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Factorial characters of some classical Lie groups

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Abstract

A definition is offered of the factorial characters of the general linear group, the symplectic group and the orthogonal group in an odd dimensional space. It is shown that these characters satisfy certain flagged Jacobi-Trudi identities. These identities are then used to give combinatorial expressions for the factorial characters: first in terms of a lattice path model and then in terms of the well known tableaux associated with the classical groups. Factorial Q-functions are then defined in terms of three sets of primed shifted tableaux, and shown to satisfy Tokuyama type identities in each case.

1 Definition of factorial characters

A class of symmetric polynomials $t_\lambda(\mathbf{x})$ labelled by partitions $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ was introduced by Biedenharn and Louck [1] as a new integral basis of the ring of all symmetric polynomials in the parameters $\mathbf{x}=(x_1,x_2,\ldots,x_n)$. These new symmetric but inhomogeneous polynomials were studied further and called factorial Schur functions by Chen and Louck [3]. They were given a more general form by Goulden and Greene [8] and by Macdonald [18] who introduced the notation $s_\lambda(\mathbf{x}\,|\,\mathbf{a})$, where $\mathbf{a}=(a_1,a_2,\ldots)$ is an infinite sequence of factorial parameters. Macdonald gave a definition of factorial Schur functions $s_\lambda(\mathbf{x}\,|\,\mathbf{a})$ as a ratio of determinants, exactly analagous to that applying to ordinary Schur functions $s_\lambda(\mathbf{x})$. Since Schur functions, through their determinantal definition, can be identified with characters of $GL(n,\mathbb{C})$, it is not unreasonable to refer to factorial Schur functions as factorial characters of $GL(n,\mathbb{C})$. What we then aim to offer here is a definition of factorial characters of the classical groups $Sp(2n,\mathbb{C})$ and $SO(2n+1,\mathbb{C})$. We defer the more difficult case of $SO(2n,\mathbb{C})$ for consideration elsewhere.

For each of the three groups $G=GL(n,\mathbb{C})$, $Sp(2n,\mathbb{C})$ and $SO(2n+1,\mathbb{C})$ there exists a finite dimensional irreducible representation V_G^λ of highest weight λ, where $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ is a partition of length $\ell(\lambda)\le n$. Its character may be denoted by $\mathrm{ch}\,V_G^\lambda(\mathbf{z})$ where $\mathbf{z}=(z_1,z_2,\ldots,z_N)$ is a suitable parametrisation of the N eigenvalues of the group elements of $G=GL(n,\mathbb{C})$, $Sp(2n,\mathbb{C})$ and $SO(2n+1,\mathbb{C})$ with $N=n$, $2n$ and $2n+1$, respectively. Setting $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ and $\overline{\mathbf{x}}=(\overline{x}_1,\overline{x}_2,\ldots,\overline{x}_n)$ with $\overline{x}_i=x_i^{-1}$ for

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$i = 1, 2, \ldots, n$, for use throughout this paper, we adopt a notation akin to that used for Schur functions, whereby $s_\lambda(\mathbf{x}) = \mathrm{ch}\, V^\lambda_{GL(n,\mathbb{C})}(\mathbf{x})$, $sp_\lambda(\mathbf{x}, \overline{\mathbf{x}}) = \mathrm{ch}\, V^\lambda_{Sp(2n,\mathbb{C})}(\mathbf{x}, \overline{\mathbf{x}})$ and $so_\lambda(\mathbf{x}, \overline{\mathbf{x}}, 1) = \mathrm{ch}\, V^\lambda_{SO(2n+1,\mathbb{C})}(\mathbf{x}, \overline{\mathbf{x}}, 1)$.

In the case of $GL(n, \mathbb{C})$ the transition from a Schur function, $s_\lambda(\mathbf{x})$, to a factorial Schur function $s_\lambda(\mathbf{x}\,|\,\mathbf{a})$ involves an infinite sequence of factorial parameters $\mathbf{a} = (a_1, a_2, \ldots)$. The transition is effected by replacing each non-negative power x_i^m of x_i by its factorial power $(x_i\,|\,\mathbf{a})^m$ defined by:

$$(x_i\,|\,\mathbf{a})^m = \begin{cases} (x_i + a_1)(x_i + a_2) \cdots (x_i + a_m) & \text{if } m > 0; \\ 1 & \text{if } m = 0. \end{cases} \tag{1.1}$$

In the case of the other groups in order to accomodate negative powers $x_i^{-m} = \overline{x}_i^m$ of x_i it is convenient to let

$$(\overline{x}_i\,|\,\mathbf{a})^m = \begin{cases} (\overline{x}_i + a_1)(\overline{x}_i + a_2) \cdots (\overline{x}_i + a_m) & \text{if } m > 0; \\ 1 & \text{if } m = 0. \end{cases} \tag{1.2}$$

We then propose the following definition of factorial characters of the classical Lie groups:

Definition 1.1 *For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and any $\mathbf{a} = (a_1, a_2, \ldots)$ let*

$$s_\lambda(\mathbf{x}\,|\,\mathbf{a}) = \frac{|\,(x_i\,|\,\mathbf{a})^{\lambda_j+n-j}\,|}{|\,(x_i\,|\,\mathbf{a})^{n-j}\,|}; \tag{1.3}$$

$$sp_\lambda(\mathbf{x}, \overline{\mathbf{x}}\,|\,\mathbf{a}) = \frac{|\,x_i(x_i\,|\,\mathbf{a})^{\lambda_j+n-j} - \overline{x}_i(\overline{x}_i\,|\,\mathbf{a})^{\lambda_j+n-j}\,|}{|\,x_i(x_i\,|\,\mathbf{a})^{n-j} - \overline{x}_i(\overline{x}_i\,|\,\mathbf{a})^{n-j}\,|}; \tag{1.4}$$

$$so_\lambda(\mathbf{x}, \overline{\mathbf{x}}, 1\,|\,\mathbf{a}) = \frac{\left|\,x_i^{1/2}(x_i\,|\,\mathbf{a})^{\lambda_j+n-j} - \overline{x}_i^{1/2}(\overline{x}_i\,|\,\mathbf{a})^{\lambda_j+n-j}\,\right|}{\left|\,x_i^{1/2}(x_i\,|\,\mathbf{a})^{n-j} - \overline{x}_i^{1/2}(\overline{x}_i\,|\,\mathbf{a})^{n-j}\,\right|}. \tag{1.5}$$

Setting $\mathbf{a} = \mathbf{0} = (0, 0, \ldots)$ one recovers the classical non-factorial characters of $GL(n, \mathbb{C})$, $Sp(2n, \mathbb{C})$ and $SO(2n+1, \mathbb{C})$, as given for example in [5]. That these definitions are appropriate for non-zero $\mathbf{a}$, in particular the separation out of the factors x_i, $\overline{x}_i$, $x_i^{1/2}$ and $\overline{x}_i^{1/2}$, depends to what extent the properties of these factorial characters are truly analagous to those of factorial Schur functions. We have in mind things like deriving for each of our factorial characters some sort of factorial Jacobi-Trudi identity and a combinatorial interpretation in terms of tableaux as established in the case of factorial Schur functions by Macdonald [18], and perhaps more ambitiously, the derivation of Tokuyama type identities [27] as recently derived in the factorial Schur function case by Bump, McNamara and Nakasuji [2], with an alternative derivation appearing in [10]. The key to accomplishing all this is the identification of appropriate analogues of the complete homogeneous symmetric functions $h_m(\mathbf{x})$. This is done for each group in the case of both ordinary and factorial characters in Section 2, in which merely by manipulating determinants, factorial flagged Jacobi-Trudi identities are derived for each of our factorial characters.

This is followed in Section 3 by consideration of the special cases of one part partitions $\lambda = (m)$. This enables us to build up in Section 4 a combinatorial realisation of factorial characters, first in terms of non-intersecting lattice path models and then in terms of the tableaux traditionally used to specify classical, non-factorial characters. Encouraged by this, we present in Section 5 definitions of factorial Q-functions in terms of appropriate primed shifted tableaux in the general linear, symplectic and odd orthogonal cases. Reversing our previous trajectory, we proceed in Section 6 from these primed tableaux to non-intersecting lattice path models and thence to determinantal expressions for each of our factorial Q-function. Finally, in Section 7 it is demonstrated that in the case $\ell(\lambda) = n$ each of our factorial Q-functions factorises into a simple product multiplied by a factorial character, thereby generalising Tokuyama's classical identity [27] to this factorial context.

2 Factorial flagged Jacobi-Trui identities

In the study of symmetric functions a key role is played by the complete homogeneous symmetric functions $h_m(\mathbf{x})$, with $h_\lambda(\mathbf{x}) = h_{\lambda_1}(\mathbf{x})h_{\lambda_2}(\mathbf{x})\dots h_{\lambda_n}(\mathbf{x})$ forming a multiplicative basis of the ring of symmetric functions, in terms of which we have the Jacobi-Trudi identity $s_\lambda(\mathbf{x}) = |h_{\lambda_j - j + i}(\mathbf{x})|$. In order to try to establish factorial Jacobi-Trudi identites we need analogues $h_m(\mathbf{x}\,|\,\mathbf{a})$ of $h_m(\mathbf{x})$ that are appropriate not only to the case of the other group characters in the case $\mathbf{a} = \mathbf{0}$ but also to the case of our factorial characters for general $\mathbf{a}$. Just as is done classically for $h_m(\mathbf{x})$ it is convenient to define all these analogues $h_m(\mathbf{x}\,|\,\mathbf{a})$ by means of generating functions. Our notation is such that each generating function $F(\mathbf{z}, \mathbf{a}; t)$ may be expanded as a power series in t, and we denote the coefficient of t^m in such an expansion by $[t^m]\ F(\mathbf{z}, \mathbf{a}; t)$ for all integers m. In our factorial situation we make the following definitions in terms of generating functions $F_m(\mathbf{z}, \mathbf{a}; t)$ that are truncated in the sense that the power m of $[t^m]$ appears in an upper limit of the associated generating function.

