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The Stable Marriage Problem with Master Preference Lists

Robert W. Irving*, David F. Manlove† and Sandy Scott‡

Department of Computing Science, University of Glasgow, Glasgow G12 8QQ, UK

Abstract

We study variants of the classical stable marriage problem in which the preferences of the men or the women, or both, are derived from a *master* preference list. This models real-world matching problems in which participants are ranked according to some objective criteria. The master list(s) may be strictly ordered, or may include ties, and the lists of individuals may involve ties and may include all, or just some, of the members of the opposite sex. In fact, ties are almost inevitable in the master list if the ranking is done on the basis of a scoring scheme with a relatively small range of distinct values. We show that many of the interesting variants of stable marriage that are NP-hard remain so under very severe restrictions involving the presence of master lists, but a number of special cases can be solved in polynomial time. Under this master list model, versions of the stable marriage problem that are already solvable in polynomial time typically yield to faster and/or simpler algorithms, giving rise to simple new structural characterisations of the solutions in these cases.

1 Introduction and background

The classical stable marriage problem. The Stable Marriage problem (SM) was introduced in the seminal paper of Gale and Shapley [3]. In its classical form, an instance of SM involves n men and n women, each of whom specifies a *preference list*, which is a total order on the members of the opposite sex. A *matching* M is a set of (man, woman) pairs such that each person belongs to exactly one pair. If $(m, w) \in M$, we say that w is m's *partner* in M, and vice versa, and we write $M(m) = w$, $M(w) = m$.

We say that a person x *prefers* y to y' if y precedes y' on x's preference list. A matching M is *stable* if it admits no *blocking pair*, namely a pair (m, w) such that m prefers w to $M(m)$ and w prefers m to $M(w)$. Gale and Shapley [3] proved that every instance of SM admits a stable matching, and described an algorithm – the Gale-Shapley algorithm – that finds such a matching in time that is linear in the input size. In general, there may be many stable matchings (in fact exponentially many in n) for a given instance of SM [13].

Extensions of the classical problem. A variety of extensions of the basic problem have been studied. In the Stable Marriage problem with Incomplete lists (SMI), the numbers of men and women need not be the same, and each person p's preference list consists of a subset of the members of the opposite sex (the *acceptable* persons for p) in strict order. A pair (m, w) is *acceptable* if each member of the pair is acceptable to the other. We let a

*Supported by EPSRC grant EP/E011993/1. Email: rwi@dcs.gla.ac.uk.

†Supported by EPSRC grants EP/E011993/1 and GR/R84597/01, and an RSE/Scottish Executive Personal Research Fellowship. Email: davidm@dcs.gla.ac.uk.

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denote the total number of acceptable pairs. A *matching* M is now a set of acceptable pairs such that each person belongs to at most one pair. In this context, (m, w) is a blocking pair for a matching M if (a) (m, w) is an acceptable pair, (b) m is either unmatched or prefers w to $M(m)$, and likewise (c) w is either unmatched or prefers m to $M(w)$. As in the classical case, there is always at least one stable matching for an instance of SMI, and it is straightforward to extend the Gale-Shapley algorithm to this case. Again, there may be many different stable matchings, but Gale and Sotomayor [4] showed that every stable matching for a given SMI instance has the same size and matches exactly the same set of people. We remark that, from the point of view of finding a stable matching, we lose no generality in assuming that, given an instance of SMI, the preference lists are *consistent* (i.e., for any two persons p and q, p is acceptable to q if and only if q is acceptable to p).

The Gale-Shapley algorithm for SM or SMI can be *man-oriented* or *woman-oriented*, i.e., applied from either the men's or the women's 'side'. In the former case, it yields a stable matching – the *man-optimal* – that is simultaneously the best possible stable matching for all of the men and the worst possible for all of the women. The roles of the sexes may be reversed to give the *woman-optimal* stable matching. Some alternative, perhaps fairer, optimality criteria have been proposed. For example, a *minimum regret* stable matching is one for which $\max r(p, M(p))$ (defined as the *regret* of M) is minimised, where the maximum is taken over all persons p, and $r(x, y)$ represents the *rank* of y in the preference list of x. An *egalitarian* stable matching is one for which $\sum r(p, M(p))$ (defined as the *weight* of M) is minimised, where the sum is taken over all persons p. Finally, a *lexicographic maximum* stable matching is one in which the maximum number of people obtain their first-choice partner, and subject to this condition, the maximum number obtain their second-choice partner, and so on. More precisely, for a matching M define $r_i(M)$ to be the number of people for whom $r(p, M(p)) = i$. Then the requirement is a stable matching M for which the vector $(r_1, \ldots, r_n)$ is lexicographically maximum.

Efficient algorithms have been devised for a number of variants of SM and SMI; for example:

- all of the stable pairs (i.e., the (man, woman) pairs that belong to at least one stable matching) can be identified in $O(a)$ time [5];
- all of the stable matchings can be found in $O(a + nk)$ time [5], where k is the total number of such matchings;
- a minimum regret stable matching can be found in $O(a)$ time [5];
- an egalitarian stable matching can be found in $O(a^2)$ time [14], later improved to $O(a^{3/2})$ time [2];
- a lexicographic maximum stable matching can be found in $O(n^{1/2}a^{3/2})$ time [2].

An alternative extension of SM arises if preference lists are allowed to contain *ties*. In the Stable Marriage problem with Ties (SMT) each person's preference list is a partial order over the members of the opposite sex in which indifference is transitive. In other words, each person p's list can be viewed as a sequence of ties, each of length ≥ 1; p prefers each member of a tie to everyone in any subsequent tie, but is indifferent between the members of any single tie. In this context, three definitions of stability have been proposed [6, 11].

A matching M is

- *weakly stable* if there is no pair (m, w), each of whom prefers the other to his/her partner in M;

- *strongly stable* if there is no pair (p, q) such that p prefers q to $M(p)$ and q either prefers p to $M(q)$ or is indifferent between them (note that p may be either a man or a woman here);
- *super-stable* if there is no pair (m, w), each of whom either prefers the other to his/her partner in M or is indifferent between them.

It is immediate from the definitions that

$$\text{super-stable} \Rightarrow \text{strongly stable} \Rightarrow \text{weakly stable}.$$

For a given instance of SMT, a weakly stable matching is bound to exist, and can be found in $O(n^2)$ time by breaking all ties in an arbitrary way (i.e., by strictly ranking the members of each tie arbitrarily) and applying the Gale-Shapley algorithm. A super-stable matching may or may not exist, but there is a $O(n^2)$ algorithm to find such a matching or report that there is none [11]. Likewise, a strongly stable matching may or may not exist, but there is a $O(n^4)$ algorithm to find one or report that none exists [11]. An improved $O(n^3)$ version of this latter algorithm has been described recently [18].

In the context of SMT, some problems involving weakly stable matchings become NP-hard [20] – for example:

- determining whether a given (man, woman) pair is weakly stable;
- finding a minimum regret weakly stable matching;
- finding an egalitarian weakly stable matching.

These results remain true even if the ties occur in the preference lists of one sex only, there is at most one tie per list, and each tie is of length 2.

Note that, in order that the various forms of optimality for stable matchings are well-defined when ties are present, we extend the notion of rank, by defining $r(x, y)$ to be one plus the number of strict predecessors of y in x's preference list; so, in particular, all entries occurring in the same tie have identical rank.

If we allow both of the above extensions of the classical problem simultaneously, we obtain the Stable Marriage problem with Ties and Incomplete lists (SMTI). The three forms of stability introduced for SMT are again all meaningful (under the assumption that a person would prefer to be matched to an acceptable partner rather than to be unmatched). The algorithmic results for strongly stable and super-stable matchings can be extended from SMT to SMTI [19]. However, the situation for weakly stable matchings turns out to be even more challenging than in SMT. Once again, it is easy to find a weakly stable matching, merely by breaking all the ties in an arbitrary way and applying the Gale-Shapley algorithm. However, the ways in which ties are broken will, in general, affect the size of the resulting matching, and the natural problem of finding a maximum cardinality weakly stable matching for an instance of SMTI turns out to be NP-hard, even under quite severe restrictions on the number and lengths of ties [20]. Specifically, NP-hardness holds even if ties occur in the preference lists of one sex only, each tie is of length 2, and each tie comprises the whole of the list in which it appears [20].

The Hospitals/Residents problem. The Hospitals/Residents problem (HR) is a many-to-one generalisation of SMI, so called because of its application in centralised matching schemes for the allocation of graduating medical students, or residents, to hospitals [25]. The best known such scheme is the National Resident Matching Program (NRMP) [22] in the US, but similar schemes exist in Canada [1], in Scotland [12, 27], and in a variety of other countries and contexts. In fact, this extension of SM was also discussed by Gale and Shapley under the name of the College Admissions problem [3]. In an instance of HR, each resident has a preference list containing a subset of the hospitals, and each hospital ranks the residents for which it is acceptable. In addition, each hospital has a *quota* of available posts. In this context, a matching is a set of acceptable (resident, hospital) pairs so that each resident appears in at most one pair and each hospital in a number of pairs that is bounded by its quota. The definition of stability has a natural extension to this more general setting (see [6] for details). It is again the case that every problem instance admits at least one stable matching [3], and that all stable matchings have the same size [4]. Clearly SMI is equivalent to the special case of HR in which each hospital has a quota of 1.

