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Construction of action for heterotic string field theory including the Ramond sector

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Abstract: Extending the formulation for open superstring field theory given in arXiv:1508.00366, we attempt to construct a complete action for heterotic string field theory. The action is non-polynomial in the Ramond string field Ψ, and we construct it order by order in Ψ. Using a dual formulation in which the role of η and Q is exchanged, the action is explicitly obtained at the quadratic and quartic order in Ψ with the gauge transformations.

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1 Introduction

There are two complementary formulation of superstring field theories: the Wess-Zumino-Witten (WZW)-like formulation [1–5], and the algebraic formulation in terms of the A_∞/L_∞ structure [6, 7]. The gauge invariant actions for the Neveu-Schwarz (NS) sector (or the NS-NS sector for the type II superstring) in the former can be written in a closed form as WZW-like actions utilizing the large Hilbert space. The corresponding actions in the latter are constructed in the small Hilbert space using the string products satisfying the A_∞/L_∞ relations, whose explicit form is obtained by solving a differential equation iteratively. Now it has been clarified that two formulations for the open superstring field theory are interrelated by a partial gauge fixing [8]. In spite of this success, it had been difficult to complete the action so as to include the Ramond (R) string in covariant way for a long time.

However an important progress was recently made in the WZW-like open superstring field theory: a complete gauge invariant action was constructed [9]. Soon afterwards, a similar action realizing a cyclic A_∞ structure was also constructed [10, 11], and the relation between two was elucidated [10]. These actions contain both the NS sector and R sector, describing space-time bosons and fermions, respectively, and completely specify their interactions. Therefore we are now in a position to study various off-shell aspects of open superstring theory.[1] The purpose of this paper is to extend this progress to the case of the heterotic string field theory.

Although it has been difficult to construct a complete action, including the R sector, for heterotic string field theory, the equations of motion was already constructed both in the WZW-like formulation [14, 15] and in the algebraic formulation [16]. In contrast to those in the open superstring field theory [9, 17], these equations of motion are nonpolynomial not only in the NS string field but also in the R string field. Therefore it is natural to consider that the complete action has also to be nonpolynomial in both the NS and R string fields. This is also expected from the simple consideration on general amplitudes with external fermions. We need proper interactions, described by the restricted polyhedra [18, 19],[2] including arbitrary (even) number of R string fields to fill the complete integration region of the moduli space of such amplitudes. This makes more difficult to construct a complete gauge invariant action for the heterotic string field theory. We attempt to construct a gauge invariant action order by order in the number of R string, and obtain it up to quartic order.

This paper is organized as follows. In section 2 we will first briefly summarize the results for the open superstring field theory given in [9]. Several important ingredients to construct the complete action, which can be straightforwardly extended to the heterotic string field theory, is introduced. Then we will explain some basics of the heterotic string field theory in section 3. We will introduce a dual formulation [22] exchanging the role of η and Q, which is useful for our aim. Section 4 is the main part of the paper. After introducing R string field in the restricted Hilbert space, we will attempt to construct a complete action order by order, first in the coupling constant and then in the R string field. A gauge invariant action will be obtained at the quadratic and quartic order in the R string field, each of which is exact in the NS string field. In section 5, we will summarize our results, and provide a few hints to construct a complete action at all order in R string. In the appendix A, we gives an explicit construction of the dual string products. The appendix B is added to illustrate how the on-shell physical amplitudes are reproduced from the constructed action.

2 Complete action for open superstring field theory

In this section we summarize the results given in [9] without going into detail. Let us focus on a few key points necessary to construct a gauge invariant action of the heterotic string field theory.

[1] A closely related approach to the heterotic and type II superstring field theory has been developed by Sen [12, 13].

[2] See also [20, 21].

To begin with, we note that there are two alternative expressions of the WZW-like action for the NS sector. The original expression given in [3] is

$$S = \int_0^1 dt\ \langle \tilde{A}_t(t), \eta \tilde{A}_Q(t)\rangle, \tag{2.1}$$

where η is the zero mode of $\eta(z)$ and $\tilde{A}_t$ and $\tilde{A}_Q$ are the *left-invariant forms*

$$\tilde{A}_t(t) = g^{-1}(t)\partial_t g(t), \qquad \tilde{A}_Q(t) = g^{-1}(t)Qg(t), \tag{2.2}$$

with $g(t) = e^{\Phi(t)}$. The NS string field Φ and its one-parameter extension $\Phi(t)$ are related through the boundary conditions, $\Phi(1) = \Phi,\ \Phi(0) = 0$.

One can easily see that the action (2.1) can also be written in the dual form in which the role of η and Q is exchanged:

$$S = -\int_0^1 dt\ \langle A_t(t), QA_\eta(t)\rangle, \tag{2.3}$$

where $A_t(t)$ and $A_\eta(t)$ are the *right-invariant forms*

$$A_t(t) = (\partial_t g(t))g^{-1}(t), \qquad A_\eta(t) = (\eta g(t))g^{-1}(t). \tag{2.4}$$

As we will see shortly, the latter expression is more suitable for the complete action, in which the A_η plays a special role. This is not only suitable but essential in the heterotic string field theory in which two operators η and Q do not appear symmetrically but act differently on the closed string products.

In order to include the Ramond sector, an important key point is to restrict the Ramond string field Ψ by the conditions[3]

$$\eta\Psi = 0, \qquad XY\Psi = \Psi, \tag{2.5}$$

where X and Y are the picture changing operators acting on states in the small Hilbert space at picture number $-3/2$ and $-1/2$, respectively:

$$X = -\delta(\beta_0)\ G_0 + \delta'(\beta_0)\ b_0, \qquad Y = -c_0\ \delta'(\gamma_0). \tag{2.6}$$

They satisfy the relations

$$XYX = X, \qquad YXY = Y, \tag{2.7}$$

implying the operator XY is a projector:

$$(XY)^2 = XY. \tag{2.8}$$

The former constraint imposes that Ψ is in the small Hilbert space, and the latter restricts the form of Ψ expanded in the ghost zero-modes as

$$\Psi = \phi + (\gamma_0 + c_0 G)\psi\,. \tag{2.9}$$

This restricted form was already known to be enough to construct consistent free superstring field theory [23–25].

[3]In this paper we use the same symbol Ψ to denote the string field in the Ramond sector both for the open superstring and for the heterotic string field. We will not confuse them since two cases never appear simultaneously.

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Note here that the operator X is BRST exact in the large Hilbert space:

$$X = \{Q, \Theta(\beta_0)\}, \tag{2.10}$$

where $\Theta(x)$ is the Heaviside step function satisfying $\Theta(x)' = \delta(x)$. More generally, we introduce the following operator Ξ which is more suitable for use in the large Hilbert space [10]:

$$\Xi = \xi_0 + (\Theta(\beta_0)\eta\xi_0 - \xi_0)P_{-3/2} + (\xi_0\eta\Theta(\beta_0) - \xi_0)P_{-1/2}\,, \tag{2.11}$$

where P_n is the projector onto states at picture number n. The anti-commutator $\{Q, \Xi\}$ is not identical to X, but equal to X if it acts on a state in the small Hilbert space at picture number $-3/2$. In other words, we can use the relation $X = \{Q, \Xi\}$ on a state in the small Hilbert space at picture number $-3/2$. Using this Ξ, we can define an important linear operator $F(t)$ as

$$F(t) = \frac{1}{1 + \Xi(D_\eta(t) - \eta)} = 1 + \sum_{n=1}^{\infty}(-\Xi(D_\eta(t) - \eta))^n, \tag{2.12}$$

where

$$D_\eta(t)A \equiv \eta A - A_\eta(t)A + (-1)^A A A_\eta(t), \tag{2.13}$$

on an arbitrary Ramond string field A. This linear operator $F(t)$ satisfies the relation

$$D_\eta(t)F(t) = F(t)\eta, \tag{2.14}$$

and thus the *dressed* Ramond string field $F(t)\Psi$ with the Ramond string field Ψ restricted by the constraints (2.5) is annihilated by $D_\eta(t)$.

Now a complete gauge invariant action is given by

$$S = -\frac{1}{2}\langle\!\langle \Psi, YQ\Psi \rangle\!\rangle - \int_0^1 dt \langle A_t(t), QA_\eta(t) + (F(t)\Psi)^2 \rangle, \tag{2.15}$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is the BPZ inner product in the small Hilbert space. We can show that this is invariant under the gauge transformations [9]:

$$A_\delta = Q\Lambda + D_\eta\Omega + \{F\Psi, F\Xi(\{F\Psi, \Lambda\} - \lambda)\}, \tag{2.16}$$

$$\delta\Psi = Q\lambda + X\eta F\Xi D_\eta(\{F\Psi, \Lambda\} - \lambda), \tag{2.17}$$

where Λ and Ω are gauge parameters in the NS sector and λ is a gauge parameter in the Ramond sector satisfying

$$\eta\lambda = 0, \qquad XY\lambda = 0. \tag{2.18}$$

3 The NS sector of heterotic string field theory

Next we summarize in this section the known results in the NS sector of the heterotic string field theory [2, 3]. In particular, we provide a dual formulation [22] which plays a significant role when we will include the Ramond sector in the next section.

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3.1 Basic ingredients

In the heterotic string, the holomorphic sector and anti-holomorphic sector are described by superconformal field theory and conformal field theory, respectively. The conformal field theory for anti-holomorphic sector consists of the matter sector with $c = 26$, and the reparameterization ghosts, $(\tilde{b}(\bar{z})$,$\tilde{c}(\bar{z}))$. The superconformal field theory for holomorphic sector consists of the matter sector with $c = 15$, the reparameterization ghosts, $(b(z),\ c(z))$, and the superconformal ghosts, $(\beta(z), \gamma(z))$. An alternative description using $(\xi(z), \eta(z), \phi(z))$ is known for the superconformal ghost sector [27], related through the bosonization relation:

$$\beta(z) = \partial\xi(z)e^{-\phi(z)}, \qquad \gamma(z) = e^{\phi(z)}\eta(z). \tag{3.1}$$

Therefore, we can consider two Hilbert spaces for describing the superconformal ghost sector. One is called the *large Hilbert space*, constructed as the Fock space of $\xi(z)$, $\eta(z)$, and $\phi(z)$. The other called the *small Hilbert space* can be defined as a subspace annihilated by the zero mode of $\eta(z)$, which is equivalent to the Hilbert space constructed as the Fock space of $\beta(z)$ and $\gamma(z)$. Note that any η-exact state belongs to the small Hilbert space due to the nilpotency $\eta^2 = 0$.

Let V_1 and V_2 be a pair of heterotic string states which satisfy the closed string constraints

$$b_0 V_i = 0, \qquad L_0^- V_i = 0, \qquad (i = 1, 2), \tag{3.2}$$

and belong to the large Hilbert space. The inner product of them is given by

$$\langle V_1, V_2\rangle = \langle V_1|c_0^-|V_2\rangle, \tag{3.3}$$

where $\langle V_1|$ denotes the BPZ conjugate of $|V_1\rangle$. It is non-vanishing when the sums of the ghost number g and the picture number p of the two input states are $(g, p) = (4, -1)$. It satisfies

$$\langle V_1, V_2\rangle = (-1)^{(V_1+1)(V_2+1)}\langle V_2, V_1\rangle, \tag{3.4}$$

and

$$\langle QV_1, V_2\rangle = (-1)^{V_1}\langle V_1, QV_2\rangle, \qquad \langle \eta V_1, V_2\rangle = (-1)^{V_1}\langle V_1, \eta V_2\rangle. \tag{3.5}$$

The interactions of closed strings are described using the string products provided in [21]

$$Q, \quad [\ \cdot\ ,\ \cdot\ ], \quad [\ \cdot\ ,\ \cdot\ ,\ \cdot\ ], \quad \cdots . \tag{3.6}$$

The n-string product carries ghost number $-2n + 3$ (and picture number 0). The string products are graded symmetric upon the interchange of the arguments

$$[V_{\sigma(1)}, \dots, V_{\sigma(k)}] = (-1)^{\sigma(\{V\})}[V_1, \dots, V_k], \tag{3.7}$$

and cyclic with respect to the inner product:

$$\langle V_1, [V_2, \dots, V_{n+1}]\rangle = (-1)^{V_1+V_2+\dots+V_n}\langle [V_1, V_2, \dots, V_n], V_{n+1}\rangle\,, \tag{3.8}$$

Here σ denotes the permutation from $\{1,\dots,n\}$ to $\{\sigma(1),\dots,\sigma(n)\}$, and the factor $(-1)^{\sigma(\{V\})}$ is the sign factor of the permutation from $\{V_1,\dots,V_n\}$ to $\{V_{\sigma(1)},\dots,V_{\sigma(n)}\}$. Defining $[V] = QV$, the string products satisfy the following relations called *the* L_∞*-relations*:

$$0=\sum_{\sigma}\sum_{m=1}^{n}(-1)^{\sigma(\{V\})}\frac{1}{m!(n-m)!}[\,[V_{\sigma(1)},\dots,V_{\sigma(m)}],V_{\sigma(m+1)},\dots,V_{\sigma(n)}]\,. \tag{3.9}$$

They describe an infinite number of relations, the first few of which is given by

$$0=Q^2, \tag{3.10}$$
$$0=Q[V_1,V_2]+[QV_1,V_2]+(-1)^{V_1}[V_1,QV_2], \tag{3.11}$$
$$\begin{aligned}0=&\,Q[V_1,V_2,V_3]+[QV_1,V_2,V_3]+(-1)^{V_1}[V_1,QV_2,V_3]+(-1)^{V_1+V_2}[V_1,V_2,QV_3]\\&+[[V_1,V_2],V_3]+(-1)^{V_1(V_2+V_3)}[[V_2,V_3],V_1]+(-1)^{V_3(V_1+V_2)}[[V_3,V_1],V_2]\,.\end{aligned} \tag{3.12}$$

The operator η acts as a derivation on the string products:

$$\eta\big[V_1,\dots,V_n\big]=\sum_{i=1}^{n-1}(-1)^{1+V_1+\cdots+V_{k-1}}\big[V_1,\dots,\eta V_k,\dots,V_n\big]. \tag{3.13}$$

It is useful to introduce new string products $[\cdots]_B$,

$$[V_1,\cdots,V_n]_B\equiv\sum_{m=0}^{\infty}\frac{\kappa^m}{m!}[B^m,V_1,\cdots,V_n],\qquad(n\geq 1), \tag{3.14}$$

shifted by a Grassmann even NS string field B with ghost number 2 and picture number 0. If B satisfies the Maurer-Cartan equation

$$QB+\sum_{n=2}^{\infty}\frac{\kappa^{n-1}}{n!}[B^n]=0, \tag{3.15}$$

the shifted string products (3.14) satisfy the identical L_∞ relation to (3.9):

$$0=\sum_{\sigma}\sum_{m=1}^{n}(-1)^{\sigma(\{V\})}\frac{1}{m!(n-m)!}[\,[V_{\sigma(1)},\dots,V_{\sigma(m)}]_B,V_{\sigma(m+1)},\dots,V_{\sigma(n)}]_B\,. \tag{3.16}$$

In particular, setting $n=1$, this relation provides the nilpotency of the shifted BRST charge, $(Q_B)^2=0$, defined by

$$Q_BV\equiv[V]_B=QV+\sum_{m=1}^{\infty}\frac{\kappa^m}{m!}[B^m,V]\,. \tag{3.17}$$

3.2 WZW-like action

On the basis of the WZW-like formulation, a gauge invariant action for the NS sector of heterotic string field theory was provided in [3] by an extension of the Berkovits open superstring field theory. We use a heterotic string field $\widetilde{V}$ in the large Hilbert space for the

NS sector, which is a Grassmann-odd, and has ghost number 1 and picture number 0.[4] It also satisfies the closed string constraints

$$b_0^- \widetilde{V} = 0\,, \qquad L_0^- \widetilde{V} = 0\,. \tag{3.18}$$

We introduce a one-parameter extension $\widetilde{V}(t)$ satisfying $\widetilde{V}(0) = 0$ and $\widetilde{V}(1) = \widetilde{V}$. The operators ∂_t and δ as well as η act as derivations on the string products:

$$\mathbb{X}[\widetilde{V}_1(t), \dots, \widetilde{V}_n(t)] = \sum_{k=1}^{n} (-1)^{\mathbb{X}(1+\widetilde{V}_1+\cdots+\widetilde{V}_{k-1})} [\widetilde{V}_1(t), \dots, \mathbb{X}\widetilde{V}_k(t), \dots, \widetilde{V}_n(t)], \tag{3.19}$$

where $\mathbb{X} = \eta$, ∂_t or δ. A key ingredient in the WZW-like action is the pure-gauge string field $G(\widetilde{V}(t))$, which is a Grassmann even functional of $\widetilde{V}(t)$ with ghost number 2 and picture number 0 satisfying the Maurer-Cartan equation (3.15):

$$QG(\widetilde{V}) + \sum_{n=2}^{\infty} \frac{\kappa^{n-1}}{n!} [G(\widetilde{V})^n] = 0\,. \tag{3.20}$$

It was shown in [3] that such a functional $G(\widetilde{V})$ can be obtained by solving the differential equation

$$\partial_\tau G(\tau\widetilde{V}) = \sum_{m=0}^{\infty} \frac{\kappa^m}{m!} [G(\tau\widetilde{V})^m,\ \widetilde{V}] = Q_{G(\tau\widetilde{V})} \widetilde{V}\,, \tag{3.21}$$

iteratively with the initial condition, $G = 0$ at $\tau = 0$, and set $\tau = 1$.

Acting a derivation operator $\mathbb{X} = \eta$, ∂_t, or δ on (3.20), we have

$$Q_G(\mathbb{X}G) = 0\,. \tag{3.22}$$

Here Q_G is nilpotent due to (3.20). Since its cohomology is trivial in the large Hilbert space, one can find that $\mathbb{X}G$ is Q_G-exact and can define a functional $\Psi_{\mathbb{X}}(\widetilde{V})$, which we call an associated field, satisfying

$$\mathbb{X}G(\widetilde{V}) = (-1)^{\mathbb{X}} Q_{G(\widetilde{V})} \Psi_{\mathbb{X}}(\widetilde{V}). \tag{3.23}$$

We denote $\Psi_t(\widetilde{V})$ for $\Psi_{\partial_t}(\widetilde{V})$ for simplicity. The associated field $\Psi_\eta(\widetilde{V})$ is Grassmann-even and carries ghost number 2 and picture number -1. The associated fields $\Psi_t(\widetilde{V})$ and $\Psi_\delta(\widetilde{V})$ are Grassmann-odd and carry ghost number 1 and picture number 0. These associated fields can also be obtained by iteratively solving the differential equations

$$\partial_\tau \Psi_{\mathbb{X}}(\tau\widetilde{V}) = \mathbb{X}\widetilde{V} + \kappa\big[\widetilde{V}, \Psi_{\mathbb{X}}(\tau\widetilde{V})\big]_{G(\tau\widetilde{V})}, \tag{3.24}$$

with the initial condition, $\Psi_{\mathbb{X}} = 0$ at $\tau = 0$, and set $\tau = 1$.