Definition 2.1 *For any integer $m \geq 0$ and $\mathbf{a} = (a_1, a_2, \dots)$ let*

$$h_m(\mathbf{x}\,|\,\mathbf{a}) = [t^m]\ \prod_{i=1}^{n} \frac{1}{1 - tx_i} \prod_{j=1}^{n+m-1} (1 + ta_j)\,; \tag{2.1}$$

$$h^{sp}_m(\mathbf{x}, \overline{\mathbf{x}}\,|\,\mathbf{a}) = [t^m]\ \prod_{i=1}^{n} \frac{1}{(1 - tx_i)(1 - t\overline{x}_i)} \prod_{j=1}^{n+m-1} (1 + ta_j)\,; \tag{2.2}$$

$$h^{so}_m(\mathbf{x}, \overline{\mathbf{x}}, 1\,|\,\mathbf{a}) = [t^m]\ (1 + t) \prod_{i=1}^{n} \frac{1}{(1 - tx_i)(1 - t\overline{x}_i)} \prod_{j=1}^{n+m-1} (1 + ta_j)\,; \tag{2.3}$$

Then for $m = 0$ we have $h_0(\mathbf{x}\,|\,\mathbf{a}) = h^{sp}_0(\mathbf{x}, \overline{\mathbf{x}}\,|\,\mathbf{a}) = h^{so}_0(\mathbf{x}, \overline{\mathbf{x}}, 1\,|\,\mathbf{a}) = 1$, while for $m < 0$ we set $h_m(\mathbf{x}\,|\,\mathbf{a}) = h^{sp}_m(\mathbf{x}, \overline{\mathbf{x}}\,|\,\mathbf{a}) = h^{so}_m(\mathbf{x}, \overline{\mathbf{x}}, 1\,|\,\mathbf{a}) = 0$.

The one variable case $\mathbf{x} = (x_i)$ of these Definitions 2.1 allow us to rewrite our factorial characters in the following manner:

Lemma 2.2 *For any partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ *and* $\mathbf{a} = (a_1, a_2, \ldots)$

$$s_\lambda(\mathbf{x}\,|\,\mathbf{a}) = \frac{\left| h_{\lambda_j+n-j}(x_i\,|\,\mathbf{a}) \right|}{\left| h_{n-j}(x_i\,|\,\mathbf{a}) \right|}; \tag{2.4}$$

$$sp_\lambda(\mathbf{x},\overline{\mathbf{x}}\,|\,\mathbf{a}) = \frac{\left| h^{sp}_{\lambda_j+n-j}(x_i,\overline{x}_i\,|\,\mathbf{a}) \right|}{\left| h^{sp}_{n-j}(x_i,\overline{x}_i\,|\,\mathbf{a}) \right|}; \tag{2.5}$$

$$so_\lambda(\mathbf{x},\overline{\mathbf{x}},1\,|\,\mathbf{a}) = \frac{\left| h^{so}_{\lambda_j+n-j}(x_i,\overline{x}_i,1\,|\,\mathbf{a}) \right|}{\left| h^{so}_{n-j}(x_i,\overline{x}_i\,|\,\mathbf{a}) \right|}. \tag{2.6}$$

Proof: In the case of $s_\lambda(\mathbf{x}\,|\,\mathbf{a})$ it suffices to note that for $m \geq 0$

$$\begin{aligned} h_m(x_i\,|\,\mathbf{a}) &= [t^m]\frac{1}{1-tx_i}\prod_{j=1}^{m}(1+ta_j) = [t^m]\frac{1+ta_m}{1-tx_i}\prod_{j=1}^{m-1}(1+ta_j) \\ &= [t^m]\left(1+\frac{t(x_i+a_m)}{1-tx_i}\right)\prod_{j=1}^{m-1}(1+ta_j) = (x_i+a_m)[t^{m-1}]\frac{1}{1-tx_i}\prod_{j=1}^{m-1}(1+ta_j) \\ &= (x_i+a_m)(x_i+a_{m-1})\cdots(x_i+a_1)[t^0]\frac{1}{1-tx_i} = (x_i\,|\,\mathbf{a})^m . \end{aligned} \tag{2.7}$$

One then just uses this identity in (1.3) with $m = \lambda_j+n-j$ and $m = n-j$ in the numerator and denominator, respectively.

In the case $sp_\lambda(\mathbf{x},\overline{\mathbf{x}}\,|\,\mathbf{a})$ we have

$$\begin{aligned} h^{sp}_m(x_i,\overline{x}_i\,|\,\mathbf{a}) &= [t^m]\frac{1}{(1-tx_i)(1-t\overline{x}_i)}\prod_{j=1}^{m}(1+ta_j) \\ &= [t^m]\frac{1}{x_i-\overline{x}_i}\left(\frac{x_i}{1-tx_i}-\frac{\overline{x}_i}{1-t\overline{x}_i}\right)\prod_{j=1}^{m}(1+ta_j) \\ &= \frac{1}{x_i-\overline{x}_i}\left(x_i(x_i\,|\,\mathbf{a})^m - \overline{x}_i(\overline{x}_i\,|\,\mathbf{a})^m\right), \end{aligned} \tag{2.8}$$

where use has been made of (2.7). The required result follows by using these identities in (1.4) as before, with the cancellation between numerator and denominator of the common factors $x_i - \overline{x}_i$ for $i = 1, 2, \ldots, n$.

Similarly, in the $so_\lambda(\mathbf{x},\overline{\mathbf{x}},1\,|\,\mathbf{a})$ case we have

$$\begin{aligned} h^{so}_m(x_i,\overline{x}_i,1\,|\,\mathbf{a}) &= [t^m]\frac{1+t}{(1-tx_i)(1-t\overline{x}_i)}\prod_{j=1}^{m}(1+ta_j) \\ &= [t^m]\frac{1}{x_i^{1/2}-\overline{x}_i^{1/2}}\left(\frac{x_i^{1/2}}{1-tx_i}-\frac{\overline{x}_i^{1/2}}{1-t\overline{x}_i}\right)\prod_{j=1}^{m}(1+ta_j) \\ &= \frac{1}{x_i^{1/2}-\overline{x}_i^{1/2}}\left(x_i^{1/2}(x_i\,|\,\mathbf{a})^m - \overline{x}_i^{1/2}(\overline{x}_i\,|\,\mathbf{a})^m\right). \end{aligned} \tag{2.9}$$

Then one again uses this identity in (1.5) as before, with the cancellation this time of the common factors $x_i^{1/2} - \overline{x}_i^{1/2}$ for $i = 1, 2, \ldots, n$. □

The next step is to transform each of the expressions in Lemma 2.2 into some sort of flagged Jacobi-Trudi identity. This is accomplished by means of the following Lemma:

Lemma 2.3 *For all i and j such that $1 \le i < j \le n$ and all integers m:*

$$\begin{aligned} &h_m(x_i,\dots x_{j-1}\,|\,\mathbf{a}) - h_m(x_{i+1},\dots,x_j\,|\,\mathbf{a}) \\ &\qquad = (x_i - x_j)h_{m-1}(x_i,\dots,x_j\,|\,\mathbf{a})\,; \qquad (2.10) \\ &h^{sp}_m(x_i,\overline{x}_i,\dots,x_{j-1},\overline{x}_{j-1}\,|\,\mathbf{a}) - h^{sp}_m(x_{i+1},\overline{x}_{i+1},\dots,x_j,\overline{x}_j,\,|\,\mathbf{a}) \\ &\qquad = (x_i - x_j)(1-\overline{x}_i\overline{x}_j)h^{sp}_{m-1}(x_i,\overline{x}_i,\dots,x_j,\overline{x}_j,\,|\,\mathbf{a}); \qquad (2.11) \\ &h^{so}_m(x_i,\overline{x}_i,\dots,x_{j-1},\overline{x}_{j-1},1\,|\,\mathbf{a}) - h^{so}_m(x_{i+1},\overline{x}_{i+1},\dots,x_j,\overline{x}_j,1\,|\,\mathbf{a}) \\ &\qquad = (x_i - x_j)(1-\overline{x}_i\overline{x}_j)h^{so}_{m-1}(x_i,\overline{x}_i,\dots,x_j,\overline{x}_j,1\,|\,\mathbf{a})\,. \qquad (2.12) \end{aligned}$$

Proof: First it should be noted that all these identities are trivially true for $m < 0$ and for $m = 0$ since each h_m reduces to either 0 or 1, with each h_{m-1} reducing to 0. For $m > 0$, in the simplest case

$$h_m(x_i,\dots x_{j-1}\,|\,\mathbf{a}) - h_m(x_{i+1},\dots,x_j\,|\,\mathbf{a}) = [t^m]\ \big((1-tx_j)-(1-tx_i)\big)\prod_{\ell=i}^{j}\frac{1}{1-tx_\ell}\prod_{k=1}^{m+j-i-1}(1+ta_k)$$
$$= (x_i-x_j)\ [t^{m-1}]\ \prod_{\ell=i}^{j}\frac{1}{1-tx_\ell}\prod_{k=1}^{(m-1)+j-i}(1+ta_k) = (x_i-x_j)h_{m-1}(x_i,\dots,x_j\,|\,\mathbf{a})\,. \qquad (2.13)$$

The other two cases are essentially the same, and are illustrated by the symplectic case:

$$\begin{aligned} &h^{sp}_m(x_i,\overline{x}_i,\dots,x_{j-1},\overline{x}_{j-1}\,|\,\mathbf{a}) - h^{sp}_m(x_{i+1},\overline{x}_{i+1},\dots,x_j,\overline{x}_j,\,|\,\mathbf{a}) \\ &= [t^m]\ \big((1-tx_j)(1-t\overline{x}_j)-(1-tx_i)(1-t\overline{x}_i)\big)\prod_{\ell=i}^{j}\frac{1}{(1-tx_\ell)(1-t\overline{x}_\ell)}\prod_{k=1}^{m+j-i-1}(1+ta_k) \\ &= (x_i+\overline{x}_i-x_j-\overline{x}_j)\ [t^{m-1}]\prod_{\ell=i}^{j}\frac{1}{(1-tx_\ell)(1-t\overline{x}_\ell)}\prod_{k=1}^{(m-1)+j-i}(1+ta_k) \\ &= (x_i-x_j)(1-\overline{x}_i\overline{x}_j)\ h^{sp}_{m-1}(x_i,\overline{x}_i,\dots,x_j,\overline{x}_j,\,|\,\mathbf{a})\,. \qquad (2.14) \end{aligned}$$

Exactly the same procedure applies to the odd orthogonal case. □

Now we are a position to state and prove the following result:

Theorem 2.4 *Let $\mathbf{x} = (x_1,x_2,\dots,x_n)$ and $\overline{\mathbf{x}} = (\overline{x}_1,\overline{x}_2,\dots,\overline{x}_n)$ with $\mathbf{x}^{(i)} = (x_i,x_{i+1},\dots,x_n)$ and $\overline{\mathbf{x}}^{(i)} = (\overline{x}_i,\overline{x}_{i+1},\dots,\overline{x}_n)$ for $i = 1,2,\dots,n$. Then for any partition λ of length $\ell(\lambda) \le n$ and any $\mathbf{a} = (a_1,a_2,\dots)$ we have*

$$s_\lambda(\mathbf{x}\,|\,\mathbf{a}) = \left|\,h_{\lambda_j-j+i}(\mathbf{x}^{(i)}\,|\,\mathbf{a})\,\right|; \qquad (2.15)$$

$$sp_\lambda(\mathbf{x},\overline{\mathbf{x}}\,|\,\mathbf{a}) = \left|\,h^{sp}_{\lambda_j-j+i}(\mathbf{x}^{(i)},\overline{\mathbf{x}}^{(i)}\,|\,\mathbf{a})\,\right|; \qquad (2.16)$$

$$so_\lambda(\mathbf{x},\overline{\mathbf{x}},1\,|\,\mathbf{a}) = \left|\,h^{so}_{\lambda_j-j+i}(\mathbf{x}^{(i)},\overline{\mathbf{x}}^{(i)},1\,|\,\mathbf{a})\,\right|. \qquad (2.17)$$

Proof: In the classical non-factorial case, obtained by setting $\mathbf{a} = \mathbf{0}$, these flagged Jacobi-Trudi identities have all been obtained previously by Okada [21] by means of lattice path methods. The symplectic case was also independently obtained by this means by Chen, Li and Louck [4], while the Schur function case goes back at least to Littlewood [17] who obtained it en route to his derivation of the classical Jacobi-Trudi identity by means of simple row manipulations of determinants.

It is Littlewood's method that we use here to establish all three identities. Subtracting row $(i+1)$ from row i for $i = 1, 2, \ldots, n-1$ in the numerator of (2.4) and applying (2.10), then repeating the process for $i = 1, 2, \ldots, n-2$ and so on, yields

$$\begin{aligned}
\left| h_{\lambda_j+n-j}(x_i \,|\, \mathbf{a}) \right| &= \left| \begin{array}{c} h_{\lambda_j+n-j}(x_i \,|\, \mathbf{a}) - h_{\lambda_j+n-j}(x_{i+1} \,|\, \mathbf{a}) \\ h_{\lambda_j+n-j}(x_n \,|\, \mathbf{a}) \end{array} \right| \\
&= \prod_{i=1}^{n-1} (x_i - x_{i+1}) \left| \begin{array}{c} h_{\lambda_j+n-j-1}(x_i, x_{i+1} \,|\, \mathbf{a}) \\ h_{\lambda_j+n-j}(x_n \,|\, \mathbf{a}) \end{array} \right| \\
&= \prod_{i=1}^{n-1} (x_i - x_{i+1}) \left| \begin{array}{c} h_{\lambda_j+n-j-1}(x_i, x_{i+1} \,|\, \mathbf{a}) - h_{\lambda_j+n-j-1}(x_{i+1}, x_{i+2} \,|\, \mathbf{a}) \\ h_{\lambda_j+n-j-1}(x_{n-1}, x_n \,|\, \mathbf{a}) \\ h_{\lambda_j+n-j}(x_n \,|\, \mathbf{a}) \end{array} \right| \\
&= \prod_{i=1}^{n-1} (x_i - x_{i+1}) \prod_{i=1}^{n-2} (x_i - x_{i+2}) \left| \begin{array}{c} h_{\lambda_j+n-j-2}(x_i, x_{i+1}, x_{i+2} \,|\, \mathbf{a}) \\ h_{\lambda_j+n-j-1}(x_{n-1}, x_n \,|\, \mathbf{a}) \\ h_{\lambda_j+n-j}(x_n \,|\, \mathbf{a}) \end{array} \right| \\
&= \cdots = \prod_{1 \le i < j \le n} (x_i - x_j) \left| h_{\lambda_j+n-j-(n-i)}(x_i, x_{i+1}, \ldots, x_n \,|\, \mathbf{a}) \right| \\
&= \prod_{1 \le i < j \le n} (x_i - x_j) \left| h_{\lambda_j-j+i}(\mathbf{x}^{(i)} \,|\, \mathbf{a}) \right| .
\end{aligned} \tag{2.18}$$

In the special case $\lambda = (0)$ this yields the denominator identity

$$\left| h_{n-j}(x_i \,|\, \mathbf{a}) \right| = \prod_{1 \le i < j \le n} (x_i - x_j) \left| h_{-j+i}(\mathbf{x}^{(i)} \,|\, \mathbf{a}) \right| = \prod_{1 \le i < j \le n} (x_i - x_j) \tag{2.19}$$

since the determinant $\left| h_{-j+i}(\mathbf{x}^{(i)} \,|\, \mathbf{a}) \right|$ is lower-triangular with all its diagonal elements equal to $h_0(\mathbf{x}^{(i)} \,|\, \mathbf{a}) = 1$. Taking the ratio of these two formulae implies that $s_\lambda(x \,|\, \mathbf{a}) = \left| h_{\lambda_j-j+i}(\mathbf{x}^{(i)} \,|\, \mathbf{a}) \right|$, as required in (2.15).

The other two cases (2.16) and (2.17) are obtained in an identical manner. The only difference is that instead of extracting factors $(x_i - x_j)$ as dictated by (2.10), one extracts factors $(x_i - x_j)(1 - \overline{x}_i \overline{x}_j)$ as dictated by both (2.11) and (2.12). □

3 Explicit formulae in the case $\lambda = (m)$

As a consequence of Theorem 2.4 it should be noted that in the case of a one-part partition $\lambda = (m, 0, \ldots, 0) = (m)$ we have

Corollary 3.1 *For all non-negative integers m*

$$s_{(m)}(\mathbf{x} \,|\, \mathbf{a}) = h_m(\mathbf{x} \,|\, \mathbf{a}); \quad sp_{(m)}(\mathbf{x}, \overline{\mathbf{x}} \,|\, \mathbf{a}) = h_m^{sp}(\mathbf{x}, \overline{\mathbf{x}} \,|\, \mathbf{a}); \quad so_{(m)}(\mathbf{x}, \overline{\mathbf{x}}, 1 \,|\, \mathbf{a}) = h_m^{so}(\mathbf{x}, \overline{\mathbf{x}}, 1 \,|\, \mathbf{a}) . \tag{3.1}$$

Proof: On setting $\lambda = (m, 0, \ldots, 0)$ the flagged factorial Jacobi-Trudi determinants in (2.15)-(2.17) are reduced to lower-triangular form since each $h_{-j+i} = 0$ for $i < j$. Moreover for $i > 1$ the diagonal entries are all of the form $h_0 = 1$, while the $(1,1)$ entry is just h_m with $\mathbf{x}^{(1)} = \mathbf{x}$ and $\overline{\mathbf{x}}^{(1)} = \overline{\mathbf{x}}$. □

Factorial characters in the one-part partition case may then be evaluated directly from the generating function formulae of Definition 2.1. Before doing so it is convenient, following Macdonald [18], to introduce the shift operator τ defined by

$$\tau^r \mathbf{a} = (a_{r+1}, a_{r+2}, \ldots) \quad \text{for any integer } r \text{ and any } \mathbf{a} = (a_1, a_2, \ldots)\,. \tag{3.2}$$

In the Schur function case with $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{x}' = (x_1, x_2, \ldots, x_{n-1})$ the generating function (2.4) yields

$$\begin{aligned} h_m(\mathbf{x}\,|\,\mathbf{a}) &= [t^m] \prod_{i=1}^{n} \frac{1}{1 - tx_i} \prod_{k=1}^{m+n-1} (1 + ta_k) \\ &= [t^m] \left(\frac{1 + ta_{m+n-1}}{1 - tx_n}\right) \prod_{i=1}^{n-1} \frac{1}{1 - tx_i} \prod_{k=1}^{m+n-2} (1 + ta_k) \\ &= [t^m] \left(1 + \frac{t(x_n + a_{n+m-1})}{1 - tx_n}\right) \prod_{i=1}^{n-1} \frac{1}{1 - tx_i} \prod_{k=1}^{m+n-2} (1 + ta_k) \\ &= h_m(\mathbf{x}'\,|\,\mathbf{a}) + (x_n + a_{m+n-1}) h_{m-1}(\mathbf{x}\,|\,\mathbf{a})\,. \end{aligned} \tag{3.3}$$

Iterating this recursion relation gives

$$h_m(\mathbf{x}\,|\,\mathbf{a}) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_m \le n} (x_{i_1} + a_{i_1})(x_{i_2} + a_{i_2+1}) \cdots (x_{i_m} + a_{i_m+m-1})\,. \tag{3.4}$$

This result can be exploited in the symplectic case, where it might be noted first that if we introduce dummy parameters $a_\ell = 0$ for $\ell = 0, -1, -2, \ldots$ then it follows from (2.5) that

$$\begin{aligned} h_m^{sp}(\mathbf{x}, \overline{\mathbf{x}}\,|\,\mathbf{a}) &= [t^m] \prod_{i=1}^{n} \frac{1}{(1 - tx_i)(1 - t\overline{x}_i)} \prod_{k=1}^{m+n-1} (1 + ta_k) \\ &= [t^m] \prod_{i=1}^{n} \frac{1}{(1 - tx_i)(1 - t\overline{x}_i)} \prod_{k=1-n}^{m+2n-1-n} (1 + ta_k) = h_m(\mathbf{x}, \overline{\mathbf{x}}\,|\,\tau^{-n}\mathbf{a}) = h_m(\mathbf{z}\,|\,\tau^{-n}\mathbf{a}) \end{aligned} \tag{3.5}$$

where $\tau^{-n}\mathbf{a} = (a_{-n+1}, \ldots, a_{-1}, a_0, a_1, a_2 \ldots)$, and it is convenient to order the indeterminates in $\mathbf{z}$ so that $\mathbf{z} = (x_1, \overline{x}_1, x_2, \overline{x}_2, \ldots, x_n, \overline{x}_n)$. It then follows that

$$h_m^{sp}(\mathbf{x}, \overline{\mathbf{x}}\,|\,\mathbf{a}) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_m \le 2n} (z_{i_1} + a_{i_1-n})(z_{i_2} + a_{i_2-n+1}) \cdots (z_{i_m} + a_{i_m-n+m-1})\,. \tag{3.6}$$

with

$$z_{i_j} + a_{i_j-n+j-1} = \begin{cases} x_k + a_{2k-n+j-2} & \text{if } i_j = 2k-1; \\ \overline{x}_k + a_{2k-n+j-1} & \text{if } i_j = 2k, \end{cases} \qquad \text{with } a_\ell = 0 \text{ if } \ell \le 0. \tag{3.7}$$

