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Expansions of relative operator entropies and operator valued α-divergence

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1. Introduction

Throughout this paper, an operator means a bounded linear operator on a Hilbert space H. An operator T on H is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and an operator T is said to be strictly positive (denoted by $T > 0$) if T is invertible and positive.

For strictly positive operators A and B, and for $x \in \mathbf{R}$, a path passing through A and B is defined as follows ([4], [5], [11] etc.):

$$A \natural_x B \equiv A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^x A^{\frac{1}{2}}.$$

If $x \in [0, 1]$, then the path becomes weighted geometric operator mean denoted by $A \sharp_x B$. Weighted arithmetic operator mean is defined as $A \nabla_x B \equiv (1 - x)A + xB$ for $x \in [0, 1]$.

Fujii and Kamei [3] defined relative operator entropy as follows:

$$S(A|B) \equiv A^{\frac{1}{2}} \left(\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}.$$

Furuta [8] gave a generalized form of relative operator entropy. Furuta's one is called generalized relative operator entropy and is defined as follows:

$$S_t(A|B) \equiv A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t \left(\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}, \ t \in \mathbf{R}.$$

We remark that $S_0(A|B) = S(A|B)$ holds.

Yanagi, Kuriyama and Furuichi [13] defined Tsallis relative operator entropy as follows:

$$T_t(A|B) = \frac{A \sharp_t B - A}{t}, \ t \in (0, 1].$$

By replacing $A \sharp_t B$ with $A \natural_t B$, Tsallis relative operator entropy can be extended as the notion for $t \in \mathbf{R}$. Since $\lim_{t \to 0} \frac{a^t - 1}{t} = \log a$ holds for $a > 0$, we have $T_0(A|B) \equiv \lim_{t \to 0} T_t(A|B) = S(A|B)$.

For these relative operator entropies, we can give geometrical interpretations. By the derivative of the path with respect to x at t, we can get generalized relative operator entropy, that is, the following holds:

$$\frac{d}{dx} A \natural_x B \Big|_{x=t} = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t \left(\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}} = S_t(A|B).$$

Therefore, we can regard $S_t(A|B)$ as the slope of the tangent line at $x = t$ of the path. From this interpretation, we can regard $S(A|B)$ as the slope of the tangent line at $x = 0$ of the path. Tsallis relative operator entropy can be regarded as the slope of the line passing through points A and $A \natural_t B$ on the path.

Amari [1] defined α-divergence as a notion to measure the difference between two probability distributions. Based on this notion, Fujii [2] defined operator valued α-divergence as follows: For strictly positive operators A and B, and for $\alpha \in (0,1)$,

$$D_\alpha(A|B) \equiv \frac{A \; \nabla_\alpha \; B - A \; \natural_\alpha \; B}{\alpha(1 - \alpha)}.$$

Petz [12] introduced the operator divergence $D_{FK}(A|B) \equiv B - A - S(A|B)$. Fujii et al. [6, 7] showed the following relation between $D_{FK}(A|B)$ and operator valued α-divergences at end points for interval $(0,1)$.

$$D_0(A|B) \equiv \lim_{t \to 0} D_t(A|B) = B - A - S(A|B),$$
$$D_1(A|B) \equiv \lim_{t \to 1} D_t(A|B) = A - B + S_1(A|B).$$

For the quantity $D_0(A|B)$, we give a geometrical interpretation shown in Figure 1.
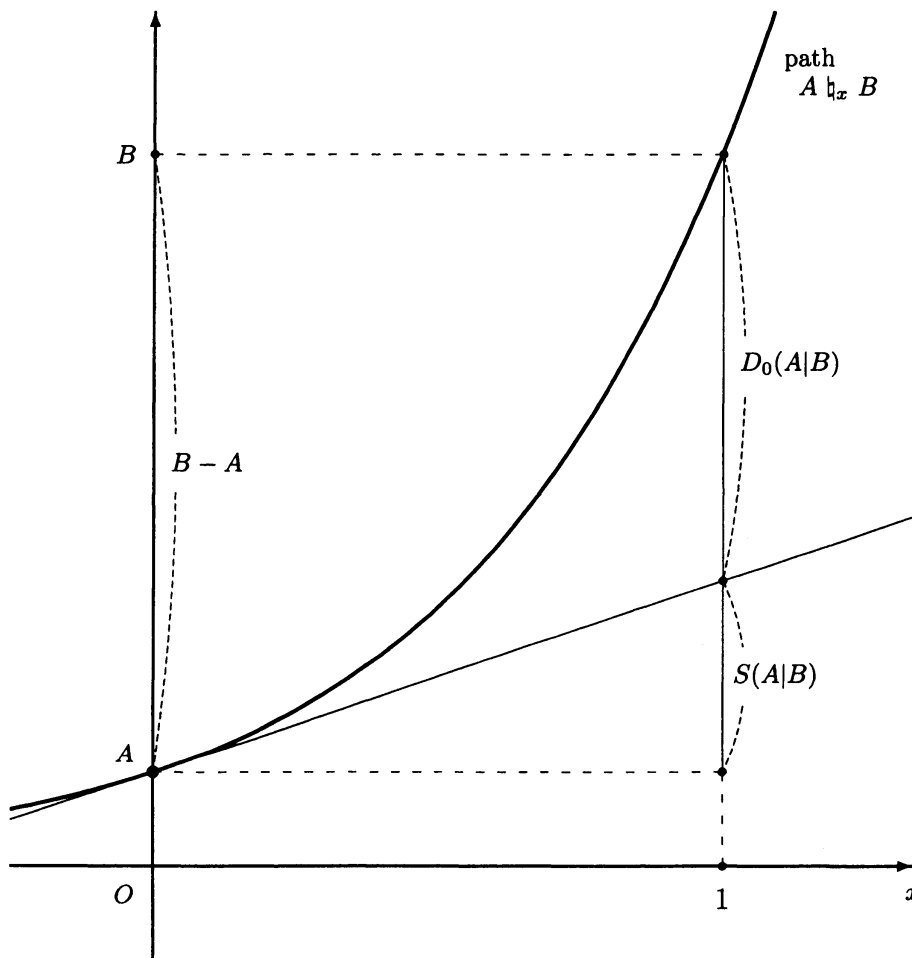

