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Title	Regular operator mappings and multivariate geometric means (Theory of operator means and related topics)
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Citation	(2015), 1935: 59-77
Issue Date	2015-04
URL	http://hdl.handle.net/2433/223687
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

Kyoto University

Regular operator mappings
and
multivariate geometric means

Frank Hansen

Abstract

We introduce the notion of regular operator mappings of several variables generalising the notion of spectral function. This setting is convenient for studying maps more general than what can be obtained from the functional calculus, and it allows for Jensen type inequalities and multivariate non-commutative perspectives.

As a main application of the theory we consider geometric means of k operator variables extending the geometric mean of k commuting operators and the geometric mean of two arbitrary positive definite matrices. We propose different types of updating conditions that seems natural in many applications and prove that each of these conditions, together with a few other natural axioms, uniquely defines the geometric mean for any number of operator variables. The means defined in this way are given by explicit formulas and are computationally tractable.[1]

MSC2010 classification: 26B25; 47A64

Key words and phrases: regular operator mapping; perspective; multivariate geometric mean.

1   Introduction

The geometric mean of two positive definite operators was introduced by Pusz and Woronowicz [13], and their definition was soon put into the context of

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[1] An earlier version of this paper was posted in the ArXiv on March 15, 2014 with the title Geometric means of several variables.

the axiomatic approach to operator means developed by Kubo and Ando [9]. Subsequently a number of authors [8, 1, 12, 2, 11, 10] have suggested several ways of defining means of operators for several variables as extensions of the geometric mean of two operators.

There is no satisfactory definition of a geometric mean of several operator variables that is both computationally tractable and satisfies a number of natural conditions put forward in the influential paper by Ando, Li, and Mathias [1]. We put the emphasis on methods to extend a geometric mean of k variables to a mean of $k + 1$ variables, and in the process we challenge one of the requirements to a geometric mean put forward by Ando, Li, and Mathias.

The symmetry condition of a geometric mean is mathematically very appealing, but the condition makes no sense in a number of applications. If for example positive definite matrices $A_1, A_2, \ldots, A_k$ correspond to measurements made at times $t_1 < t_2 < \cdots < t_k$ then there is no way of permuting the matrices since time only goes forward. It makes more sense to impose an updating condition

$$(1) \qquad G_{k+1}(A_1, \ldots, A_k, 1) = G_k(A_1, \ldots, A_k)^{k/(k+1)}$$

when moving from a mean G_k of k variables to a mean G_{k+1} of $k + 1$ variables. The condition corresponds to taking the geometric mean of k copies of $G_k(A_1, \ldots, A_k)$ and one copy of the unit matrix. A variant condition would be to impose the equality

$$(2) \qquad G_{k+1}(A_1, \ldots, A_k, 1) = G_k(A_1^{k/(k+1)}, \ldots, A_k^{k/(k+1)})$$

when updating from k to $k + 1$ variables. It is an easy exercise to realise that if we set $G_1(A) = A$, then either of the conditions (1) or (2) together with homogeneity uniquely defines the geometric mean of k commuting operators.

We furthermore prove that by setting $G_1(A) = A$ and by demanding homogeneity and a few more natural conditions, then either of the updating conditions (1) or (2) leads to unique but different solutions to the problem of defining a geometric mean of k operators. The means defined in this way are given by explicit formulas, and they are computationally tractable. They possess all of the attractive properties associated with geometric operator means discussed in [1] with the notable exception of symmetry. If one emphasises either of the updating conditions (1) or (2) we are thus forced to abandon symmetry.

Efficient averaging techniques of positive definite matrices are important in many practical applications; for example in radar imaging, medical imaging, and the analysis of financial data.

2 Regular operator mappings

2.1 Spectral functions

Let $B(\mathcal{H})$ denote the set of bounded linear operators on a Hilbert space $\mathcal{H}$. A function $F\colon \mathcal{D} \to B(\mathcal{H})$ defined in a convex domain $\mathcal{D}$ of self-adjoint operators in $B(\mathcal{H})$ is called a spectral function, if it can be written on the form $F(x) = f(x)$ for some function f defined in a real interval I, where $f(x)$ is obtained by applying the functional calculus.

The definition contains some hidden assumptions. The domain $\mathcal{D}$ should be invariant under unitary transformations and

$$(3) \qquad\qquad F(u^*xu) = u^*F(x)u \qquad x \in \mathcal{D}$$

for every unitary transformation u on $\mathcal{H}$. Furthermore, to pairs of mutually orthogonal projections p and q acting on $\mathcal{H}$, the element $pxp + qxq$ should be in $\mathcal{D}$ and the equality

$$(4) \qquad\qquad F(pxp + qxq) = pF(pxp)p + qF(qxq)q$$

should hold for any $x \in B(\mathcal{H})$ such that pxp and qxq are in $\mathcal{D}$. An operator function is a spectral function if and only if (3) and (4) are satisfied, cf. [3, 7].

The notion of spectral function is not immediately extendable to functions of several variables. However, we may consider the two properties of spectral functions noticed by C. Davis as a kind of regularity conditions, and they are readily extendable to functions of more than one variable.

The notion of a regular map of two operator variables were studied by Effros and the author in [4], cf. also [6].

Definition 2.1. *Let $F\colon \mathcal{D} \to B(\mathcal{H})$ be a mapping of k variables defined in a convex domain $\mathcal{D} \subseteq B(\mathcal{H}) \times \cdots \times B(\mathcal{H})$. We say that F is regular if*

(i) The domain $\mathcal{D}$ is invariant under unitary transformations of $\mathcal{H}$ and

$$F(u^*x_1u, \ldots, u^*x_ku) = u^*F(x_1, \ldots, x_k)u$$

for every $x = (x_1, \ldots, x_k) \in \mathcal{D}$ and every unitary u on $\mathcal{H}$.