The Hospitals / Residents problem with Ties (HRT) allows arbitrary ties in the preference lists. Since HRT is clearly an extension of SMTI, the hardness results for weak stability problems in the latter extend to the former. On the other hand, the polynomial-time algorithms for strongly stable matchings and super-stable matchings in SMTI can be extended to the analogous variants of HRT [16, 17, 18]. These results have potentially important implications for large-scale real-world matching schemes. It is unreasonable to expect, say, a large hospital to rank in strict order all of its many applicants, and any artificial strict rankings, whether submitted by the hospitals themselves, or imposed by the matching scheme administrators, may have significant implications for the number of residents assigned in a stable matching.

Master lists. In this paper, we focus on special cases of these variants of SM in which the preference lists on one or both sides may be derived from a single *master list*, which may or may not contain ties. To be more precise, a *master list* of men consists of a single list containing all of the men, which may or may not contain ties; each woman's preference list contains her acceptable partners ranked precisely according to the master list. In other words, the preference list of a woman w is precisely the master list of men, except that each man m that w finds unacceptable is deleted (so in general, the deletions that give rise to w's preference list could be made from any part of the master list). An analogous meaning is attached to a master list of women. Hence, in SM and SMT, the existence of a master list for one sex implies that all preference lists for the members of the opposite sex are identical, but of course this is not necessarily the case for SMI and SMTI. It does, however, follow from the construction that the individual preference lists are consistent, given an instance of SMI or SMTI with master lists.

The study is motivated by the fact that, in a practical matching scheme, applicants for posts might be ranked, strictly or with ties, on the basis of some more or less objective criteria, such as academic performance. A particular instance of this has occurred recently in the context of MTAS (Medical Training Application Service), for allocating junior doctors to medical posts in the UK. The applicants were assigned a numerical score based on a combination of their academic record and an evaluation of their completed application forms for the scheme, and a master preference list, inevitably containing many ties of substantial size, was derived from these scores. Given the numbers of posts involved (in 2006-07, applicants were competing for 6,000 foundation posts and 19,000 specialty posts), MTAS gives the largest example of a centralised matching scheme involving master preference lists that we are currently aware of. (For various reasons, not directly associated with the concept of a master list, the whole MTAS system became the subject of national controversy [8], and has since been largely abandoned in its present form. However it is likely that its successor, to be used during 2007-08, will also involve a master list of applicants, albeit based on different criteria.) A case study of a different kind involving master lists is reported by Perach et al. [24]. The authors describe a method for allocating students to dormitories at the Technion-Israel Institute of Technology. The assignment produced is a stable matching according to the preferences of students over dormitories; students are ranked using master lists that refer to socio-economic characteristics as well as academic excellence.

We consider a range of possible variants of SM in which there may be a master list on just one side or on both sides, and in which the master list(s), and the lists of individuals, may or may not contain ties. For convenience, we use the extensions -2ML and -1ML to denote problem variants involving master lists on both sides and on one side respectively; for example, SMTI-2ML represents the Stable Marriage problem with Ties and Incomplete Lists with a Master List on both sides – i.e., a master list of men from which the women's preferences are derived, and a master list of women from which the men's preferences are derived. In the case of just one master list, in what follows it will not usually be necessary to specify whether this list involves men or women, hence the suffix -1ML does not include this information.

The main results in this paper, and their organisation, are as follows. Section 2 contains an algorithm and a theorem showing that every instance of SMI-1ML has a unique stable matching that can be found in linear time. The next three sections cover variants of SMT involving master lists under the weak stability condition. Section 3 deals with SMTI, and shows that existing hardness results for problems involving finding maximum weakly stable matchings and weakly stable pairs (i.e, (man, woman) pairs that belong to some weakly stable matching) typically apply even in the much more restricted case of SMTI-2ML. Section 4 covers the case of SMT-2ML, and shows that the various problems that arise, such as finding weakly stable pairs, generating weakly stable matchings, and finding optimal (egalitarian, minimum regret and lexicographic maximum) weakly stable matchings, can typically be solved by simpler and more efficient algorithms than in the general case. Section 5 deals with SMT-1ML, and proves a number of hardness results involving finding weakly stable pairs and types of optimal weakly stable matchings. We then switch our attention in Section 6 to variants of SMT with master lists under the strong and super-stability criteria. We give algorithms that are simpler, and in the case of strong stability, faster, than those for the general SMTI case, to find a super-stable or a strongly stable matching or report that none exists. We conclude with a summary, some additional observations, and some open questions in Section 7.

2 SMI with master lists

Let I be an instance of SMI-1ML (or SMI-2ML). We demonstrate in this section that a greedy algorithm, reminiscent of, but simpler than, the man-oriented version of the Gale-Shapley algorithm, may be used to find the unique stable matching in I. Without loss of generality suppose that there is a master list of men, say $m_1\ m_2\ \ldots\ m_n$. Consider the greedy algorithm Greedy-SMI-1ML as described in Figure 1.

```
M = ∅;
for i in 1...n
    if m_i's list contains an unmatched woman {
        w := first unmatched woman on m_i's list;
        M := M ∪ {(m_i, w)};
    }
return M;
```

Figure 1: Algorithm Greedy-SMI-1ML.

Theorem 2.1. *For a given instance of* SMI-*1*ML *(or* SMI-*2*ML*) there is a unique stable matching, which may be found in linear time using Algorithm* Greedy-SMI-1ML.

Proof. It is immediate that the set of pairs returned by Algorithm Greedy-SMI-1ML is a matching M. To see that M is stable, suppose that (m_i, w) is a blocking pair. Then w must be matched to some m_j where $j < i$, for otherwise Algorithm Greedy-SMI-1ML would have matched m_i to w. But m_j precedes m_i on the master list of men, giving a contradiction to the assumed blocking pair.

Assume that there is a second stable matching $M' \neq M$, and let i be the smallest index such that $M'(m_i) \neq M(m_i)$. Then each of $m_1, \dots, m_{i-1}$ has the same partner in M' as he has in M. Hence, and since Algorithm Greedy-SMI-1ML gave m_i his best available partner, it follows that m_i is either unmatched in M' or prefers $w = M(m_i)$ to $M'(m_i)$. Moreover, w cannot be matched in M' with any of $m_1, \dots, m_{i-1}$, since again, each such man has the same partner in M' as he has in M. Hence w is either unmatched in M' or prefers $M(w)$ to $M'(w)$, so that (m_i, w) is a blocking pair for M' – a contradiction.

We finally remark that it is straightforward to verify that Algorithm Greedy-SMI-1ML runs in time linear in the length of the given preference lists. □

3 SMTI with master lists

3.1 Maximum cardinality weakly stable matchings

In this section we present hardness results for the problem of finding a maximum weakly stable matching, given an instance of SMTI-2ML. We firstly note that an instance of SMTI-2ML can have weakly stable matchings of different sizes. For, consider the SMTI instance I shown in Figure 2, where, as throughout this paper, entries within parentheses in a given person's preference list are tied.

Men's preferences		Women's preferences	
m_1 :	w_1	w_1 :	$(m_1\ \ m_2)$
m_2 :	$w_1\ \ w_2$	w_2 :	m_2

Figure 2: SMTI instance with weakly stable matchings of different sizes.

Firstly, it is straightforward to verify that $w_1\ w_2$ is a master list of the women, whilst $(m_1\ m_2)$ is a master list of the men. Each of the matchings $M_1 = \{(m_2, w_1)\}$ and $M_2 = \{(m_1, w_1), (m_2, w_2)\}$ is weakly stable in I. Hence, this observation leads naturally to the problem of finding a maximum cardinality weakly stable matching (henceforth a maximum weakly stable matching), given an instance of SMTI-2ML.

Define MAX SMTI-2ML to be the problem of finding a maximum weakly stable matching, given an instance of SMTI-2ML, and let MAX SMTI-2ML-D denote the decision version of MAX SMTI-2ML. In this section we show that MAX SMTI-2ML-D is NP-complete, even under various restrictions involving the positions and lengths of ties in the master lists, and involving the lengths of individual preference lists. In fact our first two results establish NP-completeness for the special case of MAX SMTI-2ML-D in which the number of men n is equal to the number of women, and the target size of weakly stable matching is equal to n. We refer to this restriction as COMPLETE SMTI-2ML. Given an instance of this problem, we refer to a weakly stable matching of size n as a *complete weakly stable matching.*

We firstly show that COMPLETE SMTI-2ML is NP-complete even in the case that there is a single tie in one of the master lists. To do this, we show that the transformation of Lemma 1 of [20] can have master lists imposed directly. As in that lemma, we use a reduction from a problem involving matchings in graphs. A matching M in a graph G is said to be *maximal* if no proper superset of M is a matching in G. Define MIN MM (respectively EXACT MM) to be the problem of deciding, given a graph G and integer K, whether G admits a maximal matching of size at most (respectively exactly) K. MIN MM is NP-complete, even for subdivision graphs [10]. (Given a graph G, the *subdivision graph* of G, denoted by $S(G)$, is obtained by subdividing each edge $\{u, w\}$ of G in order to obtain two edges $\{u, v\}$ and $\{v, w\}$ of $S(G)$, where v is a new vertex.) It turns out that EXACT MM is NP-complete for the same class of graphs [21].

We now show how to modify the transformation of Lemma 1 of [20] to show that COMPLETE SMTI-2ML is NP-complete even if there is a single tie in one of the master lists. For completeness we present the proof of correctness of the reduction in its entirety.

