabelian, but all its proper subgroups are abelian. In [5] Berkovich formulated 22 questions concerning p-groups. In Question 15 (respectively Question 20 from [4]) he proposed to describe the automorphism groups of $\mathcal{A}_1$-groups. The following lemma gives the classification of $\mathcal{A}_1$-groups.

Lemma 1.1. (L. Redei) *Let G be a minimal nonabelian p-group. Then $G = \langle x, y\rangle$ and one of the following holds*

(1) $x^{p^m} = y^{p^n} = z^p = 1,\ [x, y] = z,\ [x, z] = [y, z] = 1,\ m, n \in \mathbb{N},\ m \geqslant n \geqslant 1$*; where in case $p = 2$ we must have $m > 1$;*
(2) $x^{p^m} = y^{p^n} = 1,\ [x, y] = x^{p^{m-1}},\ m, n \in \mathbb{N}, m \geqq 2,\ n \geqq 1$*;*
(3) $a^4 = 1,\ a^2 = b^2,\ [a, b] = a^2,\ G \cong Q_8$.

In this paper we find the complete structure for the automorphism groups of metacyclic minimal nonabelian 2-groups. This, together with [6, 7], gives the complete answer to the Question 15 from [5] (respectively Question 20 from [4]) in the case of metacyclic groups. In Section 2 we generalize the results from [13] and we specify a method of finding relations in an automorphism group, that we will use in the next Sections. In the first part of Section 3 we state some results from [13], that we will use in the next part of the note, but also we specify the exact statements. Unfortunately we must point out that in Section 3 of [13] in Case A the expression "$C_K(G')$" should be replaced by "$\Omega_{m-r}(K)$." In the end of Section 3 we state the relations in the automorphism group of a split metacyclic 2-group. In this way we remove some inaccuracies from Theorem 3.7 in [8] (see Example 1 in [13]). In Section 4 we find the complete structure of the automorphism group of a metacyclic minimal nonabelian 2-group. These relations were not considered in [8].

If L is a subgroup of a group G, then $C_{\mathrm{Aut}\, G}(L)$ denotes the group of those automorphisms of G that centralize L and $N_{\mathrm{Aut}\, G}(L)$ denotes the group of those automorphisms of G that normalize L. If M and N are normal subgroups of a group G, then $\mathrm{Aut}_N(G) = C_{\mathrm{Aut}(G)}(G/N)$ denotes the group of all automorphisms of G normalizing N and centralizing G/N. Also $\mathrm{Aut}_N^M(G)$ denotes $\mathrm{Aut}_N(G) \cap C_{\mathrm{Aut}\, G}(M)$. If L is a subgroup of a p-group G and $l \in \mathbb{N}$ then we set $\Omega_l(L) = \langle g \in L \mid g^{p^l} = 1\rangle$ and $\mho_l(L) = \langle g^{p^l} \mid g \in L\rangle$.

In [15] the authors investigated the automorphism group of a semidirect product $G = H \rtimes K$. They defined the following subgroups

$$A = \{\theta \in \mathrm{Aut}\, G \mid [K, \theta] = 1 \text{ and } H^\theta = H\},$$
$$B = \{\theta \in \mathrm{Aut}\, G \mid [H, \theta] = 1 \text{ and } [K, \theta] \subseteqq H\},$$
$$C = \{\theta \in \mathrm{Aut}\, G \mid [K, \theta] = 1 \text{ and } [H, \theta] \subseteqq K\},$$
$$D = \{\theta \in \mathrm{Aut}\, G \mid [H, \theta] = 1 \text{ and } K^\theta = K\}.$$

By definition, we have $BD = B \rtimes D \leqq C_{\mathrm{Aut}\, G}(K)$ and $AC = C \rtimes A \leqq C_{\mathrm{Aut}\, G}(H)$.

2. Crossed homomorphisms and automorphisms

We call an ordered triple (Q, N, θ) data if N is an abelian group, Q is a group and $\theta : Q \to \operatorname{Aut} N$ is a homomorphism. If θ is a homomorphism of Q into $\operatorname{Aut} N$, then Q acts on N when we define, for each $x \in Q$ and $a \in N$, a^x is the image of a under x^θ. If N is a normal subgroup of G, then the action of G/N on $Z(N)$ is given by $a^{gN} = a^{(gN)^\theta} = a^g$. Given data (Q, N, θ) a crossed homomorphism is a function $\lambda : Q \to N$ such that $(xy)^\lambda = (x^\lambda)^y y^\lambda$ for all $x, y \in Q$. We denote the set of such crossed homomorphisms by $Z^1(Q, N)$. It forms a group under the operation $q^{\lambda_1+\lambda_2} = q^{\lambda_1} q^{\lambda_2}$; if θ is trivial, then $Z^1(Q, N) = \operatorname{Hom}(Q, N)$.

We recall a known result ([11], Satz I,17.1) needed in the sequel:

Lemma 2.1. *Let N be a normal subgroup of G. Then there is a natural isomorphism from $Z^1(G/N, Z(N))$ to $Aut_N^N(G)$ sending each crossed homomorphism $f : G/N \to Z(N)$ to the automorphism $\varphi_f : x \mapsto x(xN)^f$ of G.*

Lemmas 2.2–2.3 are more general versions of Lemma 2.5 and Theorem 2.6 (see also [13]).

