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On SCT automorphism groups of divisible designs

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In this talk we consider automorphism groups SCTs of divisible designs acting regularly on the set of point classes and determine the relations among SCTs, RDSs and λ-planar functions.

§1 Divisible Designs and class regularity

A *divisible design* (m, u, k, λ)-DD is an incidence structure $(\mathbb{P}, \mathbb{B})$, where

(i) $\mathbb{P}$ is a set of mu points partitioned into m classes $\mathscr{C}$ (called *point classes*), each of size u,

(ii) $\mathbb{B}$ is a collection of k-subsets of $\mathbb{P}$ (called blocks),

(iii) Any two distinct points in the same point class are incident with no blocks and any two points in distinct point classes are incident with exactly λ blocks.

We can show the following : $|\mathbb{P}| = mu$, $|\mathbb{B}| = u^2m(m-1)\lambda/k(k-1)$

An (m, u, k, λ)-DD with $k = m$ is called a *transversal design* and denoted by $\mathrm{TD}_\lambda(k, u)$. A $\mathrm{TD}_\lambda(k, u)$ is called a *symmetric transversal design* and denoted by $\mathrm{STD}_\lambda(k, u)$ with $k = u\lambda$ if its dual is also a $\mathrm{TD}_\lambda(k, u)$. We note that an $(m, 1, k, \lambda)$-DD is just a 2-(m, k, λ) design.

Partial difference matrices

Definition. (Jungnickel [2]) Let U be a group of order u. An $m \times t$ matrix $D = [d_{ij}]$ with entries from $U \cup \{0\}$ is called an (m, u, k, λ)-*partial difference matrix* (PDM) over U if the following conditions are satisfied :

(i) Each column of D has exactly k nonzero entries.

(ii) $\sum_{1 \le j \le t} d_{ij} d_{\ell j}{}^{-1} = \lambda U$, $\forall i \neq \ell$, where $0^{-1} = 0$, $0 \cdot g = g \cdot 0 = 0$ $\forall g \in U$ and $t = |\mathbb{B}|/|G| = m(m-1)u\lambda/k(k-1))$.

An (m, u, k, λ)-PDM with $m = k$ over a group U of order u is called a (u, k, λ)-*difference matrix* (DM). Moreover, a $(u, u\lambda, \lambda)$-DM, denoted by $\mathrm{GH}(u, \lambda)$, is called *a generalized Hadamard matrix.*

Example. Set $U = \langle a \rangle \simeq \mathbb{Z}_3$.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & a & 0 & a^2 \\ 1 & a & 1 & a^2 & 0 \\ a & 1 & 0 & a^2 & a \\ 1 & 0 & a^2 & 1 & a \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & a & a & a^2 & a^2 \\ 1 & 1 & a^2 & a^2 & a & a \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}$$

$(5, 3, 4, 1)$-PDM $\qquad$ $(3, 3, 2)$-DM $\qquad$ $\mathrm{GH}(3, 1)$

Class regularity

Following results are known.

Result. (D. Jungnickel [3]) The existence of an (m, u, k, λ)-DD admitting a class regular automorphism group U

$\Longleftrightarrow$ The existence of a (m, u, k, λ)-partial difference matrix over U

Result. (D.A. Drake [2]) Assume U is a group of even order u and $2 \nmid \lambda$. If a Sylow 2-subgroup of U is cyclic then there exists no (u, k, λ)-DM over U for $k \geq 3$.

We now consider *the regular action* of a subgroup G of $\mathrm{Aut}(\mathcal{D})$ on the set of point classes $\mathscr{C} = \{\mathcal{C}_i \mid i \in I_m\}$, where $I_m = \{1, 2, \cdots, m\}$.

§2 SCT groups and SCT matrices

Let $(\mathbb{P}, \mathbb{B})$ be a (m, u, k, λ)-DD and $G \leq \mathrm{Aut}(\mathbb{P}, \mathbb{B})$. We say G is an $SCT(m, u, k, \lambda)$ *group* if G is semiregular on $\mathbb{P} \cup \mathbb{B}$ and regular on the set of point classes $\mathscr{C} = \{\mathcal{C}_1, \cdots, \mathcal{C}_m\}$. (Note that $|G| = m$.)