(ii) Let p and q be mutually orthogonal projections acting on $\mathcal{H}$ and take arbitrary k-tuples $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ of operators in $B(\mathcal{H})$ such that the compressed tuples

$$(px_1p, \ldots, px_kp) \quad and \quad (qy_1q, \ldots, qy_kq)$$

are in the domain $\mathcal{D}$. Then the k-tuple of diagonal block matrices

$$(px_1p + qy_1q, \ldots, px_kp + qy_kq)$$

is also in the domain $\mathcal{D}$ and

$$F(px_1p + qy_1q, \ldots, px_kp + qy_kq)$$

$$= pF(px_1p, \ldots, px_kp)p + qF(qy_1q, \ldots, qy_kq)q.$$

By choosing q as the zero projection in the second condition in the above definition we obtain

$$F(px_1p, \ldots, px_kp) = pF(px_1p, \ldots, px_kp)p,$$

which shows that F for any orthogonal projection p on $\mathcal{H}$ may be considered as a regular operator mapping

$$F \colon \mathcal{D}_p \to B(p\mathcal{H}),$$

where the compressed domain

$$\mathcal{D}_p = \{(x_1, \ldots, x_k) \in \bigoplus_{m=1}^{k} B(p\mathcal{H}) \mid (x_1 \oplus 0(1-p), \ldots, x_k \oplus 0(1-p)) \in \mathcal{D}\}.$$

With this interpretation we may unambiguously calculate block matrices by the formula

$$F\left(\begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_k & 0 \\ 0 & y_k \end{pmatrix}\right) = \begin{pmatrix} F(x_1, \ldots, x_k) & 0 \\ 0 & F(y_1, \ldots, y_k) \end{pmatrix}$$

which is well-known from mappings generated by the functional calculus.

2.2 Jensen's inequality for regular operator mappings

We consider throughout this paper the domain

$$\mathcal{D}^k = \{(A_1, \ldots, A_k) \mid A_1, \ldots, A_k \geq 0\}$$

of k-tuples of positive semi-definite operators acting on an infinite dimensional Hilbert space $\mathcal{H}$. It is convenient to consider an infinite dimensional Hilbert space since in this case $\mathcal{H}$ is isomorphic to $\mathcal{H} \oplus \mathcal{H}$ which allows us to use block matrix techniques without imposing dimension conditions.

Theorem 2.2. *Consider a convex regular mapping*

$$F \colon \mathcal{D}^k \to B(\mathcal{H})_{sa}$$

of $\mathcal{D}^k$ into self-adjoint operators acting on $\mathcal{H}$.

(i) Let C be a contraction on $\mathcal{H}$. If $F(0, \ldots, 0) \leq 0$ then the inequality

$$F(C^*A_1C, \ldots, C^*A_kC) \leq C^*F(A_1, \ldots, A_k)C$$

holds for k-tuples $(A_1, \ldots, A_k)$ in $\mathcal{D}^k$.

(ii) Let X and Y be operators acting on $\mathcal{H}$ with $X^*X + Y^*Y = 1$. Then the inequality

$$F\big(X^*A_1X + Y^*B_1Y, \ldots, X^*A_kX + Y^*B_kY\big)$$

$$\leq X^*F(A_1, \ldots, A_k)X + Y^*F(B_1, \ldots, B_k)Y$$

holds for k-tuples $(A_1, \ldots, A_k)$ and $(B_1, \ldots, B_k)$ in $\mathcal{D}^k$.

Proof. By setting $T = (1 - C^*C)^{1/2}$ and $S = (1 - CC^*)^{1/2}$ we obtain that the block matrices

$$U = \begin{pmatrix} C & S \\ T & -C^* \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} C & -S \\ -T & -C^* \end{pmatrix}$$

are unitary operators on $\mathcal{H} \oplus \mathcal{H}$. Furthermore,

$$\frac{1}{2}U^* \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} U + \frac{1}{2}V^* \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} V = \begin{pmatrix} C^*AC & 0 \\ 0 & SAS \end{pmatrix}$$

for any operator $A \in B(\mathcal{H})$. By using that F is a convex regular map we obtain

$$\begin{pmatrix} F(C^*A_1C,\ldots,C^*A_kC) & 0 \\ 0 & F(SA_1S,\ldots,SA_kS) \end{pmatrix}$$

$$= F\left(\begin{pmatrix} C^*A_1C & 0 \\ 0 & SA_1S \end{pmatrix}, \ldots, \begin{pmatrix} C^*A_kC & 0 \\ 0 & SA_kS \end{pmatrix} \right)$$

$$= F\left(\tfrac{1}{2}U^* \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} U + \tfrac{1}{2}V^* \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} V, \ldots, \tfrac{1}{2}U^* \begin{pmatrix} A_k & 0 \\ 0 & 0 \end{pmatrix} U + \tfrac{1}{2}V^* \begin{pmatrix} A_k & 0 \\ 0 & 0 \end{pmatrix} V \right)$$

$$\leq \tfrac{1}{2}F\left(U^* \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} U, \ldots, U^* \begin{pmatrix} A_k & 0 \\ 0 & 0 \end{pmatrix} U \right) + \tfrac{1}{2}F\left(V^* \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} V, \ldots, V^* \begin{pmatrix} A_k & 0 \\ 0 & 0 \end{pmatrix} V \right)$$