Lemma 2.2. *Let N be an normal subgroup of G. Let M be a normal subgroup of G such that $M \leq Z(G)$. Assume that that $L = \{\lambda \in Z^1(G/N, Z(N)) \mid (G/N)^\lambda \subseteqq M\}$ and $A = N_{\operatorname{Aut} G}(M) \cap N_{\operatorname{Aut} G}(N)$. Then*

(1) *$A \leqq \operatorname{Aut}(G)$ and $L \leqq Z^1(G/N, Z(N))$.*
(2) *If $\alpha \in A$ and $\lambda \in L$ then the function $\mu : G/N \to Z(N)$ defined by $\mu : gN \mapsto ((g^{\alpha^{-1}} N)^\lambda)^\alpha$ is a crossed homomorphism and $\mu \in L$.*

Proof. The first part of (1) is obvious.

(2) Assume that $\alpha \in A$ and $\lambda \in L$. First let $Ng_1 = Ng_2$, then $g_2 = g_1 h$ for some $h \in N$. Then

$$(g_2N)^\mu = ((g_2^{\alpha^{-1}} N)^\lambda)^\alpha = (((g_1h)^{\alpha^{-1}} N)^\lambda)^\alpha = ((g_1^{\alpha^{-1}} N)^\lambda)^\alpha = (g_1N)^\mu$$

since N is normalized by α. So μ is well defined.

Let $g_1N, g_2N \in G/N$. We have

$$\begin{aligned}(g_1N \cdot g_2N)^\mu &= (g_1g_2N)^\mu = (((g_1g_2)^{\alpha^{-1}} N)^\lambda)^\alpha \\ &= ((g_1^{\alpha^{-1}} N g_2^{\alpha^{-1}} N)^\lambda)^\alpha = (((g_1^{\alpha^{-1}} N)^\lambda)^{g_2^{\alpha^{-1}}} ((g_2^{\alpha^{-1}} N)^\lambda))^\alpha \\ &= (((g_1^{\alpha^{-1}} N)^\lambda)^\alpha)^{g_2} ((g_2^{\alpha^{-1}} N)^\lambda))^\alpha = ((g_1N)^\mu)^{g_2N} \cdot (g_2N)^\mu.\end{aligned}$$

It is evident that $\mu \in L$ since $(G/N)^\mu \subseteqq M$. ∎

Lemma 2.3. *Let G, N, M, L and A be as in Lemma 2.2. Assume that $E := \{\varphi \in \operatorname{Aut}_N^N(G) \mid [G, \varphi] \subseteqq M\}$. Then*

(1) *$E \leqq \operatorname{Aut} G$ and there is a natural isomorphism from L to E sending each crossed homomorphism $f : G/N \to M$ to the automorphism φ_f : $x \mapsto x(xN)^f$ of G;*
(2) *if $\alpha \in A$ and $\varphi \in E$ is determined by the crossed homomorphism $\lambda \in L$, then $\alpha^{-1}\lambda\alpha$ is determined by the crossed homomorphism $\mu \in L$ defined by $\mu : gN \mapsto ((g^{\alpha^{-1}}N)^\lambda)^\alpha$.*
(3) *A normalizes E and $AE \leqq \operatorname{Aut} G$.*

Proof. (1) It is evident that $E \leqq \operatorname{Aut} G$. By definitions of M, L, E and Lemma 2.1 we get the second part of the statement.

(2)-(3) Assume that $\alpha \in A$ and $\beta \in E$. By (1) there exists $\lambda \in Z^1(G/N, Z(N))$ such that $h^\beta = h(hN)^\lambda$ ($h \in G$) and $(hN)^\lambda \in M$ for all $h \in G$. If $h \in G$ then

$$h^{\alpha^{-1}\beta\alpha} = ((h^{\alpha^{-1}})^\beta)^\alpha = (h^{\alpha^{-1}}(h^{\alpha^{-1}}N)^\lambda)^\alpha = h((h^{\alpha^{-1}}N)^\lambda)^\alpha$$

and $((h^{\alpha^{-1}}N)^\lambda)^\alpha \in M$. Hence by Lemmas 2.1 and 2.2 $\alpha^{-1}\beta\alpha \in E$, so A normalizes E. Now it is clear that $AE \leqq \operatorname{Aut} G$. ∎

For the sake of completeness we recall some results from [13]. We will use them in this note.

Lemma 2.4 ([13]). *Let N be an normal subgroup of G such that G/N is cyclic of order n. Assume that g is an element of G with $G = \langle N, g\rangle$.*

(1) *If $a \in Z(N)$ and $a^{g^{n-1}+\cdots+g+1} = 1$, then the function $\lambda : G/N \to Z(N)$, defined by $(g^iN)^\lambda = a^{g^{i-1}+\cdots+g+1}$ $(i \in \mathbb{N})$ and $N^\lambda = 1$, is a crossed homomorphism.*
(2) *If $\lambda \in Z^1(G/N, Z(N))$ then there exists $a \in Z(N)$ such that $a^{g^{n-1}+\cdots+g+1} = 1$, $(g^iN)^\lambda = a^{g^{i-1}+\cdots+g+1}$ $(i \in \mathbb{N})$ and $N^\lambda = 1$.*

Lemma 2.5 ([13]). *Let G, N, g be as in Lemma 2.4. Let M be a normal subgroup of G such that $M \leqq Z(N)$ and for all $a \in M$ $a^{g^{n-1}+\cdots+g+1} = 1$. Assume that $L = \{\lambda \in Z^1(G/N, Z(N)) \mid (G/N)^\lambda \subseteqq M\}$ and $A = N_{\operatorname{Aut} G}(N) \cap N_{\operatorname{Aut} G}(M)$. Then*

(1) *$A \leqq \operatorname{Aut}(G)$ and $L \leqq Z^1(G/N, Z(N))$; moreover $L \cong M$.*
(2) *If $\alpha \in A$ and $\lambda \in L$ then the function $\mu : G/N \to Z(N)$ defined by $\mu : hN \mapsto ((h^{\alpha^{-1}}N)^\lambda)^\alpha$ is a crossed homomorphism and $\mu \in L$.*