Assume that G is an $\mathrm{SCT}(m, u, k, \lambda)$ group of a (m, u, k, λ)-DD $\mathcal{D}(= (\mathbb{P}, \mathbb{B}))$. Choose a point class $\mathcal{C} = \{p_1, \cdots, p_u\} \in \mathscr{C}$. Then $\mathbb{P} = \bigcup_{i \in I_u} {p_i}^G$ and $\mathbb{B} = \bigcup_{j \in I_s} {B_j}^G$, where $s = |\mathbb{B}|/|G|$.

A $u \times s$ matrix $M_{\mathcal{D}} = [D_{ij}]$ $(D_{ij} \subset G)$ over G is defined by

$$D_{ij} = \{g \in G \mid p_i^g \in B_j\} \quad (i \in I_u,\ j \in I_s)$$

Theorem 1. The following holds.

$$\sum_{j \in I_s} D_{ij} {D_{\ell j}}^{(-1)} = \begin{cases} \rho + \lambda(G-1) & \text{if } i = \ell, \\ \lambda(G-1) & \text{otherwise,} \end{cases}$$

where $\rho = (m-1)u\lambda/(k-1)$.

$$\sum_{i \in I_u} |D_{ij}| = k \qquad \forall j \in I_s$$

Definition. Let G be a group of order m. Let $u, s \in \mathbb{N}$. For subsets $D_{ij} \subset G$ $(i \in I_u, j \in I_s)$ we call a $u \times s$ matrix $\begin{bmatrix} D_{11} & \cdots & D_{1s} \\ \vdots & \vdots & \vdots \\ D_{u1} & \cdots & D_{us} \end{bmatrix}$ an $\mathrm{SCT}(m, u, k, \lambda)$-

matrix over G if it satisfies the following for some $\rho \in \mathbb{N}$.

$$\sum_{j \in I_s} D_{ij} D_{\ell j}{}^{(-1)} = \begin{cases} \rho + \lambda(G-1) & \text{if } i = \ell, \\ \lambda(G-1) & \text{otherwise,} \end{cases}$$

$$\sum_{i \in I_u} |D_{ij}| = k \qquad \forall j \in I_s$$

Remark. (i) $s = (m-1)u^2\lambda/k(k-1)$, $\rho = (m-1)u\lambda/(k-1)$
(ii) An SCT$(m, 1, k, \lambda)$-matrix is just an (m, k, λ)-difference family.

$\star$ An incidence structure $\mathcal{D}(\mathbb{P}, \mathbb{B})$ defined by the following is an (m, u, k, λ)-DD admitting G as an SCT group under the action $(i, w)g = (i, wg)$ for $i \in \{1, \cdots, u\}$ and $w, g \in G$.
$\mathbb{P} = \{1, 2, \cdots, u\} \times G$
$\mathbb{B} = \{B_{j,g} \mid j \in I_s,\ g \in G\}$, where $B_{j,g} = \bigcup_{i \in I_u} (i, D_{ij}g)$
$\star$ (m, u, k, λ)-DD with SCT-group $\Longleftrightarrow$ SCT(m, u, k, λ)-matrix

Example. (i) The following is an SCT$(9, 2, 9, 9)$ matrix over $G := \langle a, b \rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$:

$$\begin{bmatrix} \langle a \rangle & \langle b \rangle & G - \langle ab \rangle & G - \langle ab^2 \rangle \\ G - \langle a \rangle & G - \langle b \rangle & \langle ab \rangle & \langle ab^2 \rangle \end{bmatrix}$$

This matrix gives a $\mathrm{TD}_9(9, 2)$, which is not obtained from any difference matrix by Drake's result.