[4]We put $\widetilde{\ }$ on the field V to distinguish it from the field in the dual formulation introduced in the next subsection. Two fields $\widetilde{V}$ and V are identical at the leading order in the coupling constant κ, but different at order κ^2 [22].

Utilizing these functionals G and $\Psi_{\mathbb{X}}$, a gauge-invariant action can be written in the WZW-like form:

$$S_{\text{WZW}} = -\int_0^1 dt \langle \Psi_t(t), \eta G(t) \rangle, \tag{3.25}$$

with $\Psi_{\mathbb{X}}(t) \equiv \Psi_{\mathbb{X}}(\widetilde{V}(t))$ and $G(t) \equiv G(\widetilde{V}(t))$. One can show that the variation of the integrand becomes a total derivative in t

$$\delta \big\langle \Psi_t(t), \eta G(t) \big\rangle = \partial_t \big\langle \Psi_\delta(t), \eta G(t) \big\rangle, \tag{3.26}$$

and thus the variation of the action is given by

$$\delta S_{\text{WZW}} = -\langle \Psi_\delta(\widetilde{V}), \eta G(\widetilde{V}) \rangle, \tag{3.27}$$

since $\widetilde{V}(0) = 0$, and $\Psi_{\mathbb{X}}(0) = G(0) = 0$. From (3.27) we find that the equation of motion is given by

$$\eta G(\widetilde{V}) = 0, \tag{3.28}$$

and the action (3.25) is invariant under the gauge transformations[5]

$$\Psi_\delta = Q_G \widetilde{\Lambda} + \eta \widetilde{\Omega}, \tag{3.29}$$

where the gauge parameters $\widetilde{\Lambda}$ and $\widetilde{\Omega}$ are Grassmann even with ghost number 0, and carry picture number 0 and 1, respectively. The gauge invariance follows from the nilpotency of Q_G and η, and one of the relations (3.23): $\eta G = -Q_G \Psi_\eta$.

3.3 Dual formulation

Then we provide a dual formulation for the heterotic string field theory given in [22], which is suitable and useful to include the Ramond sector. It is dual in the sense that the role of η and Q is exchanged, and natural extension of (2.3) and (2.4) for the open superstring field theory, on the basis of which a complete action in [9] is constructed. An explicit construction and more detailed discussion on the dual formulation is explained in appendix A.

In the dual formulation, an L_∞-structure starting with η plays a central role. Note that, in the case of the open string, a set of products $\{\eta, -*\}$ satisfy the A_∞-relations: η is nilpotent, η acts as a derivation on the star product, and the star product is associative. As a natural extension of $\{\eta, -*\}$, we introduce a set of products satisfying L_∞-relations, which we call the *dual sting products*:

$$\eta, \quad [\,\cdot\,,\,\cdot\,]^\eta, \quad [\,\cdot\,,\,\cdot\,,\,\cdot\,]^\eta, \quad \cdots. \tag{3.30}$$

The dual string products are graded commutative upon the interchange of the input string field, and cyclic:

$$[V_{\sigma(1)}, \ldots, V_{\sigma(k)}]^\eta = (-1)^{\sigma(\{V\})} [V_1, \ldots, V_k]^\eta, \tag{3.31}$$

$$\langle V_1, [V_2, \cdots, V_{n+1}]^\eta \rangle = (-1)^{V_1+V_2+\cdots+V_n} \langle [V_1, \cdots, V_n]^\eta, V_{n+1} \rangle\,. \tag{3.32}$$

[5]Note that Ψ_δ is invertible as a function of $\delta\widetilde{V}$. See also [3] and [28].

They satisfy the L_∞ relations:

$$\sum_\sigma \sum_{k=1}^{n} \frac{1}{k!(n-k)!}(-1)^{\sigma(\{V\})}\big[[V_{\sigma(1)},\dots,V_{\sigma(k)}]^\eta, V_{\sigma(k+1)},\dots,V_{\sigma(n)}\big]^\eta = 0, \tag{3.33}$$

where we denote ηV_i as $[V_i]^\eta$. The sign factor $(-1)^{\sigma(\{V\})}$ is that of the permutation from $\{V_1,\dots,V_n\}$ to $\{V_{\sigma(1)},\dots,V_{\sigma(n)}\}$. The n-th dual string product carries ghost number $3-2n$ and picture number $n-2$. We also require that the BRST operator Q acts as a derivation on the dual string products:

$$Q\big[V_1,\dots,V_n\big]^\eta + \sum_{k=1}^{n}(-1)^{V_1+\cdots+V_{k-1}}\big[V_1,\dots,QV_k,\dots,V_n\big]^\eta = 0. \tag{3.34}$$

We can actually construct such dual string products from the well-known string products, ξ_0 and the picture changing operator $X_0 = \{Q,\xi_0\}$, details of which is given in [22] or appendix A. For later use, we introduce a one parameter extension $V(t)$ satisfying $V(0) = 0$ and $V(1) = V$. The operators $\mathbb{X} = Q$, ∂_t, or δ acts as a derivation on the dual string products:

$$\mathbb{X}[V_1,\dots,V_n]^\eta = \sum_{k=1}^{n}(-1)^{\mathbb{X}(1+V_1+\cdots+V_{k-1})}[V_1,\dots,\mathbb{X}V_k,\dots,V_n]^\eta\,. \tag{3.35}$$

Utilizing the dual string products, we can provide an alternative gauge-invariant action in the dual manner to that for the WZW-like action reviewed in the previous subsection. In the dual formulation, we denote the NS string field as V, which is a Grassmann-odd state in the large Hilbert space with ghost number 1 and picture number 0. It satisfies the closed string constraint:

$$b_0^- V = 0\,, \qquad L_0^- V = 0\,. \tag{3.36}$$

Pure-gauge string field $G_\eta(V)$ in the dual formulation is defined as a functional of $V(t)$ with ghost number 2 and picture number -1 satisfying the Maurer-Cartan equation dual to (3.20):

$$0 = \eta G_\eta(V) + \sum_{n=2}^{\infty}\frac{\kappa^{n-1}}{n!}[G_\eta(V)^n]^\eta. \tag{3.37}$$

As with G satisfying (3.20), G_η can be obtained by solving the differential equation

$$\partial_\tau G_\eta(\tau V) = \sum_{m=0}^{\infty}\frac{\kappa^m}{m!}[G_\eta(\tau V)^m, V]^\eta, \tag{3.38}$$

iteratively with $G_\eta(0) = 0$, and setting $\tau = 1$. We define the shifted products of dual string products as

$$[V_1, V_2,\cdots, V_n]^\eta_{G_\eta} \equiv \sum_{m=0}^{\infty}\frac{\kappa^m}{m!}[(G_\eta)^m, V_1, V_2,\cdots, V_n]^\eta, \qquad (n\geq 1), \tag{3.39}$$

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which are also graded commutative and cyclic. In particular, it is useful to define the shifted η-operator D_η as the shifted one-string product $[\,\cdot\,]^\eta_{G_\eta}$:

$$
\begin{aligned}
D_\eta V \equiv [V]^\eta_{G_\eta} &= \sum_{m=0}^{\infty} \frac{\kappa^m}{m!} [(G_\eta)^m, V]^\eta \\
&= \eta V + \sum_{m=1}^{\infty} \frac{\kappa^m}{m!} [(G_\eta)^m, V]^\eta \,.
\end{aligned}
\tag{3.40}
$$

The shifted dual string products satisfy the L_∞ relation:

$$
\sum_{\sigma} \sum_{k=1}^{n} \frac{1}{k!(n-k)!} (-1)^{\sigma(\{V\})} \big[[V_{\sigma(1)}, \ldots, V_{\sigma(k)}]^\eta_{G_\eta}, V_{\sigma(k+1)}, \ldots, V_{\sigma(n)}\big]^\eta_{G_\eta} = 0 \,. \tag{3.41}
$$

Their lowest two relations represent that D_η is nilpotent and acts as a derivation on the dual shifted two string products:

$$
(D_\eta)^2 V_1 = 0, \tag{3.42}
$$

$$
D_\eta [V_1, V_2]^\eta_{G_\eta} = -[D_\eta V_1, V_2]^\eta_{G_\eta} - (-1)^{V_1} [V_1, D_\eta V_2]^\eta_{G_\eta} \,. \tag{3.43}
$$

The operator $\mathbb{X} = Q$, ∂_t, or δ acts on the shifted dual products as

$$
\begin{aligned}
\mathbb{X}[V_1, \ldots, V_n]^\eta_{G_\eta} = &\sum_{k=1}^{n} (-1)^{\mathbb{X}(V_1 + \cdots + V_{k-1} + 1)} [V_1, \ldots, \mathbb{X}V_k, \ldots, V_n]^\eta_{G_\eta} \\
&+ (-1)^{\mathbb{X}} \kappa [\mathbb{X}G_\eta, V_1, \ldots, V_n]^\eta_{G_\eta}.
\end{aligned}
\tag{3.44}
$$

In particular,

$$
\mathbb{X} D_\eta V_1 = (-1)^{\mathbb{X}} D_\eta \mathbb{X} V_1 + (-1)^{\mathbb{X}} \kappa [\mathbb{X}G_\eta, V_1]^\eta_{G_\eta}. \tag{3.45}
$$

Acting $\mathbb{X}$ on the Maurer-Cartan equation (3.37) we have

$$
D_\eta \mathbb{X} G_\eta(V) = 0 \,. \tag{3.46}
$$

Thus, since D_η cohomology is trivial, $\mathbb{X}G_\eta(V)$ is D_η-exact and can be written as

$$
\mathbb{X} G_\eta(V) = (-1)^{\mathbb{X}} D_\eta B_{\mathbb{X}}(V), \tag{3.47}
$$

by introducing associated fields $B_{\mathbb{X}}(V)$. The associated field B_Q is Grassmann-even and carries ghost number 2 and picture number 0, and B_t ($\equiv B_{\partial_t}$) and B_δ are Grassmann-odd and carry ghost number 1 and picture number 0. They are obtained by solving the differential equations

$$
\partial_\tau B_{\mathbb{X}}(\tau V) = \mathbb{X} V + \kappa \big[V, B_{\mathbb{X}}(\tau V)\big]^\eta_{G_\eta(\tau V)} \,, \tag{3.48}
$$

iteratively with $B_{\mathbb{X}} = 0$ at $\tau = 0$, and then set $\tau = 1$. By multiplying $0 = \mathbb{X}\mathbb{Y} - (-1)^{\mathbb{X}\mathbb{Y}} \mathbb{Y}\mathbb{X}$ to G_η for $\mathbb{X}, \mathbb{Y} = Q$, ∂_t, or δ, we can show that the identity

$$
D_\eta \big(\mathbb{X} B_{\mathbb{Y}} - (-1)^{\mathbb{X}\mathbb{Y}} \mathbb{Y} B_{\mathbb{X}} - (-1)^{\mathbb{X}} \kappa [B_{\mathbb{X}}, B_{\mathbb{Y}}]^\eta_{G_\eta} \big) = 0 \,, \tag{3.49}
$$

which is useful later, holds using (3.47), (3.45), and (3.43). We can also show

$$\langle D_\eta V_1, V_2\rangle = (-1)^{V_1}\langle V_1, D_\eta V_2\rangle, \tag{3.50}$$

from the definition (3.40).

An alternative gauge invariant action, which we call the dual WZW-like action, is given using these functionals $G_\eta(V)$ and $B_t(V)$ by

$$S = \int_0^1 dt\ \langle B_t(t), QG_\eta(t)\rangle\,, \tag{3.51}$$

with $B_t(t) \equiv B_t(V(t))$ and $G_\eta(t) \equiv G_\eta(V(t))$. The variation of the action can be calculated as

$$\delta S = \langle B_\delta(V), QG_\eta(V)\rangle\,, \tag{3.52}$$

in a completely parallel manner with the original formulation in [3]. Thus the equation of motion is given by

$$QG_\eta(V) = 0\,, \tag{3.53}$$

and the action is invariant under the gauge transformation

$$B_\delta = Q\Lambda + D_\eta\Omega\,. \tag{3.54}$$

The gauge parameters Λ and Ω having ghost number 0 carry picture number 0 and 1, respectively. The gauge invariance follows from the nilpotency $(D_\eta)^2 = Q^2 = 0$, and $QG_\eta = -D_\eta B_Q$.

Another important property of the dual string products is their *Q-exactness*. The dual n-string products for $n \geq 3$ themselves written as a BRST variation of some products $(\cdots)^\eta$ which we call the *dual gauge products*:

$$[V_1,\cdots,V_n]^\eta = Q(V_1,\cdots,V_n)^\eta - \sum_{k=1}^{n}(-1)^{V_1+\cdots+V_{k-1}}(V_1,\cdots,QV_k,\cdots,V_n)^\eta, \quad (n\geq 3)\,, \tag{3.55}$$

which is consistent with the fact that Q acts as a derivation on the dual string products. Since the dual string products are Grassmann odd, the dual gauge products are Grassmann-even. The n-th dual gauge product carries ghost number $-2n+2$ and picture number $n-2$, and is commutative and cyclic:

$$(V_{\sigma(1)},\cdots,V_{\sigma(n)})^\eta = (-1)^{\sigma(\{V\})}(V_1,\cdots,V_n)^\eta, \tag{3.56}$$

$$\langle V_1,(V_2,\cdots,V_{n+1})^\eta\rangle = (-1)^{V_2+\cdots+V_n+1}\langle (V_1,\cdots,V_n)^\eta, V_{n+1}\rangle\,, \tag{3.57}$$

where $(-1)^{\sigma(\{V\})}$ is the sign factor of the permutation from $\{V_1,\ldots,V_n\}$ to $\{V_{\sigma(1)},\ldots,V_{\sigma(n)}\}$. Two operators $\mathbb{X} = \partial_t$, or δ act as a derivation also on this product,

$$\mathbb{X}(V_1,\cdots,V_n)^\eta = \sum_{i=1}^{n-1}(-1)^{(V_1+\cdots+V_{i-1})}(V_1,\cdots,\mathbb{X}V_i,\cdots,V_n)^\eta\,. \tag{3.58}$$

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It is useful again to define the *shifted dual gauge products* $(\cdots)^{\eta}_{G_\eta}$ by

$$(V_1, \cdots, V_n)^{\eta}_{G_\eta} = \sum_{m=0}^{\infty} \frac{\kappa^m}{m!}((G_\eta)^m, V_1, \cdots, V_n)^{\eta}, \qquad (n \geq 3). \tag{3.59}$$

Note that the n-th shifted dual product contains all the dual products higher than n. The shifted dual products are cyclic, which follows from the cyclicity of the dual products:

$$\langle V_1, (V_2, \cdots, V_{n+1})^{\eta}_{G_\eta} \rangle = (-1)^{V_2+\cdots+V_n+1} \langle (V_1, \cdots, V_n)^{\eta}_{G_\eta}, V_{n+1} \rangle. \tag{3.60}$$

They are related to the shifted dual products $[V_1, \ldots, V_n]^{\eta}_{G_\eta}$ as follows.

$$\begin{aligned}
[V_1, \cdots, V_n]^{\eta}_{G_\eta} &= \sum_{m=0}^{\infty} \frac{\kappa^m}{m!}[(G_\eta)^m, \cdots, V_1, \cdots, V_n]^{\eta} \\
&= \sum_{m=0}^{\infty} \frac{\kappa^m}{m!}\Big(Q((G_\eta)^m, V_1, \cdots, V_n)^{\eta} - m((G_\eta)^{m-1}, QG_\eta, V_1, \cdots, V_n)^{\eta} \\
&\qquad\qquad - \sum_{k=1}^{n}(-1)^{V_1+\ldots+V_{k-1}}((G_\eta)^m, V_1, \cdots, QV_k, \cdots, V_n)^{\eta}\Big) \\
&= Q(V_1, \cdots, V_n)^{\eta}_{G_\eta} - \sum_{k=1}^{n}(-1)^{V_1+\cdots+V_{k-1}}(V_1, \cdots, QV_k, \cdots, V_n)^{\eta}_{G_\eta} \\
&\quad - \kappa(QG_\eta, V_1, \cdots, V_n)^{\eta}_{G_\eta}.
\end{aligned} \tag{3.61}$$

Due to the shift, the operators $\mathbb{X} = \partial_t$, or δ do not act as a derivation on the shifted dual gauge product but satisfies the relation

$$\mathbb{X}(V_1, \cdots, V_n)^{\eta}_{G_\eta} = \sum_{k=1}^{n}(V_1, \cdots, \mathbb{X}V_k, \cdots, V_n)^{\eta}_{G_\eta} + \kappa(\mathbb{X}G_\eta, V_1, \cdots, V_n)^{\eta}_{G_\eta}. \tag{3.62}$$

4 Inclusion of the Ramond sector

Now let us include the Ramond sector. In this section, after introducing the Ramond string field constrained into the restricted Hilbert space, we attempt to construct a gauge invariant action order by order in the coupling constant κ. The result can easily be extended to the full order for the part of the action quadratic in fermion, which has the form of a natural extension of the complete action for the open superstring field theory. In the heterotic string case, however, it is not gauge invariant, and necessary to include the interactions containing arbitrary even number of Ramond string fields. We determine the quartic term explicitly.

