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# Mixed Duality for Nonsmooth Multiobjective Fractional Programming without a Constraint Qualification

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## 1 Introduction and Preliminaries

For a nonempty subset  $C$  of  $\mathbb{R}^n$ , we denote  $C^*$ , the dual cone of  $C$  and defined by

$$C^* = \{u \in \mathbb{R}^n \mid u^T x \geq 0, \forall x \in C\}.$$

Further, for  $x^* \in C$ ,  $N_C(x^*)$  denotes the normal cone to  $C$  at  $x^*$  defined by

$$N_C(x^*) = \{d \in \mathbb{R}^n \mid \langle d, x - x^* \rangle \leq 0, \forall x \in C\},$$

clearly,  $(C - x^*)^* = -N_C(x^*)$ .

We consider the following multiobjective nonsmooth fractional programming problem,

$$\begin{aligned} \text{(FP)} \quad & \text{Minimize} \quad \frac{f(x) + s(x|D)}{g(x)} \\ & \text{subject to} \quad h(x) \leq 0, \\ & \quad \quad \quad x \in C, \end{aligned}$$

where  $C$  is a convex set and  $D_i, i = 1, \dots, p$  are compact convex sets of  $\mathbb{R}^n$  and  $f_i, g_i, h_j, i = 1, \dots, p, j = 1, \dots, m$  are real valued locally Lipschitz functions defined on  $C$ . The index sets are  $P = \{1, 2, \dots, p\}$ ,  $M =$

$\{1, 2, \dots, m\}$ . We denote the feasible set  $\{x \in C | h_j(x) \leq 0, j = 1, \dots, m\}$  by  $F$ . Let  $I(x^*) = \{j \in M | h_j(x^*) = 0\}$  denote the index set of active constraints at  $x^*$ . We assume that for each  $i \in P$ ,  $f_i(x) + s(x|D_i) \geq 0$ ,  $g_i(x) > 0$ .

The minimal index set of active constraints for  $F$  is denoted by

$$I^= = \{j \in M | x \in F \rightarrow h_j(x) = 0\}.$$

We also denote

$$I^<(x^*) = I(x^*) \setminus I^= = \{j \in I(x^*) | x \in F \text{ such that } h_j(x) < 0\}.$$

**Definition 1.1** A feasible solution  $x^*$  for (FP) is efficient for (FP) if and only if there is no other feasible  $x$  for (FP) such that

$$\frac{f_{i_0}(x) + s(x|D_{i_0})}{g_{i_0}(x)} < \frac{f_{i_0}(x^*) + s(x^*|D_{i_0})}{g_{i_0}(x^*)}, \text{ for some } i_0 \in P,$$

and

$$\frac{f_i(x) + s(x|D_i)}{g_i(x)} \leq \frac{f_i(x^*) + s(x^*|D_i)}{g_i(x^*)}, \forall i \in P.$$

**Definition 1.2** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a locally Lipschitz function. Then

(i) it is said to be generalized convex at  $x$  if for any  $y$

$$f(y) - f(x) \geq \langle \xi, y - x \rangle, \forall \xi \in \partial_c f(x),$$

(ii) it is said to be generalized quasiconvex at  $x$  if for any  $y$  such that  $f(y) \leq f(x)$ ,

$$\langle \xi, y - x \rangle \leq 0, \forall \xi \in \partial_c f(x),$$

(iii) it is said to be generalized strictly quasiconvex at  $x$  if for any  $y$  such that  $f(y) \leq f(x)$ ,

$$\langle \xi, y - x \rangle < 0, \forall \xi \in \partial_c f(x),$$

## 2 Optimality Conditions

We will establish necessary and sufficient optimality conditions for the fractional problem (FP).

We consider the following nonlinear programming problem:

$$\begin{aligned}
 (\text{FP}_u) \quad & \text{Minimize} \quad (f_1(x) + s(x|D_1) - u_1 g_1(x), \\
 & \quad \quad \quad \dots, f_p(x) + s(x|D_p) - u_p g_p(x)) \\
 & \text{subject to} \quad h_j(x) \leq 0, \quad j \in M, \quad x \in C;
 \end{aligned}$$

where for each  $u = (u_1, \dots, u_p) \in \mathbb{R}_+^n$ .

**Lemma 2.1** *If  $x^*$  is an efficient solution for (FP), then  $x^*$  is an efficient solution for  $(\text{FP}_{u^*})$  where  $u^* = \frac{f_i(x^*) + s(x^*|D_i)}{g_i(x^*)}$ .*

We denote  $\phi_i(x^*) = \frac{f_i(x^*) + s(x^*|D_i)}{g_i(x^*)}$ .

**Theorem 2.1** *If  $x^*$  be an efficient solution of (FP) and  $h_j(\cdot), j \in M$  are generalized strictly quasiconvex at  $x^*$ , then there exist  $\tau^* \in \mathbb{R}^p, \mu^* \in \mathbb{R}^m$ , and  $z_i^* \in \mathbb{R}^n, i \in P$  such that*

$$\begin{aligned}
 0 \in & \sum_{i=1}^p \tau_i^* [\partial_c(f_i(x^*) + (x^*)^T z_i) - \phi_i(x^*) \partial_c g_i(x^*)] \\
 & + \sum_{j \in I^<(x^*)} \mu_j^* \partial_c h_j(x^*) + N_c(x^*), \quad (*) \\
 & \tau_i^* > 0, \quad i \in P, \quad \mu_j^* \geq 0, \quad j \in M, \quad \sum_{i=1}^p \tau_i^* = 1.
 \end{aligned}$$