$$= \tfrac{1}{2}U^*F\left(\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} A_k & 0 \\ 0 & 0 \end{pmatrix} \right) U + \tfrac{1}{2}V^*F\left(\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} A_k & 0 \\ 0 & 0 \end{pmatrix} \right) V$$

$$= \tfrac{1}{2}U^* \begin{pmatrix} F(A_1,\ldots,A_k) & 0 \\ 0 & F(0,\ldots,0) \end{pmatrix} U + \tfrac{1}{2}V^* \begin{pmatrix} F(A_1,\ldots,A_k) & 0 \\ 0 & F(0,\ldots,0) \end{pmatrix} V$$

$$\leq \tfrac{1}{2}U^* \begin{pmatrix} F(A_1,\ldots,A_k) & 0 \\ 0 & 0 \end{pmatrix} U + \tfrac{1}{2}V^* \begin{pmatrix} F(A_1,\ldots,A_k) & 0 \\ 0 & 0 \end{pmatrix} V$$

$$= \begin{pmatrix} C^*F(A_1,\ldots,A_k)C & 0 \\ 0 & SF(A_1,\ldots,A_k)S \end{pmatrix},$$

where we used convexity in the first inequality, and in the second inequality used $F(0,\ldots,0) \leq 0$. The first statement now follows.

In order to prove (ii) we define the map

$$G(A_1,\ldots,A_k) = F(A_1,\ldots,A_k) - F(0,\ldots,0) \qquad (A_1,\ldots,A_k) \in \mathcal{D}^k.$$

Unitary invariance of F implies that $F(0,\ldots,0)$ is a multiple of the unit operator and thus commutes with all projections. Therefore G is regular and convex with $G(0,\ldots,0) = 0$. We then define block matrices

$$C = \begin{pmatrix} X & 0 \\ Y & 0 \end{pmatrix} \qquad \text{and} \qquad Z_m = \begin{pmatrix} A_m & 0 \\ 0 & B_m \end{pmatrix}, \qquad m = 1,\ldots,k$$

and notice that

$$C^*Z_mC = \begin{pmatrix} X^*A_mX + Y^*B_mY & 0 \\ 0 & 0 \end{pmatrix}$$

for $m = 1, \ldots, k$. Finally we use (i) to obtain

$$\begin{pmatrix} G(X^*A_1X + Y^*B_1Y, \ldots, X^*A_kX + Y^*B_kY) & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} G(X^*A_1X + Y^*B_1Y, \ldots, X^*A_kX + Y^*B_kY) & 0 \\ 0 & G(0, \ldots, 0) \end{pmatrix}$$

$$= G(C^*Z_1C, \ldots, C^*Z_kC)$$

$$\leq C^*G(Z_1, \ldots, Z_k)C = C^* \begin{pmatrix} G(A_1, \ldots, A_k) & 0 \\ 0 & G(B_1, \ldots, B_k) \end{pmatrix} C$$

$$= \begin{pmatrix} X^*G(A_1, \ldots, A_k)X + Y^*G(B_1, \ldots, B_k)Y & 0 \\ 0 & 0 \end{pmatrix}$$

from which we deduce that

$$F(X^*A_1X + Y^*B_1Y, \ldots, X^*A_kX + Y^*B_kY)$$

$$= G(X^*A_1X + Y^*B_1Y, \ldots, X^*A_kX + Y^*B_kY) + F(0, \ldots, 0)$$

$$\leq X^*G(A_1, \ldots, A_k)X + Y^*G(B_1, \ldots, B_k)Y + F(0, \ldots, 0)$$

$$= X^*F(A_1, \ldots, A_k)X + Y^*F(B_1, \ldots, B_k)Y$$
$$- X^*F(0, \ldots, 0)X - Y^*F(0, \ldots, 0)Y + F(0, \ldots, 0).$$

Since as above $F(0, \ldots, 0) = c \cdot 1$ for some real constant c we obtain

$$-X^*F(0, \ldots, 0)X - Y^*F(0, \ldots, 0)Y + F(0, \ldots, 0)$$

$$= -c(X^*X + Y^*Y) + c \cdot 1 = 0,$$

and the statement of the theorem follows. **QED**

We shall for $k = 1, 2, \ldots$ consider the convex domain

$$\mathcal{D}_+^k = \{(A_1, \ldots, A_k) \mid A_1, \ldots, A_k > 0\}$$

of positive definite and invertible operators acting on the Hilbert space $\mathcal{H}$.

Proposition 2.3. *Let F be a regular map of $\mathcal{D}_+^k$ into self-adjoint operators acting on $\mathcal{H}$. We assume that*

(i) F *is convex*

(ii) $F(tA_1, \ldots, tA_k) = tF(A_1, \ldots, A_k)$ $t > 0$, $(A_1, \ldots, A_k) \in \mathcal{D}_+^k$.

Then

$$F(C^* A_1 C, \ldots, C^* A_k C) = C^* F(A_1, \ldots, A_k) C$$

for any invertible operator C on $\mathcal{H}$ and $(A_1, \ldots, A_k) \in \mathcal{D}_+^k$.

Proof. Assume first that C is an invertible contraction on $\mathcal{H}$. Jensen's sub-homogeneous inequality is only available for regular mappings defined in $\mathcal{D}^k$. To $\varepsilon > 0$ we therefore consider the mapping $F_\varepsilon \colon \mathcal{D}^k \to B(\mathcal{H})$ by setting

$$F_\varepsilon(A_1, \ldots, A_k) = F(\varepsilon + A_1, \ldots, \varepsilon + A_k) - F(\varepsilon, \ldots, \varepsilon).$$

By unitary invariance of F we realise that $F(\varepsilon, \ldots, \varepsilon)$ is a multiple of the unity. Therefore, F_ε is regular and convex with $F_\varepsilon(0, \ldots, 0) = 0$.