4.1 Ramond string field and restricted Hilbert space

Following the case of the open superstring field theory [9], we introduce a string field Ψ constrained in the restricted Hilbert space,

$$\eta\Psi = 0, \qquad XY\Psi = \Psi, \tag{4.1}$$

for the Ramond sector. It is a Grassmann even state with the ghost number 2 and the picture number $-1/2$, and satisfies the closed string constraint

$$b_0^-\Psi = L_0^-\Psi = 0. \tag{4.2}$$

The picture changing operators X and Y are defined by

$$X = -\delta(\beta_0)G_0 + \delta'(\beta_0)b_0, \qquad Y = -2c_0^+\delta'(\gamma_0), \tag{4.3}$$

which act on states in the small Hilbert space with the picture number $-3/2$ and $-1/2$, respectively. These operators are inverse each other in the sense that they satisfy

$$XYX = X, \qquad YXY = Y\,, \tag{4.4}$$

which make the operator XY a projector:

$$(XY)^2 = XY. \tag{4.5}$$

In addition, X is commutative with the BRST charge Q, $[Q, X] = 0$. These are enough to guarantee the compatibility of the restriction with the BRST cohomology, that is, if $XY\Psi_1 = \Psi_1$ then $XYQ\Psi_1 = Q\Psi_1$, which can be shown as

$$XYQ\Psi_1 = XYQXY\Psi_1 = XYXQY\Psi_1 = XQY\Psi_1 = QXY\Psi_1 = Q\Psi_1. \tag{4.6}$$

The operator Y is chosen to be commutative with b_0^- so that all the constraints in (4.1) and (4.2) are consistent. Expanding the ghost zero modes, the restricted Ramond string field has the form

$$\Psi = \phi + (\gamma_0 + 2c_0^+G)\psi, \tag{4.7}$$

with

$$L_0^-\phi = b_0^+\phi = \beta_0\phi = 0, \qquad L_0^-\psi = b_0^+\psi = \beta_0\psi = 0, \tag{4.8}$$

where $G = G_0 + 2b_0\gamma_0$.

The appropriate inner product in the restricted Hilbert space is given by[6]

$$\langle\langle \Psi_1, Y\Psi_2\rangle\rangle, \tag{4.9}$$

using the BPZ inner product $\langle\langle A, B\rangle\rangle$ in the small Hilbert space restricted by (4.2):

$$\langle\langle A, B\rangle\rangle = \langle\langle A|c_0^-|B\rangle\rangle. \tag{4.10}$$

The state $\langle\langle A|$ is the BPZ conjugate of $|A\rangle\rangle$. Using this inner product we take the free action for the Ramond sector to be

$$S_0 = -\frac{1}{2}\langle\langle \Psi, YQ\Psi\rangle\rangle, \tag{4.11}$$

which is invariant under the gauge transformation

$$\delta\Psi = Q\lambda\,. \tag{4.12}$$

field	V	Ψ	Λ	Ω	λ
Grassmann	odd	even	even	even	odd
$(\boldsymbol{g}, \boldsymbol{p})$	$(1,0)$	$(2,-1/2)$	$(0,0)$	$(0,1)$	$(1,-1/2)$

Table 1. Properties of the string fields and the gauge parameters with the ghost number $\boldsymbol{g}$ and the picture number $\boldsymbol{p}$.

The gauge parameter λ also satisfies the same constraints as Ψ:

$$b_0^-\lambda = L_0^-\lambda = \eta\lambda = 0, \qquad XY\lambda = \lambda. \tag{4.13}$$

The properties of string fields and gauge parameters are summarized in table 1.

In order to prove the gauge invariance of the action, we need to note that the operator X is BRST trivial in the large Hilbert space [26]:

$$X = \{Q, \Theta(\beta_0)\}, \tag{4.14}$$

with the Heaviside step function $\Theta(x)$. More general operator Ξ suitable in the large Hilbert space is defined by [10]

$$\Xi = \xi + (\Theta(\beta_0)\eta\xi - \xi)P_{-3/2} + (\xi\eta\Theta(\beta_0) - \xi)P_{-1/2}\,, \tag{4.15}$$

where P_n is a projector onto the states with picture number n. We can show that this operator Ξ is BPZ even for the BPZ inner product in the large Hilbert space (3.3):

$$\langle\Xi V_1, V_2\rangle = (-1)^{V_1+1}\langle V_1, \Xi V_2\rangle. \tag{4.16}$$

Then we generalize the operator X to the one given by

$$X = \{Q, \Xi\}\,, \tag{4.17}$$

which is identical to X in (4.3) on states in the small Hilbert space with the picture number $-3/2$. Hereafter we only use the new operator, so we denote it by the same symbol X for simplicity. The operator X is BPZ even with respect to the inner product in the small Hilbert space:

$$\langle\!\langle XV_1, V_2\rangle\!\rangle = \langle\!\langle V_1, XV_2\rangle\!\rangle. \tag{4.18}$$

4.2 Perturbative construction

A complete action including interactions between the NS sector and the Ramond sector can be expanded in powers of fermion:

$$S = \sum_{n=0}^{\infty} S^{(2n)}. \tag{4.19}$$

For the NS sector, $S_{NS} \equiv S^{(0)}$, we adopt the dual WZW-like action defined in (3.51). The remaining part, $S_R \equiv \sum_{n=1}^{\infty} S^{(2n)}$, contains the kinetic term of the Ramond sector (4.11)

[6] We assume that both Ψ_1 and Ψ_2 have picture number $-1/2$, which is enough to define the action (4.11).

and interaction terms between two sectors. We can further expand the action in the coupling constant κ:

$$S_{NS} = S_0^{(0)} + \kappa S_1^{(0)} + \kappa^2 S_2^{(0)} + O\left(\kappa^3\right), \tag{4.20}$$

$$S_R = S_0^{(2)} + \kappa S_1^{(2)} + \kappa^2 \left(S_2^{(2)} + S_2^{(4)}\right) + O\left(\kappa^3\right). \tag{4.21}$$

The gauge transformations can also be expanded in κ as

$$\delta_\Lambda V = \delta_\Lambda V_0^{(0)} + \kappa \delta_\Lambda V_1^{(0)} + \kappa^2 \left(\delta_\Lambda V_2^{(0)} + \delta_\Lambda V_2^{(2)}\right) + O(\kappa^3), \tag{4.22}$$

$$\delta_\Lambda \Psi = \delta_\Lambda \Psi_0^{(1)} + \kappa \delta_\Lambda \Psi_1^{(1)} + \kappa^2 \delta_\Lambda \Psi_2^{(1)} + O(\kappa^3), \tag{4.23}$$

with

$$\delta_\Lambda V_0^{(0)} = Q\Lambda, \qquad \delta_\Lambda \Psi_0^{(1)} = 0, \tag{4.24}$$

where Λ is a gauge parameter in the NS sector,

$$\delta_\Omega V = \delta_\Omega V_0^{(0)} + \kappa \delta_\Omega V_1^{(0)} + \kappa^2 \left(\delta_\Omega V_2^{(0)} + \delta_\Omega V_2^{(2)}\right) + O(\kappa^3), \tag{4.25}$$

$$\delta_\Omega \Psi = \delta_\Omega \Psi_0^{(1)} + \kappa \delta_\Omega \Psi_1^{(1)} + \kappa^2 \delta_\Omega \Psi_2^{(1)} + O(\kappa^3), \tag{4.26}$$

with

$$\delta_\Omega V_0^{(0)} = \eta\Omega, \qquad \delta_\Omega \Psi_0^{(1)} = 0, \tag{4.27}$$

where Ω is another gauge parameter in the NS sector, and

$$\delta_\lambda V = \delta_\lambda V_0^{(2)} + \kappa \delta_\lambda V_1^{(2)} + \kappa^2 \delta_\lambda V_2^{(2)} + O(\kappa^3), \tag{4.28}$$

$$\delta_\lambda \Psi = \delta_\lambda \Psi_0^{(1)} + \kappa \delta_\lambda \Psi_1^{(1)} + \kappa^2 \left(\delta_\lambda \Psi_2^{(1)} + \delta_\lambda \Psi_2^{(3)}\right) + O(\kappa^3), \tag{4.29}$$

with

$$\delta_\lambda V_0^{(2)} = 0, \qquad \delta_\lambda \Psi_0^{(1)} = Q\lambda, \tag{4.30}$$

where λ is a gauge parameter in the Ramond sector. The number in the parentheses in the superscript of gauge transformations denotes the number of fields in the Ramond sector included. Starting from the kinetic terms

$$S_0^{(0)} = \frac{1}{2}\langle V, Q\eta V\rangle, \tag{4.31}$$

$$S_0^{(2)} = -\frac{1}{2}\langle\!\langle \Psi, YQ\Psi\rangle\!\rangle, \tag{4.32}$$

let us first attempt to construct the action S, and simultaneously the gauge transformations, order by order in κ by requiring the gauge invariance.

For the NS sector, we can obtain the cubic and quartic terms simply by expanding the action (3.51):

$$S_1^{(0)} = \frac{1}{3!}\langle V, Q[V, \eta V]^\eta\rangle, \tag{4.33a}$$

$$S_2^{(0)} = \frac{1}{4!}\langle V, Q[V, (\eta V)^2]^\eta\rangle + \frac{1}{4!}\langle V, Q[V, [V, \eta V]^\eta]^\eta\rangle. \tag{4.33b}$$

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Expanding the gauge transformation (3.54), one can also obtain

$$\delta_\Lambda V_1^{(0)} = -\frac{1}{2}[V, Q\Lambda]^\eta, \qquad \delta_\Lambda V_2^{(0)} = -\frac{1}{3}[V, \eta V, Q\Lambda]^\eta + \frac{1}{12}[V, [V, Q\Lambda]], \tag{4.34}$$

$$\delta_\Omega V_1^{(0)} = \frac{1}{2}[V, \eta\Omega]^\eta, \qquad \delta_\Omega V_2^{(0)} = \frac{1}{3!}[V, \eta V, \eta\Omega]^\eta + \frac{1}{12}[V, [V, \eta\Omega]], \tag{4.35}$$

which keep the action (4.33) invariant at each order in κ:

$$(\delta_\Lambda)_0^{(0)} S_1^{(0)} + (\delta_\Lambda)_1^{(0)} S_0^{(0)} = 0, \qquad (\delta_\Lambda)_0^{(0)} S_2^{(0)} + (\delta_\Lambda)_1^{(0)} S_1^{(0)} + (\delta_\Lambda)_2^{(0)} S_0^{(0)} = 0, \tag{4.36}$$

$$(\delta_\Omega)_0^{(0)} S_1^{(0)} + (\delta_\Omega)_1^{(0)} S_0^{(0)} = 0, \qquad (\delta_\Omega)_0^{(0)} S_2^{(0)} + (\delta_\Omega)_1^{(0)} S_1^{(0)} + (\delta_\Omega)_2^{(0)} S_0^{(0)} = 0. \tag{4.37}$$

The number in the parentheses in the superscript of δ denotes the difference of the number of the Ramond field after and before the transformation: (# of R fields after transformation) − (# of R fields before transformation).

4.2.1 Cubic interaction in S_R

Let us consider the cubic interaction in the Ramond action S_R. We start from a natural candidate of cubic interaction term given by

$$S_1^{(2)} = \alpha_1 \langle \Psi, [V, \Psi]^\eta \rangle, \tag{4.38}$$

with a constant α_1 to be determined, and find $\delta V_1^{(2)} (= \delta_1^{(2)} V)$ and $\delta\Psi_1^{(1)} (= \delta_1^{(0)} \Psi)$ requiring the gauge invariances in this order

$$\delta_0^{(0)} S_1^{(2)} + \delta_1^{(0)} S_0^{(2)} + \delta_1^{(2)} S_0^{(0)} = 0\,. \tag{4.39}$$

Note that $\delta_0^{(2)}$ does not appear and that $\delta_1^{(2)}$ appear only for the transformation with λ, which follows from just the counting of the Ramond fields. The variation of $S_1^{(2)}$ under the gauge transformation $\delta_\Lambda V_0^{(0)}$ in (4.24) is calculated as

$$(\delta_\Lambda)_0^{(0)} S_1^{(2)} = \alpha_1 \langle Q\Lambda, [\Psi^2]^\eta \rangle = -2\alpha_1 \langle \Lambda, [\Psi, Q\Psi]^\eta \rangle = -2\alpha_1 \langle [\Psi, \Lambda]^\eta, Q\Psi \rangle. \tag{4.40}$$

This can be cancelled by $(\delta_\Lambda)_1^{(0)} S_0^{(2)}$ if we take

$$\delta_\Lambda \Psi_1^{(1)} = -2\alpha_1 X\eta [\Psi, \Lambda]^\eta, \tag{4.41}$$

in a similar manner given in [9]. Similarly, the variation of $S_1^{(2)}$ under the gauge transformation $\delta_\Omega V_0^{(0)}$ in (4.27) is given by

$$(\delta_\Omega)_0^{(0)} S_1^{(2)} = \alpha_1 \langle \eta\Omega, [\Psi^2]^\eta \rangle = -2\alpha_1 \langle \Omega, [\Psi, \eta\Psi]^\eta \rangle = 0, \tag{4.42}$$

because of $\eta\Psi = 0$, and so we have

$$\delta_\Omega \Psi_1^{(1)} = 0\,. \tag{4.43}$$

Under the gauge transformation $\delta_\lambda \Psi_0^{(1)}$ in (4.30), the variation of $S_1^{(2)}$ is given by

$$\begin{aligned}
(\delta_\lambda)_0^{(0)} S_1^{(2)} &= 2\alpha_1 \langle Q\lambda, [V, \Psi]^\eta \rangle \\
&= 2\alpha_1 \langle \lambda, [QV, \Psi]^\eta \rangle - 2\alpha_1 \langle \lambda, [V, Q\Psi]^\eta \rangle \\
&= 2\alpha_1 \langle \Xi\lambda, [Q\eta V, \Psi]^\eta \rangle + 2\alpha_1 \langle \Xi\lambda, [\eta V, Q\Psi]^\eta \rangle \\
&= 2\alpha_1 \langle [\Psi, \Xi\lambda]^\eta, Q\eta V \rangle + 2\alpha_1 \langle [\eta V, \Xi\lambda]^\eta, Q\Psi \rangle,
\end{aligned} \tag{4.44}$$

where we used the fact that a relation,

$$\langle \lambda, B\rangle = \langle \eta\Xi\lambda, B\rangle = \langle \Xi\lambda, \eta B\rangle, \tag{4.45}$$

holds for general string field B since the parameter λ is in the small Hilbert space. This variation (4.44) can be canceled by $(\delta_\lambda)_1^{(2)} S_0^{(0)} + (\delta_\lambda)_1^{(0)} S_0^{(2)}$ with

$$\delta_\lambda V_1^{(2)} = -2\alpha_1[\Psi, \Xi\lambda]^\eta, \qquad \delta_\lambda \Psi_1^{(1)} = 2\alpha_1 X\eta[\eta V, \Xi\lambda]^\eta. \tag{4.46}$$

4.2.2 Quartic interaction in S_R

Let us move to the next order. In order to narrow down the form of quartic interaction terms, let us first consider the variation of $S_1^{(2)}$ under the gauge transformations $\delta_\Lambda V_1^{(0)}$ and $\delta_\Lambda \Psi_1^{(1)}$, which is calculated as

$$\begin{aligned}(\delta_\Lambda)_1^{(0)} S_1^{(2)} &= -\frac{\alpha_1}{2}\langle [V, Q\Lambda]^\eta, [\Psi^2]^\eta\rangle - 4\alpha_1^2\langle X\eta[\Psi,\Lambda]^\eta, [V,\Psi]^\eta\rangle \\ &= \frac{\alpha_1}{2}\langle Q\Lambda, [V, [\Psi^2]^\eta]^\eta\rangle - 4\alpha_1^2\langle [\Psi,\Lambda]^\eta, X[\eta V,\Psi]^\eta\rangle\,. \end{aligned} \tag{4.47}$$

Using $X = \{Q, \Xi\}$, we further calculate the second term $\langle [\Psi,\Lambda]^\eta, X[\eta V,\Psi]^\eta\rangle$ as follows:

$$\begin{aligned}\langle [\Psi,\Lambda]^\eta, X[\eta V,\Psi]^\eta\rangle &= \langle [\Psi,\Lambda]^\eta, \{Q,\Xi\}[\eta V,\Psi]^\eta\rangle \\ &= \langle [Q\Psi,\Lambda]^\eta, \Xi[\eta V,\Psi]^\eta\rangle + \langle [\Psi, Q\Lambda]^\eta, \Xi[\eta V,\Psi]^\eta\rangle \\ &\quad - \langle \Xi[\Psi,\Lambda]^\eta, [Q\eta V,\Psi]^\eta\rangle - \langle \Xi[\Psi,\Lambda]^\eta, [\eta V, Q\Psi]^\eta\rangle \\ &= -\langle [\Xi[\eta V,\Psi]^\eta,\Lambda]^\eta, Q\Psi\rangle - \langle Q\Lambda, [\Psi, \Xi[\eta V,\Psi]^\eta]^\eta\rangle \\ &\quad - \langle [\Psi, \Xi[\Psi,\Lambda]^\eta]^\eta, Q\eta V\rangle - \langle [\eta V, \Xi[\Psi,\Lambda]^\eta]^\eta, Q\Psi\rangle\,. \end{aligned} \tag{4.48}$$

Then we find

$$\begin{aligned}(\delta_\Lambda)_1^{(0)} S_1^{(2)} = &\frac{\alpha_1}{2}\langle Q\Lambda, [V, [\Psi^2]^\eta]^\eta\rangle + 4\alpha_1^2\langle Q\Lambda, [\Psi, \Xi[\eta V,\Psi]^\eta]^\eta\rangle \\ &+ 4\alpha_1^2\langle [\Psi, \Xi[\Psi,\Lambda]^\eta]^\eta, Q\eta V\rangle \\ &+ 4\alpha_1^2\langle [\Xi[\eta V,\Psi]^\eta,\Lambda]^\eta, Q\Psi\rangle + 4\alpha_1^2\langle [\eta V, \Xi[\Psi,\Lambda]^\eta]^\eta, Q\Psi\rangle\,. \end{aligned} \tag{4.49}$$