Theorem 2.6 ([13]). *Let G, N, L, M, g and A be as in Lemma 2.5. Assume that $E := \{\varphi \in \operatorname{Aut}_N^N(G) \mid [G, \varphi] \subseteqq M\}$. Then $E \leqq \operatorname{Aut} G$, $L \cong E \cong M$, A normalizes E, $AE \leqq \operatorname{Aut} G$ and $A \cap E \cong \{g^{-1}g^\varphi \mid \varphi \in A \cap E\}$.*

We will need the following lemma:

LEMMA 2.7. *Let G be a group, $g, h, z \in G$ and $[h, g] = z, [g, z] = 1 = [h, z]$. Assume that $i, j \in \mathbb{N}$ and $\alpha \in \operatorname{Aut} G$. Then*

(1) $h^{g^{i-1}+\cdots+g+1} = h^i z^{\frac{i(i-1)}{2}}$;
(2) *if* $g^\alpha = g, h^\alpha = h^j, z^\alpha = z$, *then* $(h^{g^{i-1}+\cdots+g+1})^\alpha = h^{ij} z^{\frac{i(i-1)}{2}}$;
(3) *if* $g^\alpha = g, h^\alpha = h^j, z^\alpha = z^j$, *then* $(h^{g^{i-1}+\cdots+g+1})^\alpha = h^{ij} z^{j\frac{i(i-1)}{2}}$;
(4) *if* $g^\alpha = g^j, h^\alpha = h, z^\alpha = z^j$, *then* $(h^{g^{i-1}+\cdots+g+1})^\alpha = h^{i} z^{j\frac{i(i-1)}{2}}$;
(5) *if* $g^\alpha = g^j, h^\alpha = h, z^\alpha = z$, *then* $(h^{g^{i-1}+\cdots+g+1})^\alpha = h^{i} z^{\frac{i(i-1)}{2}}$.

By Lemmas 2.3, 2.4 and 2.7 we get

LEMMA 2.8. *Let G, N, M, E, g be as in Theorem 2.6 and $i, j \in \mathbb{N}$, $i = j^{-1} \bmod n$. Assume that $\lambda \in Z^1(G/N, Z(N))$, $(gN)^\lambda = h$ for some $h \in M$ and $\beta \in E$ is an automorphism determined by λ. Assume also that $\alpha \in \operatorname{Aut} G$, $[h, g] = z$ and $[g, z] = 1$. Then*

(1) *if* $g^\alpha = g^j, h^\alpha = h, z^\alpha = z^j$, *then* $((g^{\alpha^{-1}}N)^\lambda)^\alpha = h^i z^{j\frac{i(i-1)}{2}}$;
in particular if $z = 1$, *then* $\beta^\alpha = \beta^i$;
(2) *if* $g^\alpha = g^j, h^\alpha = h, z^\alpha = z$, *then* $((g^{\alpha^{-1}}N)^\lambda)^\alpha = h^i z^{\frac{i(i-1)}{2}}$;
in particular if $z = 1$, *then* $\beta^\alpha = \beta^i$;
(3) *if* $g^\alpha = g, h^\alpha = h^j$, *then* $((g^{\alpha^{-1}}N)^\lambda)^\alpha = h^j$ *and* $\beta^\alpha = \beta^j$.

3. A SPLIT METACYCLIC 2-GROUP

Let $G = H \rtimes K$ be a split metacyclic 2-group, where $H = \langle x \rangle$ and $K = \langle y \rangle$ and let A, B, C and D be the subgroups of $\operatorname{Aut} G$ defined in the introduction. In this section we refer to the appropiate cases of the split metacyclic 2-groups from [8], but occasionally we repeat some known results for readers' convenience. In fact we consider only Case A.

Let $G = H \rtimes K = \langle x, y \mid x^{2^m} = y^{2^n} = 1,\ x^y = x^{1+2^{m-r}} \rangle$, where $m \geqq 3,\ n \geqq 1,\ 1 \leqq r \leqq \min\{m-2, n\}$.

It is convenient to consider G in the following three subcases (see [8])

$$\text{(I)}\quad m \leqq n, \qquad \text{(II)}\quad n \leqq m-r < m, \qquad \text{(III)}\quad m-r < n < m.$$

Moreover there exist two special cases. They are case (II), when $m = 2r$, $n = r = m-r \geqq 2$ and $G = \langle x, y \mid x^{2^{2r}} = y^{2^r} = 1,\ x^y = x^{1+2^r} \rangle$ and case (III), when $r = n > m-n \geqq 2$ and $G = \langle x, y \mid x^{2^m} = y^{2^n} = 1,\ x^y = x^{1+2^{m-n}} \rangle$. These are referred to as exceptional cases. We will also need the following number theoretic result (see [8, 13]), which is easily established by induction.

LEMMA 3.1. *Let m, n and r be positive integers.*

(1) *For all* $m \geqq 2, n \geqq 1$, $(1+2^m)^{2^n} \equiv 1 + 2^{m+n} \pmod{2^{2m+n-1}}$
and $(1+2^m)^{2^{n-1}} \equiv 1 + 2^{m+n-1} \pmod{2^{m+n}}$.

(2) *For* $n \geqq 2, r \geqq 1$ *and* $m = n + r$, *let* $S = 1 + u + \cdots + u^{2^r-1}$, *where* $u \equiv 1 \pmod{2^n}$. *Then* $S \equiv 2^r + 2^{m-1} \pmod{2^m}$ *if* $u \not\equiv 1 \pmod{2^{n+1}}$ *and* $S \equiv 2^r \pmod{2^m}$ *if* $u \equiv 1 \pmod{2^{n+1}}$.