We may thus use Jensen's sub-homogeneous inequality for regular mappings and obtain

$$F_\varepsilon(C^* A_1 C, \ldots, C^* A_k C) \leq C^* F_\varepsilon(A_1, \ldots, A_k) C,$$

where we now restrict $(A_1, \ldots, A_k)$ to the domain $\mathcal{D}_+^k$ and rearrange the inequality to

$$F(\varepsilon + C^* A_1 C, \ldots, \varepsilon + C^* A_k C)$$

$$\leq C^* F(A_1 + \varepsilon, \ldots, A_k + \varepsilon) C + F(\varepsilon, \ldots, \varepsilon) - C^* F(\varepsilon, \ldots, \varepsilon) C.$$

Since F is positively homogeneous the term $F(\varepsilon, \ldots, \varepsilon) = \varepsilon F(1, \ldots, 1)$ is vanishing for $\varepsilon \to 0$ and we obtain

$$(5) \qquad F(C^* A_1 C, \ldots, C^* A_k C) \leq C^* F(A_1, \ldots, A_k) C$$

for invertible C. Again using homogeneousness we obtain inequality (5) also for arbitrary invertible C. Then by repeated application of (5) we obtain

$$F(A_1, \ldots, A_k) \leq C^{*-1} F(C^* A_1 C, \ldots, C^* A_k C) C^{-1} \leq F(A_1, \ldots, A_k)$$

and the statement follows. **QED**

3   The perspective of a regular map

Definition 3.1. *Let* $F\colon \mathcal{D}_+^k \to B(\mathcal{H})$ *be a regular mapping. The perspective map* $\mathcal{P}_F$ *is the mapping defined in the domain* $\mathcal{D}_+^{k+1}$ *by setting*

$$\mathcal{P}_F(A_1,\ldots,A_k,B) = B^{1/2}F(B^{-1/2}A_1 B^{-1/2},\ldots,B^{-1/2}A_k B^{-1/2})B^{1/2}$$

for positive invertible operators $A_1,\ldots,A_k$ *and* B *acting on* $\mathcal{H}$.

It is a small exercise to prove that the perspective $\mathcal{P}_F$ is a regular mapping which is positively homogeneous in the sense that

$$\mathcal{P}_F(tA_1,\ldots,tA_k,tB) = t\mathcal{P}_F(A_1,\ldots,A_k,B)$$

for arbitrary $(A_1,\ldots,A_k,B) \in \mathcal{D}_+^{k+1}$ and real numbers $t > 0$. The following theorem generalises a result of Effros [5, Theorem 2.2] for functions of one variable.

Theorem 3.2. *The perspective* $\mathcal{P}_F$ *of a convex regular map* $F\colon \mathcal{D}_k^+ \to B(\mathcal{H})$ *is convex.*

Proof. Consider tuples $(A_1,\ldots,A_{k+1})$ and $(B_1,\ldots,B_{k+1})$ in $\mathcal{D}_+^{k+1}$ and take $\lambda \in [0,1]$. We define the operators

$$
\begin{aligned}
C &= \lambda A_{k+1} + (1-\lambda)B_{k+1}\\
X &= \lambda^{1/2}A_{k+1}^{1/2}C^{-1/2}\\
Y &= (1-\lambda)^{1/2}B_{k+1}^{1/2}C^{-1/2}
\end{aligned}
$$

and calculate that

$$X^*X + Y^*Y = C^{-1/2}\lambda A_{k+1}C^{-1/2} + C^{-1/2}(1-\lambda)B_{k+1}C^{-1/2} = 1$$

and

$$
\begin{aligned}
X^*A_{k+1}^{-1/2}&A_i A_{k+1}^{-1/2}X + Y^*B_{k+1}^{-1/2}B_i B_{k+1}^{-1/2}Y\\
&= C^{-1/2}\lambda^{1/2}A_{k+1}^{1/2}A_{k+1}^{-1/2}A_i A_{k+1}^{-1/2}\lambda^{1/2}A_{k+1}^{1/2}C^{-1/2}\\
&\quad + C^{-1/2}(1-\lambda)^{1/2}B_{k+1}^{1/2}B_{k+1}^{-1/2}B_i B_{k+1}^{-1/2}(1-\lambda)^{1/2}B_{k+1}^{1/2}C^{-1/2}\\
&= C^{-1/2}(\lambda A_i + (1-\lambda)B_i)C^{-1/2}
\end{aligned}
$$

for $i = 1, \ldots, k$. We thus obtain

$$\mathcal{P}_F(\lambda A_1 + (1 - \lambda)B_1, \ldots, \lambda A_{k+1} + (1 - \lambda)B_{k+1})$$

$$= C^{1/2}F\big(C^{-1/2}(\lambda A_1 + (1 - \lambda)B_1)C^{-1/2}, \ldots,$$

$$C^{-1/2}(\lambda A_k + (1 - \lambda)B_k)C^{-1/2}\big)C^{1/2}$$

$$= C^{1/2}F\big(X^* A_{k+1}^{-1/2} A_1 A_{k+1}^{-1/2} X + Y^* B_{k+1}^{-1/2} B_1 B_{k+1}^{-1/2} Y, \ldots,$$

$$X^* A_{k+1}^{-1/2} A_k A_{k+1}^{-1/2} X + Y^* B_{k+1}^{-1/2} B_k B_{k+1}^{-1/2} Y\big)C^{1/2}$$