Turning to the odd orthogonal case and using (2.6) we have

$$
\begin{aligned}
h_m^{so}(\mathbf{x},\overline{\mathbf{x}},1\,|\,\mathbf{a}) &= [t^m]\ (1+t)\prod_{i=1}^{n}\frac{1}{(1-tx_i)(1-t\overline{x}_i)}\prod_{k=1}^{m+n-1}(1+ta_k)\\
&= [t^m]\ ((1+ta_{m+n})+t(1-a_{m+n}))\prod_{i=1}^{n}\frac{1}{(1-tx_i)(1-t\overline{x}_i)}\prod_{k=1}^{m+n}(1+ta_k)\\
&= h_m^{sp}(\mathbf{x},\overline{\mathbf{x}}\,|\,\tau\mathbf{a})+(1-a_{m+n})\ h_{m-1}^{sp}(\mathbf{x},\overline{\mathbf{x}}\,|\,\tau\mathbf{a})\\
&= h_m(\mathbf{x},\overline{\mathbf{x}}\,|\,\tau^{1-n}\mathbf{a})+(1-a_{m+n})\ h_{m-1}(\mathbf{x},\overline{\mathbf{x}}\,|\,\tau^{1-n}\mathbf{a})\,, \qquad (3.8)
\end{aligned}
$$

where dummy parameters $a_\ell = 0$ for $\ell = 0,-1,-2,\ldots$ are once again involved.

It follows that

$$
\begin{aligned}
h_m^{so}(\mathbf{x},\overline{\mathbf{x}},1\,|\,\mathbf{a}) &= \sum_{1\le i_1\le i_2\le\cdots\le i_m\le 2n}(z_{i_1}+a_{i_1+1-n})(z_{i_2}+a_{i_2+2-n})\cdots(z_{i_m}+a_{i_m-n+m})\\
&+\sum_{1\le i_1\le i_2\le\cdots\le i_{m-1}\le 2n}(z_{i_1}+a_{i_1+1-n})(z_{i_2}+a_{i_2+2-n})\cdots(z_{i_{m-1}}+a_{i_{m-1}+m-1-n})(1-a_{m+n}) \qquad (3.9)
\end{aligned}
$$

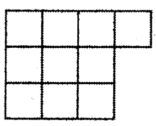
with

$$
z_{i_j}+a_{i_j-n+j} = \begin{cases} x_k + a_{2k-n+j-1} & \text{if } i_j = 2k-1;\\ \overline{x}_k + a_{2k-n+j} & \text{if } i_j = 2k, \end{cases} \qquad \text{with } a_\ell = 0 \text{ if } \ell \le 0. \qquad (3.10)
$$

4 Combinatorial realisation of factorial characters

The significance of these results is that they offer an immediate lattice path model of each of the relevant one-part partition factorial characters. Then by making use of n-tuples of such lattice paths in the interpretation of the factorial flagged Jacobi-Trudi identities of Theorem 2.4 one arrives at a non-intersecting lattice path model of factorial characters specified by any partition λ of length $\ell(\lambda) \le n$. This leads inexorably to a further realisation of factorial characters in terms of certain appropriately weighted tableaux. The tableaux themselves are none other than those already associated with Schur functions, symplectic group characters and odd orthogonal group characters in the classical non-factorial case.

Restricting our attention to fixed n and partitions $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n)$ of length $\ell(\lambda) \le n$, each such partition defines a Young diagram F^λ consisting of $|\lambda| = \lambda_1+\lambda_2+\cdots+\lambda_n$ boxes arranged in $\ell(\lambda)$ rows of lengths λ_i for $i = 1,2,\ldots,\ell(\lambda)$. We adopt the English convention as used by Macdonald [19] whereby the rows are left-adjusted to a vertical line and are weakly decreasing in length from top to bottom. For example in the case $\lambda = (4,3,3)$, for which $\ell(\lambda) = 3$ and

$F^{433} =$ 

More precisely we define $F^\lambda = \{(i,j)\,|\,1\le i\le \ell(\lambda); 1\le j\le \lambda_i\}$ and refer to (i,j) as being the box in the ith row and jth column of F^λ. Assigning an entry T_{ij} taken from some alphabet to each box (i,j) of F^λ in accordance with various rules gives rise to tableaux T of shape λ that may be used, as we shall see, to express both ordinary and factorial characters in a combinatorial manner.

Definition 4.1 (Littlewood [16]) *Let $\mathcal{T}_\lambda$ be the set of all semistandard Young tableaux T of shape λ that are obtained by filling each box (i,j) of F^λ with an entry T_{ij} from the alphabet*

$$\{1 < 2 < \cdots < n\}$$

in such a way that: **(T1)** *entries weakly increase across rows from left to right;* **(T2)** *entries weakly increase down columns from top to bottom;* **(T3)** *no two identical non-zero entries appear in the same column.*

Definition 4.2 (King [14]) *Let $\mathcal{T}_\lambda^{sp}$ be the set of all symplectic tableaux T of shape λ that are obtained by filling each box (i,j) of F^λ with an entry T_{ij} from the alphabet*

$$\{1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n}\}$$

in such a way that conditions **(T1)-(T3)** *are satisfied, together with:* **(T4)** *neither k nor $\overline{k}$ appear lower than the kth row.*

Definition 4.3 (Sundaram [26]) *Let $\mathcal{T}_\lambda^{so}$ be the set of all odd orthogonal tableaux T of shape λ that are obtained by filling each box (i,j) of F^λ with an entry T_{ij} from the alphabet*

$$\{1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n} < 0\}$$

in such a way that conditions **(T1)-(T4)** *are satisfied, together with:* **(T5)** *in any row 0 appears at most once.*

These definitions are exemplified for $GL(4)$, $Sp(8)$ and $SO(9)$ in turn as shown below from left to right:

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&4\\ \hline 2&3&3\\ \cline{1-3} 4&4&4\\ \cline{1-3}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1&\overline{1}&2&\overline{4}\\ \hline \overline{3}&4&4\\ \cline{1-3} 4&\overline{4}&\overline{4}\\ \cline{1-3}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1&\overline{1}&2&\overline{4}\\ \hline \overline{3}&4&0\\ \cline{1-3} 4&\overline{4}&0\\ \cline{1-3}\end{array} \tag{4.1}$$

These definitions allow us to provide combinatorial expressions for factorial characters as follows:

Theorem 4.4 *For each g and $\mathbf{z}$ as tabulated below, and any $\mathbf{a} = (a_1, a_2, \ldots)$*

$$g_\lambda(\mathbf{z}\,|\,\mathbf{a}) = \sum_{T\in\mathcal{T}_\lambda^g} \prod_{(i,j)\in F^\lambda} \mathrm{wgt}(T_{ij})\,. \tag{4.2}$$

where

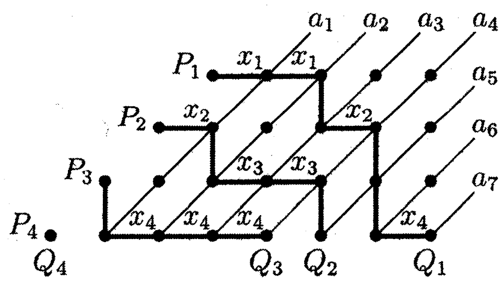
$g_\lambda(\mathbf{z}\,\|\,\mathbf{a})$	T_{ij}	$\mathrm{wgt}(T_{ij})$	
$s_\lambda(\mathbf{x}\,\|\,\mathbf{a})$	k	$x_k + a_{k+j-i}$	
$sp_\lambda(\mathbf{x},\overline{\mathbf{x}}\,\|\,\mathbf{a})$	k	$x_k + a_{2k-1-n+j-i}$	$a_m = 0$ *for* $m \leq 0$
	$\overline{k}$	$\overline{x}_k + a_{2k-n+j-i}$	
$so_\lambda(\mathbf{x},\overline{\mathbf{x}},1\,\|\,\mathbf{a})$	k	$x_k + a_{2k-n+j-i}$	$a_m = 0$ *for* $m \leq 0$
	$\overline{k}$	$\overline{x}_k + a_{2k+1-n+j-i}$	
	0	$1 - a_{n+1+j-i}$	

(4.3)

with $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ and λ any partition of length $\ell(\lambda) \leq n$.

Proof: In the Schur function case, as in [10], we adopt matrix coordinates (k,ℓ) for lattice points with $k=1,2,\ldots,n$ specifying row labels from top to bottom, and $\ell=1,2,\ldots,\lambda_1+n$ specifying column labels from left to right. Each lattice path that we are interested in is a continuous path from some $P_i=(i,n-i+1)$ to some $Q_j=(n,n-j+1+\lambda_j)$ with $i,j\in\{1,2,\ldots,n\}$. Such a path consists of a sequence of horizontal or vertical edges and is associated with a contribution to $h_{\lambda_j-j+i}(\mathbf{x}^{(i)}\,|\,\mathbf{a})$ in the form of a summand of (2.1) with $m=\lambda_j-j+i$ and $\mathbf{x}$ replaced by $\mathbf{x}^{(i)}$. Taking into account the restriction of the alphabet from $\mathbf{x}$ to $\mathbf{x}^{(i)}$, the weight assigned to horizontal edge from $(k,\ell-1)$ to (k,ℓ) is $x_k+a_{k+\ell-n-1}$. Thanks to the Lindström-Gessel-Viennot theorem [15, 6, 7] the only surviving contributions to the determinantal expression for $s_\lambda(\mathbf{x}\,|\,\mathbf{a})$ in the flagged factorial Jacobi-Trudi identity (2.15) are those corresponding to an n-tuple of non-intersecting lattice paths from P_i to Q_i for $i=1,2,\ldots,n$. Such n-tuples are easily seen [23] to be in bijective correspondence with semistandard Young tableaux T of shape λ as in Definition 4.1, with the jth horizontal edge at level k on the path from P_i to Q_i giving an entry $T_{ij}=k$ in T for $i=1,2,\ldots,n$ and $j=1,2,\ldots,\lambda_i$. To complete the proof of Theorem 4.4 in the factorial Schur function case it only remains to note that the weight $\mathrm{wgt}(T_{ij})$ to be assigned to T_{ij} is that of the edge from $(k,\ell-1)$ to (k,ℓ) given by $x_k+a_{k+\ell-n-1}=x_k+a_{k+j-i}$ with $j=\ell-(n-i+1)$ since this is the number of horizontal steps from P_i to column ℓ on the lattice path from P_i to Q_i. This is exemplified in Figure 1 in the case $n=4$ and $\lambda=(4,3,3)$.