In order to cancel the first two terms on the right hand side, we introduce quartic interaction terms with two Ramond strings as

$$S_2^{(2)} = \alpha_2\langle \Psi, [V, \eta V, \Psi]^\eta\rangle + \alpha_3\langle \Psi, [V, \Xi[\eta V,\Psi]^\eta]^\eta\rangle, \tag{4.50}$$

with constants α_2 and α_3 to be determined. The first term is a genuine four-string interaction filling a missing region in, for example, the moduli space of four-string amplitude with two fermions. The variation of $S_2^{(2)}$ under the gauge transformation $(\delta_\Lambda)_0^{(0)} V$ can be straightforwardly calculated as follows. The variation of the first term is given by

$$\begin{aligned}(\delta_\Lambda)_0^{(0)}(\alpha_2\langle \Psi, [V,\eta V,\Psi]^\eta\rangle) &= \alpha_2\langle \Psi, [Q\Lambda, \eta V, \Psi]^\eta\rangle + \alpha_2\langle \Psi, [V, \eta Q\Lambda, \Psi]^\eta\rangle \\ &= \alpha_2\langle Q\Lambda, [\eta V, \Psi^2]^\eta\rangle + \alpha_2\langle \eta Q\Lambda, [V, \Psi^2]^\eta\rangle \\ &= 2\alpha_2\langle Q\Lambda, [\eta V, \Psi^2]^\eta\rangle - \alpha_2\langle Q\Lambda, [V, [\Psi^2]^\eta]^\eta\rangle \\ &\quad + 2\alpha_2\langle Q\Lambda, [\Psi, [V,\Psi]^\eta]^\eta\rangle \\ &= -\alpha_2\langle Q\Lambda, [V, [\Psi^2]^\eta]^\eta\rangle + 2\alpha_2\langle Q\Lambda, [\Psi, [V,\Psi]^\eta]^\eta\rangle \\ &\quad - 2\alpha_2\langle [\Psi^2, \Lambda]^\eta, Q\eta V\rangle - 4\alpha_2\langle [\eta V, \Psi, \Lambda]^\eta, Q\Psi\rangle\,. \end{aligned} \tag{4.51}$$

The variation of the second term in $S_2^{(2)}$ can similarly be calculated as

$$(\delta_\Lambda)_0^{(0)}(\alpha_3\langle\Psi,[V,\Xi[\eta V,\Psi]^\eta]^\eta\rangle) = \alpha_3\langle Q\Lambda,[\Psi,\Xi[\eta V,\Psi]^\eta]^\eta\rangle + \alpha_3\langle\Psi,[V,\Xi[\eta Q\Lambda,\Psi]^\eta]^\eta\rangle,$$
$$= \alpha_3\langle Q\Lambda,[\Psi,[V,\Psi]^\eta]^\eta\rangle + 2\alpha_3\langle Q\Lambda,[\Psi,\Xi[\eta V,\Psi]^\eta]^\eta\rangle. \quad (4.52)$$

Therefore, in total, we have

$$\begin{aligned}(\delta_\Lambda)_0^{(0)}S_2^{(2)} = &-\alpha_2\langle Q\Lambda,[V,[\Psi^2]^\eta]^\eta\rangle + (2\alpha_2+\alpha_3)\langle Q\Lambda,[\Psi,[V,\Psi]^\eta]^\eta\rangle\\ &+ 2\alpha_3\langle Q\Lambda,[\Psi,\Xi[\eta V,\Psi]^\eta]^\eta\rangle\\ &- 2\alpha_2\langle[\Psi^2,\Lambda]^\eta,Q\eta V\rangle - 4\alpha_2\langle[\eta V,\Psi,\Lambda]^\eta,Q\Psi\rangle.\end{aligned} \quad (4.53)$$

From (4.49) and (4.53), we find that the constants α_1, α_2 and α_3 should be chosen to be

$$\alpha_1 = \frac{1}{2}, \qquad \alpha_2 = \frac{1}{4}, \qquad \alpha_3 = -\frac{1}{2}. \quad (4.54)$$

Then we have

$$\begin{aligned}(\delta_\Lambda)_1^{(0)}S_1^{(2)} + (\delta_\Lambda)_0^{(0)}S_2^{(2)} = &-\frac{1}{2}\langle[\Psi^2,\Lambda]^\eta,Q\eta V\rangle + \langle[\Psi,\Xi[\Psi,\Lambda]^\eta]^\eta,Q\eta V\rangle\\ &- \langle[\eta V,\Psi,\Lambda]^\eta,Q\Psi\rangle + \langle[\eta V,\Xi[\Psi,\Lambda]^\eta]^\eta,Q\Psi\rangle\\ &+ \langle[\Xi[\eta V,\Psi]^\eta,\Lambda]^\eta,Q\Psi\rangle\,.\end{aligned} \quad (4.55)$$

These terms can be cancelled by $(\delta_\Lambda)_2^{(2)}S_0^{(0)}$ and $(\delta_\Lambda)_2^{(0)}S_0^{(2)}$ if we choose

$$\delta_\Lambda V_2^{(2)} = \frac{1}{2}[\Psi^2,\Lambda]^\eta - [\Psi,\Xi[\Psi,\Lambda]^\eta]^\eta, \quad (4.56)$$
$$\delta_\Lambda\Psi_2^{(1)} = -X\eta[\eta V,\Psi,\Lambda]^\eta + X\eta[\eta V,\Xi[\Psi,\Lambda]^\eta]^\eta + X\eta[\Xi[\eta V,\Psi]^\eta,\Lambda]^\eta. \quad (4.57)$$

Note that $\delta_\Lambda V_2^{(2)} = (\delta_\Lambda)_2^{(2)}V$ and $\delta_\Lambda\Psi_2^{(1)} = (\delta_\Lambda)_2^{(0)}\Psi$. Thus the gauge invariance under transformation with the parameter Λ in this order holds:

$$(\delta_\Lambda)_1^{(0)}S_1^{(2)} + (\delta_\Lambda)_0^{(0)}S_2^{(2)} + (\delta_\Lambda)_2^{(2)}S_0^{(0)} + (\delta_\Lambda)_2^{(0)}S_0^{(2)} = 0\,. \quad (4.58)$$

Then the variation under the gauge transformations with the parameter Ω at this order can easily be calculated as

$$(\delta_\Omega)_1^{(0)}S_1^{(2)} = \frac{1}{4}\langle[\Psi^2]^\eta,[V,\eta\Omega]^\eta\rangle = -\frac{1}{4}\langle\eta\Omega,[V,[\Psi^2]^\eta]^\eta\rangle = -\frac{1}{4}\langle\Omega,[\eta V,[\Psi^2]^\eta]^\eta\rangle\,, \quad (4.59)$$
$$(\delta_\Omega)_0^{(0)}S_2^{(2)} = \frac{1}{4}\langle\eta\Omega,[\eta V,\Psi^2]^\eta\rangle - \frac{1}{2}\langle\eta\Omega,[\Psi,\Xi[\eta V,\Psi]^\eta]^\eta\rangle = \frac{1}{4}\langle\Omega,[\eta V,[\Psi^2]^\eta]^\eta\rangle\,, \quad (4.60)$$

and hence

$$(\delta_\Omega)_1^{(0)}S_1^{(2)} + (\delta_\Omega)_0^{(0)}S_2^{(2)} = 0\,. \quad (4.61)$$

The correction at this order is not necessary:

$$\delta_\Omega V_2^{(2)} = 0\,, \qquad \delta_\Omega\Psi_2^{(1)} = 0\,. \quad (4.62)$$

Let us finally calculate variations of the action under the gauge transformation with the parameter λ. The variations $(\delta_\lambda)_1^{(2)} S_1^{(0)}$ and $(\delta_\lambda)_1^{(0)} S_1^{(2)}$ are calculated as

$$(\delta_\lambda)_1^{(2)} S_1^{(0)} = -\frac{1}{2}\langle [QV, \eta V]^\eta, (\delta_\lambda)_1^{(2)} V\rangle = \frac{1}{2}\langle \Xi\lambda, [\Psi, [QV, \eta V]^\eta]^\eta\rangle\,, \tag{4.63}$$

and

$$(\delta_\lambda)_1^{(0)} S_1^{(2)} = -\langle [V, \Psi]^\eta, X\eta[\eta V, \Xi\lambda]^\eta\rangle = \langle \Xi\lambda, [\eta V, X[\eta V, \Psi]^\eta]^\eta\rangle, \tag{4.64}$$

respectively, where we used a relation

$$\begin{aligned}\langle A, X\eta B\rangle &= \langle \Xi\eta A, X\eta B\rangle = \langle\!\langle \eta A, X\eta B\rangle\!\rangle \\ &= \langle\!\langle X\eta A, \eta B\rangle\!\rangle = \langle \Xi X\eta A, \eta B\rangle \\ &= (-1)^A \langle X\eta A, B\rangle\,.\end{aligned} \tag{4.65}$$

The variation $(\delta_\lambda)_0^{(0)} S_2^{(2)}$ is given by

$$(\delta_\lambda)_0^{(0)} S_2^{(2)} = \frac{1}{2}\langle Q\lambda, [V, \eta V, \Psi]^\eta\rangle - \frac{1}{2}\langle Q\lambda, [V, \Xi[\eta V, \Psi]^\eta]^\eta\rangle - \frac{1}{2}\langle Q\lambda, [\eta V, \Xi[V, \Psi]^\eta]^\eta\rangle\,. \tag{4.66}$$

Substituting the relation $Q\lambda = Q\eta\Xi\lambda$, this can further be calculated as

$$\begin{aligned}(\delta_\lambda)_0^{(0)} S_2^{(2)} = &-\frac{1}{2}\langle \Xi\lambda, [\Psi, [QV, \eta V]^\eta]^\eta\rangle - \langle \Xi\lambda, [\eta V, X[\eta V, \Psi]^\eta]^\eta\rangle \\ &+ \langle [\eta V, \Psi, \Xi\lambda]^\eta, Q\eta V\rangle - \frac{1}{2}\langle [V, [\Psi, \Xi\lambda]^\eta]^\eta, Q\eta V\rangle - \langle [\Xi[\eta V, \Psi]^\eta, \Xi\lambda]^\eta, Q\eta V\rangle \\ &+ \frac{1}{2}\langle [(\eta V)^2, \Xi\lambda]^\eta, Q\Psi\rangle - \langle [\eta V, \Xi[\eta V, \Xi\lambda]^\eta]^\eta, Q\Psi\rangle - \langle [\Psi, \Xi[\eta V, \Xi\lambda]^\eta]^\eta, Q\eta V\rangle \\ &+ \frac{1}{2}\langle [[V, \eta V]^\eta, \Xi\lambda]^\eta, Q\Psi\rangle.\end{aligned} \tag{4.67}$$

In total we have

$$\begin{aligned}(\delta_\lambda)_1^{(2)} S_1^{(0)} &+ (\delta_\lambda)_1^{(0)} S_1^{(2)} + (\delta_\lambda)_0^{(0)} S_2^{(2)} \\ &= \langle [\eta V, \Psi, \Xi\lambda]^\eta, Q\eta V\rangle - \frac{1}{2}\langle [V, [\Psi, \Xi\lambda]^\eta]^\eta, Q\eta V\rangle - \langle [\Psi, \Xi[\eta V, \Xi\lambda]^\eta]^\eta, Q\eta V\rangle \\ &\quad - \langle [\Xi[\eta V, \Psi]^\eta, \Xi\lambda]^\eta, Q\eta V\rangle + \frac{1}{2}\langle [(\eta V)^2, \Xi\lambda]^\eta, Q\Psi\rangle - \langle [\eta V, \Xi[\eta V, \Xi\lambda]^\eta]^\eta, Q\Psi\rangle \\ &\quad + \frac{1}{2}\langle [[V, \eta V]^\eta, \Xi\lambda]^\eta, Q\Psi\rangle\,.\end{aligned} \tag{4.68}$$

This can be cancelled by $(\delta_\lambda)_2^{(2)} S_0^{(0)}$ and $(\delta_\lambda)_2^{(0)} S_0^{(2)}$ if we take $(\delta_\lambda)_2^{(2)} V$ and $(\delta_\lambda)_2^{(0)} \Psi$ as

$$\delta_\lambda V_2^{(2)} = -[\eta V, \Psi, \Xi\lambda]^\eta + \frac{1}{2}[V, [\Psi, \Xi\lambda]^\eta]^\eta + [\Psi, \Xi[\eta V, \Xi\lambda]^\eta]^\eta + [\Xi[\eta V, \Psi]^\eta, \Xi\lambda]^\eta, \tag{4.69}$$

$$\delta_\lambda \Psi_2^{(1)} = \frac{1}{2}X\eta[(\eta V)^2, \Xi\lambda]^\eta - X\eta[\eta V, \Xi[\eta V, \Xi\lambda]^\eta]^\eta + \frac{1}{2}X\eta[[V, \eta V]^\eta, \Xi\lambda]^\eta. \tag{4.70}$$

So far so good: in this way the gauge invariance with the parameter λ holds at quadratic order in both coupling constant and the Ramond fields:

$$(\delta_\lambda)_1^{(2)} S_1^{(0)} + (\delta_\lambda)_1^{(0)} S_1^{(2)} + (\delta_\lambda)_0^{(0)} S_2^{(2)} + (\delta_\lambda)_2^{(2)} S_0^{(0)} + (\delta_\lambda)_2^{(0)} S_0^{(2)} = 0\,. \tag{4.71}$$

Let us move to the quartic order in the Ramond string field. The non-trivial contribution absent in the open superstring field theory comes from $(\delta_\lambda)_1^{(2)}S_1^{(2)}$ given by

$$(\delta_\lambda)_1^{(2)}S_1^{(2)} = \frac{1}{2}\langle[\Psi^2]^\eta, \delta V_1^{(2)}\rangle = -\frac{1}{2}\langle\Xi\lambda, [\Psi, [\Psi^2]^\eta]^\eta\rangle = \frac{1}{6}\langle\lambda, [\Psi^3]^\eta\rangle, \tag{4.72}$$

which requires to add Ψ^4 interaction $S_2^{(4)}$ to the action. Note that $\delta_\lambda\Psi_1^{(3)}(= (\delta_\lambda)_1^{(2)}\Psi)$ never appears. Let us consider (4.72) in further detail. As was given in (3.55), $[\Psi^3]^\eta$ can be written in a *BRST exact* form as,

$$[\Psi^3]^\eta = Q(\Psi^3)^\eta - 3(\Psi^2, Q\Psi)^\eta, \tag{4.73}$$

with the dual gauge product $(\cdots)^\eta$ given in (A.30). Using this relation, (4.72) can be rewritten as

$$\begin{aligned}(\delta_\lambda)_1^{(2)}S_1^{(2)} &= \frac{1}{6}\langle\lambda, Q(\Psi^3)^\eta\rangle - \frac{1}{2}\langle\lambda, (\Psi^2, Q\Psi)^\eta\rangle \\ &= -\frac{1}{6}\langle Q\lambda, (\Psi^3)^\eta\rangle + \frac{1}{2}\langle(\Psi^2, \lambda)^\eta, Q\Psi\rangle.\end{aligned} \tag{4.74}$$

From the form of the first term, we can suppose that $S_2^{(4)}$ have the form

$$S_2^{(4)} = \alpha_4\langle\Psi, (\Psi^3)^\eta\rangle, \tag{4.75}$$

with a constant α_4, whose variation under $\delta_\lambda\Psi_0^{(1)}$ becomes

$$(\delta_\lambda)_0^{(0)}S_2^{(4)} = 4\alpha_4\langle Q\lambda, (\Psi^3)^\eta\rangle\,. \tag{4.76}$$

Then we have

$$(\delta_\lambda)_1^{(2)}S_1^{(2)} + (\delta_\lambda)_0^{(0)}S_2^{(4)} = \frac{1}{2}\langle(\Psi^2, \lambda)^\eta, Q\Psi\rangle\,, \tag{4.77}$$

by setting

$$\alpha_4 = \frac{1}{4!}\,. \tag{4.78}$$

The remaining terms can be cancelled by $(\delta_\lambda)_2^{(2)}S_0^{(2)}$ if we take

$$\delta_\lambda\Psi_2^{(3)} = \frac{1}{2}X\eta(\Psi^2, \lambda)^\eta. \tag{4.79}$$

That is, the gauge invariance at this order holds:

$$(\delta_\lambda)_1^{(2)}S_1^{(2)} + (\delta_\lambda)_0^{(0)}S_2^{(4)} + (\delta_\lambda)_2^{(2)}S_0^{(2)} = 0\,. \tag{4.80}$$

Let us summarize the results up to this order. The action in the NS sector is given by

$$\begin{aligned}S_{NS} &= S^{(0)} \\ &= S_0^{(0)} + \kappa S_1^{(0)} + \kappa^2 S_2^{(0)} + O(\kappa^3),\end{aligned} \tag{4.81}$$

where

$$S_0^{(0)} = -\frac{1}{2}\langle QV, \eta V\rangle, \tag{4.82}$$

$$S_1^{(0)} = -\frac{1}{3!}\langle QV, [V, \eta V]^\eta\rangle, \tag{4.83}$$

$$S_2^{(0)} = -\frac{1}{4!}\langle QV, [V, \eta V, \eta V]^\eta\rangle - \frac{1}{4!}\langle QV, [V, [V, \eta V]^\eta]^\eta\rangle. \tag{4.84}$$

The action in the Ramond sector is given by

$$S_R = S_0^{(2)} + \kappa S_1^{(2)} + \kappa^2 (S_2^{(2)} + S_2^{(4)}) + O(\kappa^3), \tag{4.85}$$

where

$$S_0^{(2)} = -\frac{1}{2}\langle\!\langle \Psi, YQ\Psi\rangle\!\rangle\,, \tag{4.86}$$

$$S_1^{(2)} = \frac{1}{2}\langle \Psi, [V, \Psi]^\eta\rangle\,, \tag{4.87}$$

$$S_2^{(2)} = \frac{1}{4}\langle \Psi, [V, \eta V, \Psi]^\eta\rangle - \frac{1}{2}\langle \Psi, [V, \Xi[\eta V, \Psi]^\eta]^\eta\rangle\,,$$

$$S_2^{(4)} = \frac{1}{4!}\langle \Psi, (\Psi^3)^\eta\rangle\,. \tag{4.88}$$

The gauge transformation with the gauge parameter for Λ in the NS sector is given by

$$\delta_\Lambda V = \delta_\Lambda V_0^{(0)} + \kappa \delta_\Lambda V_1^{(0)} + \kappa^2 \left(\delta_\Lambda V_2^{(0)} + \delta_\Lambda V_2^{(2)}\right) + O(\kappa^3)\,, \tag{4.89}$$

$$\delta_\Lambda \Psi = \delta_\Lambda \Psi_0^{(1)} + \kappa \delta_\Lambda \Psi_1^{(1)} + \kappa^2 \delta_\Lambda \Psi_2^{(1)} + O(\kappa^3)\,, \tag{4.90}$$

where

$$\delta_\Lambda V_0^{(0)} = Q\Lambda\,, \tag{4.91}$$

$$\delta_\Lambda V_1^{(0)} = -\frac{1}{2}[V, Q\Lambda]^\eta\,, \tag{4.92}$$

$$\delta_\Lambda V_2^{(0)} = -\frac{1}{3}[V, \eta V, Q\Lambda]^\eta + \frac{1}{12}[V, [V, Q\Lambda]^\eta]^\eta\,, \tag{4.93}$$

$$\delta_\Lambda V_2^{(2)} = \frac{1}{2}[\Psi, \Psi, \Lambda]^\eta - [\Psi, \Xi[\Psi, \Lambda]^\eta]^\eta\,, \tag{4.94}$$

$$\delta_\Lambda \Psi_0^{(1)} = 0\,, \tag{4.95}$$

$$\delta_\Lambda \Psi_1^{(1)} = -X\eta[\Psi, \Lambda]^\eta\,, \tag{4.96}$$

$$\delta_\Lambda \Psi_2^{(1)} = -X\eta[\eta V, \Psi, \Lambda]^\eta + X\eta[\eta V, \Xi[\Psi, \Lambda]^\eta]^\eta + X\eta[\Xi[\eta V, \Psi]^\eta, \Lambda]^\eta\,. \tag{4.97}$$