Using Lemma 3.1 the following lemmas are easily established.

LEMMA 3.2.
(1) $C_H(K) = \langle x^{2^r} \rangle$, (2) $C_K(H) = \langle y^{2^r} \rangle$,
(3) $G' = [H,K] = \langle x^{2^{m-r}} \rangle$, (4) G *is nil* 2 <=> $2r \leqq m$.

LEMMA 3.3. $\Omega_{m-r}(K), [H, \Omega_{m-r}(K)]$ *are given in the three cases as follows:*

(I) $\Omega_{m-r}(K) = \langle y^{2^{n-m+r}} \rangle \leqq Z(G)$, $[H, \Omega_{m-r}(K)] = 1$;
(II) $\Omega_{m-r}(K) = \langle y \rangle = C_K(G')$, $[H, \Omega_{m-r}(K)] = \langle x^{2^{m-r}} \rangle = G' \leqq Z(G)$;
(III) $\Omega_{m-r}(K) = \langle y^{2^{n-m+r}} \rangle \leqq C_K(G')$, $[H, \Omega_{m-r}(K)] = \langle x^{2^n} \rangle \leqq Z(G)$.

As in [14] when p was odd or by considering matrices of maps from [8] one could find the effect of an automorphism φ on the generators of G.

LEMMA 3.4. *Let* G, x, y *be as above.*

(1) *Assume that* $n \neq r$. *Then a map* $\varphi : G \to G$ *is an automorphism if and only if* $x^{-1}x^{\varphi} \in \mho_1(H)\Omega_{m-r}(K)$, $y^{\varphi}y^{-1} \in \Omega_n(H)C_K(H)$;
(2) *Assume that* $n = r$. *Then a map* $\varphi : G \to G$ *is an automorphism if and only if either* $x^{-1}x^{\varphi} \in \mho_1(H)\mho_1(\Omega_{m-r}(K)), y^{\varphi}y^{-1} \in \Omega_n(H)$ *or* $x^{-1}x^{\varphi} \in \mho_1(H)\Omega_{m-r}(K) \backslash \mho_1(H)\mho_1(\Omega_{m-r}(K)), y^{\varphi}y^{-1} \in \Omega_n(H)y^{2^{r-1}}$.

By Theorem 2.6 and the definitions of A, B and D we get the following lemma.

LEMMA 3.5. *Let* G, A, B, D *be as above. Then*

(1) $B \cong \mathrm{Aut}_H^H(G)$,
(2) $AD = A \times D$ *normalizes* B,
(3) $B \cap D = 1$.

For the proofs of Theorem 3.6 and Lemma 3.7 see [13].

THEOREM 3.6. *Let* G *be as above.*

(1) $\mathrm{Aut}\, G = C_{\mathrm{Aut}\, G}(H) C_{\mathrm{Aut}\, G}(K)$ *if and only if* $r \neq n$;
(2) $C_{\mathrm{Aut}\, G}(H) = BD$;
(3) $C_{\mathrm{Aut}\, G}(K) = AC$ *if and only if* $m \leqq n$.

We set $M := [H, \Omega_{m-r}(K)]\Omega_{m-r}(K)$, $N := G'K$ and

$$E := \{\varphi \in \mathrm{Aut}_N^N(G) \mid [H, \varphi] \subseteqq M\} \subseteqq \mathrm{Aut}_N^N(G).$$

LEMMA 3.7. *Let* G, M *be as above and* $n \neq r$.

(1) M *is abelian and normal in* G.
(2) *If* $a \in M$ *then* $a^{x^{2^{m-r}-1} + \cdots + x + 1} = 1$.

LEMMA 3.8. *Let G, A, D, E be as above and $n \neq r$. Then*
(1) $E \leqq \operatorname{Aut} G$; (2) $E \cong M$;
(3) $AD = A \times D$ *normalizes* E; (4) $E \cap A \cong [H, \Omega_{m-r}(K)]$;
(5) $C_{\operatorname{Aut} G}(K) = AE$; (6) $D \cong \operatorname{Aut}_{C_K(H)}(K)$.

PROOF. In the proof of Lemma 3.9 in [13] we put $\Omega_{m-r}(K)$ instead of $C_K(G')$. □

We define $\mathrm{c} \in \operatorname{Aut} G$ by setting $x^{\mathrm{c}} = xy$, when $m - r \geqq n \neq r$, and $x^{\mathrm{c}} = xy^{2^{n-m+r}}$, when $m - r < n \neq r$, $y^{\mathrm{c}} = y$. We also set $F := \langle \mathrm{c} \rangle \leqq E$.

THEOREM 3.9. *Let G, E, A, F be as above and $n \neq r$. Then*
(1) $F \cong \Omega_{m-r}(K)$, $AF = AE$ *and* $A \cap F = 1$;
(2) $\operatorname{Aut} G = BDAF$ *and* $|\operatorname{Aut} G| = |B||D||A||F|$.

PROOF. In the proof of Theorem 3.10 in [13] we put $\Omega_{m-r}(K)$ instead of $C_K(G')$. □

By Theorem 3.9 and Lemma 3.4 it is obvious that

THEOREM 3.10. *Let G, A, B, D, F, T be as above. Then*
(1) $A \cong AutH \cong C_2 \times C_{2^{m-2}}$ *and* $B \cong \Omega_n(H) \cong C_{2^{\min\{m,n\}}}$;
(2) $D \cong C_K(H) \cong C_{2^{n-r}}$ *except if* $n > 1 = r$ *when* $D \cong AutK \cong C_2 \times C_{2^{n-2}}$;
(3) *If* $n \neq r$, *then* $F \cong \Omega_{m-r}(K) \cong C_{2^{\min\{m-r,n\}}}$;
(4) *Assume that* $n = r$. *Then* $T \cong \Omega_{m-r}(K) \cong C_{2^{\min\{m-r,n\}}}$ *except if* $r = 2$ *when* $T \cong C_2 \times C_2$.

