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A NOTE ON CHARACTER SQUARE

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Abstract. We study the finite groups with an irreducible character χ satisfying the following hypothesis: χ^2 has exactly two distinct irreducible constituents, and one of which is linear, and then obtain a result analogous to the Zhmud's ([8]).

1. Introduction

Let χ be a character of a finite group G. Clearly χ^2 is a character of G, and if χ^2 has few irreducible constituents then the group structure is necessary restricted. For example, Isaacs and Zisser ([5]) have studied the groups G with a faithful irreducible character χ such that $\chi^2 = a\psi$ or $\chi^2 = a\psi + b\overline{\psi}$, where a, b are positive integers, $\psi \in \mathrm{Irr}(G)$ and $\overline{\psi}$ is the complex conjugate of ψ; while Zhmud ([8]) has considered the finite groups with a faithful irreducible character χ satisfying the following hypothesis.

Hypothesis 1. *χ is real, and χ^2 has just two distinct irreducible constituents.*

Theorem A ([8]). *Let G be a finite group with a faithful irreducible character χ satisfying Hypothesis 1. Then $G \in \{SL(2,3), \widehat{S_4}, SL(2,5)\}$, where $\widehat{S_4}$ is one of the two representation groups of the symmetric group S_4 with generalized quaternion Sylow 2-subgroup.*

Observe that $\chi \in \mathrm{Irr}(G)$ is real if and only if $[\chi^2, 1_G] = 1$. Hence the above hypothesis is equivalent to the following one, there exist a positive integer b and an irreducible character ψ of G such that $\chi^2 = 1_G + b\psi$.

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In this note, we will study the finite groups G with a faithful $\chi \in \mathrm{Irr}(G)$ satisfying the following hypothesis.

HYPOTHESIS 2. *χ^2 has just two distinct irreducible constituents, and one of which is linear.*

THEOREM B. *A finite group G has a faithful irreducible character χ satisfying Hypothesis 2 if and only if $G' \cap Z(G) \cong \mathbb{Z}_2$ and G has a faithful and primitive irreducible character of degree 2 .*

Note that if a finite group G has a faithful and primitive irreducible character of degree 2, then $Z(G)$ is cyclic and (see [1, Theorem 18.1]) $G/Z(G)$ is isomorphic to A_4, S_4 or $PSL(2,5)$, also G can be viewed as a finite subgroup of the general linear group $GL(2,\mathbb{C})$. A classification of such groups can be found in [2, Section 3] or [3] for example. It is clear that there are many groups satisfying the conditions in Theorem B.

EXAMPLE 1.1. Let $H \cong GL(2,3)$ be one of the two representation groups of S_4 with semidihedral Sylow 2-subgroup. Let us present an irreducible character χ of H satisfying Hypothesis 1.2. It follows from the character table of H that it has three irreducible characters χ, λ and ψ, taking on the corresponding G-classes the following values:

$$\chi : 2, 0, -1, 0, -2, i\sqrt{2}, 1, -i\sqrt{2},$$
$$\lambda : 1, -1, 1, 1, 1, -1, 1, -1,$$
$$\psi : 3, 1, 0, -1, 3, -1, 0, -1.$$

Then

$$\chi^2 : 4, 0, 1, 0, 4, -2, 1, -2, \text{ and so } \chi^2 = \lambda + \psi.$$

Throughout this note, G always denotes a finite group, all characters are complex characters. For a character χ of a finite group G, we denote by $\mathrm{Irr}(\chi)$ the set of irreducible constituents of χ. In general, we use Isaacs [4] as a source for standard notations and results from character theory.

2. PROOFS

For any irreducible character χ of G, we define $\chi^{(2)}$ by $\chi^{(2)}(g) = \chi(g^2)$ for any $g \in G$. It is known that $\chi^{(2)}$ is a generalized character of G and that $\mathrm{Irr}(\chi^{(2)}) \subseteq \mathrm{Irr}(\chi^2)$ (see [1, §4.6]).