Figure 1: An interpretation for $D_0(A|B) = B - A - S(A|B)$.

In [11], Kamei showed that relative operator entropy has some kind of additivity as follows: For strictly positive operators A and B, and for $s \in \mathbf{R}$,

$$(\star) \qquad\qquad S(A|A \; \natural_s \; B) = sS(A|B).$$

In [9], we gave a viewpoint of operator valued distance for $S(A|B)$. Here, we give the following geometrical interpretation for this relation: The second component $A \, \natural_s \, B$ of the left hand side is an arbitrary point on the path. So, we can regard the relation $(\star)$ as relative operator entropy for a fixed point A and any point on the path.

For strictly positive operators A and B, and for $t \in [0, 1]$ and $r \in [-1, 1]$, operator power mean is defined as follows:

$$A \, \natural_{t,r} \, B \equiv A^{\frac{1}{2}} \left\{ (1-t)I + t \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^r \right\}^{\frac{1}{r}} A^{\frac{1}{2}} = A \, \natural_{\frac{1}{r}} \, \{ A \, \nabla_t \, (A \, \natural_r \, B) \}.$$

To preserve $(1-t)I + t \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^r \geq 0$, we have to impose t in $[0, 1]$. Operator power mean compounds the arithmetic, geometric and harmonic means, that is, the following holds.

arithmetic operator mean
$A \, \nabla_t \, B = (1-t)A + tB$

$\uparrow_{r=1}$

$A \, \natural_{t,r} \, B \xrightarrow[r \to 0]{}$

geometric operator mean
$A \, \natural_t \, B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t A^{\frac{1}{2}}$

$\downarrow_{r=-1}$

harmonic operator mean
$A \, \Delta_t \, B = (A^{-1} \, \nabla_t \, B^{-1})^{-1}$

We treat this operator power mean as an expanded path which links point A with point B. As the corresponding notions to relative operator entropies and operator valued α-divergence, we introduce expanded relative operator entropy, expanded Tsallis relative operator entropy and expanded operator valued α-divergence.

In this report, we aim at getting results on these expanded notions. In section 2, we show some results on relative operator entropies for two points on the path $A \, \natural_x \, B$. In section 3, we show results on expanded relative operator entropies, and in section 4, we show results on expanded operator valued α-divergence.

2. Relative operator entropies

Based on the relation $(\star)$, we show some basic results on relative operator entropies for two points on the path $A \, \natural_x \, B$.

To show the results in this section, we prepare the following properties of the path.

Lemma 2.1. *Let A and B be strictly positive operators. Then,*

(1) $\qquad\qquad\qquad A \, \natural_t \, (A \, \natural_s \, B) \;=\; A \, \natural_{st} \, B,$

(2) $\qquad\qquad\qquad (A \, \natural_t \, B) \, \natural_s \, A \;=\; A \, \natural_{(1-s)t} \, B$

hold for $s, \, t \in \mathbf{R}$.

Lemma 2.2. (Lemma 3.6 in [10]) *Let A and B be strictly positive operators. Then,*

$$(A \, \natural_u \, B) \, \natural_w \, (A \, \natural_{u+v} \, B) = A \, \natural_{u+vw} \, B$$

holds for $u, \, v, \, w \in \mathbf{R}$.

For generalized relative operator entropy and Tsallis relative operator entropy, we have the following result corresponding to the relation $(\star)$.

Theorem 2.3. *Let A and B be strictly positive operators. Then,*

(1) $\qquad\qquad S_t(A|A\ \natural_s\ B) \;=\; sS_{st}(A|B),$

(2) $\qquad\qquad T_t(A|A\ \natural_s\ B) \;=\; sT_{st}(A|B)$

hold for $s,\ t \in \mathbf{R}$.