$$\leq C^{1/2}\big(X^* F(A_{k+1}^{-1/2} A_1 A_{k+1}^{-1/2}, \ldots, A_{k+1}^{-1/2} A_k A_{k+1}^{-1/2})X$$

$$+ Y^* F(B_{k+1}^{-1/2} B_1 B_{k+1}^{-1/2}, \ldots, B_{k+1}^{-1/2} B_k B_{k+1}^{-1/2})Y\big)C^{1/2}$$

$$= \lambda A_{k+1}^{1/2} F(A_{k+1}^{-1/2} A_1 A_{k+1}^{-1/2}, \ldots, A_{k+1}^{-1/2} A_k A_{k+1}^{-1/2})A_{k+1}^{1/2}$$

$$+ (1 - \lambda)B_{k+1}^{1/2} F(B_{k+1}^{-1/2} B_1 B_{k+1}^{-1/2}, \ldots, B_{k+1}^{-1/2} B_k B_{k+1}^{-1/2})B_{k+1}^{1/2}$$

$$= \lambda \mathcal{P}_F(A_1, \ldots, A_{k+1}) + (1 - \lambda)\mathcal{P}_F(B_1, \ldots, B_{k+1}),$$

where we used Jensen's inequality for regular mappings. **QED**

Proposition 3.3. *Let* $F \colon \mathcal{D}_+^{k+1} \to B(\mathcal{H})$ *be a convex and positively homogeneous regular mapping. Then* F *is the perspective of its restriction* G *to* $\mathcal{D}_+^k$ *given by*

$$G(A_1, \ldots, A_k) = F(A_1, \ldots, A_k, 1)$$

for positive invertible operators $A_1, \ldots, A_k$ *acting on* $\mathcal{H}$.

Proof. Since F is a convex and positively homogeneous regular mapping we may apply Proposition 2.3. Then by setting $C = A_{k+1}^{-1/2}$ we obtain

$$A_{k+1}^{-1/2} F(A_1, \ldots, A_k, A_{k+1})A_{k+1}^{-1/2}$$

$$= F(A_{k+1}^{-1/2} A_1 A_{k+1}^{-1/2}, \ldots, A_{k+1}^{-1/2} A_k A_{k+1}^{-1/2}, 1).$$

By rearranging this equation we obtain

$$F(A_1, \ldots, A_k, A_{k+1}) = A_{k+1}^{1/2} G(A_{k+1}^{-1/2} A_1 A_{k+1}^{-1/2}, \ldots, A_{k+1}^{-1/2} A_k A_{k+1}^{-1/2})A_{k+1}^{1/2}$$

which is the statement to be proved. **QED**

The result in the above proposition may be reformulated in the following way: The perspective $\mathcal{P}_G$ of a convex regular mapping $G \colon \mathcal{D}_+^k \to B(\mathcal{H})$ is the unique extension of G to a positively homogeneous convex regular mapping $F \colon \mathcal{D}_+^{k+1} \to B(\mathcal{H})$.

4 The construction of geometric means

We construct a sequence of multivariate geometric means $G_1, G_2, \ldots$ by the following general procedure.

(i) We begin by setting $G_1(A) = A$ for each positive definite invertible operator A.

(ii) To each geometric mean G_k of k variables we associate an auxiliary mapping $F_k \colon \mathcal{D}_+^k \to B(\mathcal{H})$ such that

(a) F_k is a regular map,

(b) F_k is concave,

(c) $F_k(t_1, \ldots, t_k) = \left(t_1 \cdots t_k \right)^{1/(k+1)}$ for positive numbers $t_1, \ldots, t_k$.

(iii) We define the geometric mean $G_{k+1} \colon \mathcal{D}_+^{k+1} \to B(\mathcal{H})$ of $k + 1$ variables as the perspective

$$G_{k+1}(A_1, \ldots, A_{k+1}) = \mathcal{P}_{F_k}(A_1, \ldots, A_{k+1})$$

of the auxiliary map F_k.

Geometric means defined by this very general procedure are concave and positively homogeneous regular mappings by Theorem 3.2 and the preceding remarks. They also satisfy

$$(6) \qquad G_k(A_1, \ldots, A_k) = \left(A_1 \cdots A_k \right)^{1/k}$$

for commuting operators. Indeed, since G_k is the perspective of F_{k-1} and this map satisfies (c) in condition (ii), we obtain $G_k(t_1, \ldots, t_k) = \left(t_1 \cdots t_k \right)^{1/k}$ for positive numbers. Equality (6) then follows since G_k is regular. The geometric mean of two variables

$$(7) \qquad G_2(A_1, A_2) = A_2^{1/2} \left(A_2^{-1/2} A_1 A_2^{-1/2} \right)^{1/2} A_2^{1/2}$$

coincides with the geometric mean of two variables $A_1 \# A_2$ introduced by Pusz and Woronowicz. This is so since G_2 is the perspective of F_1 and $F_1(A) = A^{1/2}$. The last statement is obtained since F_1 is a regular mapping and satisfies $F_1(t) = t^{1/2}$ for positive numbers by (c) in condition (ii).

There are many ways to associate the auxiliary map F_k in the above procedure, so we should not in general expect much similarity between the geometric means for different number of variables.

4.1 The inductive geometric mean

We define the auxiliary mapping $F_k \colon \mathcal{D}_+^k \to B(\mathcal{H})$ by setting

$$F_k(A_1, \ldots, A_k) = G_k(A_1, \ldots, A_k)^{k/(k+1)}$$

for $k = 1, 2, \ldots$.

Theorem 4.1. *The means G_k constructed in section 4 then have the following properties:*

(i) $G_k \colon \mathcal{D}_+^k \to B(\mathcal{H})_+$ *is a regular map for each* $k = 1, 2, \ldots$.