LEMMA 2.1 ([7, Lemma 1.1]). *Let $\chi \in \mathrm{Irr}(G)$ and g be a p-element of G for some prime p. If $|\chi(g)|^2 = a$ is a rational integer, then p divides $\chi(1)^2 - a$. In particular, if $\chi(g) = 0$ then p divides $\chi(1)$; and if $|\chi(g)| = 1$ then p does not divide $\chi(1)$.*

LEMMA 2.2. *Let G be a finite group with an irreducible character χ (not necessary faithful) satisfying Hypothesis 2. Then the following statements hold.*

(1) $\chi^2 = \lambda + \psi$, *where* $\lambda, \psi \in \mathrm{Irr}(G)$, *and* $\lambda(1) = 1$.
(2) *The restriction of* χ *on* G' *is irreducible.*

PROOF. (1) By the hypothesis, there exist positive integers a, b and $\lambda, \psi \in \mathrm{Irr}(G)$ such that
$$\chi^2 = a\lambda + b\psi,\ \lambda(1) = 1.$$
Since λ is linear and $0 < [\chi^2, \lambda] = [\chi, \lambda\overline{\chi}]$, we have $a = [\chi^2, \lambda] = [\chi, \lambda\overline{\chi}] = 1$. Clearly χ is nonlinear, and it follows by Burnside's Theorem ([4, Theorem 3.15]) that $\chi(g_0) = 0$ for some $g_0 \in G$. Then $0 = \chi^2(g_0) = \lambda(g_0) + b\psi(g_0)$, hence the algebraic integer
$$\overline{\lambda(g_0)}\psi(g_0) = -\overline{\lambda(g_0)}\lambda(g_0)/b = -1/b,$$
and this implies that $b = 1$, so $\chi^2 = \lambda + \psi$ as desired.

(2) Suppose that the restriction of χ on G' is reducible. By [4, Theorem 6.22], there exists a normal subgroup M of G such that $|G : M|$ is a prime p and χ is induced by some $\chi_1 \in \mathrm{Irr}(M)$. Since $M \trianglelefteq G$, by the definition of induced character, we have $\chi(g) = 0$ whenever $g \in G - M$, and hence
$$\psi(g) = \chi^2(g) - \lambda(g) = -\lambda(g)$$
is a root of unit. Let $\theta = \psi_{_M}$. Then
$$[\theta, \theta] = \frac{\sum_{g\in G} |\psi(g)|^2 - \sum_{g\in G-M} |\psi(g)|^2}{|M|} = \frac{|G| - (|G| - |M|)}{|M|} = 1,$$
hence θ is irreducible. Write $\chi_{_M} = \chi_1 + \cdots + \chi_p$, where $\chi_1, \cdots, \chi_p$ are distinct G-conjugates of χ_1. Observe that
$$\cup_{1\le i,j\le p}\mathrm{Irr}(\chi_i\chi_j) = \mathrm{Irr}((\chi_{_M})^2) = \mathrm{Irr}((\chi^2)_{_M}) = \mathrm{Irr}(\lambda_{_M} + \psi_{_M}) = \{\lambda_{_M}, \theta\}.$$
It follows that for some $i, j \in \{1, \cdots, p\}$, $[\theta, \chi_i\chi_j] > 0$, and then
$$\psi(1) = \theta(1) \le \chi_i(1)\chi_j(1) = \chi(1)^2/p^2 = (1 + \psi(1))/p^2,$$
a contradiction. Hence $\chi_{_{G'}}$ is necessary irreducible. ∎

The following result is due to E.M. Zhmud (see [8, page 89]), we restate its short proof for the reader's convenience.

LEMMA 2.3 ([8]). *Suppose that a finite group* G *has an irreducible character* χ *such that* $\chi^2 = 1_G + \psi$ *for some* $\psi \in \mathrm{Irr}(G)$. *Then* $\chi(1) = 2$.

PROOF. As $\mathrm{Irr}(\chi^{(2)}) \subseteq \mathrm{Irr}(\chi^2)$, we have $\chi^{(2)} = m1_G + n\psi$, $m, n \in \mathbb{Z}$. Clearly χ is real, hence $m = [\chi^{(2)}, 1_G] = \pm 1$ by Theorem 4.6.2 in [1]. Then
$$1 + \psi(1) = \chi^2(1) = (\chi^{(2)}(1))^2 = (m + n\psi(1))^2 = 1 + 2mn\psi(1) + n^2\psi^2(1),$$
so $2mn + n^2\psi(1) = 1$. It follows that $n = \pm 1$, hence $\psi(1) = (1 - 2mn)/n^2 = 3$, and $\chi(1) = 2$. ∎

LEMMA 2.4. *Let* G *be a finite group with an irreducible character* χ *(not necessary faithful) satisfying Hypothesis 2. Then* $\chi(1) = 2$.

PROOF. By Lemma 2.2,

$$\chi^2 = \lambda + \psi,$$

where $\lambda, \psi \in \mathrm{Irr}(G)$, and $\lambda(1) = 1$. We work by induction first on $|G : ker\lambda|$ and second on $|G|$.

If $\lambda = 1_G$, then by Lemma 2.3, $\chi(1) = 2$ and we are done.