Proof. (1) If $s = 0$, then it is obvious that the both sides equal zero, and if $t = 0$, then this equality becomes the relation $(\star)$. Otherwise, we get

$$S_t(A|A\ \natural_s\ B)$$
$$= A^{\frac{1}{2}}\left\{A^{-\frac{1}{2}}A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^s A^{\frac{1}{2}}A^{-\frac{1}{2}}\right\}^t \log\left\{A^{-\frac{1}{2}}A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^s A^{\frac{1}{2}}A^{-\frac{1}{2}}\right\}A^{\frac{1}{2}}$$
$$= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{st}\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^s A^{\frac{1}{2}} \;=\; sS_{st}(A|B).$$

(2) By (1) in Lemma 2.1, we have

$$T_t(A|A\ \natural_s\ B) \;=\; \frac{A\ \natural_t\ (A\ \natural_s\ B) - A}{t} \;=\; s\,\frac{A\ \natural_{st}\ B - A}{st} \;=\; sT_{st}(A|B).$$

$\square$

For the relation (1) in Theorem 2.3, we can give a geometrical interpretation shown in Figure 2. We remark that two tangent lines drawn in this figure intersect on the axis of the vertical direction.

In [10], we showed the following result on translation of generalized relative operator entropy.

Proposition 2.4. (Proposition 3.1 in [10]) *Let A and B be strictly positive operators. Then,*

$$S_{u+v}(A|B) = (A\ \natural_{u+v}\ B)(A\ \natural_u\ B)^{-1}S_u(A|B)$$

holds for $u,\ v \in \mathbf{R}$.

When we regard $S_u(A|B)$ and $S_{u+v}(A|B)$ as tangent vectors at u and $u + v$ on the path $A\ \natural_w\ B$, respectively, Proposition 2.4 means that $S_{u+v}(A|B)$ is parallelly transferring $S_u(A|B)$ by v along the path.

Here, we define the following noncommutative ratio on the path $A\ \natural_w\ B$.

Definition 2.5. *For strictly positive operators A and B, and for $u,\ v \in \mathbf{R}$, noncommutative ratio on the path $A\ \natural_w\ B$ is defined as follows:*

$$\mathcal{R}(u,v;A,B) \equiv (A\ \natural_{u+v}\ B)(A\ \natural_u\ B)^{-1}.$$

For the noncommutative ratio, the following property holds.

Proposition 2.6. (Proposition 3.3 in [10]) *Let A and B be strictly positive operators. Then,*

$$(A\ \natural_{u+v}\ B)(A\ \natural_u\ B)^{-1} = (A\ \natural_v\ B)A^{-1},$$

that is,

$$\mathcal{R}(u,v;A,B) = \mathcal{R}(0,v;A,B) = (A\ \natural_v\ B)A^{-1}$$
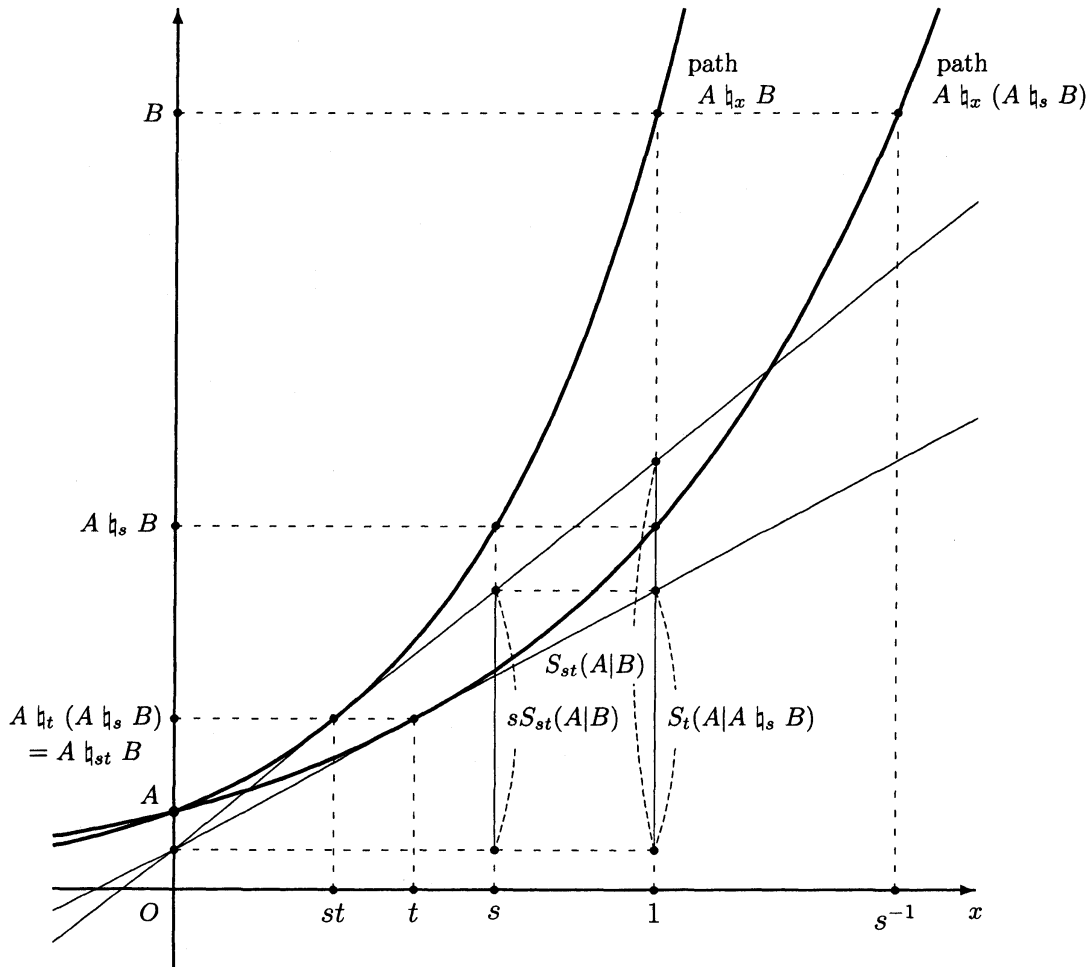
holds for $u,\ v \in \mathbf{R}$.