We define automorphisms of G on generators as follows

$$x^{\mathrm{a}_1} = x^{-1}, \quad x^{\mathrm{a}_2} = x^5, \quad y^{\mathrm{a}_1} = y^{\mathrm{a}_2} = y;$$

$$x^{\mathrm{b}} = x, \quad y^{\mathrm{b}} = \begin{cases} xy, & n \geqq m \\ x^{2^{m-n}}y, & n < m \end{cases};$$

$$x^{\mathrm{c}} = \begin{cases} xy, & m - r \geqq n, \\ xy^{2^{n-m+r}}y, & m - r < n \end{cases}, \quad y^{\mathrm{c}} = y.$$

Now we assume that $n \neq r$ and $r \geqq 2$. In this case we define

$$x^{\mathrm{d}} = x, \quad y^{\mathrm{d}} = y^{1+2^r}.$$

By Theorem 3.6, 3.9 and Lemma 3.8 it is clear that $\operatorname{Aut} G = FABD$ and each automorphism φ of G can be presented uniquely as $\varphi = \alpha\beta\gamma\delta$, where $\alpha \in F, \beta \in A, \gamma \in B, \delta \in D$. It is clear that $A = \langle \mathrm{a}_1, \mathrm{a}_2 \rangle$, $B = \langle \mathrm{b} \rangle$, $D = \langle \mathrm{d} \rangle$ and AD is abelian. It is evident that $G = HK = KH$, so if $g \in G$, then $g = kh$ for some $k \in K, h \in H$. In the proof of Lemma 3.11 (2) we will use this reverse notation of elements of G.

We define i, j, k, s, t, u, w, z are such that

$i = 0$ in (I), $5^i = 1 + 2^{m-r} \bmod 2^m$ in (II), $5^i = 1 + 2^n \bmod 2^m$ in (III),

$j = 0$ in (I), $5^j = 1 - 2^{m-r+1} \bmod 2^m$ in (II),

$5^j = 1 - 2^{n+1} \bmod 2^m$ in (III),

$k = 1 + 2^r + 2^{m-1}$ in (I), $k = 1 + 2^r$ in (II)&(III),

$u = 1 - 2^{n-m+r}$ in (I), $u = 1 - 2^{m-n}$ in (II), $u = 1 - 2^r$ in (III),

$5^t = (1 - 2^{n-1})u^{-1} \mod 2^n$ in (I),

$5^t = (1 - 2^{2m-r-n-1})u^{-1} \mod 2^m$ in (II),

$5^t = (1 - 2^{m-1})u^{-1} \mod 2^m$ in (III),

$s = u^{-1} \mod 2^n$ in (I), $s = u^{-1} \mod 2^m$ in (II)&(III),

$(1 + 2^r)^w = u \mod 2^n$,

$z = -2^{n-m+r} + 2^{n-1}$ in (I), $z = -2^{m-n} + 2^{m-r+1}$ in (II),

$z = -2^r + 2^{n-1}$ in (III).

LEMMA 3.11. *Let* $\mathrm{a}_1, \mathrm{a}_2, \mathrm{b}, \mathrm{c}, \mathrm{d}$ *be as above. Assume that* $n \neq r$ *and* $r \geqq 2$. *Then*

(1) $\mathrm{c}^{\mathrm{a}_1} = \mathrm{c}^{-1} a_2^i$, $\mathrm{c}^{\mathrm{a}_2^{-1}} = \mathrm{c}^5 a_2^j, \mathrm{c}^{\mathrm{d}} = \mathrm{c}^{1+2^r}$;
(2) $\mathrm{b}^{\mathrm{a}_1} = \mathrm{b}^{-1}$, $\mathrm{b}^{\mathrm{a}_2} = \mathrm{b}^5, \mathrm{b}^{\mathrm{d}^{-1}} = \mathrm{b}^k$;
(3) $\mathrm{c}^{\mathrm{b}} = \mathrm{c}^s {\mathrm{a}_2}^t \mathrm{b}^z \mathrm{d}^w$.

PROOF. (1) Let $N = G'K$ and $M = [H, \Omega_{m-r}(K)]\Omega_{m-r}(K)$. Then $\mathrm{a}_1, \mathrm{a}_2, \mathrm{d} \in N_{\mathrm{Aut}\, G}(N) \cap N_{\mathrm{Aut}\, G}(M)$, $\mathrm{c} \in \mathrm{Aut}_N^N(G)$ and $h := x^{-1}x^{\mathrm{c}} \in M$. By Lemmas 2.1, 2.4, 3.7 and 3.3 we get $g^{\mathrm{c}} = g(gN)^\lambda$ $(g \in G)$, $(x^i N)^\lambda = h^{x^{i-1}+\ldots+x+1}$ $(i \in \mathbb{N})$. By Lemma 2.8 (3) we get the last relation. Now we use Lemma 2.8 (1) to get the first two relations: in (I) we have $[h, x] = [y^{2^{n-m+r}}, x] = 1$; in (II) since $[h, x] = [y, x] = x^{-2^{m-r}}$, we obtain