The gauge transformation with the gauge parameter Ω in the NS sector is given by

$$\delta_\Omega V = \delta_\Omega V_0^{(0)} + \kappa \delta_\Omega V_1^{(0)} + \kappa^2 \delta_\Omega V_2^{(0)} + O(\kappa^3)\,, \tag{4.98}$$

$$\delta_\Omega \Psi = \delta_\Omega \Psi_0^{(1)} + \kappa \delta_\Omega \Psi_1^{(1)} + \kappa^2 \delta_\Omega \Psi_2^{(1)} + O(\kappa^3)\,, \tag{4.99}$$

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where

$$\delta_\Omega V_0^{(0)} = \eta\Omega\,, \tag{4.100}$$

$$\delta_\Omega V_1^{(0)} = \frac{1}{2}[V, \eta\Omega]^\eta\,, \tag{4.101}$$

$$\delta_\Omega V_2^{(0)} = \frac{1}{6}[V, \eta V, \eta\Omega]^\eta + \frac{1}{12}[V, [V, \eta\Omega]^\eta]^\eta\,, \tag{4.102}$$

$$\delta_\Omega \Psi_0^{(1)} = 0\,, \tag{4.103}$$

$$\delta_\Omega \Psi_1^{(1)} = 0\,, \tag{4.104}$$

$$\delta_\Omega \Psi_2^{(1)} = 0\,. \tag{4.105}$$

The gauge transformation with the gauge parameter λ in the Ramond sector is given by

$$\delta_\lambda V = \delta_\lambda V_0^{(2)} + \kappa\delta_\lambda V_1^{(2)} + \kappa^2\delta_\lambda V_2^{(2)} + O(\kappa^3)\,, \tag{4.106}$$

$$\delta_\lambda \Psi = \delta_\lambda \Psi_0^{(1)} + \kappa\delta_\lambda \Psi_1^{(1)} + \kappa^2\left(\delta_\lambda \Psi_2^{(1)} + \delta_\lambda \Psi_2^{(3)}\right) + O(\kappa^3)\,, \tag{4.107}$$

where

$$\delta_\lambda V_0^{(2)} = 0\,, \tag{4.108}$$

$$\delta_\lambda V_1^{(2)} = -[\Psi, \Xi\lambda]^\eta\,, \tag{4.109}$$

$$\delta_\lambda V_2^{(2)} = -[\eta V, \Psi, \Xi\lambda]^\eta + \frac{1}{2}[V, [\Psi, \Xi\lambda]^\eta]^\eta + [\Psi, \Xi[\eta V, \Xi\lambda]^\eta]^\eta + [\Xi[\eta V, \Psi]^\eta, \Xi\lambda]^\eta\,, \tag{4.110}$$

$$\delta_\lambda \Psi_0^{(1)} = Q\lambda\,, \tag{4.111}$$

$$\delta_\lambda \Psi_1^{(1)} = X\eta[\eta V, \Xi\lambda]^\eta\,, \tag{4.112}$$

$$\delta_\lambda \Psi_2^{(1)} = \frac{1}{2}X\eta[\eta V, \eta V, \Xi\lambda]^\eta - X\eta[\eta V, \Xi[\eta V, \Xi\lambda]^\eta]^\eta + \frac{1}{2}X\eta[[V, \eta V]^\eta, \Xi\lambda]^\eta\,, \tag{4.113}$$

$$\delta_\lambda \Psi_2^{(3)} = \frac{1}{2}X\eta(\Psi^2, \lambda)^\eta\,. \tag{4.114}$$

4.3 Fermion expansion

As a next step to the complete action let us consider the fermion expansion. We extend the above results to all order in the NS string field at each order in the Ramond string field.

Suppose that an arbitrary variation of $S^{(2n)}$, the action at $O(\Psi^{2n})$, has the form

$$\delta S^{(2n)} = -\langle\!\langle \delta\Psi, YE^{(2n-1)}\rangle\!\rangle + \langle B_\delta, E^{(2n)}\rangle\,. \tag{4.115}$$

The equations of motion are therefore given by

$$E^{(0)} + E^{(2)} + E^{(4)} + \cdots = 0\,, \tag{4.116}$$

$$E^{(1)} + E^{(3)} + E^{(5)} + \cdots = 0\,, \tag{4.117}$$

for the NS and the Ramond string fields, respectively. We can also expand the gauge transformation in powers of the Ramond string field as

$$B_\delta = B_\delta^{(0)} + B_\delta^{(2)} + B_\delta^{(4)} + \cdots\,, \tag{4.118}$$

$$\delta\Psi = \delta\Psi^{(1)} + \delta\Psi^{(3)} + \delta\Psi^{(5)} + \cdots\,. \tag{4.119}$$

Note that the superscript also counts the Ramond gauge parameter λ. We will determine the action and the gauge transformations order by order in the Ramond string field by requiring the gauge-invariance at each order:

$$0 = -\sum_{k=1}^{n} \langle\!\langle \delta\Psi^{(2n-2k+1)}, YE^{(2k-1)} \rangle\!\rangle + \sum_{k=0}^{n} \langle B_\delta^{(2n-2k)}, E^{(2k)} \rangle \,. \tag{4.120}$$

In particular, at the lowest order in fermion, corresponding to $n = 0$, it reduces

$$0 = \langle B_\delta^{(0)}, E^{(0)} \rangle. \tag{4.121}$$

For the dual WZW-like action (3.51), $E^{(0)} = QG_\eta(V)$, and (4.121) requires $B_\delta^{(0)} = Q\Lambda + D_\eta\Omega$ as summarized in the previous section.

4.3.1 Quadratic in fermion

We first provide the action $S^{(2)}$ and the gauge transformation $B_\delta^{(2)}$ and $\delta\Psi^{(1)}$ so that the action is gauge invariant at quadratic order in the Ramond string field:

$$0 = -\langle\!\langle \delta\Psi^{(1)}, YE^{(1)} \rangle\!\rangle + \langle B_\delta^{(2)}, E^{(0)} \rangle + \langle B_\delta^{(0)}, E^{(2)} \rangle. \tag{4.122}$$

From the results in the perturbative expansion, we can deduce that the action $S^{(2)}$ is given by the same form of that for the open superstring (2.15):

$$S^{(2)} = -\frac{1}{2}\langle\!\langle \Psi, YQ\Psi \rangle\!\rangle + \frac{\kappa}{2}\int_0^1 dt \big\langle B_t(t), [F(t)\Psi, F(t)\Psi]^\eta_{G_\eta(t)} \big\rangle, \tag{4.123}$$

where $F(t)$ is the linear operator defined by

$$F(t) = \frac{1}{1 + \Xi(D_\eta(t) - \eta)} = 1 + \sum_{n=1}^{\infty} \big(-\Xi(D_\eta(t) - \eta) \big)^n. \tag{4.124}$$

We should note that this has the same form as (2.12) but D_η defined in (3.40) contains infinite number of terms with arbitrary power of G_η. In what follows, we show that the variation of the action $S^{(2)}$ becomes

$$\begin{aligned} \delta S^{(2)} &= -\langle\!\langle \delta\Psi, Y(E^{(1)}) \rangle\!\rangle + \big\langle B_\delta, E^{(2)} \big\rangle \\ &= -\langle\!\langle \delta\Psi, Y(Q\Psi + X\eta F\Psi) \rangle\!\rangle + \frac{\kappa}{2}\big\langle B_\delta, [F\Psi, F\Psi]^\eta_{G_\eta} \big\rangle \,, \end{aligned} \tag{4.125}$$

and prove that the action $S^{(0)} + S^{(2)}$ is invariant at quadratic order in the Ramond string field under the following gauge transformations at this order

$$B_\delta^{(2)} = \frac{\kappa^2}{2}[F\Psi, F\Psi, \Lambda]^\eta_{G_\eta} - \kappa^2 [F\Psi, F\Xi[F\Psi, \Lambda]^\eta_{G_\eta}]^\eta_{G_\eta} - \kappa[F\Psi, F\Xi\lambda]^\eta_{G_\eta} \,, \tag{4.126}$$

$$\delta\Psi^{(1)} = -\kappa X\eta F\Xi D_\eta [F\Psi, \Lambda]^\eta_{G_\eta} + Q\lambda - X\eta F\lambda \,. \tag{4.127}$$

Let us first summarize the properties of $F(t)$, we will use. The linear map $F(t)$ satisfies $F(0) = 1$ since it depends on t only through $G_\eta(t)$ and $G_\eta(0) = 0$. It is invertible and

$$F^{-1}(t) = 1 + \Xi(D_\eta(t) - \eta) = \eta\Xi + \Xi D_\eta(t) \,. \tag{4.128}$$

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Multiplying it by η from the left or by D_η from the right, we have

$$\eta F^{-1}(t) = F^{-1}(t)D_\eta(t) = \eta\Xi D_\eta(t)\,, \tag{4.129}$$

We further have

$$F(t)\eta = D_\eta(t)F(t)\,, \qquad \{D_\eta(t), F(t)\Xi\} = 1\,. \tag{4.130}$$

The former can be obtained by multiplying the first equation in (4.129) by F(t) from both left and right and using $\eta^2 = D_\eta(t)^2 = 0$. Then the latter can be derived as

$$\begin{aligned} D_\eta(t)F(t)\Xi + F(t)\Xi D_\eta(t) &= F(t)\eta\Xi + F(t)\Xi D_\eta(t) \\ &= F(t)\big(1 + \Xi(D_\eta(t) - \eta)\big) = 1\,. \end{aligned} \tag{4.131}$$

It is also shown

$$\langle F(t)\Xi V_1, V_2\rangle = (-1)^{V_1+1}\langle V_1, F(t)\Xi V_2\rangle\,, \tag{4.132}$$

from the definition (4.124) and the BPZ properties (3.5), (3.50), and (4.16). The commutator of $F(t)$ and the derivation $\mathbb{X} = Q$, ∂_t, or δ on the dual string products is given by

$$\begin{aligned} [\mathbb{X}, F(t)]V_1 &= -F(t)[\mathbb{X}, F^{-1}(t)]F(t)V_1 \\ &= -F(t)(\mathbb{X}\Xi - (-1)^{\mathbb{X}}\Xi\mathbb{X})(D_\eta(t) - \eta)F(t)V_1 \\ &\quad - \kappa F(t)\Xi[\mathbb{X}G_\eta(t), F(t)V_1]^\eta_{G_\eta(t)}\,. \end{aligned} \tag{4.133}$$

We also summarize the properties of $F(t)\Psi$ for later use. Since $F(t)\eta = D_\eta(t)F(t)$ and $\eta\Psi = 0$, $F(t)\Psi$ is $D_\eta(t)$-exact:

$$F(t)\Psi = F(t)\{\eta, \Xi\}\Psi = D_\eta(t)F(t)\Xi\Psi. \tag{4.134}$$

Acting with $QF(t)$ on Ψ, (4.133) leads to

$$\begin{aligned} QF(t)\Psi &= F(t)\big(Q\Psi + X\eta F(t)\Psi\big) - \kappa F(t)\Xi[QG_\eta(t), F(t)\Psi]^\eta_{G_\eta(t)} \\ &= D_\eta(t)F\Xi(Q\Psi + X\eta F(t)\Psi) - \kappa F(t)\Xi[QG_\eta(t), F(t)\Psi]^\eta_{G_\eta(t)}. \end{aligned} \tag{4.135}$$

For $\mathbb{X} = \partial_t$, or δ, which commute with Ξ, $\mathbb{X}F(t)\Psi$ can be transformed into the following form:

$$\begin{aligned} \mathbb{X}F(t)\Psi &= F(t)\mathbb{X}\Psi + (-1)^{\mathbb{X}}\kappa F(t)\Xi D_\eta(t)[B_{\mathbb{X}}(t), F(t)\Psi]^\eta_{G_\eta(t)} && (4.136) \\ &= F(t)\mathbb{X}\Psi + (-1)^{\mathbb{X}}\kappa[B_{\mathbb{X}}(t), F(t)\Psi]^\eta_{G_\eta(t)} \\ &\quad - (-1)^{\mathbb{X}}\kappa D_\eta(t)F(t)\Xi[B_{\mathbb{X}}(t), F(t)\Psi]^\eta_{G_\eta(t)}, && (4.137) \end{aligned}$$

where we used $\mathbb{X}G_\eta(t) = (-1)^{\mathbb{X}}D_\eta(t)B_{\mathbb{X}}(t)$ and $\{D_\eta(t), F(t)\Xi\} = 1$.

Now let us consider the variation of $S^{(2)}$:

$$\delta S^{(2)} = -\langle\!\langle \delta\Psi, YQ\Psi\rangle\!\rangle + \frac{\kappa}{2}\int_0^1 dt\ \delta\langle B_t(t), [F(t)\Psi, F(t)\Psi]^\eta_{G_\eta(t)}\rangle. \tag{4.138}$$

From here to (4.143), we omit the t-dependence for notational brevity. The variation of the integrand of the interaction term,

$$\frac{\kappa}{2}\delta\big\langle B_t,[F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle=\frac{\kappa}{2}\big\langle\delta B_t,[F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle+\kappa\big\langle B_t,[\delta F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle \\ +\frac{\kappa^2}{2}\big\langle B_t,[\delta G_\eta,F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle, \tag{4.139}$$

can be calculated as follows. Since $[F\Psi,F\Psi]^{\eta}_{G_\eta}$ is D_η-exact, we can use (3.49) for the first term, and obtain

$$\begin{aligned}\frac{\kappa}{2}\big\langle\delta B_t,[F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle&=\frac{\kappa}{2}\big\langle\partial_t B_\delta+\kappa[B_\delta,B_t]^{\eta}_{G_\eta},[F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle\\&=\frac{\kappa}{2}\big\langle\partial_t B_\delta,[F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle+\frac{\kappa^2}{2}\big\langle B_\delta,[B_t,[F\Psi,F\Psi]^{\eta}_{G_\eta}]^{\eta}_{G_\eta}\big\rangle.\end{aligned} \tag{4.140}$$

For the second term, utilizing (4.137), we find

$$\begin{aligned}\kappa\big\langle B_t,[\delta F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle&=-\kappa\big\langle[B_t,F\Psi]^{\eta}_{G_\eta},F\delta\Psi+\kappa[B_\delta,F\Psi]^{\eta}_{G_\eta}-\kappa D_\eta F\Xi[B_\delta,F\Psi]^{\eta}_{G_\eta}\big\rangle\\&=-\kappa\big\langle[B_t,F\Psi]^{\eta}_{G_\eta},\kappa[B_\delta,F\Psi]^{\eta}_{G_\eta}+D_\eta F\Xi\left(\delta\Psi-\kappa[B_\delta,F\Psi]^{\eta}_{G_\eta}\right)\big\rangle\\&=-\kappa^2\big\langle[B_t,F\Psi]^{\eta}_{G_\eta},[B_\delta,F\Psi]^{\eta}_{G_\eta}\big\rangle-\big\langle\partial_t F\Psi,\delta\Psi-\kappa[B_\delta,F\Psi]^{\eta}_{G_\eta}\big\rangle\\&=-\kappa^2\big\langle B_\delta,[F\Psi,[B_t,F\Psi]^{\eta}_{G_\eta}]^{\eta}_{G_\eta}\big\rangle+\big\langle\delta\Psi,\partial_t F\Psi\big\rangle\\&\quad+\kappa\big\langle B_\delta,[\partial_t F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle.\end{aligned} \tag{4.141}$$

For the third term, utilizing the L_∞-relation of G_η-shifted dual string products, we obtain

$$\begin{aligned}\frac{\kappa^2}{2}\big\langle B_t,[\delta G_\eta,F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle&=\frac{\kappa^2}{2}\big\langle B_t,[D_\eta B_\delta,F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle\\&=\frac{\kappa^2}{2}\big\langle B_t,\Big(-D_\eta[B_\delta,F\Psi,F\Psi]^{\eta}_{G_\eta}+[B_\delta,[F\Psi,F\Psi]^{\eta}_{G_\eta}]^{\eta}_{G_\eta}\\&\qquad-2[F\Psi,[B_\delta,F\Psi]^{\eta}_{G_\eta}]^{\eta}_{G_\eta}\Big)\big\rangle\\&=\frac{\kappa^2}{2}\big\langle B_\delta,[\partial_t G_\eta,F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle-\frac{\kappa^2}{2}\big\langle B_\delta,[B_t,[F\Psi,F\Psi]^{\eta}_{G_\eta}]^{\eta}_{G_\eta}\big\rangle\\&\quad+\kappa^2\big\langle B_\delta,[F\Psi,[B_t,F\Psi]^{\eta}_{G_\eta}]^{\eta}_{G_\eta}\big\rangle\,.\end{aligned} \tag{4.142}$$

Then the total variation is given by

$$\begin{aligned}&\frac{\kappa}{2}\delta\big\langle B_t,[F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle\\&=\frac{\kappa}{2}\big\langle\partial_t B_\delta,[F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle+\frac{\kappa^2}{2}\big\langle B_\delta,[\partial_t G_\eta,F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle+\kappa\big\langle B_\delta,[\partial_t F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle+\big\langle\delta\Psi,\partial_t F\Psi\big\rangle\\&=\partial_t\Big(-\langle\!\langle\delta\Psi,YX\eta F\Psi\rangle\!\rangle+\frac{\kappa}{2}\big\langle B_\delta,[F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle\Big),\end{aligned} \tag{4.143}$$

where we assumed that $\delta\Psi$ also satisfies the constraints (4.1). Using $B_\delta(0)=0$ and $\eta F(0)\Psi=\eta\Psi=0$, we eventually find

$$\delta S^{(2)}=-\langle\!\langle\delta\Psi,Y(Q\Psi+X\eta F\Psi)\rangle\!\rangle+\frac{\kappa}{2}\big\langle B_\delta,[F\Psi,F\Psi]^{\eta}_{G_\eta}\big\rangle\,, \tag{4.144}$$

and hence

$$E^{(1)} = Q\Psi + X\eta F\Psi\,, \qquad E^{(2)} = \frac{\kappa}{2}[F\Psi, F\Psi]^{\eta}_{G_\eta}\,. \tag{4.145}$$