Figure 2: An interpretation for $S_t(A|A \natural_s B) = sS_{st}(A|B)$.

By Proposition 2.6, $\mathcal{R}(u, v; A, B)$ does not depend on u. So, we denote $\mathcal{R}(u, v; A, B)$ by $\mathcal{R}(v)$ in the rest of this report simply. We call multiplying by $\mathcal{R}(v)$ from the left side noncommutative ratio translation.

From Proposition 2.4 and Definition 2.5, we get the following immediately.

Corollary 2.7. (Corollary 3.4 in [10]) *Let A and B be strictly positive operators. Then,*

$$S_{u+v}(A|B) = \mathcal{R}(v)S_u(A|B)$$

holds for u, $v \in \mathbf{R}$.

Remark 1. *By putting $u = 0$ in Corollary 2.7, we have $S_v(A|B) = \mathcal{R}(v)S(A|B)$.*

The following is an extension of the relation ($\star$). This is a result of generalized relative operator entropy for any two points on the path.

Proposition 2.8. (Proposition 3.7 in [10]) *Let A and B be strictly positive operators. Then,*

$$S_t(A \natural_v B|A \natural_{v+s} B) = sS_{v+st}(A|B)$$

*hold for s, t, v ∈ **R**.*

By using noncommutative ratio, we can represent the results of relative operator entropies for any two points on the path as follows:

Theorem 2.9. (Theorem 3.11 in [10]) *Let A and B be strictly positive operators. Then,*

(1) $\qquad\qquad S_t(A \, \natural_v \, B | A \, \natural_{v+s} \, B) = s\mathcal{R}(v)S_{st}(A|B),$

(2) $\qquad\qquad T_t(A \, \natural_v \, B | A \, \natural_{v+s} \, B) = s\mathcal{R}(v)T_{st}(A|B),$

(3) $\qquad\qquad S(A \, \natural_v \, B | A \, \natural_{v+s} \, B) = s\mathcal{R}(v)S(A|B)$

*hold for s, t, v ∈ **R**.*

Proof. **(1)** By Proposition 2.8 and Corollary 2.7, we have

$$S_t(A \, \natural_v \, B | A \, \natural_{v+s} \, B) \;=\; sS_{v+st}(A|B) \;=\; s\mathcal{R}(v)S_{st}(A|B).$$

(2) By Lemma 2.2 and Proposition 2.6, we get

$$T_t(A \, \natural_v \, B | A \, \natural_{v+s} \, B) \;=\; \frac{(A \, \natural_v \, B) \, \natural_t \, (A \, \natural_{v+s} \, B) - A \, \natural_v \, B}{t}$$

$$= \frac{A \, \natural_{v+st} \, B - A \, \natural_v \, B}{t} \;=\; s\frac{(A \, \natural_v \, B)A^{-1}(A \, \natural_{st} \, B) - (A \, \natural_v \, B)A^{-1}A}{st}$$

$$= s(A \, \natural_v \, B)A^{-1}T_{st}(A|B) \;=\; s\mathcal{R}(v)T_{st}(A|B).$$

(3) This equality can be obtained by putting $t = 0$ for (1).