Theorem 3.1. COMPLETE SMTI-*2*ML *is NP-complete, even if there is only a single tie that occurs in one of the master lists.*

Proof. Clearly COMPLETE SMTI-2ML is in NP. We transform from EXACT MM for subdivision graphs, which is NP-complete as indicated above. Hence let G (a subdivision graph of some graph) and K (a positive integer) be an instance of EXACT MM. Then G is a bipartite graph, so that $G = (U, W, E)$, where, without loss of generality, each vertex in U has degree 2. Further, without loss of generality, we assume that $|U| = |W|$. For if $|U| = |W| + r$ for some $r > 0$, then we may add r vertices $a_1, \ldots, a_r$ to U, and $2r$ vertices $b_1, \ldots, b_r$, $c_1, \ldots, c_r$ to W, where a_i is adjacent to b_i and c_i for each i ($1 \le i \le r$). Clearly every vertex in the new set U has degree 2 in the new graph, and G has a maximal matching of size K if and only if the transformed graph has a maximal matching of size $K + r$. (A similar transformation can be carried out if $|W| = |U| + r'$ for some $r' > 0$.) Finally, without loss of generality, we may assume that $K \le n$, where $n = |U| = |W|$.

Let $U = \{m_1, \ldots, m_n\}$ and let $W = \{w_1, \ldots, w_n\}$. We construct an instance I of COMPLETE SMTI-2ML as follows: let $U \cup U' \cup X$ be the set of men in I, and let $W \cup Y \cup Z$ be the set of women in I, where $U' = \{m'_1, \ldots, m'_n\}$, $X = \{x_1, \ldots, x_{n-K}\}$, $Y = \{y_1, \ldots, y_n\}$ and $Z = \{z_1, \ldots, z_{n-K}\}$. For each vertex $m_i \in U$, let w_{j_i} and w_{k_i} be the two vertices adjacent to m_i in G, where $j_i < k_i$. For each woman $w_j \in W$, let $U_j = \{m_i : \{m_i, w_j\} \in E\}$, and let $U'_j = \{m'_i : \{m_i, w_j\} \in E \wedge j = k_i\}$. The preference lists of the people in I are as follows:

$$\begin{array}{rlr}
m_i : & y_i \;\; w_{j_i} \;\; w_{k_i} \;\; z_1 \;\; z_2 \;\ldots\; z_{n-K} & (1 \le i \le n) \\
m'_i : & y_i \;\; w_{k_i} & (1 \le i \le n) \\
x_i : & w_1 \;\; w_2 \;\ldots\; w_n & (1 \le i \le n-K) \\
w_j : & (U_j \cup U'_j) \;\; x_1 \;\; x_2 \;\ldots\; x_{n-K} & (1 \le j \le n) \\
y_j : & (m_j \;\; m'_j) & (1 \le j \le n) \\
z_j : & (m_1 \;\; m_2 \;\ldots\; m_n) & (1 \le j \le n-K)
\end{array}$$

It is straightforward to verify that

$$(m_1 \;\; m_2 \;\ldots\; m_n \;\; m'_1 \;\; m'_2 \;\ldots\; m'_n) \;\; x_1 \;\; x_2 \;\ldots\; x_{n-K}$$

is a master list of the men in I, and

$$y_1 \;\; y_2 \;\ldots\; y_n \;\; w_1 \;\; w_2 \;\ldots\; w_n \;\; z_1 \;\; z_2 \;\ldots\; z_{n-K}$$

is a master list of the women in I. Clearly there is a single tie in the master list of men, whilst the master list of women is strictly ordered. We claim that G has a maximal matching of size K if and only if I has a complete weakly stable matching.

For, suppose that G has a maximal matching M, where $|M| = K$. We construct a matching M' in I as follows. For each edge $\{m_i, w_j\}$ in M, if $j = j_i$, then we add (m_i, w_{j_i})

and (m'_i, y_i) to M'. If $j = k_i$, then we add (m'_i, w_{k_i}) and (m_i, y_i) to M'. There remain $2(n-K)$ men of the form m_{p_i}, m'_{p_i} $(1 \le i \le n-K)$ who are as yet unmatched. Add (m_{p_i}, z_i) and (m'_{p_i}, y_{p_i}) to M' $(1 \le i \le n-K)$. Similarly there remain $n-K$ women of the form w_{q_i} $(1 \le i \le n-K)$ who are as yet unmatched. Add (x_i, w_{q_i}) to M' $(1 \le i \le n-K)$. Clearly every man and woman in I is matched in M'.

It is straightforward to verify that no woman in $Y \cup Z$ can be involved in a blocking pair of M', and hence neither can a man in U'. No man in X can be involved in a blocking pair, since the women in W are matched in M' to men in X in increasing indicial order. Finally, suppose that (m_i, w_j) is a blocking pair of M', where $m_i \in U$ and $w_j \in W$. Then $(m_i, z_k) \in M'$ for some $z_k \in Z$, and $(x_l, w_j) \in M'$ for some $x_l \in X$. Thus no edge of M is incident to m_i or w_j in G. Hence $M \cup \{\{m_i, w_j\}\}$ is a matching in G, contradicting the maximality of M. Thus M' is a complete weakly stable matching in I.

Conversely, suppose that M' is a complete weakly stable matching in I. For each i $(1 \le i \le n)$, exactly one of m_i, m'_i is matched in M' to y_i, for otherwise y_i is unmatched in M', a contradiction. Hence at most one of m_i, m'_i is matched in M' to a woman in W. It follows that

$$M = \big\{\{m_i, w_j\} \in E : (m_i, w_j) \in M' \vee (m'_i, w_j) \in M'\big\}$$

is a matching in G. There are exactly $n-K$ men m_{r_i} $(1 \le i \le n-K)$ who have a partner from Z in M'. Since $M'(m'_{r_i}) = y_{r_i}$ $(1 \le i \le n-K)$ by the above remark, it follows that $|M| = K$.

To complete the proof, it remains to show that M is maximal. For, suppose not. Then there is some edge $\{m_i, w_j\}$ in G such that no edge of M is incident to either m_i or w_j. Thus $(m_i, z_k) \in M'$ for some $z_k \in Z$, and $(x_l, w_j) \in M'$ for some $x_l \in X$. But then (m_i, w_j) is a blocking pair of M', a contradiction. Hence M is maximal in G. □

The proof of Theorem 3.1 shows that the problem of finding a maximum cardinality weakly stable matching for an instance of SMTI is NP-hard, even if there is a master list on both sides, one of the master lists is strictly ordered, and the other has a single tie at its head.

It is interesting to observe that the situation is quite different if the tie is at the tail of the list. In fact, if there is such a master list on one side, and on the other side all preferences are strict (with or without a master list), we can find a maximum cardinality weakly stable matching in polynomial time. Suppose that the master list is of men. Process the men, in order, from the strictly ranked part of the master list, matching each man in turn with his favourite unmatched woman (if any). It is easy to see that all the pairs formed in this way must belong to every weakly stable matching. On reaching the master list tie, form a maximum cardinality matching of the men contained in it with unmatched acceptable partners, and adjoin these new pairs to those formed previously to obtain a maximum cardinality weakly stable matching.

This is a case where a master list makes the problem easier. The variant of SMTI (without master lists) in which all the ties are on one side and at the ends of the preference lists is known to be NP-hard [20], though a 5/3-approximation algorithm for this variant has recently been described [15].

Let I be the instance of SMTI as created by the proof of Theorem 3.1. We note that the proof is unchanged if the men in X are tied in the master list (and consequently in the preference list of each $w_j \in W$). Therefore in this modified instance, each woman has either one or two ties in her preference list. Moreover, if S_1 is the set of men who collectively occupy the first tie in the women's lists (or the only tie, in the case that there is just one), and S_2 is the set of men who occupy the second tie in the lists of the women who have two ties, then it follows that $S_1 = U \cup U'$, $S_2 = X$, and consequently $S_1 \cap S_2 = \emptyset$. We use this restriction for the transformation in the next theorem, which demonstrates the NP-completeness of COMPLETE SMTI-2ML for another restricted case. The transformation itself is similar to that of Theorem 2 in [20].

Theorem 3.2. COMPLETE SMTI-*2*ML *is NP-complete, even if the ties occur in one master list only and are of length 2.*

Proof. As in Theorem 3.1, COMPLETE SMTI-2ML is in NP. We transform from the version of COMPLETE SMTI-2ML as described in the paragraph preceding this theorem. Let I be an instance of this problem, where $U = \{m_1, \ldots, m_n\}$ is the set of men and $W = \{w_1, \ldots, w_n\}$ is the set of women. Without loss of generality let $w_1\ w_2 \ldots w_n$ be the master list of women. Recall that every woman has one or two ties in her list. For each woman $w_j \in W$, let U_j^h denote the set of men who appear in the first tie in w_j's list (or the only tie, in the case that there is just one), and let U_j^t denote the set of men who appear in the second tie in w_j's list ($U_j^t = \emptyset$ in the case that there is no such tie). Let $U_j^h = \{m_{k_{j,1}}, \ldots, m_{k_{j,h_j}}\}$ for some $h_j > 0$, and let $U_j^t = \{m_{l_{j,1}}, \ldots, m_{l_{j,t_j}}\}$ for some $t_j \geq 0$. We form an instance I' of COMPLETE SMTI-2ML as follows: let $U \cup X \cup Z$ be the set of men in I', and let $\bigcup_{j=1}^n W_j \cup Y$ be the set of women in I', where $X = \bigcup_{i=1}^n X_i$, $Y = \bigcup_{j=1}^n Y_j$ and $Z = \bigcup_{i=1}^n Z_i$, and

$$
\begin{array}{rcll}
W_j & = & \{w_{j,r} : 1 \leq r \leq h_j + t_j\} & (1 \leq j \leq n) \\
X_i & = & \{x_{i,r} : 1 \leq r \leq h_i + t_i\} & (1 \leq i \leq n) \\
Y_j & = & \{y_{j,r} : 1 \leq r \leq h_j + t_j\} & (1 \leq j \leq n) \\
Z_i & = & \{z_{i,r} : 1 \leq r \leq h_i + t_i - 1\} & (1 \leq i \leq n).
\end{array}
$$