Now suppose that $\lambda \neq 1_G$. Let $N = ker\lambda$, then $N < G$. Write $|G/N| = 2^u v$, where v is odd. Let

$$e = (v-1)/2,\ \chi_0 = \chi\lambda^e,\ \lambda_0 = \lambda^v,\ \psi_0 = \psi\lambda^{2e}.$$

Then

$$\lambda^{2e+1} = \lambda_0,\ \chi_0^2 = (\chi\lambda^e)^2 = \chi^2\lambda^{2e} = \lambda^{2e+1} + \psi\lambda^{2e} = \lambda_0 + \psi_0.$$

Hence we may replace χ, λ, ψ by χ_0, λ_0, ψ_0 respectively. Since $|G : ker\lambda_0| = 2^u$, by induction we may assume that G/N is a cyclic 2-group. Let M be a maximal subgroup of G with $N \leq M < G$, and let t be a 2-element outside M.

We claim that ψ_M is irreducible. Suppose that this is not true. Since $|G : M| = 2$, ψ vanishes on $G - M$ by [4, Lemma 2.29], and in particular $\psi(t) = 0$. Then $\chi^2(t) = \lambda(t) + \psi(t) = \lambda(t)$. Now $\chi(t)$ is a root of unit, it follows by Lemma 2.1 that $\chi(1)^2 - 1$ is even, so $\chi(1)$ is odd. By [6, Theorem A], we may take an odd order element $g \in G$ (and hence $g \in N$) such that $\chi(g) = 0$. Observe that $\chi(g) = 0$ is equivalent to $\chi(g^2) = 0$ for the odd order element g. Since $\mathrm{Irr}(\chi^{(2)}) \subseteq \mathrm{Irr}(\chi^2)$, we may write the generalized character $\chi^{(2)}$ as

$$\chi^{(2)} = m\lambda + n\psi,$$

where m, n are integers. Then

$$0 = \chi(g^2) = \chi^{(2)}(g) = m\lambda(g) + n\psi(g) = m + n\psi(g),$$

$$0 = \chi^2(g) = \lambda(g) + \psi(g) = 1 + \psi(g).$$

Now we have

$$m = n,\ \chi(1) = \chi^{(2)}(1) = m + m\psi(1),\ m \geq 1.$$

However $\chi^2(1) = 1+\psi(1)$, and this implies that $\chi^2(1) \leq \chi(1)$, a contradiction. Hence ψ_M is irreducible as claimed.

Now

$$(\chi_M)^2 = \chi_M^2 = \lambda_M + \psi_M.$$

Since χ_M is irreducible by Lemma 2.2, we may replace G, χ, λ, ψ by M, χ_M, λ_M and ψ_M respectively. Clearly $|M : ker\lambda_M| \leq |G : ker\lambda|$ and $|M| < |G|$, it follows by induction that $\chi(1) = \chi_M(1) = 2$, and we are done. $\square$

In what follows, we will use the following easy results: (1) $\mathrm{cd}(A_4) = \{1, 3\}$, $\mathrm{cd}(S_4) = \{1, 2, 3\}$, $\mathrm{cd}(PSL(2, 5)) = \{1, 3, 4, 5\}$; (2) if D is a normal subgroup of G with $G' \cap D = 1$, then any irreducible (linear) character of D is extendible to G.

PROOF OF THE NECESSITY IN THEOREM B. Let χ be a faithful irreducible character of G satisfying Hypothesis 2. By Lemma 2.2 and 2.4, $\chi(1) = 2$ and $\chi_{G'}$ is irreducible.

We claim that χ is primitive. Otherwise, $\chi = \vartheta^G$, where $\vartheta \in \mathrm{Irr(H)}$, $H < G$. Since $\chi(1) = 2$, we have $|G : H| = 2$ and hence $G' \leq H$. Since $\chi_{G'}$ is irreducible, χ_H is also irreducible, this is a contradiction. Thus χ is primitive as claimed.

Now by [1, Theorem 18.1] $G/Z(G)$ is isomorphic to A_4, S_4 or $PSL(2, 5)$. We first assume that $G' \cap Z(G) = 1$. Suppose that $G/Z(G) \cong A_4$. Then $G' \cong (G' \times Z(G))/Z(G) = (G/Z(G))' \cong (A_4)'$ is abelian, and this contradicts the fact that $\chi_{G'}$ is irreducible and of degree 2. Suppose that $G/Z(G) \cong S_4$. Since $(G/Z(G))' = A_4$, we have $G' \cong A_4$. But $\mathrm{cd}(A_4) = \{1, 3\}$, this violates the fact that $\chi(1) = 2$ and $\chi_{G'}$ is irreducible. Suppose that $G/Z(G) \cong PSL(2, 5)$, then $G = PSL(2, 5) \times Z(G)$, this is impossible since $\chi(1) = 2$. Therefore $G' \cap Z(G) > 1$.