$$((x^{\mathrm{a}_1{}^{-1}} N)^\lambda)^{\mathrm{a}_1} = y^{-1} x^{2^{m-r}(2^m-1)(2^{m-1}-1)} = y^{-1} x^{2^{m-r}},$$
$$((x^{\mathrm{a}_2} N)^\lambda)^{\mathrm{a}_2{}^{-1}} = y^5 x^{-2^{m-r+1}};$$

in (III) since $[h, x] = [y^{2^{n-m+r}}, x] = x^{-2^n}$, by Lemma 2.8 (1) we obtain

$$((x^{\mathrm{a}_1{}^{-1}} N)^\lambda)^{\mathrm{a}_1} = x^{2^n} y^{-2^{n-m+r}}, \quad ((x^{\mathrm{a}_2} N)^\lambda)^{\mathrm{a}_2{}^{-1}} = x^{-2^{n+1}} y^{5\cdot 2^{n-m+r}}.$$

(2) Note that $x^{\mathrm{b}} = x$ and $y^{\mathrm{b}} = yx^{1+2^{m-r}}$ in (I), $y^{\mathrm{b}} = yx^{2^{m-n}}$ in (II), $y^{\mathrm{b}} = yx^{2^{m-n}+2^{2m-n-r}}$ in (III). Let $Q = \langle x \rangle$. Then $\mathrm{a}_1, \mathrm{a}_2, \mathrm{d} \in N_{\mathrm{Aut}\, G}(Q), \mathrm{b} \in \mathrm{Aut}_Q^Q(G)$ and $h := y^{-1}y^{\mathrm{b}} \in Q$. By Lemmas 2.1, 2.4, 3.7 and 3.3 we get $g^{\mathrm{b}} = g(gQ)^\lambda$ $(g \in G)$, $(y^i Q)^\lambda = h^{y^{i-1}+\ldots+y+1}$ $(i \in \mathbb{N})$. By Lemma 2.8 (3)

we obtain the first two relations. Now we use Lemma 2.8 (2) to get the last relation: in (I) since $[h,y]=[x^{1+2^{m-r}},y]=x^{2^{m-r}(1+2^{m-r})}$ we obtain

$$\begin{aligned}((y^{\mathrm{d}}N)^{\lambda})^{\mathrm{d}^{-1}} &= (x^{1+2^{m-r}})^{1+2^r}\cdot x^{2^{m-r}(1+2^{m-r})2^{r-1}(2^r+1)}\\ &= x^{(1+2^{m-r})(1+2^r+2^{m-1})};\end{aligned}$$

in (II) we get $[h,y]=[x^{2^{m-n}},y]=1$; in (III) since $[h,y]=[x^{2^{m-n}+2^{2m-n-r}},y]$ $=x^{2^{m-r}(2^{m-n}+2^{2m-n-r})}$ we obtain

$$\begin{aligned}((y^{\mathrm{d}}N)^{\lambda})^{\mathrm{d}^{-1}} &= (x^{2^{m-n}+2^{2m-n-r}})^{1+2^r}x^{2^{m-r}(2^{m-n}+2^{2m-n-r})(2^r+1)2^{r-1}}\\ &= (x^{2^{m-n}+2^{2m-n-r}})^{1+2^r}.\end{aligned}$$

(3) The direct computations with the help of Lemma 3.1 give the relation. □

Theorem 3.12. *Let G be as above and $m \geqq 3$, $n \geqq 1$, $1 \leqq r \leqq \min\{m-2,n\}$, $n \neq r$ and $r \geqq 2$. Then* $\operatorname{Aut}G$ *can be given by the following presentation, where the relations with commuting generators are omitted:* $\operatorname{Aut}G=\langle \mathrm{a}_1,\mathrm{a}_2,\mathrm{b},\mathrm{c},\mathrm{d}\mid {\mathrm{a}_1}^2={\mathrm{a}_2}^{2^{m-2}}=\mathrm{b}^{2^{\min\{m,n\}}}=\mathrm{c}^{2^{\min\{m-r,n\}}}=\mathrm{d}^{2^{n-r}}=1, \mathrm{c}^{\mathrm{a}_1}=\mathrm{c}^{-1}a_2^i,\ \mathrm{c}^{\mathrm{a}_2^{-1}}=\mathrm{c}^5a_2^j, \mathrm{c}^{\mathrm{d}}=\mathrm{c}^{1+2^r}, \mathrm{b}^{\mathrm{a}_1}=\mathrm{b}^{-1},\ \mathrm{b}^{\mathrm{a}_2}=\mathrm{b}^5, \mathrm{b}^{\mathrm{d}^{-1}}=\mathrm{b}^k, \mathrm{c}^{\mathrm{b}}=\mathrm{c}^s{\mathrm{a}_2}^t\mathrm{b}^z\mathrm{d}^w\rangle$.

4. Metacyclic minimal nonabelian 2-groups

In this section we will deal with groups $G=\langle x,y\mid x^{2^m}=y^{2^n}=1,\ x^y=x^{1+2^{m-1}}\rangle$; where $m,n\in\mathbb{N},\ m\geqq 2, n\geqq 1$. So $G=H\rtimes K$ is a split metacyclic 2-group, where $H=\langle x\rangle$ and $K=\langle y\rangle$.