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Remark 2. *We can get Theorem 2.3 by putting $v = 0$ for the relations (1) and (2) in Theorem 2.9.*

3. Expanded relative operator entropies

In this section, we show the results of expanded relative operator entropies for two points on the expanded path $A \, \natural_{t,r} \, B$. Similarly to $S_t(A|B)$, in [9], expanded relative operator entropy $S_{t,r}(A|B)$ is defined by the derivative of expanded path with respect to x at t as follows: For strictly positive operators A and B, and for $t \in [0,1]$ and $r \in [-1,1]$,

$$S_{t,r}(A|B) \;\equiv\; \frac{d}{dx} \, A \, \natural_{x,r} \, B \, \Big|_{x=t}$$

$$= \; A^{\frac{1}{2}} \left[\left\{ (1-t)I + t\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r \right\}^{\frac{1}{r}-1} \frac{\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r - I}{r} \right] A^{\frac{1}{2}}$$

$$= \; \left[A \, \natural_{\frac{1}{r}-1} \, \{A \, \nabla_t \, (A \, \natural_r \, B)\} \right] A^{-1}T_r(A|B).$$

We remark that expanded relative operator entropy has the following relations [9]:

$$B - A$$

$$\uparrow_{r=1}$$

$$T_r(A|B) \quad \xleftarrow[t=0]{} \quad S_{t,r}(A|B) \quad \xrightarrow[t=1]{} \quad -T_r(B|A)$$

$$\downarrow_{r\to 0}$$

$$S_t(A|B)$$

By replacing weighted geometric operator mean with operator power mean, we obtain the definition of expanded Tsallis relative operator entropy [9]: For strictly positive operators A and B, and for $t \in (0,1]$ and $r \in [-1,1]$,

$$T_{t,r}(A|B) \equiv \frac{A \; \natural_{t,r} \; B - A}{t}.$$

We remark that expanded Tsallis relative operator entropy also has the following relations:

$$B - A$$

$$\uparrow_{r=1}$$

$$T_r(A|B) \quad \xleftarrow[t\to 0]{} \quad T_{t,r}(A|B) \quad \xrightarrow[t=1]{} \quad B - A$$

$$\downarrow_{r\to 0}$$

$$T_t(A|B)$$

For expanded relative operator entropy, we can show the following result corresponding to (1) in Theorem 2.3.

Theorem 3.1. *Let A and B be strictly positive operators. Then,*

$$S_{t,r}(A|A \; \natural_{s,r} \; B) = s(A \; \natural_{st,r} \; B) \left\{ A \; \nabla_{st} \; (A \; \natural_r \; B) \right\}^{-1} T_r(A|B) = s S_{st,r}(A|B)$$

holds for $t, \; s \in [0,1]$ and $r \in [-1,1]$.

In cases of $t \in \{0, 1\}$, we get the following relations.

Corollary 3.2. *Let A and B be strictly positive operators. Then,*

$$
\begin{aligned}
\text{(1)} \qquad S_{0,r}(A|A \; \natural_{s,r} \; B) &= T_r(A|A \; \natural_{s,r} \; B) = s T_r(A|B), \\
\text{(2)} \qquad S_{1,r}(A|A \; \natural_{s,r} \; B) &= -T_r(A \; \natural_{s,r} \; B|A) \\
&= s(A \; \natural_{s,r} \; B) \left\{ A \; \nabla_s \; (A \; \natural_r \; B) \right\}^{-1} T_r(A|B)
\end{aligned}
$$

hold for $s \in [0,1]$ and $r \in [-1,1]$.

To prove Theorem 3.1, we prepare the following lemmas. We omit their proofs.

Lemma 3.3. *Let A and B be strictly positive operators. Then,*

(1)
$$A \natural_r (A \sharp_{t,r} B) = A \nabla_t (A \natural_r B),$$

(2)
$$A \sharp_{t,r} (A \sharp_{s,r} B) = A \sharp_{st,r} B,$$

(3)
$$A \sharp_{t,r} B = B \sharp_{1-t,r} A$$

hold for $t, s \in [0,1]$ and $r \in [-1,1]$.

Lemma 3.4. *Let A and B be strictly positive operators. Then,*

$$S_{t,r}(A|B) = (A \sharp_{t,r} B)\{A \nabla_t (A \natural_r B)\}^{-1} T_r(A|B)$$
$$= \left[A \natural_{\frac{1}{r}} \{A \nabla_t (A \natural_r B)\}\right] \{A \nabla_t (A \natural_r B)\}^{-1} T_r(A|B)$$

holds for $t \in [0,1]$ and $r \in [-1,1]$.

Proof of Theorem 3.1. By Lemma 3.3 and Lemma 3.4, the following holds:

$$S_{t,r}(A|A \sharp_{s,r} B) = \{A \sharp_{t,r} (A \sharp_{s,r} B)\}[A \nabla_t \{A \natural_r (A \sharp_{s,r} B)\}]^{-1} T_r(A|A \sharp_{s,r} B)$$
$$= (A \sharp_{st,r} B)[A \nabla_t \{A \nabla_s (A \natural_r B)\}]^{-1} T_r(A|A \sharp_{s,r} B)$$
$$= (A \sharp_{st,r} B)\{A \nabla_{st} (A \natural_r B)\}^{-1} \frac{A \natural_r (A \sharp_{s,r} B) - A}{r}$$
$$= (A \sharp_{st,r} B)\{A \nabla_{st} (A \natural_r B)\}^{-1} \frac{A \nabla_s (A \natural_r B) - A}{r}$$
$$= (A \sharp_{st,r} B)\{A \nabla_{st} (A \natural_r B)\}^{-1} \frac{(1-s)A + s(A \natural_r B) - A}{r}$$
$$= s(A \sharp_{st,r} B)\{A \nabla_{st} (A \natural_r B)\}^{-1} T_r(A|B) = sS_{st,r}(A|B).$$

$\square$

The following (1) in Theorem 3.5 is a corresponding result to (2) in Theorem 2.3.