(ii) $G_k(tA_1, \ldots, tA_k) = tG_k(A_1, \ldots, A_k)$ *for* $t > 0$, $(A_1, \ldots, A_k) \in \mathcal{D}_+^k$ *and* $k = 1, 2, \ldots$.

(iii) $G_k \colon \mathcal{D}_+^k \to B(\mathcal{H})$ *is concave for each* $k = 1, 2, \ldots$.

(iv) $G_{k+1}(A_1, \ldots, A_k, 1) = G_k(A_1, \ldots, A_k)^{k/(k+1)}$ *for* $(A_1, \ldots, A_k) \in \mathcal{D}_+^k$ *and* $k = 1, 2, \ldots$.

Any sequence of mappings $\tilde{G}_k$ beginning with $\tilde{G}_1(A) = A$ and satisfying the above conditions coincide with the means G_k for $k = 1, 2, \ldots$.

Proof. Each map G_k is for $k = 2, 3, \ldots$ the perspective of a regular map and this implies (i) and (ii). The assertion of concavity for G_1 is immediate. Suppose now G_k is concave for some k. Since the map $t \to t^p$ is both operator monotone and operator concave for $0 \le p \le 1$, we realise that the auxiliary mapping

$$F_k(A_1, \ldots, A_k) = G_k(A_1, \ldots, A_k)^{k/(k+1)}$$

is concave, and since G_{k+1} is the perspective of F_k we then obtain by Theorem 3.2 that also G_{k+1} is concave. Since the first map G_1 is concave we have thus proved by induction that G_k is concave for all $k = 1, 2, \ldots$. The last property (iv) follows since G_{k+1} is the perspective of $G_k^{k/(k+1)}$.

Let finally $\tilde{G}_k$ be a sequence of mappings satisfying (i) to (iv). Since each $\tilde{G}_{k+1}$ is concave and homogeneous it follows by Proposition 3.3 that $\tilde{G}_{k+1}$ is the perspective of its restriction $\tilde{G}_{k+1}(A_1, \ldots, A_k, 1)$. Because of (iv) we then realise that $\tilde{G}_{k+1}$ is the perspective of the map

$$\tilde{F}_k(A_1, \ldots, A_k) = \tilde{G}_k(A_1, \ldots, A_k)^{k/(k+1)}$$

constructed from $\tilde{G}_k$. The $\tilde{G}_k$ mappings are thus constructed by the same algorithm as the mappings G_k for every $k \geq 2$, and since $\tilde{G}_1 = G_1$ they must all coincide. **QED**

In addition to the properties listed in the above theorem the means G_k enjoy a number of other properties that we list below.

Theorem 4.2. *The means G_k constructed in section 4 have the following additional properties:*

(i) *The means G_k are increasing in each variable for $k = 1, 2 \ldots$.*

(ii) *The means G_k are congruence invariant. For any invertible operator C on $\mathcal{H}$ the identity*

$$G_k(C^*A_1C, \ldots, C^*A_kC) = C^*G_k(A_1, \ldots, A_k)C$$

holds for $(A_1, \ldots, A_k) \in \mathcal{D}_+^k$ and $k = 1, 2, \ldots$.

(iii) *The means G_k are jointly homogeneous in the sense that*

$$G_k(t_1A_1, \ldots, t_kA_k) = (t_1 \cdots t_k)^{1/k}G_k(A_1, \ldots, A_k)$$

for scalars $t_1, \ldots, t_k > 0$, operators $(A_1, \ldots, A_k) \in \mathcal{D}_+^k$ and $k = 1, 2, \ldots$.

(iv) *The means G_k are self-dual in the sense that*

$$G_k(A_1^{-1}, \ldots, A_k^{-1}) = G_k(A_1, \ldots, A_k)^{-1}$$

for $(A_1, \ldots, A_k) \in \mathcal{D}_+^k$ and $k = 1, 2, \ldots$.

(v) *When restricted to positive definite matrices the determinant identity*

$$\det G_k(A_1, \ldots, A_k) = (\det A_1 \cdots \det A_k)^{1/k}$$

holds for $k = 1, 2 \ldots$.

Proof. The first property follows by the following standard argument for positive concave mappings. Consider positive definite invertible operators $A_m \leq B_m$ for $m = 1, \ldots, k$. By first assuming that the difference $B_m - A_m$ is invertible we may take $\lambda \in (0, 1)$ and write

$$\lambda B_m = \lambda A_m + (1 - \lambda)C_m \qquad m = 1, \ldots, k,$$

where $C_m = \lambda(1 - \lambda)^{-1}(B_m - A_m)$ is positive definite and invertible. By using concavity we then obtain

$$G_k(\lambda B_1, \ldots, \lambda B_k) \geq \lambda G(A_1, \ldots, A_k) + (1 - \lambda)G_k(C_1, \ldots, C_k)$$

$$\geq \lambda G(A_1, \ldots, A_k).$$

Letting $\lambda \to 1$ we obtain $G_k(B_1, \ldots, B_k) \geq G_k(A_1, \ldots, A_k)$ by continuity. In the general case we choose $0 < \mu < 1$ such that

$$\mu A_m < A_m \leq B_m \qquad m = 1, \ldots, k$$

and obtain $G_k(\mu A_1, \ldots, \mu A_k) \leq G_k(B_1, \ldots, B_k)$. By letting $\mu \to 1$ we then obtain $G_k(A_1, \ldots, A_k) \leq G_k(B_1, \ldots, B_k)$ which shows (i).