(ii) The following is an SCT$(12, 5, 11, 2)$ matrix over $\mathrm{Alt}(4) = N \rtimes H$, $N = \{1, a, b, c\} \simeq E_4$, $H = \{1, d, d^2\} \simeq \mathbb{Z}_3$:

$$\begin{bmatrix} 0 & \alpha & \beta & \gamma & \delta \\ \alpha & \beta & \gamma & \delta & 0 \\ \beta & \gamma & \delta & 0 & \alpha \\ \gamma & \delta & 0 & \alpha & \beta \\ \delta & 0 & \alpha & \beta & \gamma \end{bmatrix}, \text{ where } \begin{cases} \alpha = ad + cd^2 \\ \beta = d + bd^2 + d^2 + cd \\ \gamma = b + c \\ \delta = ad^2 + bd + a \end{cases}$$

From this we obtain a $(12, 5, 11, 2)$-DD with the full automorphism group isomorphic to $\mathrm{Alt}(5)$ $(\geq \mathrm{Alt}(4) \simeq N \rtimes H)$. This DD is not class regular, hence not obtained from any partial difference matrix.

Relations among SCT aut. , Class regular aut. and RDS

$\exists$ SCT aut. $\Longleftrightarrow$ $\exists$ SCT mat.
$\Downarrow$
Divisible design $\supset$ Transversal design
$\Uparrow$ $\Uparrow$
$\exists$ class regular aut. $\Longleftrightarrow$ $\exists$ partial DM $\supset$ DM $\supset$ GH mat.

$\exists$ SCT aut. & $\exists$ class regular aut. $\Longleftrightarrow$ $\exists$ splitting relative difference set

Difference families and SCT matrices

A family of k-subsets $\{D_1, \cdots, D_n\}$ of a group G of order v is called an n-(v, k, λ) *difference family* if

$$D_1 D_1^{(-1)} + \cdots + D_n D_n^{(-1)} = kn + \lambda(G-1).$$

From an n-(v, k, λ) difference family in a group G we obtain a 2-(v, k, λ) design $(\mathbb{P}, \mathbb{B})$: $\mathbb{P} = G$, $\mathbb{B} = \{D_i x \mid i \in I_n,\ x \in G\}$. In the following we give a relation between difference families and SCT matrices with $u = 2$.

Theorem 2. Let $\{D_1, \cdots, D_{4d}\}$ be a $4d$-$(m, k, d(4k-m))$ difference family in a group G of order m. Set $C_i = G - D_i$ for $i \in I_{4d}$. Then the following is an SCT$(m, 2, m, dm)$ matrix corresponding to a TD$_{dm}(m, 2)$.

$$M = \begin{bmatrix} D_1 & \cdots & D_{2d} & C_{2d+1} & \cdots & C_{4d} \\ C_1 & \cdots & C_{2d} & D_{2d+1} & \cdots & D_{4d} \end{bmatrix}$$

$$\because\ C_i C_i^{(-1)} = D_i D_i^{(-1)} + (m-2k)G$$
$$D_i C_i^{(-1)} = C_i D_i^{(-1)} = kG - D_i D_i^{(-1)}$$

Some theorems on difference families

The following results on difference families are known.

Result. (Leung-Ma-Schmidt [4]) Let q be a prime power and $d > 0$ an integer. Suppose, either (i) $q \equiv 2d-1 \pmod{4d}$ and $2 \nmid d$ or (ii) $q \equiv 4d-1 \pmod{8d}$. Then there exists a $4d$-$(q^2, (q^2-q)/2, dq^2-2dq)$ difference family in $(GF(q^2), +)$.

Result. (Q. Xiang [6]) Let q be a power of a prime and b, c positive integers such that $q+1 = 2^c b$ and $c \geq 2$ with $2 \nmid b$. Then there exists a 2^c-$(q^2, (q^2-q)/2, 2^{c-2}(q^2-2q))$ difference family in $(GF(q^2), +)$.

Remark. Set $d = 2^{c-2}$ in the above result. Then 2^c-$(q^2, (q^2-q)/2, 2^{c-2}(q^2-2q))$ is identical with $4d$-$(q^2, (q^2-q)/2, dq^2-2dq)$.

We now apply Theorem 2 to the above results for $m = q^2, k = (q^2-q)/2$.