First assume that $n \geqq m \geqq 3$. We define automorphisms of G on generators as follows

$$x^{\mathrm{a}_1}=x^{-1},\quad x^{\mathrm{a}_2}=x^5,\quad y^{\mathrm{a}_1}=y^{\mathrm{a}_2}=y;$$

$$x^{\mathrm{b}}=x,\quad y^{\mathrm{b}}=\begin{cases} xy, & n\geqq m\\ x^{2^{m-n}}y, & n<m\end{cases};$$

$$x^{\mathrm{c}}=\begin{cases} xy, & m>n\\ xy^{2^{n-m+1}}y, & m\leqq n\end{cases},\quad y^{\mathrm{c}}=y;$$

$$x^{\mathrm{d}_1}=x^{d_2}=x,\quad y^{\mathrm{d}_1}=y^{-1},\quad y^{\mathrm{d}_2}=y^5.$$

By Theorems 3.6, 3.9 and Lemma 3.8 it is clear that $\operatorname{Aut}G=FABD$ and each automorphism φ of G can be presented uniquely as $\varphi=\alpha\beta\gamma\delta$, where $\alpha\in F,\beta\in A,\gamma\in B,\delta\in D$. It is clear that $A=\langle \mathrm{a}_1,\mathrm{a}_2\rangle$, $B=\langle \mathrm{b}\rangle$, $D=\langle \mathrm{d}_1,\mathrm{d}_2\rangle$ and AD is abelian. It is evident that $G=HK=KH$, so if $g\in G$, then $g=kh$ for some $k\in K,h\in H$. In the proof of Lemma 4.1(2) we will use also this reverse notation of elements of G.

LEMMA 4.1. *Let* a_1, a_2, b, c, d_1, d_2 *be as above. Assume that* $m \geqq 3$, $n \geqq 3$. *Then*

(1) $c^{a_1} = c^{-1}{a_2}^i$, $c^{{a_2}^{-1}} = c^5, c^{d_1} = c^{-1}, c^{d_2} = c^5$, *where* $i = 0$ *when* $m > n$ *and* $i = 2^{m-3}$ *when* $m \leqq n$;

(2) $b^{a_1} = b^{-1}$, $b^{a_2} = b^5, b^{d_1} = b^{-1}, b^{{d_2}^{-1}} = b^5$;

(3) *if* $n - m \geq 1$, *then* $c^b = c^s {a_2}^t b^{-2^{n-m+1}} {d_2}^w$, *where* s, t, w *are such that* $s = 5^t = (1 - 2^{n-m+1})^{-1} \mod 2^m$, $5^w = 1 - 2^{n-m+1} \mod 2^n$;

(4) *if* $m = n$, *then* $c^b = c^{-1} a_1 {a_2}^{2^{m-3}} b^{-2+2^{m-1}} d_1$;

(5) *if* $m - n > 1$, *then* $c^b = c^s {a_2}^t b^{-2^{m-n}} d_2^w$, *where* s, t, w *are such that* $s = 5^t = (1 - 2^{m-n})^{-1} \mod 2^m$, $5^w = 1 - 2^{m-n} \mod 2^n$;

(6) *if* $m = n + 1$, *then* $c^b = c^{-1} a_1 {a_2}^{2^{m-3}} b^{-2+2^{m-2}} d_1$.

PROOF. (1) Let $N = G'K$ and $M = [H, \Omega_{m-r}(K)]\Omega_{m-r}(K)$. Then $a_k, d_k \in N_{\mathrm{Aut}\, G}(N) \cap N_{\mathrm{Aut}\, G}(M)$ $(k = 1, 2)$, $c \in \mathrm{Aut}_N^N(G)$ and $h := x^{-1}x^c \in M$. By Lemmas 2.1, 2.4, 3.7 and 3.3 we get $g^c = g(gN)^\lambda$ $(g \in G)$, $(x^i N)^\lambda = h^{x^{i-1} + \ldots + x + 1}$ $(i \in \mathbb{N})$. For the first two relations see the proof of Lemma 3.11 (1) with $r = 1$. By Lemma 2.8 (3) we obtain the last two relation.

(2) Note that $x^b = x$ and $y^b = yx^{1+2^{m-1}}$ when $n \geqq m$, $y^b = yx^{2^{m-n}}$ when $m > n$. Let $Q = \langle x \rangle$. Then $a_k, d_k \in N_{\mathrm{Aut}\, G}(Q)$ $(k = 1, 2)$, $b \in \mathrm{Aut}_Q^Q(G)$ and $y^{-1}y^b \in Q$. By Lemmas 2.1, 2.4, 3.7 and 3.3 we get $g^b = g(gQ)^\lambda$ $(g \in G)$, $(y^i Q)^\lambda = h^{y^{i-1} + \ldots + y + 1}$ $(i \in \mathbb{N})$. By Lemma 2.8 (3) we obtain the first two relations. Now we will use Lemma 2.8 (2) to get the last two relations: if $m > n$ then $[y^{2^{m-n}}, y] = 1$, so we get the last two relations; if $m \leqq n$, then $[x^{1+2^{m-1}}, y] = x^{2^{m-1}}$ and we get $((y^{d_1^{-1}} N)^\lambda)^{d_1} = (x^{1+2^{m-1}})^{2^n - 1} x^{2^{m-1}(2^n-1)(2^{n-1}-1)} = x^{-1}$ and $((y^{d_2} N)^\lambda)^{d_2^{-1}} = (x^{1+2^{m-1}})^5 x^{2^{m-1} 10} = (x^{1+2^{m-1}})^5$.

(3)-(6) The direct computations with the help of Lemma 3.1 give the relations. ∎

In the next theorems the relations with commuting generators are omitted.