$LP(T)\ =$



$T\ =$

1	1	2	4
2	3	3	
4	4	4	

$\mathrm{wgt}(T)\ =$

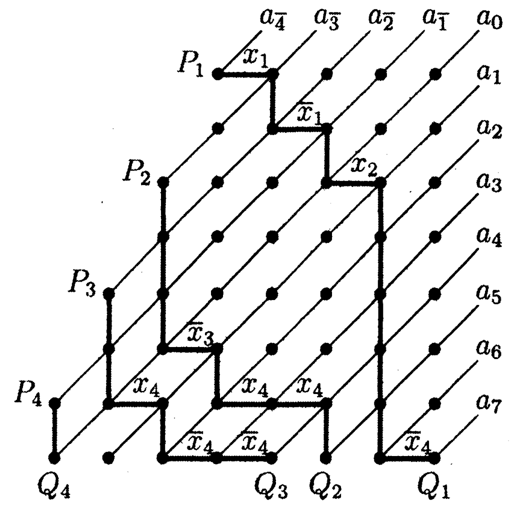
x_1+a_1	x_1+a_2	x_2+a_4	x_4+a_7
x_2+a_1	x_3+a_3	x_3+a_4	
x_4+a_2	x_4+a_3	x_4+a_4	

Figure 1: Contribution to $s_{433}(\mathbf{x}\,|\,\mathbf{a})$ from T and $LP(T)$.

Thanks to (3.5), the lattice path proof in the factorial symplectic case proceeds exactly as in the Schur function case with the alphabet extended to include both x_k and $\overline{x}_k$ for $k=1,2,\ldots,n$, and with $\mathbf{a}$ replaced by $\tau^{-n}\mathbf{a}$. The starting points are now $P_i=(2i-1,n-i+1)$ thereby ensuring that condition **(T4)** is satisfied, and the end points are $Q_j=(2n,n-j+1+\lambda_j)$. Once again it is only the n-tuples of non-intersecting lattice paths from P_i to Q_i that contribute to $sp_\lambda(\mathbf{x},\overline{\mathbf{x}}\,|\,\mathbf{a})$ and these are in bijective correspondence with the symplectic tableaux of Definition 4.2 of shape λ with entries from $\{1<\overline{1}<\cdots<n<\overline{n}\}$. This is exemplified in Figure 2 for $n=4$ and $\lambda=(4,3,3)$.

Finally, in the factorial odd orthogonal case the alphabet is extended to include not only both x_k and $\overline{x}_k$ for $k=1,2,\ldots,n$, but also 1, and $\mathbf{a}$ is replaced this time by $\tau^{1-n}\mathbf{a}$ as dictated by (3.8). The starting points are $P_i=(2i-1,n-i+1)$, ensuring as in the symplectic case that the condition **(T4)** is satisfied, and the end points are $Q_j=(2n+1,n-j+1+\lambda_j)$

$LP(T)$ =



T =

1	$\bar{1}$	2	$\bar{4}$
$\bar{3}$	4	4	
4	$\bar{4}$	$\bar{4}$	

wgt(T) =

x_1	$\bar{x}_1$	x_2+a_1	$x_{\bar{4}}+a_7$
$x_{\bar{3}}+a_1$	x_4+a_3	x_4+a_4	
x_4+a_1	$x_{\bar{4}}+a_3$	$x_{\bar{4}}+a_4$	

Figure 2: Contribution to $sp_\lambda(\mathbf{x},\overline{\mathbf{x}}\,|\,\mathbf{a})$ from T and $LP(T)$, where $a_m = 0$ for $m \leq 0$.

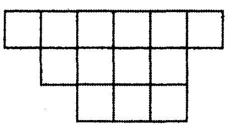
since the alphabet is now of length $2n+1$. To take into account the last factor $(1 - a_{m+n})$ appearing in (3.9) the lattice paths may now include a final diagonal step. The fact that it is diagonal ensures that there is at most one of these steps on each lattice path. Once again it is only the n-tuples of non-intersecting lattice paths from P_i to Q_i that contribute to $so_\lambda(\mathbf{x},\overline{\mathbf{x}},1\,|\,\mathbf{a})$ and these are in bijective correspondence with the odd orthogonal tableaux of Definition 4.3 of shape λ with entries from $\{1 < \bar{1} < \cdots < n < \bar{n} < 0\}$. The fact that on each lattice path the final step is either vertical or diagonal, with the latter to be associated with entries 0 ensures that the condition **(T5)** is automatically satisfied. This is exemplified in Figure 3 for $n = 4$ and $\lambda = (4,3,3)$. □

5 Primed shifted tableaux and factorial Q-functions

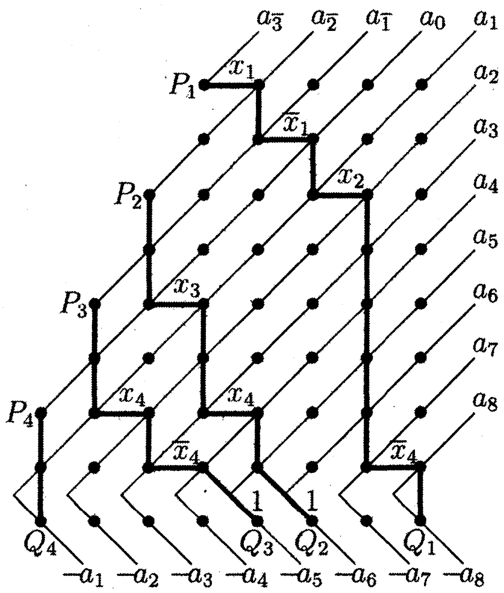
The passage from Schur functions to Schur Q-functions can be effected by replacing tableaux by primed shifted tableaux [28, 22]. We replicate this in the factorial setting by offering definitions of three types of factorial Q-functions expressed in terms of certain primed shifted tableaux. To this end we first define shifted Young diagrams.

A partition is said to be strict if its non-zero parts are distinct. Each such strict partition λ of length $\ell(\lambda) \leq n$ specifies a shifted Young diagram SF^λ consisting of rows of boxes of lengths λ_i for $i = 1, 2, \ldots, \ell(\lambda)$ left adjusted to a diagonal line. This is exemplified in the case $\lambda = (6,4,3)$ by

SF^{6531} =



$LP(T) =$



$T =$

1	$\overline{1}$	2	$\overline{4}$
3	4	0	
4	$\overline{4}$	0	

$\mathrm{wgt}(T) =$

x_1	$x_{\overline{1}}$	x_2+a_2	$x_{\overline{4}}+a_8$
x_3+a_1	x_4+a_4	$1-a_6$	
x_4+a_2	$x_{\overline{4}}+a_4$	$1-a_5$	

Figure 3: Contribution to $so_\lambda(\mathbf{x},\overline{\mathbf{x}}\,|\,\mathbf{a})$ from T and $LP(T)$, where $a_m = 0$ for $m \leq 0$.

This allows us to define various primed shifted tableaux.

Definition 5.1 *[28, 22] Let $\mathcal{P}_\lambda$ be the set of all primed shifted tableaux P of shape λ that are obtained by filling each box of SF^λ with an entry P_{ij} from the alphabet*

$$\{1' < 1 < 2' < 2 < \cdots < n' < n\}$$

with one entry in each box, in such a way that: **(Q1)** *entries weakly increase from left to right across rows;* **(Q2)** *entries weakly increase from top to bottom down columns;* **(Q3)** *no two identical unprimed entries appear in any column;* **(Q4)** *no two identical primed entries appear in any row;*

Definition 5.2 *[9] Let $\mathcal{P}^{sp}_\lambda$ be the set of all primed shifted tableaux P of shape λ that are obtained by filling each box of SF^λ with an entry P_{ij} from the alphabet*

$$\{1' < 1 < \overline{1}' < \overline{1} < 2' < 2 < \overline{2}' < \overline{2} < \cdots < n' < n < \overline{n}' < \overline{n}\}$$

with one entry in each box, in such a way that the conditions **(Q1)**-**(Q4)** *are satisfied together with:* **(Q5)** *at most one of $\{k', k, \overline{k}', \overline{k}\}$ appears on the main diagonal for each $k = 1, 2, \ldots, n$.*

Definition 5.3 *Let $\mathcal{P}^{so}_\lambda$ be the set of all primed shifted tableaux P of shape λ that are obtained by filling each box of SF^λ with an entry P_{ij} from the alphabet*

$$\{1' < 1 < \overline{1}' < \overline{1} < 2' < 2 < \overline{2}' < \overline{2} < \cdots < n' < n < \overline{n}' < \overline{n} < 0'\}$$

with one entry in each box, in such a way that the conditions **(Q1)**-**(Q5)** *are satisfied. together with:* **(Q6)** *the entry $0'$ does not appear on the main diagonal.*