Theorem 3.5. *Let A and B be strictly positive operators. Then,*

(1)
$$T_{t,r}(A|A \sharp_{s,r} B) = sT_{st,r}(A|B) \quad (t \in (0,1]),$$

(2)
$$T_{0,r}(A|A \sharp_{s,r} B) = sT_r(A|B),$$

(3)
$$T_{1,r}(A|A \sharp_{s,r} B) = sT_{s,r}(A|B),$$

(4)
$$T_{1-t,r}(A \sharp_{s,r} B|A) = s\frac{tT_{st,r}(A|B) - T_{s,r}(A|B)}{1-t} \quad (t \in [0,1))$$

hold for $s \in [0,1]$ and $r \in [-1,1]$.

Proof. (1) By (2) in Lemma 3.3, these can be shown as follows:

$$T_{t,r}(A|A \sharp_{s,r} B) = \frac{A \sharp_{t,r} (A \sharp_{s,r} B) - A}{t} = s\frac{A \sharp_{st,r} B - A}{st} = sT_{st,r}(A|B).$$

(2), (3) For (1), by putting $t = 0$ and $t = 1$, we can get these relations, respectively.
(4) Since $A \sharp_{t,r} B = B \sharp_{1-t,r} A$ holds for $t \in [0,1]$ and $r \in [-1,1]$, we have

$$T_{1-t,r}(A \sharp_{s,r} B|A) = \frac{(A \sharp_{s,r} B) \sharp_{1-t,r} A - A \sharp_{s,r} B}{1-t}$$
$$= \frac{A \sharp_{t,r} (A \sharp_{s,r} B) - A \sharp_{s,r} B}{1-t} = \frac{A \sharp_{st,r} B - A \sharp_{s,r} B}{1-t}$$
$$= \frac{(A \sharp_{st,r} B - A) - (A \sharp_{s,r} B - A)}{1-t} = s\frac{tT_{st,r}(A|B) - T_{s,r}(A|B)}{1-t}.$$

$\square$

4. Expanded operator valued α-divergence

By Theorem 2.5 in [10] and the results in [6, 7], the following relation between operator valued α-divergence and Tsallis relative operator entropy was shown.

Theorem 4.1. ([6, 7, 10])*Let A and B be strictly positive operators. Then,*

$$D_t(A|B) = -T_{1-t}(B|A) - T_t(A|B)$$

holds for $t \in [0, 1]$.

Theorem 4.1 gives a geometrical interpretation for operator valued α-divergence. Tsallis relative operator entropy $T_t(A|B)$ can be regarded as the slope of the line passing through points A and $A \,\natural_t\, B$. Since $-T_{1-t}(B|A) = -\frac{B \,\natural_{1-t}\, A - B}{1-t} = \frac{B - A \,\natural_t\, B}{1-t}$, we can regard this operator value as the slope of the line passing through points $A \,\natural_t\, B$ and B. Therefore, $D_t(A|B)$ gives the difference between the slops of these two lines. We can illustrate the quantity corresponding to $D_t(A|B)$ by bold straight line in Figure 3.
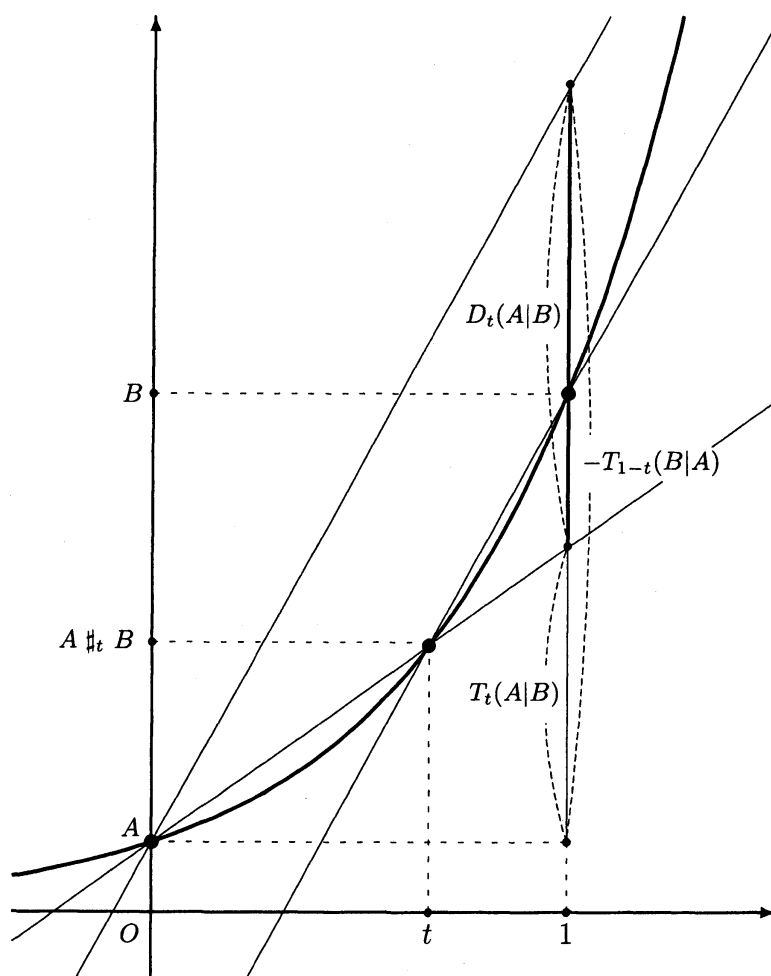