By requiring (4.122), let us determine $\delta\Psi^{(1)}$ and $B^{(2)}_{\delta}$ for each of gauge transformations with the parameters Λ, Ω and λ. Let us first consider the invariance under the transformation with the parameter Λ:

$$0 = -\langle\!\langle \delta_\Lambda\Psi^{(1)}, YE^{(1)}\rangle\!\rangle + \big\langle B^{(0)}_{\delta_\Lambda}, E^{(2)}\big\rangle + \big\langle B^{(2)}_{\delta_\Lambda}, E^{(0)}\big\rangle\,. \tag{4.146}$$

Here the second term is already known. Recalling (4.135) (at $t = 1$),

$$QF\Psi = D_\eta F\Xi E^{(1)} - \kappa F\Xi[E^{(0)}, F\Psi]^{\eta}_{G_\eta}, \tag{4.147}$$

it can be calculated as

$$\begin{aligned}
\big\langle B^{(0)}_{\delta_\Lambda}, E^{(2)}\big\rangle &= \frac{\kappa}{2}\langle Q\Lambda, [F\Psi, F\Psi]^{\eta}_{G_\eta}\rangle \\
&= \langle\Big(-\frac{\kappa^2}{2}[F\Psi, F\Psi, \Lambda]^{\eta}_{G_\eta} + \kappa^2[F\Psi, F\Xi[F\Psi, \Lambda]^{\eta}_{G_\eta}]^{\eta}_{G_\eta}\Big), E^{(0)}\rangle \\
&\quad - \kappa\langle F\Xi D_\eta[F\Psi, \Lambda]^{\eta}_{G_\eta}, E^{(1)}\rangle\,.
\end{aligned} \tag{4.148}$$

If we note that $E^{(1)}$ satisfies the constraints (4.1), (4.146) holds by taking

$$B^{(2)}_{\delta_\Lambda} = \frac{\kappa^2}{2}[F\Psi, F\Psi, \Lambda]^{\eta}_{G_\eta} - \kappa^2[F\Psi, F\Xi[F\Psi, \Lambda]^{\eta}_{G_\eta}]^{\eta}_{G_\eta}\,, \tag{4.149}$$

$$\delta_\Lambda\Psi^{(1)} = -\kappa X\eta F\Xi D_\eta[F\Psi, \Lambda]^{\eta}_{G_\eta}\,. \tag{4.150}$$

The invariance under the transformation with the parameter Ω requires

$$0 = -\langle\!\langle \delta_\Omega\Psi^{(1)}, YE^{(1)}\rangle\!\rangle + \big\langle B^{(0)}_{\delta_\Omega}, E^{(2)}\big\rangle + \big\langle B^{(2)}_{\delta_\Omega}, E^{(0)}\big\rangle\,. \tag{4.151}$$

Since the second term is again known and calculated as

$$\langle B^{(0)}_{\delta_\Omega}, E^{(2)}\rangle = \langle D_\eta\Omega, E^{(2)}\rangle = \langle \Omega, D_\eta E^{(2)}\rangle = 0\,, \tag{4.152}$$

we conclude that

$$B^{(2)}_{\delta_\Omega} = 0, \qquad \delta_\Omega\Psi^{(1)} = 0\,. \tag{4.153}$$

Finally, for the invariance under the transformation with λ:

$$\begin{aligned}
0 &= -\langle\!\langle \delta_\lambda\Psi^{(1)}, YE^{(1)}\rangle\!\rangle + \big\langle B^{(2)}_{\delta_\lambda}, E^{(0)}\big\rangle \\
&= -\langle\!\langle \delta_\lambda\Psi^{(1)}_0, YE^{(1)}\rangle\!\rangle - \langle\!\langle \tilde{\delta}_\lambda\Psi^{(1)}, YE^{(1)}\rangle\!\rangle + \big\langle B^{(2)}_{\delta_\lambda}, E^{(0)}\big\rangle\,,
\end{aligned} \tag{4.154}$$

where we decomposed $\delta_\lambda\Psi^{(1)}$ into the free part (4.30) and remaining: $\delta_\lambda\Psi^{(1)} = \delta_\lambda\Psi^{(1)}_0 + \tilde{\delta}_\lambda\Psi^{(1)}$. The known part in this case is the first term, which is calculated as

$$\begin{aligned}
-\langle\!\langle \delta_\lambda\Psi^{(1)}_0, YE^{(1)}\rangle\!\rangle &= -\langle\!\langle Q\lambda, Y(Q\Psi + X\eta F\Psi)\rangle\!\rangle = \big\langle Q\lambda, F\Psi\big\rangle \\
&= \kappa\langle [F\Psi, F\Xi\lambda]^{\eta}_{G_\eta}, E^{(0)}\rangle - \langle F\Xi D_\eta\lambda, E^{(1)}\rangle\,.
\end{aligned} \tag{4.155}$$

The invariance (4.154) holds if we take

$$B^{(2)}_{\delta_\lambda} = -\kappa[F\Psi, F\Xi\lambda]^{\eta}_{G_\eta}, \tag{4.156}$$

$$\tilde{\delta}_\lambda\Psi^{(1)} = -X\eta F\Xi D_\eta\lambda = X\eta F\lambda\,. \tag{4.157}$$

Thus, in total, the gauge transformation at this order becomes (4.127).

4.3.2 Quartic in fermion

So far we have determined the complete action at the quadratic order in fermion (4.123), which has the same form as that of the open superstring field theory, and thus is its straightforward extension. For the heterotic string field theory, however, this is not the end of story. At the next order in the fermion expansion, the gauge invariance further requires

$$0 = -\langle\langle\delta\Psi^{(1)}, YE^{(3)}\rangle\rangle - \langle\langle\delta\Psi^{(3)}, YE^{(1)}\rangle\rangle + \langle B_\delta^{(0)}, E^{(4)}\rangle + \langle B_\delta^{(2)}, E^{(2)}\rangle + \langle B_\delta^{(4)}, E^{(0)}\rangle\,, \tag{4.158}$$

in which, in particular, we find

$$\langle B_\delta^{(2)}, E^{(2)}\rangle \neq 0\,. \tag{4.159}$$

Thus it is necessary to add the action $S^{(4)}$ quartic in fermion, and determine $B_\delta^{(4)}$ and $\delta\Psi^{(3)}$ so that the equation (4.158) is satisfied.

Let us begin with considering the transformation with the parameter λ, which is the most efficient way to find out $S^{(4)}$ as shown in the following. From (4.145) and (4.157) we have

$$\begin{aligned}\langle B_{\delta_\lambda}^{(2)}, E^{(2)}\rangle &= \frac{\kappa^2}{6}\langle F\Xi\lambda, D_\eta[F\Psi, F\Psi, F\Psi]^\eta_{G_\eta}\rangle\\ &= \frac{\kappa^2}{6}\langle F\lambda, [F\Psi, F\Psi, F\Psi]^\eta_{G_\eta}\rangle\,.\end{aligned} \tag{4.160}$$

Here, from the Q-exactness of the dual string products (3.61), we can rewrite

$$\begin{aligned}[F\Psi, F\Psi, F\Psi]^\eta_{G_\eta} &= Q(F\Psi, F\Psi, F\Psi)^\eta_{G_\eta} - 3(F\Psi, F\Psi, QF\Psi)^\eta_{G_\eta}\\ &\quad - \kappa(QG_\eta, F\Psi, F\Psi, F\Psi)^\eta_{G_\eta}\,,\end{aligned} \tag{4.161}$$

and thus

$$\begin{aligned}\langle B_{\delta_\lambda}^{(2)}, E^{(2)}\rangle &= \frac{\kappa^2}{6}\langle F\lambda, Q(F\Psi, F\Psi, F\Psi)^\eta_{G_\eta}\rangle - \frac{\kappa^2}{2}\langle F\lambda, (F\Psi, F\Psi, QF\Psi)^\eta_{G_\eta}\rangle\\ &\quad - \frac{\kappa^3}{6}\langle F\lambda, (QG_\eta, F\Psi, F\Psi, F\Psi)^\eta_{G_\eta}\rangle\\ &= -\frac{\kappa^2}{6}\langle QF\lambda, (F\Psi, F\Psi, F\Psi)^\eta_{G_\eta}\rangle + \frac{\kappa^2}{2}\langle (F\Psi, F\Psi, F\lambda)^\eta_{G_\eta}, QF\Psi\rangle\\ &\quad + \frac{\kappa^3}{6}\langle (F\Psi, F\Psi, F\Psi, F\lambda)^\eta_{G_\eta}, QG_\eta\rangle\,.\end{aligned} \tag{4.162}$$

Using (4.133), the first and second terms can further be calculated as

$$\begin{aligned}-\frac{\kappa^2}{6}\langle QF\lambda, (F\Psi, F\Psi, F\Psi)^\eta_{G_\eta}\rangle &= \frac{\kappa^2}{6}\langle\langle (Q\lambda + X\eta F\lambda), YX\eta F\Xi D_\eta(F\Psi, F\Psi, F\Psi)^\eta_{G_\eta}\rangle\rangle\\ &\quad - \frac{\kappa^3}{6}\langle [F\Xi(F\Psi, F\Psi, F\Psi)^\eta_{G_\eta}, F\lambda]^\eta_{G_\eta}, QG_\eta\rangle\,,\end{aligned} \tag{4.163}$$

and

$$\begin{aligned}\frac{\kappa^2}{2}\langle (F\Psi, F\Psi, F\lambda)^\eta_{G_\eta}, QF\Psi\rangle &= \frac{\kappa^2}{2}\langle\langle X\eta F\Xi D_\eta(F\Psi, F\Psi, F\lambda)^\eta_{G_\eta}, Y(Q\Psi + X\eta F\Psi)\rangle\rangle\\ &\quad - \frac{\kappa^3}{2}\langle [F\Psi, F\Xi(F\Psi, F\Psi, F\lambda)^\eta_{G_\eta}]^\eta_{G_\eta}, QG_\eta\rangle\,,\end{aligned} \tag{4.164}$$

respectively, and we eventually have

$$\begin{aligned}\langle B^{(2)}_{\delta_\lambda}, E^{(2)}\rangle =& \frac{\kappa^2}{6}\langle\!\langle \delta_\lambda\Psi^{(1)}, YX\eta F\Xi D_\eta(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}\rangle\!\rangle\\ &+\frac{\kappa^2}{2}\langle\!\langle X\eta F\Xi D_\eta(F\Psi,F\Psi,F\lambda)^\eta_{G_\eta}, YE^{(1)}\rangle\!\rangle\\ &+\langle\Big(\frac{\kappa^3}{6}(F\Psi,F\Psi,F\Psi,F\lambda)^\eta_{G_\eta}-\frac{\kappa^3}{2}[F\Psi,F\Xi(F\Psi,F\Psi,F\lambda)^\eta_{G_\eta}]^\eta_{G_\eta}\\ &\qquad-\frac{\kappa^3}{6}[F\Xi(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta},F\lambda]^\eta_{G_\eta}\Big),E^{(0)}\rangle\,. \end{aligned}\tag{4.165}$$

Substituting this into (4.158), and taking into account $B^{(0)}_{\delta_\lambda}=0$, we obtain

$$E^{(3)} = \frac{\kappa^2}{6}X\eta F\Xi D_\eta(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}\,,\tag{4.166}$$

$$\begin{aligned}B^{(4)}_{\delta_\lambda} =& -\frac{\kappa^3}{6}(F\Psi,F\Psi,F\Psi,F\lambda)^\eta_{G_\eta}+\frac{\kappa^3}{6}[F\Xi(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta},F\lambda]^\eta_{G_\eta}\\ &+\frac{\kappa^3}{2}[F\Psi,F\Xi(F\Psi,F\Psi,F\lambda)^\eta_{G_\eta}]^\eta_{G_\eta}\,,\end{aligned}\tag{4.167}$$

$$\delta_\lambda\Psi^{(3)} = \frac{\kappa^2}{2}X\eta F\Xi D_\eta(F\Psi,F\Psi,F\lambda)^\eta_{G_\eta}\,.\tag{4.168}$$

From this form of $E^{(3)}$, the action $S^{(4)}$ has to satisfy

$$\begin{aligned}\delta S^{(4)} &= -\langle\!\langle\delta\Psi, YE^{(3)}\rangle\!\rangle\\ &= -\frac{\kappa^2}{6}\Big\langle\!\Big\langle\delta\Psi, YX\eta F\Xi D_\eta(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}\Big\rangle\!\Big\rangle\\ &= \frac{\kappa^2}{6}\Big\langle F\delta\Psi,(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}\Big\rangle\,,\end{aligned}\tag{4.169}$$

under an arbitrary variation of Ψ, where we used $\delta\Psi$ satisfies the constraint (4.1) and therefore $D_\eta F\Xi\delta\Psi = F\eta\Xi\delta\Psi = F\delta\Psi$. Since the shifted dual gauge products are cyclic, we can integrate it, and obtain

$$S^{(4)} = \frac{\kappa^2}{24}\Big\langle F\Psi,(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}\Big\rangle\,.\tag{4.170}$$

We further consider the gauge transformations in the NS sector. Under an arbitrary variation of the NS string field, we have

$$\begin{aligned}\delta S^{(4)} &= \frac{\kappa^3}{6}\Big\langle -F\Xi[\delta G_\eta,F\Psi]^\eta_{G_\eta},(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}\Big\rangle+\frac{\kappa^3}{24}\Big\langle F\Psi,(\delta G_\eta,F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}\Big\rangle\\ &= \frac{\kappa^3}{6}\Big\langle B_\delta, D_\eta[F\Psi,F\Xi(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}]^\eta_{G_\eta}\Big\rangle-\frac{\kappa^3}{24}\Big\langle B_\delta,D_\eta(F\Psi,F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}\Big\rangle,\end{aligned}\tag{4.171}$$

where we used (4.133), the cyclicity of the shifted dual string product, and $\delta G_\eta = D_\eta B_\delta$. Thus we obtain

$$E^{(4)} = -\frac{\kappa^3}{24}D_\eta(F\Psi,F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}+\frac{\kappa^3}{6}D_\eta[F\Psi,F\Xi(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}]^\eta_{G_\eta}\,.\tag{4.172}$$

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Let us consider the invariance under the parameter Ω first. The action is invariant if we can determine $B^{(4)}_{\delta_\Omega}$ and $\delta_\Omega\Psi^{(3)}$ so that they satisfy

$$0 = -\langle\!\langle \delta_\Omega\Psi^{(3)}, Y(Q\Psi + X\eta F\Psi)\rangle\!\rangle + \langle D_\eta\Omega, E^{(4)}\rangle + \langle B^{(4)}_{\delta_\Omega}, QG_\eta\rangle. \tag{4.173}$$

However, since the second term vanishes,

$$\langle D_\eta\Omega, E^{(4)}\rangle = \langle \Omega, D_\eta E^{(4)}\rangle = 0\,, \tag{4.174}$$

we can consistently take

$$B^{(4)}_{\delta_\Omega} = 0\,, \qquad \delta_\Omega\Psi^{(3)} = 0\,. \tag{4.175}$$

Finally, let us consider the gauge invariances under the transformation with Λ. We show that one can determine $\delta_\Lambda\Psi^{(3)}$ and $B^{(4)}_{\delta_\Lambda}$ so that the condition (4.120) at quartic order,

$$\begin{aligned}0 = &-\langle\!\langle \delta_\Lambda\Psi^{(1)}, YE^{(3)}\rangle\!\rangle + \langle B^{(0)}_{\delta_\Lambda}, E^{(4)}\rangle + \langle B^{(2)}_{\delta_\Lambda}, E^{(2)}\rangle\\ &-\langle\!\langle \delta_\Lambda\Psi^{(3)}, YE^{(1)}\rangle\!\rangle + \langle B^{(4)}_{\delta_\Lambda}, E^{(0)}\rangle\end{aligned} \tag{4.176}$$

holds, where the first three terms are already determined. What we have to show is that these terms vanishes up to terms containing $E^{(1)} = Q\Psi + X\eta F\Psi$ and $E^{(0)} = QG_\eta$, which can be compensated by appropriately determining $\delta_\Lambda\Psi^{(3)}$ and $B^{(4)}_{\delta_\lambda}$, respectively:

$$0 \cong \langle B^{(0)}_{\delta_\Lambda}, E^{(4)}\rangle + \langle B^{(2)}_{\delta_\Lambda}, E^{(2)}\rangle - \langle\!\langle \delta_\Lambda\Psi^{(1)}, YE^{(3)}\rangle\!\rangle\,, \tag{4.177}$$

where $A \cong B$ denotes that A equals to B except for terms containing $E^{(1)}$ and $E^{(0)}$. It is useful to note that

$$QF\Psi \cong 0\,, \tag{4.178}$$

$$Q[B_1,\ldots,B_n]^\eta_{G_\eta} \cong \sum_{k=1}^{n}(-1)^{1+B_1+\ldots+B_{k-1}}[B_1,\ldots,QB_k,\ldots,B_n]^\eta_{G_\eta}\,, \tag{4.179}$$

$$\{Q, D_\eta\} \cong 0\,, \tag{4.180}$$

$$Q(F\Psi,\ldots,F\Psi)^\eta_{G_\eta} \cong [F\Psi,\ldots,F\Psi]^\eta_{G_\eta}\,. \tag{4.181}$$