Finally let $W_j^t = \bigcup_{r=h_j+1}^{h_j+t_j} \{w_{j,r}\}$. The preference lists in I' are formed as follows: each man $m_i \in U$ starts with his preference list from I. Let $w_j \in W$ be an arbitrary woman on m_i's list in I. If $m_i \in U_j^h$ then $m_i = m_{k_{j,a}}$ for some a ($1 \leq a \leq h_j$), while if $m_i \in U_j^t$ then $m_i = m_{l_{j,b}}$ for some b ($1 \leq b \leq t_j$). In the former case we replace w_j by the women in $W_j^t \cup \{w_{j,a}\}$ in increasing order of the second subscript, while in the latter case we replace w_j by $w_{j,b+h_j}$. The remaining preference lists are as follows:

$$
\begin{array}{rlll}
x_{i,r}: & (w_{i,r}\ \ y_{i,r}) & (1 \leq i \leq n) & (1 \leq r \leq h_i + t_i) \\
z_{i,r}: & y_{i,1}\ \ y_{i,2}\ \cdots\ y_{i,h_j+t_j} & (1 \leq i \leq n) & (1 \leq r \leq h_i + t_i - 1) \\
w_{j,r}: & x_{j,r}\ \ m_{k_{j,r}} & (1 \leq j \leq n) & (1 \leq r \leq h_j) \\
w_{j,r+h_j}: & x_{j,r+h_j}\ \ m_{k_{j,1}}\ \ m_{k_{j,2}}\ \cdots\ m_{k_{j,h_j}}\ \ m_{l_{j,r}} & (1 \leq j \leq n) & (1 \leq r \leq t_j) \\
y_{j,r}: & x_{j,r}\ \ z_{j,1}\ \ z_{j,2}\ \cdots\ z_{j,h_j+t_j-1} & (1 \leq j \leq n) & (1 \leq r \leq h_j + t_j)
\end{array}
$$

Assume that $\bigcup_{j=1}^n U_j^h = \{m_{a_1}, m_{a_2}, \ldots, m_{a_c}\}$ and $\bigcup_{j=1}^n U_j^t = \{m_{b_1}, m_{b_2}, \ldots, m_{b_d}\}$, where $a_1 < a_2 < \cdots < a_c$. Then $a_i \neq b_j$ ($1 \leq i \leq c, 1 \leq j \leq d$) by the discussion preceding the theorem. Without loss of generality let the men in U_j^h be ordered such that if $1 \leq p < q \leq h_j$ then $k_{j,p} < k_{j,q}$. Then it is straightforward to verify that

$$
\begin{array}{l}
x_{1,1}\ \ x_{1,2}\ \cdots\ x_{1,h_1+t_1}\ \ x_{2,1}\ \cdots\ x_{n,h_n+t_n}\ \ m_{a_1}\ \ m_{a_2}\ \cdots\ m_{a_c}\ \ m_{b_1}\ \ m_{b_2}\ \cdots\ m_{b_d} \\
z_{1,1}\ \ z_{1,2}\ \cdots\ z_{1,h_1+t_1-1}\ \ z_{2,1}\ \cdots\ z_{n,h_n+t_n-1}
\end{array}
$$

is a master list of the men in I', and

$$
(w_{1,1}\ \ y_{1,1})\ \ (w_{1,2}\ \ y_{1,2})\ \cdots\ (w_{1,h_1+t_1}\ \ y_{1,h_1+t_1})\ \ (w_{2,1}\ \ y_{2,1})\ \cdots\ (w_{n,h_n+t_n}\ \ y_{n,h_n+t_n})
$$

is a master list of the women in I'. Clearly there are ties in only one master list, they are of length 2, and a tie forms the whole of the individual list in which it appears. We claim that I has a complete weakly stable matching if and only if I' does.

For, suppose that I has such a matching M. We form a matching M' in I' as follows. Let $(m_i, w_j) \in M$. If $m_i \in M_j^h$, then $m_i = m_{k_{j,a}}$ for some a $(1 \leq a \leq h_j)$. If $m_i \in M_j^t$, then $m_i = m_{l_{j,b}}$ for some b $(1 \leq b \leq t_j)$; let $a = b + h_j$. In both cases, add the pairs $(m_i, w_{j,a})$, $(x_{j,r}, w_{j,r})$ $(1 \leq r \leq h_j + t_j,\ r \neq a)$, $(x_{j,a}, y_{j,a})$, $(z_{j,k}, y_{j,k})$ $(1 \leq k \leq a-1)$, and $(z_{j,k-1}, y_{j,k})$ $(a+1 \leq k \leq h_j + t_j)$ to M'. It is clear that M' is a complete weakly stable matching in I'.

It is straightforward to verify that no man in X can be involved in a blocking pair of M' in I'. Hence, and since the men in Z are matched to women in Y in increasing order of their second subscript, neither can any person in $Y \cup Z$. Now suppose that $(m_i, w_{j,a})$ blocks M' in I'. Then $a > h_j$ and $m_i \in M_j^h$. Let $m_p = M'(w_{j,a})$; then $m_p = m_{l_{j,b}}$, where $b = a - h_j$. Clearly $(m_i, w_{j,a}) \notin M'$, and also $(m_i, w_{j,r}) \notin M'$ (for $1 \leq r \leq h_j + t_j$, $r \neq a$), since $(x_{j,r}, w_{j,r}) \in M'$ (for the same r). Thus $M'(m_i) \notin W_j$, so that in I, m_i strictly prefers w_j to $M(m_i)$. Also, in I, w_j strictly prefers m_i to m_p. Hence (m_i, w_j) blocks M in I, a contradiction. Thus M' is weakly stable in I'.

Conversely, suppose that M' is a complete weakly stable matching in I'. We form a matching M in I as follows. Let j $(1 \leq j \leq n)$ be given. Since $|Z_j| = h_j + t_j - 1$ and $|Y_j| = h_j + t_j$, it follows that $M'(y_{j,a}) = x_{j,a}$ for some a $(1 \leq a \leq h_j + t_j)$, and hence $M'(w_{j,a}) = m_i$, for some $m_i \in U$. Since $M'(x_{j,r}) = w_{j,r}$ (for $1 \leq r \leq h_j + t_j$, $r \neq a$), then $M' \cap (U \times W_j) = \{(m_i, w_{j,a})\}$. Let m_i be the partner of w_j in M. Clearly M is a complete matching in I.

Suppose that (m_i, w_j) blocks M in I. Let $m_p = M(w_j)$. Then in I, w_j strictly prefers m_i to m_p, so that $m_i \in U_j^h$ and $m_p \in U_j^t$. Thus $m_p = m_{l_{j,b}}$ for some b $(1 \leq b \leq t_j)$, so that $M'(m_p) = w_{j,a}$, where $a = b + h_j$. Now in I', $w_{j,a}$ strictly prefers m_i to m_p. Also in I', m_i strictly prefers $w_{j,a}$ to $M'(m_i)$ (since $M(m_i) \neq w_j$ implies that $M'(m_i) \notin W_j$). Thus $(m_i, w_{j,a})$ blocks M' in I', a contradiction. Hence M is weakly stable in I. □

We now give an inapproximability result. We show it is NP-hard to approximate MAX SMTI-2ML within δ, for some $\delta > 1$, even if the individual preference lists in the given instance are of constant length and there is only one tie in each master list. The transformation is similar to that of Theorem 6 of [7].

Theorem 3.3. *It is NP-hard to approximate* MAX SMTI-2ML *within δ, for some $\delta > 1$. The result holds even if the individual preference lists in the given instance are of constant length and there is only one tie in each master list.*

Proof. By [7, Theorem 1], it is NP-hard to approximate MIN MM for subdivision graphs of cubic graphs within δ_0, for some $\delta_0 > 1$. Let G be an instance of this problem. Then G is the subdivision graph of some cubic graph, and hence $G = (U, W, E)$ is a bipartite graph where without loss of generality each vertex in U has degree 3 and each vertex in W has degree 2. Let $U = \{m_1, \ldots, m_s\}$ and let $W = \{w_1, \ldots, w_t\}$. For each vertex $m_i \in U$, let W_i denote the three vertices adjacent to m_i in G. Similarly for each vertex $w_j \in W$, let U_j denote the two vertices adjacent to w_j in G. We construct an instance I of MAX SMTI-2ML as follows: let $U \cup X$ be the set of men and let $W \cup Y$ be the set of women, where $X = \{x_1, \ldots, x_t\}$ and $Y = \{y_1, \ldots, y_s\}$. The preference lists in I are as follows:

$$\begin{array}{llll} m_i: & (W_i)\ y_i & (1 \leq i \leq s) & \qquad w_j: & (U_j)\ x_j & (1 \leq j \leq t) \\ x_i: & w_i & (1 \leq i \leq t) & \qquad y_j: & m_j & (1 \leq j \leq s) \end{array}$$

It is straightforward to verify that

$$(m_1\ m_2\ \ldots\ m_s)\ x_1\ x_2\ \ldots\ x_t$$

is a master list of the men, and

$$(w_1\ w_2\ \ldots\ w_t)\ y_1\ y_2\ \ldots\ y_s$$

is a master list of the women in I. Moreover the length of each man's preference list is at most 4, whilst the length of each woman's preference list is at most 3.