Figure 3: An interpretation for $D_t(A|B) = -T_{1-t}(B|A) - T_t(A|B)$.

Based on Theorem 4.1, we define expanded operator valued α-divergence.

Definition 4.2. *For strictly positive operators A and B, and for $t \in [0, 1]$, $r \in [-1, 1]$, expanded operator valued α-divergence is defined as follows:*

$$D_{t,r}(A|B) \equiv -T_{1-t,r}(B|A) - T_{t,r}(A|B).$$

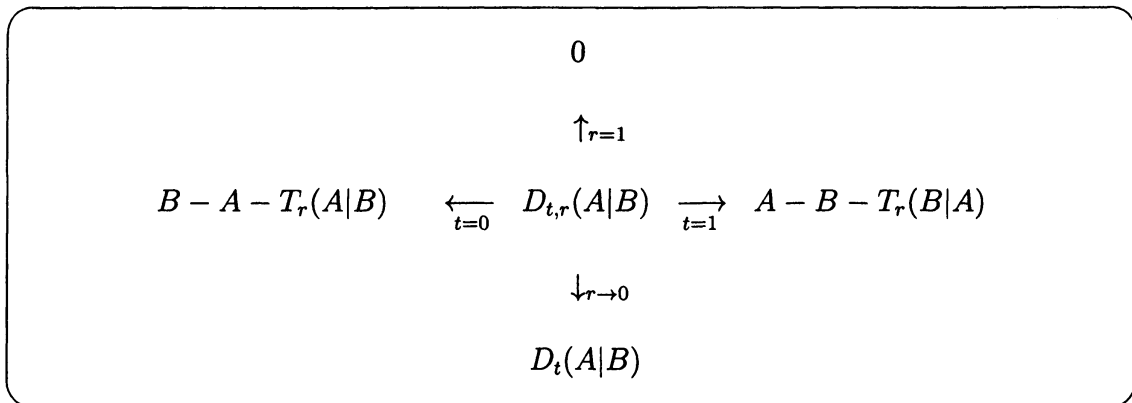**Remark 3.** *It is obvious that $D_{1-t,r}(B|A) = D_{t,r}(A|B)$ holds for $t \in (0,1)$ and $r \in [-1,1]$.*

We get the following relations for expanded operator valued α-divergence immediately.

Proposition 4.3. *Let A and B be strictly positive operators. Then,*

$$(1) \qquad D_{t,0}(A|B) \ = \ D_t(A|B),$$
$$(2) \qquad D_{t,1}(A|B) \ = \ 0,$$
$$(3) \qquad D_{0,r}(A|B) \ = \ B - A - T_r(A|B),$$
$$(4) \qquad D_{1,r}(A|B) \ = \ A - B - T_r(B|A)$$

hold for $t \in [0,1]$ and $r \in [-1,1]$.

We can illustrate the relations in Proposition 4.3 as follows:

$$0$$

$$\uparrow_{r=1}$$

$$B - A - T_r(A|B) \quad \xleftarrow[t=0]{} \quad D_{t,r}(A|B) \quad \xrightarrow[t=1]{} \quad A - B - T_r(B|A)$$

$$\downarrow_{r \to 0}$$

$$D_t(A|B)$$

We can rewrite $D_{t,r}(A|B)$ as the difference between weighted arithmetic mean and operator power mean as follows.

Theorem 4.4. *Let A and B be strictly positive operators. Then,*

$$D_{t,r}(A|B) = \frac{A \, \nabla_t \, B - A \, \natural_{t,r} \, B}{t(1-t)}$$

holds for $t \in (0,1)$ and $r \in [-1,1]$.

Proof. We can get this relation as follows:

$$D_{t,r}(A|B) \ = \ -T_{1-t,r}(B|A) - T_{t,r}(A|B) \ = \ -\frac{B \, \natural_{1-t,r} \, A - B}{1-t} - \frac{A \, \natural_{t,r} \, B - A}{t}$$

$$= \ \frac{-tA \, \natural_{t,r} \, B + tB - (1-t)A \, \natural_{t,r} \, B + (1-t)A}{t(1-t)} \ = \ \frac{A \, \nabla_t \, B - A \, \natural_{t,r} \, B}{t(1-t)}.$$

$\square$

For expanded operator valued α-divergence, we are trying to obtain similar relations to Theorem 3.1 and Theorem 3.5. The followings are relations we have obtained until now.