Let $\chi_{G' \cap Z(G)} = 2\mu$ where $\mu \in \mathrm{Irr}(G' \cap Z(G))$, and let $1 \neq g \in G' \cap Z(G)$. Clearly $Z(G)$ is cyclic and μ is faithful. Then $4 = |\chi^2(g)| = |\lambda(g) + \psi(g)| = |1 + \psi(g)|$. Since $\psi(g)$ is the sum of three roots of unity, we have $\psi(g) = 3$ and $\chi(g) = -2$. So $\mu(g) = -1$, and hence $g^2 \in ker\mu = 1$, that is, $G' \cap Z(G) \cong \mathbb{Z}_2$. □

PROOF OF THE SUFFICIENCY IN THEOREM B. Suppose that a finite group G has a faithful and primitive irreducible character χ of degree 2 and that $G' \cap Z(G) \cong \mathbb{Z}_2$. Then $Z(G)$ is cyclic (see [4, Theorem 2.32]) and $G/Z(G)$ is isomorphic to A_4, S_4 or $PSL(2, 5)$ (see [1, Theorem 18.1]). Write $Z(G) = Z$, $G' \cap Z = D$, and let $\chi_Z = 2\mu$ where $\mu \in \mathrm{Irr}(Z)$ is faithful. Clearly $\chi_Z^2 = 4\mu^2$ and μ^2 is a faithful linear character of Z/D. Let τ be any irreducible constituent of χ^2. Clearly τ is an irreducible constituent of $(\mu^2)^G$. Since $(G/D)' \cap (Z/D) = G'D/D \cap Z/D = (G' \cap Z)/D = 1$, μ^2 is extendible to $\mu_0 \in \mathrm{Irr}(G/D) \subset \mathrm{Irr}(G)$. It follows by [4, Corollary 6.17] that $\tau = \mu_0\sigma$ for some $\sigma \in \mathrm{Irr}(G/Z)$.

We claim that this χ meets the requirement. Otherwise, χ^2 has no irreducible constituent of degree 3. Assume that $G/Z(G) \cong A_4$ or S_4. Since $\mathrm{cd}(A_4) = \{1, 3\}$ and $\mathrm{cd}(S_4) = \{1, 2, 3\}$, we see that any irreducible constituent τ of χ^2 is a product of μ_0 and an irreducible character σ of G/Z with degree at most 2. Note that in the group A_4 or S_4, the Klein four group K_4 is contained in the kernel of any irreducible character of degree at most 2. Hence the Fitting subgroup $F(G)$ of G is contained in the kernel of σ. By the arbitrariness of τ, it follows that $F(G) \leq Z(\chi^2) \leq Z(G)$, a contradiction. Thus this χ satisfies Hypothesis 2. Assume that $G/Z \cong PSL(2,5)$. Since $\mathrm{cd}(PSL(2,5)) = \{1,3,4,5\}$, we get that $\chi^2 = \mu_0\sigma \in \mathrm{Irr}(G)$, where $\sigma \in \mathrm{Irr}(G/Z)$ is of degree 4. Since $\mathrm{Irr}(\chi^{(2)}) \subseteq \mathrm{Irr}(\chi^2)$, we deduce the following contradiction that $\chi^{(2)} = m\chi^2$ for some integer m. Hence the χ satisfies Hypothesis 2. ∎

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References

[1] Y. G. Berkovich and E. M. Zhmud, Characters of finite groups, Part 1 and 2, Translations of Mathematical Monographs **172** and **181**, AMS, Providence, RI, 1998.
[2] A. M. Cohen, *Finite complex reflection groups*, Ann. Sci. École Norm. Sup. (4) **9** (1976), 379–436.
[3] P. Du Val, Homographies, quaternions and rotations, Clarendon Press, Oxford, 1964.
[4] I. M. Isaacs, Character theory of finite groups, Academic Press, New York, 1976.
[5] I. M. Isaacs and I. Zisser, *Squares of characters with few irreducible constituents in finite groups*, Arch. Math. (Basel) **63** (1994), 197–207.
[6] G. Malle, G. Navarro and J. B. Olsson, *Zeros of characters of finite groups*, J. Group Theory **3** (2000), 353–368.
[7] G. Qian, W. Shi and X. You, *Conjugacy classes outside a normal subgroup*, Comm. Algebra **32** (2004), 4809–4820.
[8] E. Zhmud, *Finite groups with a faithful real-valued irreducible character whose square has exactly two distinct irreducible constituents*, Glas. Mat. Ser. III **45(65)** (2010), 85–92.

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