Suppose that M is a maximal matching in G such that $|M| = \beta_1^-(G)$, where $\beta_1^-(G)$ denotes the minimum size of a maximal matching in G. We construct a matching M' in I as follows. Initially let $M' = M$. For each man $m_i \in U$ who is unmatched in M, add (m_i, y_i) to M'. Similarly for each woman $w_j \in W$ who is unmatched in M, add (x_j, w_j) to M'. Clearly M' is a matching in I, and $|M'| = |M| + (s - |M|) + (t - |M|) = s + t - |M|$. It is straightforward to verify that the maximality of M in G implies that M' is weakly stable in I. Hence $s^+(I) \geq s + t - |M| = s + t - \beta_1^-(G)$, where $s^+(I)$ denotes the maximum size of a weakly stable matching in G.

Conversely, suppose that M' is a weakly stable matching in I, where $|M'| = s^+(I)$. Let $M = M' \cap E$. It is straightforward to verify that the weak stability of M' in I implies that M is maximal in G. Also $|M'| \leq |M| + (t - |M|) + (s - |M|) = s + t - |M|$, for every edge in M contributes one (man, woman) pair to M', and in addition, at most $(t - |M|)$ men in X can be matched in M', and at most $(s - |M|)$ women in Y can be matched in M'. Hence $s^+(I) = |M'| \leq s + t - |M| \leq s + t - \beta_1^-(G)$.

Thus $s^+(I) + \beta_1^-(G) = s + t$. Now $2t = 3s$, as G is the subdivision graph of some cubic graph. Also $n = s + t$ and $m = 2t$, where n is the number of men in I and $m = |E|$.

Theorem 1 of [7] shows that it is NP-hard to distinguish between the cases that $\beta_1^-(G) \leq c_0 m$ and $\beta_1^-(G) > \delta_0 c_0 m$, where $c_0 > 0$ is some constant. Hence if $\beta_1^-(G) \leq c_0 m$, then $s^+(I) \geq cn$, whilst if $\beta_1^-(G) > \delta_0 c_0 m$, then $s^+(I) < \delta cn$, where $c = \frac{5-6c_0}{5}$ and $\delta = \frac{5-6\delta_0 c_0}{5-6c_0}$. The result follows by Theorem 1 and Proposition 4 of [7]. ☐

3.2 Weakly stable pairs in SMTI-2ML

In this section we consider the complexity of the problem of finding all weakly stable pairs, given an instance of SMTI-2ML. Define WEAKLY STABLE PAIR OF Π to be the problem of deciding, given an instance I of Π and a (man, woman) pair (m, w), whether (m, w) is a weakly stable pair in I.

Theorem 3.4. WEAKLY STABLE PAIR OF SMTI-2ML *is NP-complete, even if there are ties in only one of the master lists.*

Proof. Clearly WEAKLY STABLE PAIR OF SMTI-2ML is in NP. To show that the problem is NP-hard, we transform from the variant of COMPLETE SMTI-2ML in which ties occur only in the master list of men, which is NP-complete by Theorem 3.1. Let I be an instance of this problem, where $U = \{m_1, \ldots, m_n\}$ and $W = \{w_1, \ldots, w_n\}$ are the sets of men and women in I respectively. Let L_m and L_w be the master lists of men and women respectively, and let P_i and Q_j be the preference lists of each $m_i \in U$ and $w_j \in W$ respectively. We construct an instance I' of WEAKLY STABLE PAIR OF SMTI-2ML as follows: the set of men in I' is $\{m_0\} \cup U$, and the set of women in I' is $\{w_0\} \cup W$. The preference lists for each person in I' are as follows:

$$\begin{array}{llll} m_0 : \ L_w \ w_0 & & w_0 : \ L_m \ m_0 & \\ m_i : \ P_i \ w_0 & (1 \leq i \leq n) & w_i : \ Q_i \ m_0 & (1 \leq i \leq n) \end{array}$$

Clearly a master list L'_m of the men in I' may be obtained by appending m_0 to L_m, whilst a master list L'_w of the women in I' may be obtained by appending w_0 to L_w. This gives an instance of SMTI-2ML in which there are no ties in the master list of women. It is straightforward to check that I admits a complete weakly stable matching if and only if I' has a weakly stable matching containing the pair (m_0, w_0). ☐

4 SMT with master lists on both sides

4.1 Finding all weakly stable pairs

We now show that, in contrast to Theorem 3.4, for an instance I of SMT-2ML, we can find all the weakly stable pairs in time $O(n+s)$, where n is the number of men and s is the number of weakly stable pairs in I. Let L_m and L_w be the master lists in I. Henceforth we assume, without loss of generality, that the men are indexed so that m_i is listed before m_j in L_m if and only if $i < j$ (note that this includes the possibility that m_i and m_j are tied in L_m), and similarly for the women on L_w. We say that a tie $T \in L_m$ *overlaps* a tie $T' \in L_w$ if and only if there is some i such that $m_i \in T$ and $w_i \in T'$, and for every such i we say that m_i and w_i are *in the overlap* between T and T'. If all the ties in a given preference list are broken in some way, making the list strictly ordered, we say that the list has been *resolved*.

Lemma 4.1. *Let I be an instance of* SMT-2ML. *The pair (m_i, w_j) is a weakly stable pair in I if and only if T and T' overlap, where T and T' are the ties in L_m and L_w containing m_i and w_j respectively.*

Proof. Suppose that T and T' overlap. Then it is possible to resolve the master lists so that m_i and w_j occupy the same position in the (strictly ordered) resolved master lists. We then match the man at position k with the woman at position k ($1 \leq k \leq n$) in these resolved lists, where n is the size of the instance. It is immediate that this matching is weakly stable.

Suppose that T and T' do not overlap. Let M be a weakly stable matching containing (m_i, w_j). Let p and q be the minimum and maximum indices of men m_k appearing in T respectively, and let r and s be the minimum and maximum indices of women w_k appearing in T' respectively. Then $[p..q] \cap [r..s] = \emptyset$. Without loss of generality suppose that $p > s$. There are $p-1$ men on L_m who are strictly preferable to m_i on L_m. As $p > s$, at least one such man, m_x say, is matched in M to a woman who is strictly inferior to w_j on L_w. Hence (m_x, w_j) is a blocking pair of M, a contradiction. □

Theorem 4.2. *Given an instance I of* SMT-2ML *we can find all the weakly stable pairs in $O(n+s)$ time, where s is the number of weakly stable pairs.*

Proof. By Lemma 4.1, a given pair (m_i, w_j) is weakly stable in I if and only if T and T' overlap, where T and T' are the ties in L_m and L_w containing m_i and w_j respectively. Consider a tie T in L_m. We can find the weakly stable partners of each man in T as follows: let $f_T = \min\{k : m_k \in T\}$, and let $l_T = \max\{k : m_k \in T\}$. Let X and Y be the ties in L_w such that $w_{f_T} \in X$, and $w_{l_T} \in Y$. Let $f_X = \min\{k : w_k \in X\}$, and let $l_Y = \max\{k : w_k \in Y\}$. Then the weakly stable partners of the men in T are $w_{f_X}, ..., w_{l_Y}$. Finding f_T and l_T for every tie T takes $O(n)$ overall time, while finding f_X and l_Y takes $O(s_T)$ time, where s_T is the number of weakly stable partners of the men in T. If we repeat this process for every tie in L_m then it follows that we can find all the weakly stable pairs in $O(n+s)$ time. □

4.2 Generation of all weakly stable matchings

We next show that we can find all the weakly stable matchings for an instance I of SMT-2ML, with sublinear time between the generation of successive matchings.

Let I be an instance of SMT-2ML of size n, and let $U = \{m_1, \ldots, m_n\}$ and $W = \{w_1, \ldots, w_n\}$ be the sets of men and women respectively in I. By Theorem 4.2, we can list the weakly stable pairs for I in $O(n+s)$ time, where s is the number of such pairs.

We construct a bipartite graph G_I, the *matching graph*, as follows. The set of vertices in G_I is $U \cup W$, and there is an edge from $m_i \in U$ to $w_j \in W$ if and only if w_j is a weakly stable partner of m_i. This construction takes $O(n+s)$ time.

Lemma 4.3. *Let I be an instance of* SMT-2ML*. Then there is a one-to-one correspondence between the weakly stable matchings for I and the perfect matchings in the matching graph G_I.*

Proof. Let $M = \{(m_1, w_{k_1}), \ldots, (m_n, w_{k_n})\}$ be a weakly stable matching for I. It is clear that $\{m_i, w_{k_i}\}$ is an edge in G_I $(1 \leq i \leq n)$, and hence M is a perfect matching in G_I.

Conversely, let $M = \{(m_1, w_{k_1}), \ldots, (m_n, w_{k_n})\}$ be a perfect matching in G_I, and suppose M is not weakly stable. Let (m, w) be a blocking pair for M. Then w prefers m to $M(w)$, so m is a strict predecessor of $M(w)$ in L_m. Since $(m, M(m)) \in M$, the tie in L_m containing m must overlap with that in L_w containing $M(m)$, by Lemma 4.1. Similarly, m prefers w to $M(m)$, so w is a strict predecessor of $M(m)$ in L_w. Since $(M(w), w) \in M$, the tie in L_w containing w must overlap with that in L_m containing $M(w)$, by Lemma 4.1. It is clear that not all of these four conditions can be satisfied simultaneously, and the result follows. □

Uno [28] describes an algorithm which, given an initial perfect matching in a bipartite graph $G = (V, E)$, generates all k perfect matchings for G in $O(k \log |V|)$ time. By Lemma 4.1, the matching produced by breaking the ties in the two master lists arbitrarily, and then matching the man at position i with the woman at position i, for each $1 \leq i \leq n$, is weakly stable, and can be produced in $O(n)$ time. Then, using Uno's algorithm, the remaining perfect matchings in G_I can be generated in $O(\log n)$ time per matching, giving overall complexity $O(n + s + k \log n)$ to generate the k perfect matchings in G_I, which, by Lemma 4.3, are exactly the weakly stable matchings for I. Thus we have the following theorem.