TD$_{dq^2}(q^2, 2)$s admitting SCT groups

Proposition. Let q be a power of a prime and d a positive integer satisfying one of the following :

(i) $q \equiv 2d-1 \pmod{4d}$.
(ii) $q \equiv 4d-1 \pmod{8d}$.
(iii) $4d \mid q+1, 8d \nmid q+1$ with d a power of 2.

Then, there exists an SCT$(q^2, 2, q^2, dq^2)$ matrix over $(GF(q^2), +)$ and the resulting TD$_{dq^2}(q^2, 2)$ admits an SCT automorphism group of order q^2.

Remark. If $2 \nmid dq$, then no TD$_{dq^2}(q^2, 2)$s are obtained from difference matrices by Drake's result.

§3 Direct product RDSs and SCTs

Let $\mathcal{G}$ be a group of order um and U its (not necessarily normal) subgroup of order u. A k-subset D of $\mathcal{G}$ is called an (m, u, k, λ)-*relative difference set* (or, RDS for short) relative to U if $DD^{(-1)} = k + \lambda(\mathcal{G} - U)$. Usually U is called the forbidden subgroup.

An (m, u, k, λ)-divisible design $\mathcal{D} = (\mathbb{P}, \mathbb{B})$ is obtained from (m, u, k, λ)-RDS in the following way : the set $\mathbb{P}$ of points are elements of $\mathcal{G}$ and the set of blocks $\mathbb{B}$ are subsets $Dx (x \in \mathcal{G})$. We note that the set of point classes are $\{Ug \mid g \in \mathcal{G}\}$.

We say $\mathcal{G}$ is *splitting* (over U) if there exists a subgroup G of $\mathcal{G}$ of order m such that $\mathcal{G} = GU$ and $G \cap U = 1$. In this case G is an SCT(m, u, k, λ) group of $\mathcal{D}$.

From now on we consider an SCT matrix obtained from a splitting abelian RDS ; $\mathcal{G} = G \times U$.

Hypothesis 3. Let $G = \{g_1, \cdots, g_m\}$ and $U = \{w_1, \cdots, w_u\}$ be abelian groups of order m and u, respectively. Suppose D is an (m, u, k, λ)-RDS in the group $\mathcal{G} = G \times U$ relative to U. Set $\mathbb{P} = \mathcal{G} = \{w_i g_j \mid i \in I_u, j \in I_m\}$ and $\mathbb{B} = \{Dw_i g_j \mid i \in I_u, j \in I_m\}$. Then $\mathcal{D}_{D,\mathcal{G}} := (\mathbb{P}, \mathbb{B})$ is a (m, u, k, λ)-DD with the set $\mathscr{C} := \{Ug_1, \cdots, Ug_m\}$ of point classes.

We now consider the action of G on $(\mathbb{P}, \mathbb{B})$ as an SCT group.

$\{w_i G \mid i \in I_u\}$: the set of G-orbits on $\mathbb{P}$,
$\{Dw_i G \mid i \in I_u\}$: the set of G-orbits on $\mathbb{B}$,
$D = G_{w_1} w_1 \cup \cdots \cup G_{w_u} w_u \quad (\exists G_{w_1}, \cdots, \exists G_{w_u} \subset G)$.

We choose a point class $\mathcal{C} = \{w_1, \cdots, w_u\} (\in \mathscr{C})$ as a set of representatives of G-orbits on $\mathbb{P}$ and $\{Dw_1, \cdots, Dw_u\} (\subset \mathbb{B})$ as a set of representatives of G-orbits on $\mathbb{B}$.

Direct product RDSs and SCTs

Under Hypothesis 3, the corresponding $u \times u$ SCT matrix $[D_{ij}]$ is given by

$$D_{ij} = \{g \in G \mid (w_i)g \in Dw_j\} = G \cap Dw_i^{-1} w_j.$$

As $D = G_{w_1} w_1 \cup \cdots \cup G_{w_u} w_u \quad (G_{w_1}, \cdots, G_{w_u} \subset G)$,
we have $[D_{ij}] = [G_{w_i w_j^{-1}}]$, which we call *an SCT matrix of standard form with respect to* $\{D, G \times U\}$.