**Corollary 2.1** *Let  $x^*$  be an efficient solution for (FP), and  $h_j, j \in M$  are generalized strictly quasiconvex at  $x^*$ , then there exist  $\tau^* \in \mathbb{R}^p, \mu^* \in \mathbb{R}^m$  and  $z_i^* \in D_i, i \in P$  such that*

$$0 \in \sum_{i=1}^p \tau_i^* [\partial_c(f_i(x^*) + (x^*)^T z_i) - \phi_i(x^*) \partial_c g_i(x^*)] + \sum_{j=1}^m \mu_j^* \partial_c h_j(x^*) + N_c(x^*),$$

$$(z_i^*)^T x^* = s(x^* | D_i), \quad z_i^* \in D_i, \quad i \in P,$$

$$\mu_j^* h_j(x^*) = 0, \quad j \in M,$$

$$\tau_i^* > 0, \quad i \in P, \quad \mu_j^* \geq 0, \quad j \in M, \quad \sum_{i=1}^p \tau_i^* = 1.$$

**Theorem 2.2** *Let  $x^*$  be a feasible solution of (FP) and assume that the conditions (\*) hold at  $x^*$ . If all of the functions  $f_i(\cdot), i = 1, \dots, p$ ,  $-g_i(\cdot)$  and  $h_j(\cdot), j = 1, \dots, m$  are generalized convex at  $x^*$ , then  $x^*$  is an efficient solution of (FP).*

### 3 Mixed Duality

We introduce a mixed type dual fractional programming problem and establish weak, strong and converse duality theorems.

$$\text{(FD) Maximize } \left( \frac{f_1(y) + y^T z_1 + \sum_{j \in M_1} \mu_j h_j(y)}{g_1(y)}, \dots, \frac{f_p(y) + y^T z_p + \sum_{j \in M_1} \mu_j h_j(y)}{g_p(y)} \right)$$

$$\text{subject to } 0 \in \sum_{i \in P} g_i(y) (\partial_c(\tau_i(f_i(y) + y^T z_i)) + \tau_i \sum_{j \in M_1} \partial_c \mu_j h_j(y)) - \sum_{i \in P} (f_i(y)$$

$$+ y^T z_i + \sum_{j \in M_1} \mu_j h_j(y)) \partial_c \tau_i g_i(y) + \sum_{j \in M_2} \partial_c \mu_j h_j(y) + N_c(y),$$

$$\mu_j h_j(y) \geq 0, \quad j \in M_2, \quad z_i \in D_i, \quad i \in P,$$

$$\mu_j \geq 0, \quad j \in M, \quad \tau_i > 0, \quad \sum_{i \in M} \lambda_i = 1, \quad y \in C.$$

where  $M_1$  is a subset of  $M = \{1, \dots, m\}$ ,  $M_2 = M \setminus M_1$ . Assume that for each  $i \in P$ ,  $f_i(y) + y^T z_i + \sum_{j \in M_1} \mu_j h_j(y) \geq 0$ ,  $g_i(y) > 0$ .

**Theorem 3.1** (Weak Duality) *Let  $x$  be a feasible solution for problem (FP) and  $(y, z, \tau, \mu)$  be a feasible solution for problem (FD). If all of the functions  $f_i(\cdot)$ ,  $g_i(\cdot)$ ,  $i \in P$ , and  $h_{M_1}(\cdot)$  are generalized convex at  $y$ , and  $\mu_j h_j(\cdot)$ ,  $j \in M_2$  are generalized quasiconvex at  $y$ , then the following cannot hold:*

$$\frac{f_{i_0}(x) + s(x|D_{i_0})}{g_{i_0}(x)} < \frac{f_{i_0}(y) + y^T z_{i_0} + \sum_{j \in M_1} \mu_j h_j(y)}{g_{i_0}(y)}, \text{ for some } i_0 \in P$$

and

$$\frac{f_i(x) + s(x|D_i)}{g_i(x)} \leq \frac{f_i(y) + y^T z_i + \sum_{j \in M_1} \mu_j h_j(y)}{g_i(y)}, \forall i \in P.$$

**Theorem 3.2** (Strong Duality) *Let  $x^*$  be an efficient solution of problem (FP). If the assumptions of Theorem 3.1 hold and  $h_j(\cdot)$ ,  $j = 1, \dots, m$  are generalized strictly quasiconvex, then there exist  $\tau^* \in \mathbb{R}_+^p$ ,  $z^* \in D_i$ ,  $i \in P$  and  $\mu^* \in \mathbb{R}_+^m$  such that  $(x^*, z^*, \tau^*, \mu^*)$  is an efficient solution of (FD),  $(x^*)^T z_i^* = s(x^*|D_i)$ ,  $i \in P$  and the optimal values of (FP) and (FD) are equal.*

**Theorem 3.3** (Strict Converse Duality) *Let  $x^*$  and  $(y^*, z^*, \tau^*, \mu^*)$  be efficient solutions of (FP) and (FD), respectively. If in addition the hypotheses of Theorem 3.2 hold, then  $y^* = x^*$ , that is,  $y^*$  solves (FP) and the objective values of (FP) and (FD) are equal.*

## 4 Special Cases

We give some special cases of our dual programming.

- (1) If  $D_i = \{0\}$ ,  $i = 1, \dots, p$  and  $I_1 = \emptyset$ , then our mixed dual problem (FD) reduce to the Mond-Weir type problem ( $D_1$ ) in Weir and Mond [13].
- (2) If  $D_i = \{0\}$ ,  $i = 1, \dots, p$  and  $I_2 = \emptyset$ , then our mixed dual problem (FD) reduce to the Wolfe type problem ( $D_2$ ) in Weir and Mond [13].
- (3) If  $D_i = \{0\}$ ,  $i = 1, \dots, p$ , then our mixed dual problem (FD) reduce to the mixed dual problem ( $MD$ ) in Nobakhtian [10].

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