Theorem 4.4. *Let I be an instance of* SMT-2ML *of size n. All the weakly stable matchings for I can be generated in $O(n+s+k \log n)$ time, where k is the number of such matchings, and s is the number of weakly stable pairs in I.*

4.3 Optimal weakly stable matchings

It turns out that finding the various kinds of optimal weakly stable matchings is straightforward in the case of SMT-2ML, as the following theorem shows. Henceforth, MINIMUM REGRET Π, EGALITARIAN Π and LEX MAX Π denote the problems of finding a minimum regret, egalitarian and lexicographic maximum weakly stable matching respectively, given an instance of Π. The decision version of each problem is obtained by appending "-D".

Theorem 4.5. MINIMUM REGRET SMT-2ML*,* EGALITARIAN SMT-2ML *and* LEX MAX SMT-2ML *can all be solved in $O(n)$ time.*

Proof. For every position i in one master list, some person from the other master list must be matched with the person at that position. It follows that every weakly stable matching for I must have the same regret and the same weight, and the numbers of people matched to their ith choice is the same in every weakly stable matching, for each value of i.

If we resolve the master lists arbitrarily, and match the entries at position i in each resolved list, for each i, then we obtain a weakly stable matching, and this can be found in $O(n)$ time. By the foregoing it is simultaneously a minimum regret, egalitarian and lexicographic maximum weakly stable matching, and the result follows. □

5 SMT with a master list on one side

In the case of SMT-1ML, it turns out that, in contrast to Section 4, various problems associated with finding weakly stable matchings become NP-hard.

5.1 Finding all weakly stable pairs

We firstly consider the problem of finding the weakly stable pairs, given an instance of SMT-1ML. We give an NP-completeness result along the lines of Theorem 3.4.

Theorem 5.1. WEAKLY STABLE PAIR OF SMT-*1*ML *is NP-complete.*

Proof. Clearly WEAKLY STABLE PAIR OF SMT-1ML is in NP. To show that the problem is NP-hard, we transform from COMPLETE SMTI-2ML, which is NP-complete by Theorem 3.1. Let I be an instance of this problem, where $U = \{m_1, \ldots, m_n\}$ and $W = \{w_1, \ldots, w_n\}$ are the sets of men and women in I respectively. Let L_m denote the master list of men, and let P_i be the preference list of m_i in I $(1 \leq i \leq n)$. We construct an instance I' of WEAKLY STABLE PAIR OF SMT-1ML as follows: the set of men in I' is $\{m_0\} \cup U$, and the set of women in I' is $\{w_0\} \cup W$. The preference lists for the men are as follows:

$$\begin{array}{llll} m_0: & -- & w_0 & \\ m_i: & P_i \;\; w_0 & -- & (1 \leq i \leq n) \end{array}$$

In a given preference list, the symbol $--$ denotes all remaining people of the opposite sex in arbitrary strict order. We obtain a master list L'_m of men in I' by appending m_0 to L_m. It is straightforward to verify that I admits a complete weakly stable matching if and only if I' has a weakly stable matching M' containing (m_0, w_0). □

Since COMPLETE SMTI-2ML is NP-complete even if ties occur only in one master list, the above reduction implies that WEAKLY STABLE PAIR OF SMT-1ML is NP-complete even if ties occur only in the master list, or if ties occur only in the lists on the other side.

5.2 Minimum regret weakly stable matchings

We next consider minimum regret weakly stable matchings in SMT-1ML. We show that the problem of finding such a matching is solvable in linear time for a specific restriction of SMT-1ML, and NP-hard in general.

Theorem 5.2. MINIMUM REGRET SMT-*1*ML *can be solved in $O(n^2)$ time if there is no tie at the tail of the master list.*

Proof. Every person must be matched in every weakly stable matching. Since there is a unique person p at the end of the master list, in every weakly stable matching p's partner must have regret n. Break all the ties arbitrarily and find a stable matching for the derived instance of SM. This matching must be weakly stable in the initial instance, and can be found in $O(n^2)$ time. □

If there is a tie at the tail of the master list, it turns out that MINIMUM REGRET SMT-1ML is NP-hard, even if there are no ties on the other side.

Theorem 5.3. MINIMUM REGRET SMT-*1*ML-D *is NP-complete if there is a tie at the tail of the master list, even if there are no ties in the lists on the other side.*

Proof. Clearly MINIMUM REGRET SMT-1ML-D is in NP. To show that the problem is NP-hard, we transform from the variant of COMPLETE SMTI-2ML in which ties occur only in the master list of men, which is NP-complete by Theorem 3.1.

Let I be an instance of this problem, where $U = \{m_1, \ldots, m_n\}$ and $W = \{w_1, \ldots, w_n\}$ are the sets of men and women in I respectively, and let P_i be the preference list of m_i in I $(1 \leq i \leq n)$. Let L_m denote the master list of men. We construct an instance I' of MINIMUM REGRET SMT-1ML as follows: the set of men in I' is $U \cup X$ and the set of women in I' is $W \cup Y$, where $X = \{x_1, \ldots, x_{n+1}\}$ and $Y = \{y_1, \ldots, y_{n+1}\}$. The preference lists of the men are as follows:

$$\begin{array}{rllr} m_i : & P_i \;\; y_1 \;\; y_2 \;\; \ldots \;\; y_{n+1} & -- & (1 \leq i \leq n) \\ x_i : & y_1 \;\; y_2 \;\; \ldots \;\; y_{n+1} & -- & (1 \leq i \leq n+1) \end{array}$$

while the master list of men becomes $L'_m : L_m \;\; (X)$. Clearly there are no ties in the men's lists. It is straightforward to verify that I admits a complete weakly stable matching if and only if I' has a weakly stable matching of regret at most $n + 1$. □

5.3 Egalitarian weakly stable matchings

We now consider egalitarian weakly stable matchings. We give an NP-hardness and approximability result for the problem of finding such a matching, given an instance of SMT-1ML.

Theorem 5.4. EGALITARIAN SMT-*1*ML-D *is NP-complete.*

Proof. Clearly EGALITARIAN SMT-1ML-D is in NP. To show that the problem is NP-hard, we transform from COMPLETE SMTI-2ML, which is NP-complete, even if one master list is strictly ordered, by Theorem 3.1. Let I be an instance of this problem, where $U = \{m_1, \ldots, m_n\}$ and $W = \{w_1, \ldots, w_n\}$ are the sets of men and women in I respectively. Let L_m denote the master list of men. Without loss of generality we assume that L_m contains no ties. Let P_i be the preference list of m_i in I $(1 \leq i \leq n)$. We construct an instance I' of EGALITARIAN SMT-1ML-D as follows: the set of men in I' is $U \cup X$ and the set of women in I' is $W \cup Y$, where $X = \{x_1, \ldots, x_{n^2}\}$ and $Y = \{y_1, \ldots, y_{n^2}\}$. The preference lists of the men are as follows:

$$\begin{array}{rllr} m_i : & P_i \;\; y_1 \;\; y_2 \;\; \ldots \;\; y_{n^2} & -- & (1 \leq i \leq n) \\ x_i : & y_i & -- & (1 \leq i \leq n^2) \end{array}$$

while the master list of men becomes:

$$L'_m : x_1 \;\; \ldots \;\; x_{n^2} \;\; L_m.$$

We show that I admits a complete weakly stable matching if and only if I' has a weakly stable matching M' where the weight of M', $w(M')$, is at most $K = \frac{(n^2+n)(n^2+n+1)}{2} + 2n^2$.

For, suppose that I admits a complete weakly stable matching M. We create a matching $M' = M \cup \{(x_i, y_i) : 1 \leq i \leq n^2\}$ in I'. It is straightforward to verify that M' is weakly stable in I'. Each of the n^2 men in X contributes 1 to the weight of M', while each of the n men in U contributes at most n to the weight of M'. Finally, there are $n^2 + n$ women in I', so, since there are no ties in L'_m, the women contribute $\sum_{i=1}^{n^2+n} i = \frac{(n^2+n)(n^2+n+1)}{2}$ to the weight of M'. It follows that $w(M') \leq K$.

Conversely suppose that I' admits a weakly stable matching of weight at most K. Suppose for a contradiction that I does not admit a complete weakly stable matching. Let M' be an arbitrary weakly stable matching in I'. Then $(x_i, y_i) \in M'$ $(1 \leq i \leq n^2)$. Also there is some man $m_i \in U$ who is not matched with a woman from P_i. Then m_i contributes at least n^2+2 to $w(M')$. Each of the remaining n^2+n-1 men contributes at least 1 to $w(M')$. Since L'_m contains no ties, it follows that $w(M') \geq \frac{(n^2+n)(n^2+n+1)}{2}+2n^2+n+1 > K$. Since M' is arbitrary, we obtain a contradiction. Hence I admits a complete weakly stable matching after all. □

The above reduction implies that EGALITARIAN SMT-1ML is NP-complete even if the master list contains no ties. We also observe that, given an instance I of SMT-1ML, any weakly stable matching M in I satisfies

$$\frac{1}{2}n^2 + \frac{3}{2}n = n + \frac{n(n+1)}{2} \leq w(M) \leq n^2 + \frac{n(n+1)}{2} = \frac{3}{2}n^2 + \frac{1}{2}n.$$

This observation leads to the following result.