Similarly, if we choose a point class $\mathcal{C} = \{w_1 g, \cdots, w_u g\} \in \mathscr{C}$ $(g \in G)$ and $\{Dw_1 g_{n_1}, \cdots, Dw_u g_{n_u}\} \subset \mathbb{B}$ $(n_1, \cdots, n_u \in I_m)$ as sets of representatives of G-orbits on $\mathbb{P}$ and $\mathbb{B}$, respectively, then we have the following.

Lemma 4. Under Hypothesis 3, set $D = G_{w_1} w_1 \cup \cdots \cup G_{w_u} w_u$, where $G_{w_1}, \cdots, G_{w_u} \subset G$. Then a $u \times u$ matrix $[G_{w_i w_j^{-1}} g^{-1} g_{n_j}]$ is an SCT(m, u, k, λ) matrix.

Let notations be as in Lemma 4. Then we have the following.

Proposition 5. Set $M = [G_{w_i w_j^{-1}}]$, the SCT matrix of standard form with respect to $\{D, G \times U\}$. Then,

(i) any SCT matrix is obtained from M by multiplication of any column by an element of G and any permutation of rows and columns;

(ii) M is circulant if u is a prime and $w_i = w^{i-1}$ for $i \in I_u$, where $U = \langle w \rangle$.

§4 Spreads and SCTs

Theorem 6. Let $q = p^e$ be a power of a prime p and let G be an elementary abelian p-group of order q^2. Let $\{H_1, \cdots, H_{q+1}\}$ be a spread of G (i.e. $|H_i| = q, |H_i \cap H_j| = 1, \ \forall i \neq j$). Set $q_0 = q/p^m$ $(= p^{e-m})$ and

$A_i = H^*_{iq_0+1} + H^*_{iq_0+2} + \cdots + H^*_{(i+1)q_0}$ $(0 \leq i \leq p^m - 2)$,

$A_{p^m-1} = H^*_{(p^m-1)q_0+1} + H^*_{(p^m-1)q_0+2} + \cdots + H^*_{p^m \cdot q_0} + H^*_{p^m \cdot q_0+1} + 1$

Let $L = [n_{ij}]$ be a Latin square of order p^m with entries from $\{0, 1, \cdots, p^m - 1\}$. Then the following is an SCT$(p^{2e}, p^m, p^{2e}, p^{2e-m})$ matrix, which gives an STD$_{q^2/p^m}(p^{2e}, p^m)$.

$$\begin{bmatrix} A_{n_{1,1}} & A_{n_{1,2}} & \cdots & A_{n_{1,p^m}} \\ A_{n_{2,1}} & A_{n_{2,2}} & \cdots & A_{n_{2,p^m}} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n_{p^m,1}} & \cdots & A_{n_{p^m,p^m-1}} & A_{n_{p^m,p^m}} \end{bmatrix}$$

Sketch of proof : (1) $\sum_{i \in I_{p^m}} A_i A_i^{(-1)} = q^2 + qq_0(G-1)$ $(\forall i \in I_{p^m})$.

(2) If $\{n_{i1}, \cdots, n_{ip^m}\} = \{n_{\ell 1}, \cdots, n_{\ell p^m}\} = I_{p^m}$ and $n_{i,1} \neq n_{\ell,1}, \cdots, n_{ip^m} \neq n_{\ell p^m}$, then

$$A_{i1} A_{\ell 1}{}^{(-1)} + \cdots + A_{ip^m} A_{\ell p^m}{}^{-1} = q_0 q(G-1)$$

An equivalence class in Latin squares of order n

We show that some of the STDs obtained in Theorem 6 admit no class regular automorphism groups. This implies that these STDs are never obtained from generalized Hadamard matrices. In order to prove this we need a lemma on the set of Latin squares.