Utilizing them, we have

$$\begin{aligned}-\langle\!\langle \delta_\Lambda\Psi^{(1)}, YE^{(3)}\rangle\!\rangle &= \frac{\kappa^3}{6}\langle F\Xi D_\eta[F\Psi,\Lambda]^\eta_{G_\eta}, X\eta F\Xi D_\eta(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}\rangle\\ &= \frac{\kappa^3}{6}\langle [F\Psi,\Lambda]^\eta_{G_\eta}, D_\eta F\Xi QF\Xi D_\eta(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}\rangle\\ &\cong \frac{\kappa^3}{6}\langle [F\Psi,\Lambda]^\eta_{G_\eta}, D_\eta F\Xi[F\Psi,F\Psi,F\Psi]^\eta_{G_\eta}\rangle\\ &\quad - \frac{\kappa^3}{6}\langle [F\Psi,\Lambda]^\eta_{G_\eta}, QD_\eta F\Xi(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}\rangle\\ &= \frac{\kappa^3}{6}\langle \Lambda, [F\Psi,[F\Psi,F\Psi,F\Psi]^\eta_{G_\eta}]^\eta_{G_\eta}\rangle\\ &\quad - \frac{\kappa^3}{6}\langle F\Xi[F\Psi,\Lambda]^\eta_{G_\eta}, D_\eta[F\Psi,F\Psi,F\Psi]^\eta_{G_\eta}\rangle\\ &\quad - \frac{\kappa^3}{6}\langle \Lambda, [F\Psi, QD_\eta F\Xi(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}]^\eta_{G_\eta}\rangle\,,\end{aligned} \tag{4.182}$$

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where we used $D_\eta F\Xi Q D_\eta F\Xi = (1 - F\Xi D_\eta) Q D_\eta F\Xi \cong Q D_\eta F\Xi$, and $\Xi^2 = 0$. Similarly, one can show that the remaining two terms become

$$\begin{aligned}
\langle B^{(0)}_{\delta_\Lambda}, E^{(4)}\rangle \cong \frac{\kappa^3}{6}\big\langle\Lambda, [F\Psi, QD_\eta F\Xi(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}]^\eta_{G_\eta}\big\rangle \\
+\frac{\kappa^3}{24}\big\langle\Lambda, D_\eta[F\Psi,F\Psi,F\Psi,F\Psi]^\eta_{G_\eta}\big\rangle\,, \qquad (4.183)\\
\langle B^{(2)}_{\delta_\Lambda}, E^{(2)}\rangle \cong \frac{\kappa^3}{4}\big\langle\Lambda, [F\Psi,F\Psi,[F\Psi,F\Psi]^\eta_{G_\eta}]^\eta_{G_\eta}\big\rangle \\
-\frac{\kappa^3}{2}\big\langle F\Xi[F\Psi,\Lambda]^\eta_{G_\eta}, [F\Psi,[F\Psi,F\Psi]^\eta_{G_\eta}]^\eta_{G_\eta}\big\rangle\,. \qquad (4.184)
\end{aligned}$$

Then we find (4.177) holds by the L_∞-relations of G_η-shifted dual products:

$$\begin{aligned}
&\frac{\kappa^3}{24}\Big\langle\Lambda, \Big(D_\eta[F\Psi,F\Psi,F\Psi,F\Psi]^\eta_{G_\eta}+4[F\Psi,[F\Psi,F\Psi,F\Psi]^\eta_{G_\eta}]^\eta_{G_\eta}+6[F\Psi,F\Psi,[F\Psi,F\Psi]^\eta_{G_\eta}]^\eta_{G_\eta}\Big)\Big\rangle\\
&\quad -\frac{\kappa^3}{6}\Big\langle F\Xi[F\Psi,\Lambda]^\eta_{G_\eta}, \Big(D_\eta[F\Psi,F\Psi,F\Psi]^\eta_{G_\eta}+3[F\Psi,[F\Psi,F\Psi]^\eta_{G_\eta}]^\eta_{G_\eta}\Big)\Big\rangle = 0\,. \qquad (4.185)
\end{aligned}$$

By picking up the terms with $E^{(1)}$ and $E^{(0)}$, the transformations $\delta_\Lambda\Psi^{(3)}$ and $B^{(4)}_{\delta_\Lambda}$ can be explicitly determined as

$$\begin{aligned}
B^{(4)}_{\delta_\Lambda} =& -\frac{\kappa^4}{24}[(F\Psi,F\Psi,F\Psi,F\Psi)^\eta_{G_\eta},\Lambda]^\eta_{G_\eta} + \frac{\kappa^4}{6}[[F\Psi,F\Xi(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}]^\eta_{G_\eta},\Lambda]^\eta_{G_\eta}\\
&+\frac{\kappa^4}{24}(F\Psi,F\Psi,F\Psi,F\Psi,D_\eta\Lambda)^\eta_{G_\eta} - \frac{\kappa^4}{6}(F\Psi,F\Psi,F\Psi,F\Xi[F\Psi,D_\eta\Lambda]^\eta_{G_\eta})^\eta_{G_\eta}\\
&-\frac{\kappa^4}{6}[F\Psi,F\Xi(F\Psi,F\Psi,F\Psi,D_\eta\Lambda)^\eta_{G_\eta}]^\eta_{G_\eta} - \frac{\kappa^4}{6}[F\Xi(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta},F\Psi,D_\eta\Lambda]^\eta_{G_\eta}\\
&+\frac{\kappa^4}{2}[F\Psi,F\Xi(F\Psi,F\Psi,F\Xi[F\Psi,D_\eta\Lambda]^\eta_{G_\eta})^\eta_{G_\eta}]^\eta_{G_\eta}\\
&+\frac{\kappa^4}{6}[F\Psi,F\Xi[F\Xi(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta},D_\eta\Lambda]^\eta_{G_\eta}]^\eta_{G_\eta}\\
&+\frac{\kappa^4}{6}[F\Xi(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta},F\Xi[F\Psi,D_\eta\Lambda]^\eta_{G_\eta}]^\eta_{G_\eta}, \qquad (4.186)\\
\delta_\Lambda\Psi^{(3)} =& -\frac{\kappa^3}{6}X\eta F\Xi D_\eta(F\Psi,F\Psi,F\Psi,D_\eta\Lambda)^\eta_{G_\eta} + \frac{\kappa^3}{2}X\eta F\Xi D_\eta(F\Xi[F\Psi,D_\eta\Lambda]^\eta_{G_\eta},F\Psi,F\Psi)^\eta_{G_\eta}\\
&+\frac{\kappa^3}{6}X\eta F\Xi D_\eta[F\Xi(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta},D_\eta\Lambda]^\eta_{G_\eta}\,. \qquad (4.187)
\end{aligned}$$

5 Summary and discussion

Using the expansion in the number of the Ramond string field, we have constructed in this paper a gauge invariant action of heterotic string field theory at the quadratic and quartic order:

$$\begin{aligned}
S = -\frac{1}{2}\langle\!\langle\Psi, YQ\Psi\rangle\!\rangle + \int_0^1 dt\big\langle B_t(t), QG_\eta(t) + \frac{\kappa}{2}[F(t)\Psi,F(t)\Psi]^\eta_{G_\eta(t)}\big\rangle\\
+\frac{\kappa^2}{24}\langle F\Psi,(F\Psi,F\Psi,F\Psi)^\eta_{G_\eta}\rangle + O(\Psi^6). \qquad (5.1)
\end{aligned}$$

This is invariant under the gauge transformations with the parameter Λ,

$$
\begin{aligned}
B_{\delta_\Lambda} =\ & Q\Lambda + \frac{\kappa^2}{2}[F\Psi, F\Psi, \Lambda]^{\eta}_{G_\eta} - \kappa^2[F\Psi, F\Xi[F\Psi, \Lambda]^{\eta}_{G_\eta}]^{\eta}_{G_\eta} \\
& - \frac{\kappa^4}{24}[(F\Psi, F\Psi, F\Psi, F\Psi)^{\eta}_{G_\eta}, \Lambda]^{\eta}_{G_\eta} + \frac{\kappa^4}{6}[[F\Psi, F\Xi(F\Psi, F\Psi, F\Psi)^{\eta}_{G_\eta}]^{\eta}_{G_\eta}, \Lambda]^{\eta}_{G_\eta} \\
& + \frac{\kappa^4}{24}(F\Psi, F\Psi, F\Psi, F\Psi, D_\eta\Lambda)^{\eta}_{G_\eta} - \frac{\kappa^4}{6}(F\Psi, F\Psi, F\Psi, F\Xi[F\Psi, D_\eta\Lambda]^{\eta}_{G_\eta})^{\eta}_{G_\eta} \\
& - \frac{\kappa^4}{6}[F\Psi, F\Xi(F\Psi, F\Psi, F\Psi, D_\eta\Lambda)^{\eta}_{G_\eta}]^{\eta}_{G_\eta} - \frac{\kappa^4}{6}[F\Xi(F\Psi, F\Psi, F\Psi)^{\eta}_{G_\eta}, F\Psi, D_\eta\Lambda]^{\eta}_{G_\eta} \\
& + \frac{\kappa^4}{2}[F\Psi, F\Xi(F\Psi, F\Psi, F\Xi[F\Psi, D_\eta\Lambda]^{\eta}_{G_\eta})^{\eta}_{G_\eta}]^{\eta}_{G_\eta} \\
& + \frac{\kappa^4}{6}[F\Psi, F\Xi[F\Xi(F\Psi, F\Psi, F\Psi)^{\eta}_{G_\eta}, D_\eta\Lambda]^{\eta}_{G_\eta}]^{\eta}_{G_\eta} \\
& + \frac{\kappa^4}{6}[F\Xi(F\Psi, F\Psi, F\Psi)^{\eta}_{G_\eta}, F\Xi[F\Psi, D_\eta\Lambda]^{\eta}_{G_\eta}]^{\eta}_{G_\eta} + O(\Psi^6),
\end{aligned} \tag{5.2}
$$

$$
\begin{aligned}
\delta_\Lambda\Psi =\ & -\kappa X\eta F\Xi D_\eta[F\Psi, \Lambda]^{\eta}_{G_\eta} \\
& - \frac{\kappa^3}{6}X\eta F\Xi D_\eta(F\Psi, F\Psi, F\Psi, D_\eta\Lambda)^{\eta}_{G_\eta} + \frac{\kappa^3}{2}X\eta F\Xi D_\eta(F\Psi, F\Psi, F\Xi[F\Psi, D_\eta\Lambda]^{\eta}_{G_\eta})^{\eta}_{G_\eta} \\
& + \frac{\kappa^3}{6}X\eta F\Xi D_\eta[F\Xi(F\Psi, F\Psi, F\Psi)^{\eta}_{G_\eta}, D_\eta\Lambda]^{\eta}_{G_\eta} + O(\Psi^5),
\end{aligned} \tag{5.3}
$$

with the parameter Ω,

$$
B_{\delta_\Omega} = D_\eta\Omega + O(\Psi^6), \tag{5.4}
$$

$$
\delta_\Omega\Psi = O(\Psi^5), \tag{5.5}
$$

and with the parameter λ,

$$
\begin{aligned}
B_{\delta_\lambda} =\ & -\kappa[F\Psi, F\Xi\lambda]^{\eta}_{G_\eta} - \frac{\kappa^3}{6}(F\Psi, F\Psi, F\Psi, F\lambda)^{\eta}_{G_\eta} + \frac{\kappa^3}{6}[F\Xi(F\Psi, F\Psi, F\Psi)^{\eta}_{G_\eta}, F\lambda]^{\eta}_{G_\eta} \\
& + \frac{\kappa^3}{2}[F\Psi, F\Xi(F\Psi, F\Psi, F\lambda)^{\eta}_{G_\eta}]^{\eta}_{G_\eta} + O(\Psi^6),
\end{aligned} \tag{5.6}
$$

$$
\delta_\lambda\Psi = Q\lambda + X\eta F\lambda + \frac{\kappa^2}{2}X\eta F\Xi D_\eta(F\Psi, F\Psi, F\lambda)^{\eta}_{G_\eta} + O(\Psi^5), \tag{5.7}
$$

except for the higher order in the Ramond string field. The equations of motion derived from this action are

$$
\begin{aligned}
E^{NS} =\ & QG_\eta + \frac{\kappa}{2}[F\Psi, F\Psi]^{\eta}_{G_\eta} \\
& - \frac{\kappa^3}{24}D_\eta\Big((F\Psi, F\Psi, F\Psi, F\Psi)^{\eta}_{G_\eta} - 4[F\Psi, F\Xi(F\Psi, F\Psi, F\Psi)^{\eta}_{G_\eta}]^{\eta}_{G_\eta}\Big) + O(\Psi^6),
\end{aligned} \tag{5.8}
$$

$$
E^{R} = Q\Psi + X\eta F\Psi + \frac{\kappa^2}{6}X\eta F\Xi D_\eta(F\Psi, F\Psi, F\Psi)^{\eta}_{G_\eta} + O(\Psi^5). \tag{5.9}
$$

Note that all of these results include all order terms in the coupling constant κ at each order in the Ramond string field. We can also confirm that the action (5.1) reproduces the four-point amplitudes with external fermions as given in the appendix B.

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The most important remaining task is to give a complete action and gauge transformation. We finally discuss two observations which may provide clues to achieve it. The first observation is a relation between the equations of motion and the gauge transformations. At the beginning, it is natural to assume that the NS string field V appears in the higher order action only in the form of G_η, since the corresponding ansatz is true for the case of the equations of motion in the dual formulation [14, 15]. If we assume this ansatz the gauge transformation with the parameter Ω does not subject to change any more. One can find that the gauge transformations $\delta_\Lambda\Psi$, $D_\eta B_{\delta_\lambda}$ and $\delta_\lambda\Psi$ are obtained by replacing fields in the equations of motion with gauge parameters:[7]

$$\Big((D_\eta\Lambda)\frac{\delta}{\delta G_\eta}\Big)E^{(2k)} = D_\eta B^{(2k)}_{\delta_\Lambda} + \kappa[E^{(2k)},\Lambda]^\eta_{G_\eta}\,, \qquad \text{for} \quad k=0,1,2\,, \tag{5.10}$$

$$\Big((D_\eta\Lambda)\frac{\delta}{\delta G_\eta}\Big)E^{(2k+1)} = \delta_\Lambda\Psi^{(2k+1)}, \qquad \text{for} \quad k=0,1\,, \tag{5.11}$$

$$-\Big(\lambda\frac{\delta}{\delta\Psi}\Big)E^{(2k)} = D_\eta B^{(2k)}_{\delta_\lambda}, \qquad \text{for} \quad k=1,2\,, \tag{5.12}$$

$$-\Big(\lambda\frac{\delta}{\delta\Psi}\Big)E^{(2k+1)} = \delta_\lambda\Psi^{(2k+1)}, \qquad \text{for} \quad k=0,1\,. \tag{5.13}$$

These relations might be an appearance of an L_∞-structure, or equivalently a Batalin-Vilkovisky structure of the action: in formulations based on the L_∞-products, the gauge transformation is given by a functional differentiation of the equation of motion. To elucidate the role of theses relations in detail remains as future work which may provide a hint to complete an action to all orders.

The second observation is the expression of the equations of motion obtained as a dual form of the first-order equations of motion obtained in [15]:

$$(\eta+Q)\widetilde{B} + \sum_{m=2}^{\infty}\frac{1}{m!}[\widetilde{B}^m]^\eta = 0, \tag{5.14}$$

where $\widetilde{B} = \sum_{n=0}^{\infty}\widetilde{B}_{(n-2)/2}$. Expanding this in the picture number, the first two equations with the picture number $P=-2$ and $-3/2$,

$$\eta\widetilde{B}_{-1} + \sum_{m=2}^{\infty}\frac{\kappa^{m-1}}{m!}[\widetilde{B}^m_{-1}]^\eta = 0\,, \tag{5.15}$$

$$D_\eta\widetilde{B}_{-1/2} = 0\,, \tag{5.16}$$

can be solved as

$$\widetilde{B}_{-1} = G_\eta\,, \qquad \widetilde{B}_{-1/2} = F\Psi\,. \tag{5.17}$$

The next two with $p=-1$ and $-1/2$,

$$QG_\eta + \frac{\kappa}{2}[\widetilde{B}_{-1/2},\widetilde{B}_{-1/2}]^\eta_{G_\eta} + D_\eta\widetilde{B}_0 = 0\,, \tag{5.18}$$

$$Q\widetilde{B}_{-1/2} + \kappa[\widetilde{B}_0,\widetilde{B}_{-1/2}]^\eta_{G_\eta} + \frac{\kappa^2}{3!}[\widetilde{B}_{-1/2},\widetilde{B}_{-1/2},\widetilde{B}_{-1/2}]^\eta_{G_\eta} + D_\eta\widetilde{B}_{1/2} = 0\,, \tag{5.19}$$

[7]One can see that similar relations hold exactly for the open superstring field theory.

can be interpreted as equations of motion with the infinitely many subsidiary equations determining the infinitely many "auxiliary fields", $\widetilde{B}_{n/2}$ $(n \geq 0)$. In the original formulation in [15], we can iteratively solve these subsidiary conditions in the fermion expansion, and obtain the equations of motion. We similarly assume here that the terms in the auxiliary fields with the lowest order in fermion are $\widetilde{B}_{n/2} = O(\Psi^{n+4})$. Then the subsidiary equations simply become

$$Q\widetilde{B}_{n/2} + \frac{\kappa^{n+3}}{(n+4)!}[\underbrace{F\Psi, F\Psi, \cdots, F\Psi}_{n+4}]^{\eta}_{G_\eta} = 0\,, \tag{5.20}$$

which can be solved as

$$\widetilde{B}^{(n+4)}_{n/2} = -\frac{\kappa^{n+3}}{(n+4)!}(\underbrace{F\Psi, F\Psi, \cdots, F\Psi}_{n+4})^{\eta}_{G_\eta}\,, \tag{5.21}$$

except for the terms proportional to the lowest order equations of motion

$$QG_\eta + \frac{\kappa}{2}[F\Psi, F\Psi]^{\eta}_{G_\eta} = 0\,, \tag{5.22}$$

$$QF\Psi = 0\,, \tag{5.23}$$

obtained from (5.18) and (5.19), respectively. Unfortunately, however, the next order equations of motion obtained by substituting (5.21) into (5.18) and (5.19) are not equivalent to our equations of motion (5.9). Although this difference can be filled by assuming that $\widetilde{B}_{-1/2}$ contains the terms with the higher order in fermion as

$$\widetilde{B}_{-1/2} = F\Psi - \frac{\kappa^2}{3!}D_\eta F\Xi(F\Psi, F\Psi, F\Psi)^{\eta}_{G_\eta} + O(\Psi^5), \tag{5.24}$$

we cannot determine it without further assumption. Although this can be absorbed in the redefinition of the Ramond string field Ψ, we have to find a way to reproduce the higher order terms in the equations of motion derived from the action, which may provides a clue for constructing a complete gauge invariant action.

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A Construction of the dual gauge product

In order to make this paper self-contained, we give in this appendix a construction of the dual string products. We follow the convention and notation of [22].