In the case $\lambda = (6,5,3)$ each of these types of shifted primed tableaux is illustrated by

$$\begin{array}{|c|c|c|c|c|c|}\hline 1' & 1 & 2' & 2 & 3' & 4 \\ \hline \multicolumn{1}{c|}{} & 2 & 3' & 3 & 3 & \multicolumn{1}{c}{} \\ \cline{2-5} \multicolumn{2}{c|}{} & 4' & 4 & 4 & \multicolumn{1}{c}{} \\ \cline{3-5} \end{array} \qquad \begin{array}{|c|c|c|c|c|c|}\hline 1 & \overline{1} & 2' & \overline{2}' & 3' & 3 \\ \hline \multicolumn{1}{c|}{} & 2' & 2 & 3 & 4' & \multicolumn{1}{c}{} \\ \cline{2-5} \multicolumn{2}{c|}{} & \overline{4}' & 4 & 4 & \multicolumn{1}{c}{} \\ \cline{3-5} \end{array} \qquad \begin{array}{|c|c|c|c|c|c|}\hline 1 & \overline{1} & 2' & \overline{2}' & 3 & 0' \\ \hline \multicolumn{1}{c|}{} & \overline{2}' & \overline{2} & 3 & 4' & \multicolumn{1}{c}{} \\ \cline{2-5} \multicolumn{2}{c|}{} & 4' & 4 & 0' & \multicolumn{1}{c}{} \\ \cline{3-5} \end{array} \tag{5.1}$$

We then propose the following definitions of factorial Q-functions:

Definition 5.4 *For* $\mathbf{a} = (a_1, a_2, \ldots)$, $a_0 = 0$, *and any strict partition* λ *of length* $\ell(\lambda) \leq n$, *let*

$$Q^g_\lambda(\mathbf{z}; \mathbf{w} \,|\, \mathbf{a}) = \sum_{P \in \mathcal{P}^g_\lambda} \prod_{(i,j) \in SF^\lambda} \mathrm{wgt}(P_{ij}) \qquad \textit{where} \tag{5.2}$$

g	$Q^g_\lambda(\mathbf{z}; \mathbf{w} \,\vert\, \mathbf{a})$
gl	$Q_\lambda(\mathbf{x}; \mathbf{y} \,\vert\, \mathbf{a})$
sp	$Q^{sp}_\lambda(\mathbf{x}, \overline{\mathbf{x}}; \mathbf{y}, \overline{\mathbf{y}} \,\vert\, \mathbf{a})$
so	$Q^{so}_\lambda(\mathbf{x}, \overline{\mathbf{x}}; \mathbf{y}, \overline{\mathbf{y}}, 1 \,\vert\, \mathbf{a})$

and

P_{ij}	$\mathrm{wgt}(P_{ij})$	P_{ij}	$\mathrm{wgt}(P_{ij})$
k	$x_k + a_{j-i}$	k'	$y_k - a_{j-i}$
$\overline{k}$	$\overline{x}_k + a_{j-i}$	$\overline{k}'$	$\overline{y}_k - a_{j-i}$
		$0'$	$1 - a_{j-i}$

(5.3)

It might be noted that for these factorial Q-functions the dependence on the factorial parameters $\mathbf{a}$ is simpler than it is for factorial characters since the factors in (5.3) are all of the form $z_k \pm a_{j-i}$ with the subscript on a completely independent of that on z.

The definition given here of $Q_\lambda(\mathbf{x}; \mathbf{y} \,|\, \mathbf{a})$ has been introduced and studied elsewhere [10]. The special case $Q_\lambda(\mathbf{x}; \mathbf{x} \,|\, -\mathbf{a})$ obtained by setting $\mathbf{y} = \mathbf{x}$ and $\mathbf{a} = -\mathbf{a}$ coincides with the generalized Q-function $Q_\lambda(\mathbf{x} \,|\, \mathbf{a})$ introduced by Ivanov [12, 13] and studied further by Ikeda, Milhalcea and Naruse [11]. If one further sets $\mathbf{a} = \mathbf{0}$ one recovers the combinatorial primed shifted tableaux formula [28, 22, 24, 20] for the original Schur Q-functions $Q_\lambda(\mathbf{x})$.

6 Determinantal expressions for factorial Q-functions

In order to establish algebraic expressions for our factorial Q-functions we follow the method of Okada [20] to construct lattice path models, in which each row of a primed shifted tableau, P, specifies a lattice path contributing to an $\ell(\lambda)$-tuple, $LP(P)$, of non-intersecting lattice paths. In these models the ith paths extend from P_i to Q_i for $i = 1, 2, \ldots, \ell(\lambda)$ as specified by

g	P_{ii}	P_i	Q_i
gl	k, k'	$(k, 0)$	(n, λ_i)
sp	$k, k', \overline{k}, \overline{k}'$	$(2k - \frac{1}{2}, 0)$	$(2n, \lambda_i)$
so	$k, k', \overline{k}, \overline{k}'$	$(2k - \frac{1}{2}, 0)$	$(2n+1, \lambda_i)$

(6.1)

It is convenient to set $d_i = k$ if $P_{ii} \in \{k, k', \overline{k}, \overline{k}'\}$ and introduce $\mathbf{d} = (d_1, d_2, \ldots, d_{\ell(\lambda)})$.

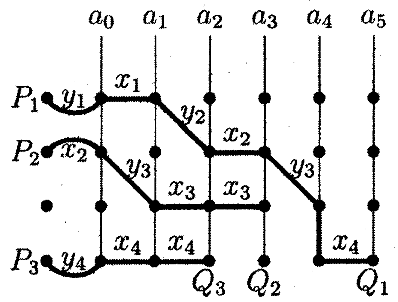
The entries P_{ij} in the ith row of P determine λ_i edges of the corresponding lattice path extending from P_i to Q_i. The nature of these edges and their corresponding weights, as determined by Definition 5.4, are as prescribed below.

gl	$P_{ii}=k$	x_k		$P_{ii}=k'$	y_k	
sp, so	$P_{ii}=k$	x_k		$P_{ii}=k'$	y_k	
sp, so	$P_{ii}=\overline{k}$	$\overline{x}_k$		$P_{ii}=\overline{k}'$	$\overline{y}_k$	
sp, so	$P_{ij}=k, i<j$	x_k+a_{j-i}		$P_{ij}=k', i<j$	y_k-a_{j-i}	
sp, so	$P_{ij}=\overline{k}, i<j$	$\overline{x}_k+a_{j-i}$		$P_{ij}=\overline{k}', i<j$	$\overline{y}_k-a_{j-i}$	
so				$P_{ij}=0'$	$1-a_{j-i}$	

(6.2)

Each edge has its rightmost vertex at (r,ℓ) in the rectangular lattice, with $\ell=0$ for the curved edges and $\ell=j-i$ for all the others, while $r=k$ for entries k and k' in the case $g=gl$, but $r=2k-1$ or $r=2k$ according as the entry k, k', $\overline{k}$ or $\overline{k}'$ is unbarred or barred in the cases $g=sp$ and so, and $r=2n+1$ if the entry is $0'$ as may only occur in the case $g=so$. The path P_iQ_i is completed by the insertion of vertical edges of weight 1. All this is illustrated in the case of our three running examples in Figures 4, 5, and 6, respectively, in the case $\lambda=(6,4,3)$, for which $\ell(\lambda)=3$.

$LP(P) =$



$P =$

1′	1	2′	2	3′	4
	2	3′	3	3	
		4′	4	4	

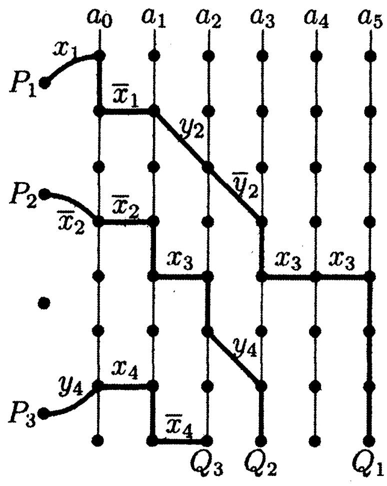
$\mathrm{wgt}(P) =$

y_1	x_1+a_1	y_2-a_2	x_2+a_3	y_3-a_4	x_4+a_5
	x_2	y_3-a_1	x_3+a_2	x_3+a_3	
		y_4	x_4+a_1	x_4+a_2	

Figure 4: Contribution to $Q_\lambda(\mathbf{x};\mathbf{y}\,|\,\mathbf{a})$ from P and $LP(P)$, where $a_0=0$.

The sets of all non-intersecting $\ell(\lambda)$-tuples of lattice paths from fixed starting points P_i to fixed end points Q_i determined by $\mathbf{d}$ and λ as tabulated in (6.1) are in bijective correspondence with sets of all the primed shifted tableaux of Definitions 5.1-5.3 with diagonal entries P_{ii} consistent with d_i for all $i=1,2,\ldots,d_{\ell(\lambda)}$. The sets of $\ell(\lambda)$-tuples of lattice paths may be extended to include those from P_i to $Q_{\pi(i)}$ for $i=1,2,\ldots,\ell(\lambda)$ with π any permutation in $S_{\ell(\lambda)}$. Assigning weights in accordance with (6.2), the sum of the contributions from all these $\ell(\lambda)$-tuples multiplied by $\mathrm{sgn}(\pi)$ then constitutes a determinant. As established by Okada [20] all contributions mutually cancel, other than those from the non-intersecting $\ell(\lambda)$-tuples for which π is the identity element. It then follows that our

$LP(P) =$



$P =$

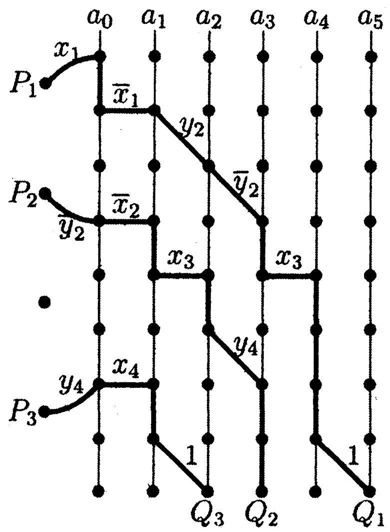
1	$\overline{1}$	$2'$	$\overline{2}'$	3	3
	$\overline{2}$	$\overline{2}$	3	$4'$	
		$4'$	4	$\overline{4}$	

$\mathrm{wgt}(P) =$

x_1	$\overline{x}_1+a_1$	y_2-a_2	$\overline{y}_2-a_3$	x_3+a_4	x_3+a_5
	$\overline{x}_2$	$\overline{x}_2+a_1$	x_3+a_2	y_4-a_3	
		y_4	x_4+a_1	$\overline{x}_4+a_2$	

Figure 5: Contribution to $Q^{sp}_{\lambda}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\,\mathbf{a})$ from P and $LP(P)$, where $a_0=0$.