Theorem 5.5. EGALITARIAN SMT-*1*ML *is approximable within a factor of 3 when the master list is strict.*

5.4 Lexicographic maximum weakly stable matchings

Our final result in this section concerns the NP-hardness of computing a lexicographic maximum weakly stable matching, given an instance of SMT-1ML. Define the following decision problem:

RESTRICTED LEX MAX SMT-1ML-D
Instance: Instance I of SMT-1ML, and in addition, a vector $\langle c_1, \ldots, c_n \rangle$, where n is the number of men in I.
Question: Is there a weakly stable matching M in I such that $r_i(M) \geq c_i$, for each i $(1 \leq i \leq n)$?

We firstly show that RESTRICTED LEX MAX SMT-1ML-D is NP-complete.

Lemma 5.6. RESTRICTED LEX MAX SMT-*1*ML-D *is NP-complete.*

Proof. Clearly RESTRICTED LEX MAX SMT-1ML-D is in NP. To show NP-hardness, we transform from EXACT MM in subdivision graphs, which is NP-complete as discussed in the preamble to Theorem 3.1. Hence let G (a subdivision graph of some graph) and K (a positive integer) be an instance of EXACT MM. Then G is a bipartite graph, so that $G = (U, W, E)$, where without loss of generality each vertex in U has degree 2. As in the proof of Theorem 3.1, we assume that $|U| = |W| = n$ and that $K \leq n$.

Let $U = \{m_1, ..., m_n\}$ and $W = \{w_1, ..., w_n\}$. We construct an instance I of RESTRICTED LEX MAX SMT-1ML-D as follows. Let $U \cup X \cup \{p\}$ be the set of men in I, and let $W \cup Y \cup \{q\}$ be the set of women in I, where $X = \{x_1, ..., x_{n-K}\}$ and $Y = \{y_1, ..., y_{n-K}\}$. For each $m_i \in U$, let W_i denote the two vertices adjacent to m_i in G. The preference lists of the men in I are as follows:

$$\begin{array}{rllll} m_i: & q \;\; (W_i) \;\; (Y) & -- & (1 \leq i \leq n) \\ x_i: & (W) \;\; -- & & (1 \leq i \leq n-K) \\ p: & q \;\; -- & & \end{array}$$

whilst each woman's preference list is derived from the following master list of men:

$$L_m: \; p \;\; (U) \;\; (X).$$

Then I has N men and N women, where $N = 2n - K + 1$. Define the vector $\langle c_1, \ldots, c_N \rangle$ where $c_1 = n - K + 2$, $c_2 = n + K$ and $c_i = 0$ $(3 \leq i \leq N)$. We claim that G has a maximal matching of size K if and only if I has a weakly stable matching M' such that $r_i(M') \geq c_i$ $(1 \leq i \leq N)$.

For, suppose that G has a maximal matching M, where $|M| = K$. We construct a matching M' in I as follows. Firstly let $M' = M \cup \{(p, q)\}$. There remain $n - K$ men in U who are as yet unmatched in M' – denote these men by m_{a_i} $(1 \leq i \leq n - K)$. Add (m_{a_i}, y_i) to M' $(1 \leq i \leq n - K)$. Similarly there remain $n - K$ women in W who are as yet unmatched in M' – denote these women by w_{b_j} $(1 \leq j \leq n - K)$. Add (x_j, w_{b_j}) to M' $(1 \leq j \leq n - K)$. It is straightforward to verify that M' is a weakly stable matching in I such that $r_i(M') \geq c_i$ $(1 \leq i \leq N)$ as required.

Conversely, suppose that M' is a weakly stable matching in I, where $r_i(M') \geq c_i$ $(1 \leq i \leq N)$. We firstly observe that $(p, q) \in M'$. It follows immediately from the value of c_1 that each man in X is matched in M' to a woman in W. Hence exactly K women in W are matched in M' to a man in U. Now suppose that $(m_i, w_j) \in M'$ where $w_j \notin W_i$. Then $r_2(M') < K + K + (n - K) = c_2$, a contradiction. Let $M = M' \cap (U \times W)$. Then M is a matching in G such that $|M| = K$. Finally the weak stability of M' in I implies that M is maximal in G. □

The main result of this section follows immediately from the above result and the observation that, in the constructed instance I of RESTRICTED LEX MAX SMT-1ML-D, any weakly stable matching M' in I satisfies $r_1(M') \leq c_1$.

Theorem 5.7. LEX MAX SMT-*1*ML *is NP-hard.*

6 Super-stable and strongly stable matchings in SMTI

In this section we describe efficient algorithms to find a super-stable and a strongly stable matching, whenever such matchings exist, for an instance of SMTI-1ML. These algorithms apply, in simplified form, to more specialised variants, such as SMTI-2ML, SMT-1ML and SMT-2ML. We also show that, in this context, if a super-stable matching exists then it is unique. There may be more than one strongly stable matching, but the set of such matchings can be clearly identified, and the algorithm that we describe can return any one of these matchings depending on how the non-determinism within it is resolved. In both cases, the algorithms use a greedy strategy, and are simpler than the algorithms for the general case [11, 19].

6.1 Super-stable matchings

Suppose, without loss of generality, that there is a master list of men. There may or may not be a master list of women, and there may be ties in the master list(s) and/or in individual preference lists.

The algorithm for a super-stable matching, Algorithm SMTI-ML-Super, appears in Figure 3. It incorporates a greedy strategy that processes each tie T in the master list in turn. (Recall that a tie may be of length 1.) The heads of the current preference lists of the men in T are examined – we refer to the woman (or women) at the head of such a man's list as the *key* woman (or women) for that man. If a man in T has more than one key woman, or if any two men in T have the same key woman, then, as we will show, no super-stable matching can exist, and the algorithm returns null. Otherwise each man in T is paired with his key woman, these pairs are added to the potential super-stable matching, and the women in question are deleted from the lists of all the other men. If the end of the master list is reached then, as we will show, the matching so constructed is the unique super-stable matching for the instance.

```
M := ∅;
for each tie T in order in the master list of men {
    S := set of men in T with a non-empty list;
    if some man in S has a tie of length ≥ 2 at the head of his list
        return null;
    elsif two men in S have the same woman at the head of their lists
        return null;
    else {
        M := M ∪ {(m, w) : m ∈ S, w at head of m's list };
        delete each such w from all other men's lists;
    }
}
return M;
```

Figure 3: Algorithm SMTI-ML-Super

Note that, if the algorithm returns null, say when processing tie T, then for any man lower than T in the master list, his set of key women is undefined. If a man's preference list becomes empty during the execution of the algorithm then the set of key women for that man is empty. In all other cases, the set of key women for a man is well-defined, and depends only on the problem instance, since the execution of the algorithm is completely deterministic.

We now establish the correctness of Algorithm SMTI-ML-Super. We require a preliminary lemma.

Lemma 6.1. *(i) Let woman w be a key woman for man m, and let M be a super-stable matching. Then $(m, w) \in M$.*
(ii) If woman w is deleted from man m's list during Algorithm SMTI-ML-Super *then (m, w) cannot be a pair in any super-stable matching.*
(iii) If man m has an empty set of key women then m cannot be matched in any super-stable matching.

Proof. (i) Suppose that M is a super-stable matching in which m is not matched to w, and that every man preceding m in the master list is matched to a key woman (necessarily his unique key woman) in every super-stable matching. We claim that (m, w) must be a blocking pair for M. First of all, m cannot have a partner in M whom he prefers to w; the fact that w is a key woman for m means that any preferred woman x must have been deleted from m's list before the master list tie containing m was processed during the algorithm's execution, hence x is the (unique) key woman for some strict predecessor p of m in the master list, and by our assumption p must be matched to x in M. Secondly, w cannot have a partner in M whom she prefers to m, for, again by our assumption, any such man p is matched in M to his unique key woman y; moreover $y \neq w$ for otherwise w would have been deleted from m's list before the master list tie containing m was processed, a contradiction. This establishes the claim, so there can be no super-stable matching in which m is not matched to w.
(ii) Suppose that w is deleted from man m's list during Algorithm SMTI-ML-Super. Then w must be a key woman for some other man p, and by part (i), must be matched to p in any super-stable matching.
(iii) Man m has an empty set of key women only if all women have been deleted from his list, so that the conclusion follows immediately from part (ii). □

Theorem 6.2. *(a) If Algorithm* SMTI-ML-Super *returns a matching M then M is the unique super-stable matching for the given instance of* SMTI-*1*ML.

(b) If Algorithm SMTI-ML-Super *returns null then there is no super-stable matching for the given instance of* SMTI-*1*ML.

Proof. (a) Suppose that the algorithm returns matching M, and that the pair (m, w) is a blocking pair of M as a super-stable matching. Suppose first that m is unmatched in M. Then at the point where the algorithm processes the master list tie T containing m, m's preference list must be empty. So w must have been removed from m's list when some earlier tie was processed, and as a consequence w must prefer $M(w)$ to m. Now suppose that m prefers w to $M(m)$. As in the previous case, w must have been removed from m's list when some earlier tie was processed, so that w prefers $M(w)$ to m. Finally, if w and $M(m)$ appear in the same tie on m's list then either the previous case applies again, or m has at least two key women, a contradiction in either case.

The fact that M is the unique super-stable matching follows at once from Lemma 6.1(i).

(b) First suppose that the algorithm returns null because two men m and p have the same sole key woman w. By Lemma 6.1, m and p must both be matched to w in any super-stable matching, so there can be no such matching.