THEOREM 4.2. *Let* G *be as above and* $m, n \geqq 3$. *Then* $\mathrm{Aut}\, G$ *can be given by the following presentation:* $\mathrm{Aut}\, G = \langle a_1, a_2, b, c, d_1, d_2 \mid {a_1}^2 = {a_2}^{2^{m-2}} = b^{2^{\min\{m,n\}}} = c^{2^{\min\{m-r,n\}}} = d^{2^{n-r}} = 1, c^{a_1} = c^{-1}{a_2}^i$, $c^{{a_2}^{-1}} = c^5, c^{d_1} = c^{-1}, c^{d_2} = c^5$, $b^{a_1} = b^{-1}$, $b^{a_2} = b^5, b^{d_1} = b^{-1}, b^{{d_2}^{-1}} = b^5$, $c^b = \alpha\rangle$, *where* i *is given in Lemma 4.1 and* α *is the appropriate relation in (3)-(4) of Lemma 4.1.*

If $m = 2$ and $n = 1$, then $G \cong \mathrm{Aut}\, G$ is dihedral of order 8.

Now assume that $m > n = 2$. We define automorphisms of G on generators as follows

$$x^{\mathrm{a}_1} = x^{-1}, \quad x^{\mathrm{a}_2} = x^5, \quad y^{\mathrm{a}_1} = y^{\mathrm{a}_2} = y; \quad x^{\mathrm{b}} = x, \quad y^{\mathrm{b}} = x^{2^{m-2}}y;$$
$$x^{\mathrm{c}} = xy, \quad y^{\mathrm{c}} = y; \quad x^{\mathrm{d}} = x, \quad y^{\mathrm{d}} = y^{-1}.$$

THEOREM 4.3. *Let G be as above and $m > n = 2$. Then* $\mathrm{Aut}\, G$ *can be given by the following presentation:*

(1) *if $m > 3$, then* $\mathrm{Aut}\, G = \langle \mathrm{a}_1, \mathrm{a}_2, \mathrm{b}, \mathrm{c}, \mathrm{d} \mid {\mathrm{a}_1}^2 = {\mathrm{a}_2}^{2^{m-2}} = \mathrm{b}^4 = \mathrm{c}^4 = \mathrm{d}^2 = 1, \mathrm{c}^{\mathrm{a}_1} = \mathrm{c}^{-1}{\mathrm{a}_2}^{2^{m-3}},\ \mathrm{c}^{\mathrm{d}} = \mathrm{c}^{-1}, \mathrm{b}^{\mathrm{a}_1} = \mathrm{b}^{-1}, \mathrm{b}^{\mathrm{d}} = \mathrm{b}^{-1}, \mathrm{b}^{\mathrm{c}} = \mathrm{b}{\mathrm{a}_2}^t \rangle$, *where* $5^t = 1 - 2^{m-2} \bmod 2^m$*;*

(2) *if $m = 3$, then* $\mathrm{Aut}\, G = \langle \mathrm{a}_1, \mathrm{a}_2, \mathrm{b}, \mathrm{c}, \mathrm{d} \mid {\mathrm{a}_1}^2 = {\mathrm{a}_2}^2 = \mathrm{b}^4 = \mathrm{c}^4 = \mathrm{d}^2 = 1, \mathrm{c}^{\mathrm{a}_1} = \mathrm{c}^{-1}\mathrm{a}_2,\ \mathrm{c}^{\mathrm{d}} = \mathrm{c}^{-1}, \mathrm{b}^{\mathrm{a}_1} = \mathrm{b}^{-1}, \mathrm{b}^{\mathrm{d}} = \mathrm{b}^{-1}, \mathrm{c}^{\mathrm{b}} = \mathrm{c}^{-1}\mathrm{a}_1\mathrm{a}_2\mathrm{d} \rangle$.

Now assume that $m \geqq 3$, $n = 1$. We define automorphisms of G on generators as follows

$$x^{\mathrm{a}_1} = x^{-1}, \quad x^{\mathrm{a}_2} = x^5, \quad y^{\mathrm{a}_1} = y^{\mathrm{a}_2} = y;$$
$$x^{\mathrm{b}} = x, \quad y^{\mathrm{b}} = x^{2^{m-1}}y; \quad x^{\mathrm{c}} = xy, \quad y^{\mathrm{c}} = y.$$

THEOREM 4.4. *Let G be as above and $m \geqq 3, n = 1$. Then* $\mathrm{Aut}\, G$ *can be given by the following presentation:* $\mathrm{Aut}\, G = \langle \mathrm{a}_1, \mathrm{a}_2, \mathrm{b}, \mathrm{c} \mid {\mathrm{a}_1}^2 = {\mathrm{a}_2}^{2^{m-2}} = \mathrm{b}^2 = \mathrm{c}^2 = 1, \mathrm{c}^{\mathrm{a}_1} = \mathrm{c}{\mathrm{a}_2}^{2^{m-3}},\ \mathrm{c}^{\mathrm{b}} = \mathrm{c}{\mathrm{a}_2}^{2^{m-3}} \rangle$.

Now assume that $m = n = 2$. We define automorphisms of G on generators as follows

$$x^{\mathrm{a}} = x^{-1}, \quad y^{\mathrm{a}} = y; \quad x^{\mathrm{b}} = x, \quad y^{\mathrm{b}} = xy;$$
$$x^{\mathrm{c}} = xy^2, \quad y^{\mathrm{c}} = y; \quad x^{\mathrm{d}} = x, \quad y^{\mathrm{d}} = y^{-1}.$$

THEOREM 4.5. *Let G be as above and $m = n = 2$. Then* $\mathrm{Aut}\, G$ *can be given by the following presentation:* $\mathrm{Aut}\, G = \langle \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d} \mid \mathrm{a}^2 = \mathrm{b}^4 = \mathrm{c}^2 = \mathrm{d}^2 = 1, \mathrm{b}^{\mathrm{a}} = \mathrm{b}^{-1},\ \mathrm{b}^{\mathrm{c}} = \mathrm{bd} \rangle$.

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