$LP(P) =$



$P =$

1	$\overline{1}$	$2'$	$\overline{2}'$	3	$0'$
	$\overline{2}'$	$\overline{2}$	3	$4'$	
		$4'$	4	0	

$\mathrm{wgt}(P) =$

x_1	$\overline{x}_1+a_1$	y_2-a_2	$\overline{y}_2-a_3$	x_3+a_4	$1-a_5$
	$\overline{y}_2$	$\overline{x}_2+a_1$	x_3+a_2	y_4-a_3	
		y_4	x_4+a_1	$1-a_2$	

Figure 6: Contribution to $Q^{so}_{\lambda}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1\,|\,\mathbf{a})$ from P and $LP(P)$, where $a_0=0$.

Q-functions may be expressed in the form

$$Q_\lambda(\mathbf{x};\mathbf{y}\,|\,\mathbf{a}) = \sum_{\mathbf{d}} \big|\,(x_{d_i}+y_{d_i})\ \tilde{q}_{\lambda_j-1}(\mathbf{x}^{(d_i)};\mathbf{y}^{(d_i+1)}\,|\,\mathbf{a})\,\big|\,; \tag{6.3}$$

$$\begin{aligned} Q^{sp}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\,\mathbf{a}) = \sum_{\mathbf{d}} \big|\,&(x_{d_i}+y_{d_i})\ \tilde{q}_{\lambda_j-1}(\mathbf{x}^{(d_i)},\ \overline{\mathbf{x}}^{(d_i)};\mathbf{y}^{(d_i+1)},\ \overline{\mathbf{y}}^{(d_i)},\,|\,\mathbf{a}) \\ &+ (\overline{x}_{d_i}+\overline{y}_{d_i})\ \tilde{q}_{\lambda_j-1}(\mathbf{x}^{(d_i+1)},\ \overline{\mathbf{x}}^{(d_i)};\mathbf{y}^{(d_i+1)},\ \overline{\mathbf{y}}^{(d_i+1)}\,|\,\mathbf{a})\,\big|\,; \end{aligned} \tag{6.4}$$

$$\begin{aligned} Q^{so}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,1\,|\,\mathbf{a}) = \sum_{\mathbf{d}} \big|\,&(x_{d_i}+y_{d_i})\ \tilde{q}_{\lambda_j-1}(\mathbf{x}^{(d_i)},\ \overline{\mathbf{x}}^{(d_i)};\mathbf{y}^{(d_i+1)},\ \overline{\mathbf{y}}^{(d_i)},\ 1\,|\,\mathbf{a}) \\ &+ (\overline{x}_{d_i}+\overline{y}_{d_i})\ \tilde{q}_{\lambda_j-1}(\mathbf{x}^{(d_i+1)},\ \overline{\mathbf{x}}^{(d_i)};\mathbf{y}^{(d_i+1)},\ \overline{\mathbf{y}}^{(d_i+1)},\ 1\,|\,\mathbf{a})\,\big|, \end{aligned} \tag{6.5}$$

where for any $m \geq 0$, and all relevant $\mathbf{u} = (u_1, u_2, \ldots, u_r)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_s)$

$$\tilde{q}_m(\mathbf{u};\mathbf{v}\,|\,\mathbf{a}) = \sum_{1 \lessdot i_1 \lessdot i_2 \lessdot \cdots \lessdot i_m \lessdot r+s} (w_{i_1}\pm a_1)(w_{i_2}\pm a_2)\cdots(w_{i_m}\pm a_m)\,, \tag{6.6}$$

with $\mathbf{w} = (w_1, w_2, \ldots, w_{r+s})$ to be identified with an alternating ordered sequence of all the elements of $\mathbf{u}$ and $\mathbf{v}$. The notation $\lessdot$ is intended to indicate that the summation allows factors $(w_i \pm a_\ell) = (u_k + a_\ell)$ or $(v_k - a_\ell)$ to appear according as $w_i = u_k$ or v_k, with several factors of the form $(u_k + a_\ell)(u_k + a_{\ell+1})\cdots$ allowed, but at most one factor $(v_k - a_\ell)$.

To present this result in a neater form it is convenient to make use of the following

Definition 6.1

$$q_m(\mathbf{x}^{(d)};\mathbf{y}^{(d+1)}\,|\,\mathbf{a}) = [t^m]\ \frac{\prod_{j=d+1}^{n}(1+ty_j)\prod_{k=1}^{m}(1+ta_k)}{\prod_{i=d}^{n}(1-tx_i)}; \tag{6.7}$$

$$q^{sp}_m(\mathbf{x}^{(d)},\overline{\mathbf{x}}^{(d)};\mathbf{y}^{(d+1)},\overline{\mathbf{y}}^{(d+1)}\,|\,\mathbf{a}) = [t^m]\ \frac{\prod_{j=d+1}^{n}((1+ty_j)(1+t\overline{y}_j))\prod_{k=1}^{m}(1+ta_k)}{\prod_{i=d}^{n}((1-tx_i)(1-t\overline{x}_i))}; \tag{6.8}$$

$$q^{so}_m(\mathbf{x}^{(d)},\overline{\mathbf{x}}^{(d)};\mathbf{y}^{(d+1)},\overline{\mathbf{y}}^{(d+1)},1\,|\,\mathbf{a}) = [t^m]\ \frac{(1+t)\prod_{j=d+1}^{n}((1+ty_j)(1+t\overline{y}_j))\prod_{k=1}^{m}(1+ta_k)}{\prod_{i=d}^{n}((1-tx_i)(1-t\overline{x}_i))}. \tag{6.9}$$

In terms of these we have

Theorem 6.2 *Let* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. *Then for any strict partition* λ *of length* $\ell(\lambda) \leq n$ *and any* $\mathbf{a} = (a_1, a_2, \ldots)$

$$Q_\lambda(\mathbf{x};\mathbf{y}\,|\,\mathbf{a}) = \sum_{\mathbf{d}} \Big|\,(x_{d_i}+y_{d_i})q_{\lambda_j-1}(\mathbf{x}^{(d_i)};\mathbf{y}^{(d_i+1)}\,|\,\mathbf{a})\,\Big|\,; \tag{6.10}$$

$$Q^{sp}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\,\mathbf{a}) = \sum_{\mathbf{d}} \Big|\,(x_{d_i}+y_{d_i}+\overline{x}_{d_i}+\overline{y}_{d_i})q^{sp}_{\lambda_j-1}(\mathbf{x}^{(d_i)},\overline{\mathbf{x}}^{(d_i)};\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i+1)}\,|\,\mathbf{a})\,\Big|\,; \tag{6.11}$$

$$Q^{so}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1\,|\,\mathbf{a}) = \sum_{\mathbf{d}} \Big|\,(x_{d_i}+y_{d_i}+\overline{x}_{d_i}+\overline{y}_{d_i})q^{so}_{\lambda_j-1}(\mathbf{x}^{(d_i)},\overline{\mathbf{x}}^{(d_i)};\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i+1)},1\,|\,\mathbf{a})\,\Big|, \tag{6.12}$$

where each determinant is $\ell(\lambda) \times \ell(\lambda)$ *and each sum is over all* $\mathbf{d} = (d_1, d_2, \ldots, d_{\ell(\lambda)})$ *such that* $1 \leq d_1 < d_2 < \cdots < d_{\ell(\lambda)} \leq n$.

Proof: It should first be noted that in the $\mathbf{a} = \mathbf{0}$ case it is known [20] that for all $\mathbf{u} = (u_1, u_2, \ldots, u_r)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_s)$ that

$$\tilde{q}_m(\mathbf{u}; \mathbf{v} \,|\, \mathbf{0}) = [t^m] \prod_{i=1}^{r} \frac{1}{1 - tu_i} \prod_{j=1}^{s} (1 + tv_j)\,. \tag{6.13}$$

More generally, consider

$$\tilde{q}_m(\mathbf{u}; \mathbf{v} \,|\, \mathbf{a}) = [t^m] \prod_{i=1}^{r} \frac{1}{1 - tu_i} \prod_{j=1}^{s} (1 + tv_j) \prod_{k=1}^{m+r-s-1} (1 + ta_k)\,. \tag{6.14}$$

By writing $(1 + ta_{m+r-s-1})/(1 - tu_r) = 1 + t(u_r + a_{m+r-s-1})/(1 - tu_r)$ and $(1 + tv_s) = (1 + ta_{m+r-s}) + t(v_s - a_{m+r-s})$ it may be verified, very much as in (3.3) and (3.8), that

$$\tilde{q}_m(\mathbf{u}; \mathbf{v} \,|\, \mathbf{a}) = \tilde{q}_m(\mathbf{u}'; \mathbf{v} \,|\, \mathbf{a}) + (u_r + a_{m+r-s-1})\tilde{q}_{m-1}(\mathbf{u}; \mathbf{v} \,|\, \mathbf{a}) \tag{6.15}$$

and

$$\tilde{q}_m(\mathbf{u}; \mathbf{v} \,|\, \mathbf{a}) = \tilde{q}_m(\mathbf{u}; \mathbf{v}' \,|\, \mathbf{a}) + (v_s - a_{m+r-s})\tilde{q}_{m-1}(\mathbf{u}; \mathbf{v}' \,|\, \mathbf{a}) \tag{6.16}$$