Definition. Let $e_1 = (1, 0, 0, \cdots, 0), e_2 = (0, 1, 0, \cdots, 0), \cdots$ be vectors of V$(n, \mathbb{C})$. For a permutation $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ r_1 & r_2 & \cdots & r_n \end{pmatrix}$ of $\Omega := \{1, 2, \cdots, n\}$, a permutation matrix P_σ is defined by $e_i P_\sigma = e_{r_i}$ for each $i \in I_n$. Let N be the group of permutation matrices of order n and $\mathscr{L}$ the set of Latin squares on Ω. We say Latin squares L_1 and L_2 in $\mathscr{L}$ are equivalent if $L_2 = PL_1Q$ for some $P, Q \in N$. Let $H := N \times N$ be the direct product and define the action of H on $\mathscr{L}$ by $L(P, Q) = P^T LQ$ for $L \in \mathscr{L}$. Then H is a permutation group on $\mathscr{L}$.

The number of Latin squares of order n

Let $\mathscr{L}_n$ be the set of Latin squares of order n on $\{1, \cdots, n\}$.
By Theorem III.1.19 of [1],

$|\mathscr{L}_n| > f(n) := (n!)^{2n}/n^{n^2}$ for $n > 1$.

$|\mathscr{L}_2| = (2-1)!2! > \lceil f(2) \rceil = 1,$
$|\mathscr{L}_3| = (3-1)!3! > \lceil f(3) \rceil = 2,$
$|\mathscr{L}_4| = 4(4-1)!4! > \lceil f(4) \rceil = 25,$
$|\mathscr{L}_5| = 56(5-1)!5! = 161280 > \lceil f(5) \rceil = 2077$
$\vdots$

Latin squares equivalent to a circulant one

$\mathscr{L}$ = the set of Latin squares on $\Omega := \{1, 2, \cdots, n\}$
N = the group of permutation matrices of order n
$N \times N$ = the permutation group on $\mathscr{L}$ defined by $L(P, Q) = P^T L Q$

Lemma. Let C be a circulant matrix of order n whose first row is $(a_1, a_2, \cdots, a_n)$ with $\{a_1, a_2, \cdots, a_n\} = \Omega$. Let $T \in N$ be a circulant permutation matrix whose first row is $(0, 1, 0, \cdots, 0)$. If $Q, R \in N$ and $QC = CR$ then $Q = R$ and $Q \in \langle T \rangle$.

Lemma 7. Assume $C \in \mathscr{L}$ and C is circulant. Then,

(i) The number of Latin squares in $\mathscr{L}$ equivalent to C is $(n!)^2/n$;

(ii) If $n \geq 4$, then there exists a Latin square of $\mathscr{L}$ not equivalent to circulant one.

$\because$ By Theorem III.1.19 of [1], $|\mathscr{L}_n| > (n!)^{2n}/n^{n^2}$.
As $(n!)^{2n}/n^{n^2} > (n-1)!(n!)^2/n$, $(n \geq 4)$, the lemma holds.

Non class regular STDs

Theorem. Let $p > 3$ be a prime and A_L the SCT$(p^{2e-1}, p^{2e}, p, p^{2e})$ matrix defined in Theorem 6. Then the STD$_{p^{2e-1}}(p^{2e}, p)$ obtained from A_L is not class regular.

Proof. By Lemma 7, there exists a Latin square L not equivalent to a circulant one. Let $(\mathbb{P}, \mathbb{B})$ be the STD$_{p^{2e-1}}(p^{2e}, p)$ obtained from A_L and let G be the SCT$(p^{2e-1}, p^{2e}, p, p^{2e})$ automorphism group of order p^{2e}. Suppose false and let U be a class regular automorphism group of $(\mathbb{P}, \mathbb{B})$. Then, as G normalizes U and $|U| = p$, G centralizes U. The direct product $\mathcal{G} := G \times U$ contains a $(p^{2e}, p, p^{2e}, p^{2e-1})$-RDS corresponding to $(\mathbb{P}, \mathbb{B})$. By Proposition 5, L must be equivalent to a circulant Latin square, a contradiction.