Now suppose that the algorithm returns null because some man m has two key women, say w and x. Then, again by Lemma 6.1(i), m must be matched in M to both w and x, so again there can be no such matching. ☐

For the complexity analysis, we make the not unreasonable assumption that every man appears in the preference list of at least one woman, so that the sum a of the lengths of the preference lists is at least n. It is not hard to see that, implemented with suitable data structures, each deletion can be accomplished in constant time, so that Algorithm SMTI-ML-Super has complexity $O(a)$.

6.2 Strongly stable matchings

The algorithm for a strongly stable matching, Algorithm SMTI-ML-Strong, appears in Figure 4. Again it is based on a greedy strategy that processes each tie T in the master list in turn. The concept of key women is defined as before. However this time, the existence of a strongly stable matching implies a weaker necessary condition on the sets of key women, namely that their union is equal in size to the set of men in T, and that they have a set of distinct representatives (SDR). If this is not the case, then the algorithm returns null, and as we will show, there cannot be a strongly stable matching. Otherwise, each of the men in T is matched with a different key woman, these pairs are added to the potential strongly stable matching, and the women in question are deleted from the lists of all other men. If the end of the master list is reached, then as we will show, the matching so constructed is a strongly stable matching, and all such matchings can be obtained by an execution of this algorithm (with suitable choices of SDRs at each stage).

We now establish the correctness of Algorithm SMTI-ML-Strong. We require a preliminary lemma whose statement and proof are analogous to those of Lemma 6.1.

Lemma 6.3. *(i) Let $w_1, \ldots, w_r$ be the set of key women for man m, and let M be a strongly stable matching. Then $(m, w_i) \in M$ for some i $(1 \leq i \leq r)$.*
(ii) If woman w is deleted from man m's list during Algorithm SMTI-ML-Strong *then (m, w) cannot be a pair in any strongly stable matching.*
(iii) If man m has an empty set of key women then m cannot be matched in any strongly stable matching.

```
M := ∅;
for each tie T in order in the master list of men {
    let {m_1, ..., m_k} be the set of men in T with a non-empty list;
    let W_i be the set of key women for m_i (1 ≤ i ≤ k);
    if |W_1 ∪ ... ∪ W_k| = k and W_1, ..., W_k have an SDR {
        let w_1, ..., w_k be an SDR for W_1, ..., W_k;
        for i in 1 ... k {
            M := M ∪ {(m_i, w_i)};
            delete w_i from the list of each successor of m_i in the master list;
        }
    }
    else
        return null;
}
return M;
```

Figure 4: Algorithm SMTI-ML-Strong

Proof. (i) Suppose that M is a strongly stable matching in which m is not matched to any of his key women, and that every man preceding m in the master list is matched to a key woman x in every strongly stable matching. Note that this assumption also implies that all of these women x are matched in M to a man for whom they are a key woman (since there are exactly the right number of them). Then at least one of m's key women, say w_i, is not matched in M to any of the men tied with m in the master list, since the number of such men must be equal to the size of the union of their sets of key women. We claim that (m, w_i) must be a blocking pair for M. First of all, m cannot have a partner in M whom he prefers to w_i; the fact that w_i is a key woman for m means that any preferred woman y must have been deleted from m's list before the master list tie containing m was processed during the algorithm's execution, hence y is a key woman for some strict predecessor(s) of m in the master list, and by our assumption y must be matched to such a predecessor in M. Nor, by our assumption, can m have a partner tied with w_i in his preference list – so m must strictly prefer w_i to his partner in M (or is unmatched in M). Also, w_i cannot have a partner in M whom she prefers to m, for, again by our assumption, any such man p is matched in M to a key woman z; moreover $z \neq w_i$ for otherwise w_i would have been deleted from m's list before the master list tie containing m was processed, a contradiction. This establishes the claim, so there can be no strongly stable matching in which m is not matched to one of his key women.
(ii) Suppose that w is deleted from man m's list during Algorithm SMTI-ML-Strong. Then w must be a key woman for some other man p, and by part (i), and the fact that there are just enough key women to be matched with the tied men at each stage, she must be matched to p, or some man tied with p in the master list, in any strongly stable matching. As m is a successor of p on the master list, it follows that (m, w) cannot belong to a strongly stable matching.
(iii) Man m has an empty set of key women only if all women have been deleted from his list, so that the conclusion follows immediately from part (ii). □

Theorem 6.4. *(a) If Algorithm* SMTI-ML-Strong *returns a matching M then M is a strongly stable matching for the given instance of* SMTI-1ML.
(b) If Algorithm SMTI-ML-Strong *returns null then there is no strongly stable matching for the given instance of* SMTI-1ML.
(c) Every strongly stable matching is returned by some execution of the algorithm with appropriate choice of SDR at each stage.

Proof. (a) Suppose that the algorithm returns matching M, and that the pair (m, w) is a blocking pair of M as a strongly stable matching. Suppose first that m is unmatched in M. Then at the point where the algorithm processes the master list tie T containing m, m's preference list must be empty. So w must have been removed from m's list when some earlier tie was processed, and as a consequence w must prefer $M(w)$ to m, a contradiction. Next suppose that m prefers w to $M(m)$. As in the previous case, w must have been removed from m's list when some earlier tie was being processed, so that w prefers $M(w)$ to m. Finally, if w and $M(m)$ appear in the same tie on m's list then either the previous case applies again, or w and $M(m)$ are both key women for m. In this latter case, the algorithm will match w to some other man in tie T, so that (m, w) does not, after all, form a blocking pair.

(b) For a tie T in the master list, let $S = \{m_1, \ldots, m_k\}$ be the set of men in T with a non-empty list when T is processed, and let W_i be the set of key women for m_i $(1 \leq i \leq k)$. Suppose that $|\bigcup W_i| < k$ or that $W_1, \ldots, W_k$ do not have an SDR. Then, in a would-be strongly stable matching M, some man m in S is not matched to a key woman, which is a contradiction to Lemma 6.3(i).

Now suppose that $|\bigcup W_i| > k$. Then if M is a strongly stable matching, there is some woman w who is a key woman for a man m in S, but who is not matched in M to a man in S. She cannot be matched to a man in $T \setminus S$ since, by Lemma 6.3(iii) such men are unmatched in M. And nor can she be matched to a man that precedes T in the master list, since she cannot be a key woman for any such man. So w prefers m to her partner in M, or is unmatched in M. Since, by Lemma 6.3(i), m must be matched in M to a key woman, and therefore does not prefer w to his partner in M, the pair (m, w) is a blocking pair of M.

(c) Let M be a strongly stable matching for the given instance. As observed earlier, M must consist of a set of pairs (m, w) where w is a key woman for m, and every man who has a key woman is matched with one of them. It is immediate that any such matching can be generated by an application of the algorithm. □

The complexity of algorithm SMTI-ML-Strong is dominated by the need to check for the existence of a system of distinct representatives. If the ties $T_1, \ldots T_k$ in the master list are of lengths $t_1, \ldots, t_k$, then all the checks for an SDR using the perfect matching algorithm of Hopcroft and Karp [9] can be achieved in $O(\Sigma\sqrt{t_i}m_i)$ time, where m_i is the number of (man, woman) pairs (m, w) such that m belongs to T_i and w is a key woman for m. Here $\Sigma t_i = n$ and $\Sigma m_i \leq a$. Hence the overall complexity of the algorithm is $O(\sqrt{n}a)$, which contrasts with the best known bound for a strongly stable matching in a general instance of SMTI, namely $O(na)$ [18].

7 Summary and conclusion

In this paper we have presented a range of algorithmic results for variants of SM where individual preference lists may be derived from master lists of the men and/or women. Many of our results refer to the weak stability criterion – these results are summarised in Table 1. The table rows labelled 'Maximum', 'Min regret', 'Egalitarian' and 'Lexicographic' refer to the problems of finding a maximum, minimum regret, egalitarian and lexicographic maximum weakly stable matching, given an instance of the problem specified in each column. The table row labelled 'Stable pair' refers to the problem of deciding whether a given (man, woman) pair is weakly stable, given an instance of the problem specified in each column. In the body of the table, 'P' denotes polynomial-time solvable, whilst 'N' denotes NP-hard. Entries corresponding to 'Min Regret', 'Egalitarian' and

'Lexicographic' in SMTI-2ML are not given, since these problems are not well-defined for SMTI (in view of the fact that weakly stable matchings can have different sizes).

	SMI-1ML	SMTI-2ML	SMT-2ML	SMT-1ML
Maximum	P	N	P	P
Stable pair	P	N	P	N
Min regret	P	–	P	N
Egalitarian	P	–	P	N
Lexicographic	P	–	P	N

Table 1: Summary of results for master list problems involving weak stability.

Many of the results presented in this paper first appeared in Chapter 8 of [26], to which we refer the interested reader for more details concerning the algorithmic complexity of variants of SMTI involving master lists.

As described in Section 1, stable matching problems with master lists arise in large-scale applications such as the assignment of junior doctors to hospitals. The NP-hardness and inapproximability results presented in this paper for variants of SMT under weak stability involving master lists clearly carry over to the corresponding variants of HRT. Furthermore, the algorithms for SMTI-1ML under strong stability and super-stability have been extended to the HRT-1ML case [23].

Also, in many practical applications, the preference lists of at least one side tend to be short. If we combine this constraint with the presence of one or two master lists, then we are led to instances of SMTI and its variants where the individual preference lists on a given side are both of bounded length and derived from a given master list. Theorem 3.3 shows that MAX SMTI-2ML is NP-hard, even if the length of each man's individual list is at most L_1 and that of each woman's individual list is at most L_2, where $L_1 = 4$ and $L_2 = 3$. This leaves open the complexity of the problem when $L_1 < 4$ and/or $L_2 < 3$.

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