Theorem 4.5. *Let A and B be strictly positive operators. Then,*

(1) $\qquad D_{t,r}(A|A \,\natural_{s,r}\, B) \;=\; s\,\dfrac{T_{s,r}(A|B) - T_{st,r}(A|B)}{1-t} \quad (t \in [0,1))$,

(2) $\qquad D_{0,r}(A|A \,\natural_{s,r}\, B) \;=\; s\,\{T_{s,r}(A|B) - T_r(A|B)\}$,

(3) $\qquad D_{1,r}(A|A \,\natural_{s,r}\, B) \;=\; s\,\{S_{s,r}(A|B) - T_{s,r}(A|B)\}$.

hold for $s \in (0,1)$ and $r \in [-1,1]$.

Proof. (1) By Theorem 3.5, we have

$$D_{t,r}(A|A \,\natural_{s,r}\, B) \;=\; -T_{1-t,r}(A \,\natural_{s,r}\, B|A) - T_{t,r}(A|A \,\natural_{s,r}\, B)$$

$$= \; s\left\{\frac{1}{1-t}T_{s,r}(A|B) - \frac{t}{1-t}T_{st,r}(A|B)\right\} - sT_{st,r}(A|B) \;=\; s\,\frac{T_{s,r}(A|B) - T_{st,r}(A|B)}{1-t}.$$

(2) We can get this result by putting $t = 0$ in (1).

(3) By Theorem 3.1 and Theorem 3.5, we have

$$D_{1,r}(A|A \,\natural_{s,r}\, B) \;=\; -T_r(A \,\natural_{s,r}\, B|A) - T_{1,r}(A|A \,\natural_{s,r}\, B)$$

$$= \; S_{1,r}(A|A \,\natural_{s,r}\, B) - T_{1,r}(A|A \,\natural_{s,r}\, B) \;=\; sS_{s,r}(A|B) - sT_{s,r}(A|B)$$

$$= \; s\,\{S_{st,r}(A|B) - T_{s,r}(A|B)\}.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

By the similar way to Theorem 4.5, we can obtain the results of operator valued α-divergence for fixed point A and any point on the path as follows:

Proposition 4.6. *Let A and B be strictly positive operators. Then,*

(1) $\qquad D_t(A|A \,\natural_s\, B) \;=\; s\,\dfrac{T_s(A|B) - T_{st}(A|B)}{1-t} \quad (t \in [0,1))$,

(2) $\qquad D_0(A|A \,\natural_s\, B) \;=\; s\,\{T_s(A|B) - S(A|B)\}$,

(3) $\qquad D_1(A|A \,\natural_s\, B) \;=\; s\,\{S_s(A|B) - T_s(A|B)\}$

hold for $s \in \mathbf{R}$.

By applying noncommutative ratio translation to the relations in Proposition 4.6, we can get the results of operator valued α-divergence for any two points on the path as follows:

Theorem 4.7. *Let A and B be strictly positive operators. Then,*

(1) $\qquad D_t(A \,\natural_v\, B|A \,\natural_{v+s}\, B) \;=\; s\mathcal{R}(v)\,\dfrac{T_s(A|B) - T_{st}(A|B)}{1-t} \quad (t \in [0,1))$,

(2) $\qquad D_0(A \,\natural_v\, B|A \,\natural_{v+s}\, B) \;=\; s\mathcal{R}(v)\,\{T_s(A|B) - S(A|B)\}$,

(3) $\qquad D_1(A \,\natural_v\, B|A \,\natural_{v+s}\, B) \;=\; s\mathcal{R}(v)\,\{S_s(A|B) - T_s(A|B)\}$

hold for $s,\, v \in \mathbf{R}$.

Proof. (1) Lemma 2.2, Proposition 4.6, and Theorem 2.9, we have

$$D_t(A \natural_v B | A \natural_{v+s} B) = D_t(A \natural_v B | (A \natural_v B) \natural_s (A \natural_{v+1} B))$$
$$= s \frac{T_s(A \natural_v B | A \natural_{v+1} B) - T_{st}(A \natural_v B | A \natural_{v+1} B)}{1-t}$$
$$= s\mathcal{R}(v) \frac{T_s(A|B) - T_{st}(A|B)}{1-t}.$$

(2) We can get this result by putting $t = 0$ in (1).
(3) By (1) and (2) in Theorem 2.9,

$$D_1(A \natural_v B | A \natural_{v+s} B) = A \natural_v B - A \natural_{v+s} B + S_1(A \natural_v B | A \natural_{v+s} B)$$
$$= S_1(A \natural_v B | A \natural_{v+s} B) - T_1(A \natural_v B | A \natural_{v+s} B) = s\mathcal{R}(v)S_s(A|B) - s\mathcal{R}(v)T_s(A|B)$$
$$= s\mathcal{R}(v) \{S_s(A|B) - T_s(A|B)\} .$$

□

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