Since G_k is concave and homogeneous we obtain (ii) from Proposition 2.3.

Property (iii) is immediate for $k = 1$ and $k = 2$. Suppose the property is verified for k, then

$$G_{k+1}(t_1 A_1, \ldots, t_k A_k, t_{k+1} A_{k+1})$$

$$= t_{k+1} A_{k+1}^{1/2} F_k\big(t_1 t_{k+1}^{-1} A_{k+1}^{-1/2} A_1 A_{k+1}^{-1/2}, \ldots, t_k t_{k+1}^{-1} A_{k+1}^{-1/2} A_k A_{k+1}^{-1/2}\big) A_{k+1}^{1/2}$$

$$= t_{k+1} A_{k+1}^{1/2} G_k\big(t_1 t_{k+1}^{-1} A_{k+1}^{-1/2} A_1 A_{k+1}^{-1/2}, \ldots, t_k t_{k+1}^{-1} A_{k+1}^{-1/2} A_k A_{k+1}^{-1/2}\big)^{k/(k+1)} A_{k+1}^{1/2}.$$

By using the induction assumption we obtain

$$G_{k+1}(t_1 A_1, \ldots, t_k A_k, t_{k+1} A_{k+1})$$

$$= t_{k+1}(t_{k+1}^{-1} t_1^{1/k} \cdots t_k^{1/k})^{k/(k+1)} G_{k+1}(A_1, \ldots, A_k, A_{k+1})$$

$$= (t_1 \cdots t_k t_{k+1})^{1/(k+1)} G_{k+1}(A_1, \ldots, A_k, A_{k+1})$$

which shows (iii).

Property (iv) is immediate for $k = 1$ and $k = 2$. Suppose the property is verified for k, then

$$G_{k+1}(A_1^{-1}, \ldots, A_k^{-1}, A_{k+1}^{-1})$$

$$= A_{k+1}^{-1/2} F_k\big(A_{k+1}^{1/2} A_1^{-1} A_{k+1}^{1/2}, \ldots, A_{k+1}^{1/2} A_k^{-1} A_{k+1}^{1/2}\big) A_{k+1}^{-1/2}$$

$$= A_{k+1}^{-1/2} G_k\big(A_{k+1}^{1/2} A_1^{-1} A_{k+1}^{1/2}, \ldots, A_{k+1}^{1/2} A_k^{-1} A_{k+1}^{1/2}\big)^{k/(k+1)} A_{k+1}^{-1/2}.$$

By using the induction assumption we obtain

$$G_{k+1}(A_1^{-1}, \ldots, A_k^{-1}, A_{k+1}^{-1})$$

$$= A_{k+1}^{-1/2} G_k \left(A_{k+1}^{-1/2} A_1 A_{k+1}^{-1/2}, \ldots, A_{k+1}^{-1/2} A_k A_{k+1}^{-1/2} \right)^{-k/(k+1)} A_{k+1}^{-1/2}$$

$$= \left(A_{k+1}^{1/2} G_k \left(A_{k+1}^{-1/2} A_1 A_{k+1}^{-1/2}, \ldots, A_{k+1}^{-1/2} A_k A_{k+1}^{-1/2} \right)^{k/(k+1)} A_{k+1}^{1/2} \right)^{-1}$$

$$= G_{k+1}(A_1, \ldots, A_k, A_{k+1})^{-1}$$

which shows (iv).

Notice that since $\det A = \exp(\mathrm{Tr} \log A)$ for positive definite A, we have $\det A^p = (\det A)^p$ for all real exponents p. Property (v) is easy to calculate for $k = 1$ and $k = 2$. Suppose the property is verified for k. Since as above

$$G_{k+1}(A_1, \ldots, A_k, A_{k+1})$$

$$= A_{k+1}^{1/2} G_k \left(A_{k+1}^{-1/2} A_1 A_{k+1}^{-1/2}, \ldots, A_{k+1}^{-1/2} A_k A_{k+1}^{-1/2} \right)^{k/(k+1)} A_{k+1}^{1/2}$$

we obtain

$$\det G_{k+1}(A_1, \ldots, A_k, A_{k+1})$$

$$= \det A_{k+1} \left(\det A_{k+1}^{-1} \det A_1 \cdots \det A_{k+1}^{-1} \det A_k \right)^{1/(k+1)}$$

$$= \left(\det A_1 \cdots \det A_k \cdot \det A_{k+1} \right)^{1/k+1}$$

which shows (v). **QED**

Theorem 4.3. *The geometric means G_k are for $k = 1, 2, \ldots$ bounded between the symmetric harmonic and arithmetic means. That is,*

$$\frac{k}{A_1^{-1} + \cdots + A_k^{-1}} \le G_k(A_1, \ldots, A_k) \le \frac{A_1 + \cdots + A_k}{k}$$

for arbitrary $(A_1, \ldots, A_k) \in \mathcal{D}_+^k$ and $k = 1, 2, \ldots$.