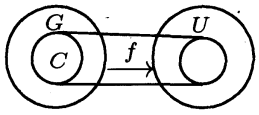
§5 RDS and λ-planar functions

In this section we define a λ-planar function as a generalization of planar functions.

Theorem. Let $\mathcal{G} = GU$ be a group of order mu and G, U its subgroups with $|G| = m, |U| = u$ and $\mathcal{G} \rhd U$. Let D be a (m, u, k, λ)-RDS in $\mathcal{G}$ relative to U. Then there exists a k-subset C of G and a function $f : C \longrightarrow U$ satisfying the following.

(i) $D = \{xf(x) \mid x \in C\}$

(ii) $\#\{x \in C \mid ax \in C,\ f(ax)^a f(x)^{-1} = b\} = \lambda$
for any $a \in G \setminus \{1\}$ and $b \in U$.



Proposition. Let G, U be groups of order m, u, respectively. Let φ be a homomorphism from G to $\mathrm{Aut}(U)$ and f a function form C to U for a k-subset C of G. Assume that for any $a \in G \setminus \{1\}$ and $b \in U$

$$(\star)\quad \#\{x \in C \mid ax \in C,\ f(ax)^{\varphi(a)} f(x)^{-1} = b\} = \lambda.$$

Then $D = \{xf(x) \mid x \in C\}$ is a (m, u, k, λ)-RDS in a semi-direct product $\mathcal{G} = GU$ of G by U with respect to φ.

Definition. Let G and U be groups. Let C be a subset of G and $\varphi \in \mathrm{Hom}(G, \mathrm{Aut}(U))$. We call a function $f : C \longrightarrow U$ a **λ-planar function** relative to (C, U, φ) if f satisfies $(\star)$. If φ is a trivial homomorphism, we say f is a λ-planar function relative to (C, U). We note that a 1-planar function relative to (G, U) is just a planar function in the usual sense (see Pott [5]).

Example. Let $q = p^e$ be a power of a prime p and set $G = F = (GF(q^2), +) \supset U = K = (GF(q), +)$. Then a function
$f(x) = x^{q+1}$ from G to U is a q-planar function relative to (G, U).

$\because$ Let $0 \neq a \in G$ and $b \in U$. Then,

$$f(a+x) - f(x) = b \iff (a^q + x^q)(a+x) - x^{q+1} = b$$
$$\iff ax^q + a^q x = b - a^{q+1} \qquad (\star\star).$$

As $ax^q + a^q x = ax^q + (ax^q)^q = \mathrm{Tr}_{F/K}(ax^q)$, $(\star\star)$ has exactly q solutions in G. Thus f is a q-planar function relative to (G, U).

λ-planar functions, SCTs, and RDSs

Theorem 8. Let G be a group of order m and U a group of order u. Let D_y be subsets of G for each $y \in U$. If a $u \times u$ matrix $D = [D_{yz^{-1}}]_{y,z \in U}$ over $\mathbb{Z}[G]$ is an SCT(m, u, k, λ) matrix, then the following holds.

(i) Set $C = \bigcup_{y \in U} D_y (\subset G)$. Then $|C| = k, G = \langle C \rangle$ and a function $f : C \longrightarrow U$ defined by $f(D_y) = y \ \ (y \in U)$ is a λ-planar function relative to (C, U).

(ii) Set $D = \{(x, f(x) \mid x \in C\}$. Then D is an (m, u, k, λ)-RDS in $G \times U$ relative to $1 \times U$.

Remark. A $(u\lambda, u, u\lambda, \lambda)$-RDS is called semiregular. It is conjectured that any forbidden subgroup of a semiregular RDS is a p-group for a prime p. Concerning this we can show the following as an application of Theorems 6 and 8.

Theorem. *Any* p-group can be a forbidden subgroup of a semiregular RDS.

As a corollary we have the following, which gives another proof of de Launey's result on generalized Hadamard matrices (cf. [1], Theorem 5.9).

Corollary There exists a GH(p^m, p^{2e-m}) matrix over *any group of oder* p^m whenever $e \geq m$.

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