where $\mathbf{u}' = (u_1, u_2, \ldots, u_{r-1})$ and $\mathbf{v}' = (v_1, v_2, \ldots, v_{s-1})$. Applying the first of these in the case $\mathbf{u} = \mathbf{x}^{(d)}$ and $\mathbf{v} = \mathbf{y}^{(d+1)}$, for which $r = s + 1$, leads to a factor $(x_n + a_m)$ in the right hand term. Repeating this process for $\tilde{q}_{m-1}(\mathbf{u}; \mathbf{v} \,|\, \mathbf{a})$ but still with $\mathbf{u} = \mathbf{x}^{(d)}$ and $\mathbf{v} = \mathbf{y}^{(d+1)}$, leads to a further factor $(x_n + a_{m-1})$ and so. In this way any dependence on x_n takes the form $(x_n + a_{\ell+1}) \cdots (x_n + a_{m-1})(x_n + a_m)$. Then applying (6.16) in the case $m = \ell$, $\mathbf{u} = \mathbf{x}'^{(d)}$ and $\mathbf{v} = \mathbf{y}^{(d+1)}$, for which $r = s$, leads to a factor $(y_n + a_\ell)$, with no further factors involving y_n allowed. Iterating these results it can be seen that $q_m(\mathbf{x}^{(d)}; \mathbf{y}^{(d+1)} \,|\, a)$, as defined in (6.7), generates the expansion (6.6) of $\tilde{q}_m(\mathbf{u}; \mathbf{v} \,|\, \mathbf{a})$ in the case $\mathbf{u} = \mathbf{x}^{(d)}$ and $\mathbf{v} = \mathbf{y}^{(d+1)}$, thereby proving (6.10). Moreover, in the symplectic case it can be shown directly from the generating functions appearing in (6.14) and (6.8) that

$$\begin{aligned} &(x_i + y_i)\, \tilde{q}_{\lambda_j - 1}(x^{(i)}, \overline{x}^{(i)}; y^{(i+1)}, \overline{y}^{(i)} \,|\, \mathbf{a}) + (\overline{x}_i + \overline{y}_i)\, \tilde{q}_{\lambda_j - 1}(x^{(i+1)}, \overline{x}^{(i)}; y^{(i+1)}, \overline{y}^{(i+1)} \,|\, \mathbf{a}) \\ &\quad = (x_i + y_i + \overline{x}_i + \overline{y}_i)\, q^{sp}_{\lambda_j - 1}(x^{(i)}, \overline{x}^{(i)}; y^{(i+1)}, \overline{y}^{(i+1)} \,|\, \mathbf{a})\,, \end{aligned} \tag{6.17}$$

thereby proving (6.11), with a similar result applying to the odd orthogonal case. □

It might be noted that each of the expressions in Theorem 6.2 in the form of a sum over determinants may be expressed directly as Pfaffian following, for example, the prescription for dealing with non-intersecting lattice paths from a selection of fixed starting points to fixed set of end points [25]. This has been done already in the case of the factorial Q-functions [13, 11], and will not be pursued here, but will be the subject of future work.

7 Tokuyama identities

Here we restrict ourselves to the case for which $\lambda = \mu + \delta$ where μ is a partition of length $\ell(\mu) \leq n$ and $\delta = (n, n-1, \ldots, 1)$ so that λ is a strict partition of length $\ell(\lambda) = n$. In such case the sums over $\mathbf{d}$ appearing in Theorem 6.2 reduce to a single term corresponding to the only possible case $\mathbf{d} = (1, 2, \ldots, n)$. Moreover, each of the surviving determinants factorises, to yield the following factorial Tokuyama type identities.

Theorem 7.1 *Let $\lambda=\mu+\delta$ with $\delta=(n,n-1,\ldots,1)$ and μ a partition of length $\ell(\mu)\le n$. Then for any $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, $\mathbf{y}=(y_1,y_2,\ldots,y_n)$ and $\mathbf{a}=(a_1,a_2,\ldots)$*

$$Q_\lambda(\mathbf{x};\mathbf{y}\,|\,\mathbf{a})=\prod_{1\le i\le j\le n}(x_i+y_j)\;s_\mu(\mathbf{x}\,|\,\mathbf{a})\,; \tag{7.1}$$

$$Q^{sp}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\,\mathbf{a})=\prod_{1\le i\le j\le n}(x_i+y_j+\overline{x}_i+\overline{y}_j)\;sp_\mu(\mathbf{x},\overline{\mathbf{x}}\,|\,\mathbf{a})\,; \tag{7.2}$$

$$Q^{so}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1\,|\,\mathbf{a})=\prod_{1\le i\le j\le n}(x_i+y_j+\overline{x}_i+\overline{y}_j)\;so_\mu(\mathbf{x},\overline{\mathbf{x}},1\,|\,\mathbf{a})\,. \tag{7.3}$$

Before embarking on the proof it is helpful to make the following definition:

Definition 7.2 *For all $1\le p\le q\le n$ let*

$$f_{m,p,q,n}(\mathbf{x};\mathbf{y}\,|\,\mathbf{a})=[t^m]\;\frac{\prod_{j=q+1}^n(1+ty_j)\prod_{k=1}^{m+q-p}(1+ta_k)}{\prod_{i=p}^n(1-tx_i)}\;; \tag{7.4}$$

$$f^{sp}_{m,p,q,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\,\mathbf{a})=[t^m]\;\frac{\prod_{j=q+1}^n((1+ty_j)(1-t\overline{y}_j))\prod_{k=1}^{m+q-p}(1+ta_k)}{\prod_{i=p}^n((1-tx_i)(1-t\overline{x}_i))}\;; \tag{7.5}$$

$$f^{so}_{m,p,q,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1\,|\,\mathbf{a})=[t^m]\;\frac{(1+t)\prod_{j=q+1}^n((1+ty_j)(1-t\overline{y}_j))\prod_{k=1}^{m+q-p}(1+ta_k)}{\prod_{i=p}^n((1-tx_i)(1-t\overline{x}_i))}\,. \tag{7.6}$$

In the special case $p=q=d$ these definitions are such that

$$f_{m,d,d,n}(\mathbf{x};\mathbf{y}\,|\,\mathbf{a})=q_m(\mathbf{x}^{(d)};\mathbf{y}^{(d+1)}\,|\,\mathbf{a})\,; \tag{7.7}$$

$$f^{sp}_{m,d,d,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\,\mathbf{a})=q^{sp}_m(\mathbf{x}^{(d)},\overline{\mathbf{x}}^{(d)};\mathbf{y}^{(d+1)},\overline{\mathbf{y}}^{(d+1)}\,|\,\mathbf{a})\,; \tag{7.8}$$

$$f^{so}_{m,d,d,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1\,|\,\mathbf{a})=q^{so}_m(\mathbf{x}^{(d)},\overline{\mathbf{x}}^{(d)};\mathbf{y}^{(d+1)},\overline{\mathbf{y}}^{(d+1)},1\,|\,\mathbf{a})\,; \tag{7.9}$$

and in the case $p=d,q=n$ they reduce to

$$f_{m,d,n,n}(\mathbf{x};\mathbf{y}\,|\,\mathbf{a})=h_m(\mathbf{x}^{(d)}\,|\,\mathbf{a})\,; \tag{7.10}$$

$$f^{sp}_{m,d,n,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\,\mathbf{a})=h^{sp}_m(\mathbf{x}^{(d)},\overline{\mathbf{x}}^{(d)}\,|\,\mathbf{a})\,; \tag{7.11}$$

$$f^{so}_{m,d,n,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1\,|\,\mathbf{a})=h^{so}_m(\mathbf{x}^{(d)},\overline{\mathbf{x}}^{(d)},1\,|\,\mathbf{a})\,; \tag{7.12}$$

Finally, for $1\le p<q\le n$

$$f_{m,p,q-1,n}(\mathbf{x};\mathbf{y}\,|\,\mathbf{a})-f_{m,p+1,q,n}(\mathbf{x};\mathbf{y}\,|\,\mathbf{a})=(x_p+y_q)f_{m-1,p,q,n}(\mathbf{x};\mathbf{y}\,|\,\mathbf{a})\,; \tag{7.13}$$

$$\begin{aligned}&f^{sp}_{m,p,q-1,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\,\mathbf{a})-f^{sp}_{m,p+1,q,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\,\mathbf{a})\\&\qquad=(x_p+y_q+\overline{x}_p+\overline{y}_q)f^{sp}_{m-1,p,q,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\,\mathbf{a})\,;\end{aligned} \tag{7.14}$$

$$\begin{aligned}&f^{so}_{m,p,q-1,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1\,|\,\mathbf{a})-f^{so}_{m,p+1,q,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1\,|\,\mathbf{a})\\&\qquad=(x_p+y_q+\overline{x}_p+\overline{y}_q)f^{so}_{m-1,p,q,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1\,|\,\mathbf{a})\,.\end{aligned} \tag{7.15}$$

Proof of Theorem 7.1: If we now focus, for example, on the symplectic case (7.2) and start by using (6.11) in the case $\ell(\lambda)=n$ then, as we have said, the sum over $\mathbf{d}$ is restricted to a single term with $d_i=i$ for $i=1,2,\ldots,n$. It follows that

$$Q^{sp}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\,\mathbf{a})=\prod_{i=1}^n(x_i+y_i+\overline{x}_i+\overline{y}_i)\left|\,f^{sp}_{\lambda_j-1;i,i,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\mathbf{a})\,\right|, \tag{7.16}$$

where we have extracted a common factor $(x_i + y_i + \overline{x}_i + \overline{y}_i)$ from the ith row for $i = 1, 2, \ldots, n$, and used (7.8). Then, by the repeated subtraction of successive rows from one another and using (7.14), precisely as in (2.18), we have

$$\left| f^{sp}_{\lambda_j-1;i,i,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\mathbf{a}) \right| = \prod_{1\leq i<j\leq n} (x_i+y_j+\overline{x}_i+\overline{y}_j) \; \left| f^{sp}_{\lambda_j-1-n+i;i,n,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\mathbf{a}) \right| . \quad (7.17)$$

We are now in a position to use (7.11) which leads directly to

$$\begin{aligned} f^{sp}_{\lambda_j-1-n+i;i,n,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\mathbf{a}) &= \left| h^{sp}_{\lambda_j-(n-i+1)}(\mathbf{x}^{(i)},\overline{\mathbf{x}}^{(i)}\,|\,\mathbf{a}) \right| \\ &= \left| h^{sp}_{\mu_j-j+i}(\mathbf{x}^{(i)},\overline{\mathbf{x}}^{(i)}\,|\,\mathbf{a}) \right| = sp_\mu(\mathbf{x},\overline{\mathbf{x}}\,|\,\mathbf{a})\,, \end{aligned} \quad (7.18)$$

as required to complete the proof of (7.2). The final steps exploit the fact that $\lambda_j = \mu_j + n - j + 1$ for $j = 1, 2, \ldots, n$, as well as the symplectic factorial flagged Jacobi-Trudi identity of Theorem 2.4. The other results (7.1) and (7.3) can be established in exactly the same way. □

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