Proof. The upper bound holds with equality for $k = 1$. Suppose that we have verified the inequality for k. Since by classical analysis

$$X^{k/(k+1)} \le 1 + \frac{k}{k+1}(X - 1)$$

for positive definite X, we obtain

$$F_k(A_1, \ldots, A_k) = G_k(A_1, \ldots, A_k)^{k/(k+1)} \leq 1 + \frac{k}{k+1}(G_k(A_1, \ldots, A_k) - 1)$$

$$\leq 1 + \frac{k}{k+1}\left(\frac{A_1 + \cdots + A_k}{k} - 1\right) = \frac{A_1 + \cdots + A_k + 1}{k+1}.$$

By taking perspectives we now obtain

$$G_{k+1}(A_1, \ldots, A_k, B) = \mathcal{P}_{F_k}(A_1, \ldots, A_k, B)$$

$$= B^{1/2}F_k(B^{-1/2}A_1B^{-1/2}, \ldots, B^{-1/2}A_kB^{-1/2})B^{1/2}$$

$$\leq B^{1/2}\frac{B^{-1/2}A_1B^{-1/2} + \cdots + B^{-1/2}A_kB^{-1/2} + 1}{k+1}B^{1/2} = \frac{A_1 + \cdots A_k + B}{k+1}$$

which proves the upper bound by induction. We next use the upper bound to obtain

$$G_k(A_1^{-1}, \ldots, A_k^{-1}) \leq \frac{A_1^{-1} + \cdots + A_k^{-1}}{k}.$$

By inversion we then obtain

$$\frac{k}{A_1^{-1} + \cdots + A_k^{-1}} \leq G_k(A_1^{-1}, \ldots, A_k^{-1})^{-1} = G_k(A_1, \ldots, A_k),$$

where we in the last equation used self-duality of the geometric mean, cf. property (iv) in Theorem 4.2. **QED**

The means studied in this section are known in the literature as the inductive means of Sagae and Tanabe [14]. By considering the power mean

$$A\#_t B = B^{1/2}(A^{-1/2}BA^{-1/2})^t B^{1/2} \qquad 0 \leq t \leq 1$$

they established the recursive relation by setting

$$G_{k+1}(A_1, \ldots, A_{k+1}) = G_k(A_1, \ldots, A_k)\#_{k/(k+1)}A_{k+1}.$$

The authors did not study the general properties of these means but established the harmonic-geometric-arithmetic mean inequality of Theorem 4.3. It is possible to prove the crucial concavity property (iii) in Theorem 4.1 by induction. It can be done without the general theory of perspectives of regular operator mappings, and it only requires the properties of an operator mean of two variables as studied by Kubo and Ando [9]. However, this is a special situation that only applies to the inductive means.

4.2 Variant geometric means

The inductive geometric means are uniquely specified within the general framework discussed in this paper by choosing the updating condition (1), cf. property (*iv*) in Theorem 4.1. We may instead construct geometric means satisfying updating condition (2) by choosing the auxiliary map

$$F_k(A_1, \ldots, A_k) = G_k(A_1^{k/(k+1)}, \ldots, A_k^{k/(k+1)})$$

for $k = 1, 2 \ldots$. It is a small exercise to realise that these means satisfy all of the properties listed in Theorem 4.1, Theorem 4.2, and Theorem 4.3 with the only exception that condition (*iv*) in Theorem 4.1 is replaced by updating condition (2). Concavity of these means cannot be reduced to concavity of operator means of two variables but relies on the general theory of regular operator mappings and Theorem 3.2.

4.3 The Karcher means

The Karcher mean $\Lambda_k(A_1, \ldots, A_k)$ of k positive definite invertible operator variables is defined as the unique positive definite solution to the equation

$$(8) \qquad \sum_{i=1}^{k} \log(X^{1/2} A_i X^{1/2}) = 0,$$

and it enjoys all of the attractive properties of an operator mean listed by Ando, Li, and Mathias, cf. [10]. The defining equation (8) immediately implies that the Karcher mean $\Lambda_k \colon \mathcal{D}_+^k \to B(\mathcal{H})$ is a regular operator mapping, and it may therefore be understood within the general framework discussed in this paper by choosing the auxiliary map

$$F_k(A_1, \ldots, A_k) = \Lambda_{k+1}(A_1, \ldots, A_k, 1).$$

The problem, however, is that we do not have any explicit expression of F_k in terms of Λ_k.

5 Other applications

Consider a regular operator mapping $F \colon \mathcal{D}_+ \to B(H)$ of k variables and let f be the numerical function such that

$$f(t_1, \ldots, t_k)I = F(t_1 I, \ldots, t_k I) \qquad t_1, \ldots, t_k > 0,$$

where I is the unit operator. Since F is regular, the restriction to products $\mathcal{A}_1 \times \cdots \times \mathcal{A}_k$ of commuting sub-C^*-algebras $\mathcal{A}_1, \ldots, \mathcal{A}_k$ of $B(H)$ is given by the functional calculus of f, that is

$$F(x_1, \ldots, x_k) = f(x_1, \ldots, x_k)$$

for tuples $(x_1, \ldots, x_n)$ of positive definite operators in $\mathcal{A}_1 \times \cdots \times \mathcal{A}_k$. Therefore, if the operator map F is convex then f is an operator convex function of k variables. This observation can be used to construct new operator convex functions of several variables.

Following this general idea one can prove that the function

$$f(t, s) = \begin{cases} \dfrac{t^{p+1} - s^{p+1}}{t - s} & t \neq s \\[3mm] \dfrac{1}{p+1} t^p & t = s \end{cases}$$

is operator concave in $(0, \infty) \times (0, \infty)$ for $0 \leq p \leq 1$. Likewise, one can prove that the function

$$f(t_1, t_2, t_3) = \begin{cases} \dfrac{t_1^{p+1} - t_2^{p+1}}{t_1 - t_2} t_3^{1-p} & t_1 \neq t_2 \\[3mm] \dfrac{1}{p+1} t_1^p t_3^{1-p} & t_1 = t_2 \end{cases}$$

is operator concave in $(0, \infty) \times (0, \infty) \times (0, \infty)$ for $0 \leq p \leq 1$. Notice that by setting $t_1 = t_2$ we recover Lieb's concavity theorem, cf. [15] for details and further examples.

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