We first introduce the coalgebraic expression of string products, which is convenient to focus on their algebraic properties [7]. The product of n closed strings is described by a multilinear map $d_n : \mathcal{H}^{\wedge n} \to \mathcal{H}$, where $\wedge$ is the *symmetrized tensor product* satisfying $\Phi_1 \wedge \Phi_2 = (-1)^{\Phi_1\Phi_2}\Phi_2 \wedge \Phi_1$. This naturally induces a map from the *symmetrized tensor* *algebra* $\mathcal{S}(\mathcal{H}) = \mathcal{H}^{\wedge 0} \oplus \mathcal{H}^{\wedge 1} \oplus \mathcal{H}^{\wedge 2} \oplus \cdots$ to $\mathcal{S}(\mathcal{H})$ itself called a *coderivation*. A coderivation $\mathbf{d}_n : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ is naturally derived from a map $d_n : \mathcal{H}^{\wedge n} \to \mathcal{H}$ as

$$\begin{aligned}\mathbf{d}_n(\Phi_1 \wedge \cdots \wedge \Phi_N) &= (d_n \wedge \mathbb{I}_{N-n})(\Phi_1 \wedge \cdots \wedge \Phi_N) \\ &= \sum_\sigma \frac{(-1)^\sigma}{n!(N-n)!} d_n(\Phi_{\sigma(1)}, \cdots, \Phi_{\sigma(n)}) \wedge \Phi_{\sigma(n+1)} \wedge \cdots \wedge \Phi_{\sigma(N)}\,, \end{aligned} \tag{A.1}$$

for $\Phi_1 \wedge \cdots \wedge \Phi_N \in \mathcal{H}^{\wedge N \geq n} \subset \mathcal{S}(\mathcal{H})$, and it vanishes when acting on $\mathcal{H}^{\wedge N<n}$. The graded commutator of two coderivations $\mathbf{b}_n$ and $\mathbf{c}_m$, $[\![\mathbf{b}_n, \mathbf{c}_m]\!]$, is a coderivation derived from the map $[\![b_n, c_m]\!] : \mathcal{H}^{\wedge n+m-1} \to \mathcal{H}$ which is defined by

$$[\![b_n, c_m]\!] = b_n(c_m \wedge \mathbb{I}_{n-1}) - (-1)^{\deg(b_n)\deg(c_m)} c_m(b_n \wedge \mathbb{I}_{m-1})\,. \tag{A.2}$$

Then the L_∞-relation can be written as

$$[\![\mathbf{L}, \mathbf{L}]\!] = 0\,, \tag{A.3}$$

where $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3 + \cdots$ and $\mathbf{L}_k$ is a coderivation derived from the k-string product.

We introduce another map on the symmetrized tensor algebra, which is called a *cohomomorphism*. From a set of multilinear maps $\{\mathsf{f}_n : \mathcal{H}^{\wedge n} \to \mathcal{H}'\}_{n=0}^\infty$, one can naturally define a cohomomorphism $\widehat{\mathsf{f}} : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H}')$, which acts on $\Phi_1 \wedge \cdots \wedge \Phi_n \in \mathcal{H}^{\wedge n} \subset \mathcal{S}(\mathcal{H})$ as

$$\begin{aligned}\widehat{\mathsf{f}}(\Phi_1 \wedge \cdots \wedge \Phi_n) = \sum_{i \leq n} \sum_{k_1 < \cdots < k_i = n} & e^{\wedge \mathsf{f}_0} \wedge \mathsf{f}_{k_1}(\Phi_1, \ldots, \Phi_{k_1}) \wedge \mathsf{f}_{k_2-k_1}(\Phi_{k_1+1}, \ldots, \Phi_{k_2}) \wedge \\ & \cdots \wedge \mathsf{f}_{k_i - k_{i-1}}(\Phi_{k_{i-1}+1}, \ldots, \Phi_n)\,. \end{aligned} \tag{A.4}$$

We also introduce a projector π_1 from the symmetrized tensor algebra to the single-state space, $\mathcal{S}(\mathcal{H}) \to \mathcal{H}$, as

$$\pi_1\big(\Phi_0 + \Phi_1 \wedge \Phi_2 + \Phi_3 \wedge \Phi_4 \wedge \Phi_5 + \ldots\big) = \Phi_0\,. \tag{A.5}$$

The NS string products for heterotic string field theory, $\mathbf{L}^{\mathrm{NS}}(\tau) = \sum_{p=0}^\infty \tau^p \mathbf{L}_{p+1}$, had been constructed in [7]. They satisfy the L_∞-relation $[\![\mathbf{L}^{\mathrm{NS}}(\tau), \mathbf{L}^{\mathrm{NS}}(\tau)]\!] = 0$, cyclicity, and (graded) commutativity with η: $[\![\eta, \mathbf{L}^{\mathrm{NS}}(\tau)]\!] = 0$. The $(p+1)$-product $\mathbf{L}^{\mathrm{NS}}_{p+1}$ carries the ghost number $1-2p$ and the picture number p. The whole string product $\mathbf{L}^{\mathrm{NS}}(\tau)$ is given by a similarity transformation of the BRST operator $\mathbf{Q}$ as

$$\mathbf{L}^{\mathrm{NS}}(\tau) = \widehat{\mathbf{G}}^{-1}(\tau)\mathbf{Q}\widehat{\mathbf{G}}(\tau)\,, \tag{A.6}$$

where $\widehat{\mathbf{G}}(\tau)$ is an invertible cohomomorphism given by the path-ordered exponential map:

$$\widehat{\mathbf{G}}(\tau) = \overleftarrow{\mathcal{P}} \exp\left(\int_0^\tau d\tau' \boldsymbol{\lambda}^{[0]}(\tau')\right)\,, \qquad \widehat{\mathbf{G}}^{-1}(\tau) = \overrightarrow{\mathcal{P}} \exp\left(-\int_0^\tau d\tau' \boldsymbol{\lambda}^{[0]}(\tau')\right)\,, \tag{A.7}$$

where $\boldsymbol{\lambda}^{[0]}(\tau) = \sum_{p=0}^\infty \tau^p \boldsymbol{\lambda}^{[0]}_{p+2}$, called gauge products, can be determined iteratively. The arrow $\leftarrow$ ($\rightarrow$) on $\mathcal{P}$ denotes that the operator at later time acts from the right (left). The

$(p+2)$-gauge product $\boldsymbol{\lambda}^{[0]}_{p+2}$ carries ghost number $-2(p+1)$ and picture number $p+1$. The cohomomorphisms $\widehat{\mathbf{G}}(\tau)$ and $\widehat{\mathbf{G}}^{-1}(\tau)$ satisfy

$$\partial_\tau \widehat{\mathbf{G}}(\tau) = \widehat{\mathbf{G}}(\tau)\boldsymbol{\lambda}^{[0]}(\tau)\,, \qquad \partial_\tau \widehat{\mathbf{G}}^{-1}(\tau) = -\boldsymbol{\lambda}^{[0]}(\tau)\widehat{\mathbf{G}}^{-1}(\tau)\,. \tag{A.8}$$

The L_∞-relations are followed from the nilpotency of $\mathbf{Q}$ as

$$\begin{aligned}[\![\mathbf{L}^{\rm NS}(\tau), \mathbf{L}^{\rm NS}(\tau)]\!] &= 2\big(\mathbf{L}^{\rm NS}(\tau)\big)^2 \\ &= 2\widehat{\mathbf{G}}^{-1}(\tau)\mathbf{Q}^2\widehat{\mathbf{G}}(\tau) = 0\,. \end{aligned}\tag{A.9}$$

The cyclicity and commutativity $[\![\boldsymbol{\eta}, \mathbf{L}^{\rm NS}(\tau)]\!] = 0$, is realized by a suitable choice of an initial gauge product $\lambda^{[0]}$, whose explicit example is given in [7].

Under these preparations, we summarize the construction of the dual string products given in [22]. We introduce a coderivation $\mathbf{L}^\eta(\tau) = \sum_{p=0}^\infty \tau^p \mathbf{L}^\eta_{p+1}$ which provides a set of the dual string products by

$$[V_1, V_2, \ldots, V_n]^\eta = \pi_1 \mathbf{L}^\eta_n (V_1 \wedge V_2 \wedge \ldots \wedge V_n)\,. \tag{A.10}$$

This n-th dual product $\mathbf{L}^\eta_n$ carries ghost number $3-2n$ and picture number $n-2$ as expected. The whole coderivation $\mathbf{L}^\eta(\tau)$ is degree odd, and can be constructed using the cohomomorphism $\widehat{\mathbf{G}}(\tau)$ appearing in the NS product $\mathbf{L}^{\rm NS}(\tau) = \widehat{\mathbf{G}}^{-1}(\tau)\mathbf{Q}\widehat{\mathbf{G}}(\tau)$ by

$$\mathbf{L}^\eta(\tau) = \widehat{\mathbf{G}}(\tau)\boldsymbol{\eta}\widehat{\mathbf{G}}^{-1}(\tau)\,. \tag{A.11}$$

By construction, they satisfy the L_∞-relation

$$[\![\mathbf{L}^\eta(\tau), \mathbf{L}^\eta(\tau)]\!] = 0\,. \tag{A.12}$$

The anti-commutativity $[\![\mathbf{Q}, \mathbf{L}^\eta(\tau)]\!] = 0$ follows from $[\![\boldsymbol{\eta}, \mathbf{L}^{\rm NS}(\tau)]\!] = 0$ as

$$[\![\mathbf{Q}, \mathbf{L}^\eta(\tau)]\!] = [\![\mathbf{Q}, \widehat{\mathbf{G}}(\tau)\boldsymbol{\eta}\widehat{\mathbf{G}}^{-1}(\tau)]\!] = \widehat{\mathbf{G}}(\tau)[\![\mathbf{L}^{\rm NS}(\tau), \boldsymbol{\eta}]\!]\widehat{\mathbf{G}}^{-1}(\tau) = 0\,. \tag{A.13}$$

The cyclicity of $L^\eta(\tau)$ again follows from that of the gauge product $\boldsymbol{\lambda}^{[0]}$. We can give an explicit expression of the dual string product using the bosonic string product if necessary, for example,

$$[V_1, V_2]^\eta = -[V_1, V_2], \tag{A.14}$$

$$\begin{aligned}
[V_1, V_2, V_3]^\eta = -\frac{1}{4}\bigg(& X_0[V_1, V_2, V_3] + [X_0V_1, V_2, V_3] + [V_1, X_0V_2, V_3] + [V_1, V_2, X_0V_3] \\
& + (-1)^{V_1}\xi_0[V_1, [V_2, V_3]] - (-1)^{V_1}[\xi_0V_1, [V_2, V_3]] \\
& + [V_1, [\xi_0V_2, V_3]] + (-1)^{V_2}[V_1, [V_2, \xi_0V_3]] \\
& + (-1)^{V_1(V_2+V_3)}\Big((-1)^{V_2}\xi_0[V_2, [V_3, V_1]] - (-1)^{V_2}[\xi_0V_2, [V_3, V_1]] \\
& \qquad + [V_2, [\xi_0V_3, V_1]] + (-1)^{V_3}[V_2, [V_3, \xi_0V_1]]\Big) \\
& + (-1)^{V_3(V_1+V_2))}\Big((-1)^{V_3}\xi_0[V_3, [V_1, V_2]] - (-1)^{V_3}[\xi_0V_3, [V_1, V_2]] \\
& \qquad + [V_3, [\xi_0V_1, V_2]] + (-1)^{V_1}[V_3, [V_1, \xi_0V_2]]\Big)\bigg),
\end{aligned}\tag{A.15}$$

where $X_0 = \{Q, \xi_0\}$. These dual string products are defined on the basis of $\mathbf{L}^{NS}(\tau)$ in the NS sector. However, we can extend it to include the Ramond sector simply by considering V_i is either the NS string field or the Ramond string field, which preserves the necessary properties, the L_∞-relation, cyclicity and the commutativity with $\mathbf{Q}$, to construct the gauge invariant action. It should be emphasized here that it is not necessary to introduce any special picture changing operator only for the Ramond sector to define the dual string products.

The dual string product $\mathbf{L}^\eta$ is commutative with $\mathbf{Q}$, and its second derivative with respect to τ can be written as the commutator of $\mathbf{Q}$ and a product $\boldsymbol{\rho}$:

$$\partial_\tau^2 \mathbf{L}^\eta(\tau) = [\![\mathbf{Q}, \boldsymbol{\rho}(\tau)]\!] = \sum_{n=0}^{\infty} \tau^n [\![\mathbf{Q}, \boldsymbol{\rho}_{n+3}]\!] . \tag{A.16}$$

The dual gauge products can be read from $\boldsymbol{\rho}$ as

$$(V_1, V_2, \ldots, V_n)^\eta = \frac{1}{(n+1)(n+2)} \pi_1 \boldsymbol{\rho}_n (V_1 \wedge V_2 \wedge \ldots \wedge V_n) , \qquad (n \geq 3) . \tag{A.17}$$

In order to obtain an explicit expression of $\boldsymbol{\rho}$, we introduce a coderivation $\mathbf{L}^{[1]}(\tau) = \sum_{n=0}^{\infty} \tau^n \mathbf{L}^{[1]}_{n+2}$ which is an intermediate products with deficit picture 1 given in [7]. It is related to the gauge products $\boldsymbol{\lambda}^{[0]}(\tau)$ as

$$[\![\boldsymbol{\eta}, \boldsymbol{\lambda}^{[0]}(\tau)]\!] = \mathbf{L}^{[1]}(\tau) , \tag{A.18}$$

and satisfies $\mathbf{L}^{[1]}(\tau = 0) = \mathbf{L}^{\mathrm{B}}_2$, where $\mathbf{L}^{\mathrm{B}}_2$ is a coderivation derived from the simple two-string product for closed string without any insertion of superconformal ghost. We also introduce a coderivation $\boldsymbol{\lambda}^{[1]}(\tau) = \sum_{n=0}^{\infty} \tau^n \boldsymbol{\lambda}^{[1]}_{n+3}$ derived from a set of intermediate gauge products with deficit picture 1 [7]. It satisfies the relation,

$$\partial_\tau \mathbf{L}^{[1]}(\tau) = [\![\mathbf{L}^{[1]}(\tau), \boldsymbol{\lambda}^{[0]}(\tau)]\!] + [\![\mathbf{L}^{[0]}(\tau), \boldsymbol{\lambda}^{[1]}(\tau)]\!] , \tag{A.19}$$

with $\mathbf{L}^{[0]} = \mathbf{L}^{NS}$. Then, utilizing these products and their path-ordered exponential maps, we can rewrite $\mathbf{L}^\eta$ as

$$\begin{aligned} \mathbf{L}^\eta(\tau) &= \widehat{\mathbf{G}}(\tau) \boldsymbol{\eta} \widehat{\mathbf{G}}^{-1}(\tau) \\ &= \boldsymbol{\eta} + \widehat{\mathbf{G}}(\tau) [\![\boldsymbol{\eta}, \widehat{\mathbf{G}}^{-1}(\tau)]\!] \\ &= \boldsymbol{\eta} - \int_0^\tau d\tau' \widehat{\mathbf{G}}(\tau') \mathbf{L}^{[1]}(\tau') \widehat{\mathbf{G}}^{-1}(\tau') . \end{aligned} \tag{A.20}$$

The integrand in the second term becomes

$$\begin{aligned} \widehat{\mathbf{G}}(\tau') \mathbf{L}^{[1]}(\tau') \widehat{\mathbf{G}}^{-1}(\tau') &= \widehat{\mathbf{G}}(0) \mathbf{L}^{[1]}(0) \widehat{\mathbf{G}}^{-1}(0) + \int_0^{\tau'} d\tau'' \, \partial_{\tau''} \Big(\widehat{\mathbf{G}}(\tau'') \mathbf{L}^{[1]}(\tau'') \widehat{\mathbf{G}}^{-1}(\tau'') \Big) \\ &= \mathbf{L}^{\mathrm{B}}_2 + \int_0^{\tau'} d\tau'' [\![\mathbf{Q}, \widehat{\mathbf{G}}(\tau'') \boldsymbol{\lambda}^{[1]}(\tau'') \widehat{\mathbf{G}}^{-1}(\tau'')]\!] , \end{aligned} \tag{A.21}$$

where we used $\widehat{\mathbf{G}}(0)=\mathbb{I}$, $\mathbf{L}^{[1]}(0)=\mathbf{L}_2^{\mathrm{B}}$, and

$$\begin{aligned}\partial_{\tau''}\Big(\widehat{\mathbf{G}}(\tau'')\mathbf{L}^{[1]}(\tau'')\widehat{\mathbf{G}}^{-1}(\tau'')\Big) &= \widehat{\mathbf{G}}(\tau'')[\![\mathbf{L}^{[0]}(\tau''),\boldsymbol{\lambda}^{[1]}(\tau'')]\!]\widehat{\mathbf{G}}^{-1}(\tau'')\\ &= [\![\mathbf{Q},\widehat{\mathbf{G}}(\tau'')\boldsymbol{\lambda}^{[1]}(\tau'')\widehat{\mathbf{G}}^{-1}(\tau'')]\!]\,, \end{aligned} \tag{A.22}$$

which follows from the differential equations (A.8) and (A.19), and (A.6) with $\mathbf{L}^{[0]}=\mathbf{L}^{NS}$. Then eventually the dual string products $\mathbf{L}^{\eta}$ can be written as

$$\begin{aligned}\mathbf{L}^{\eta}(\tau) &= \boldsymbol{\eta}-\int_0^{\tau}d\tau'\Big(\mathbf{L}_2^{\mathrm{B}}+\int_0^{\tau'}d\tau''[\![\mathbf{Q},\widehat{\mathbf{G}}(\tau'')\boldsymbol{\lambda}^{[1]}(\tau'')\widehat{\mathbf{G}}^{-1}(\tau'')]\!]\Big)\\ &= \boldsymbol{\eta}-\tau\mathbf{L}_2^{\mathrm{B}}-\int_0^{\tau}d\tau''(\tau-\tau'')[\![\mathbf{Q},\widehat{\mathbf{G}}(\tau'')\boldsymbol{\lambda}^{[1]}(\tau'')\widehat{\mathbf{G}}^{-1}(\tau'')]\!]\,. \end{aligned} \tag{A.23}$$

Here we used $\int_0^{\tau}d\tau'\int_0^{\tau'}d\tau''=\int_0^{\tau}d\tau''\int_{\tau''}^{\tau}d\tau'$ and carried out τ'-integral. In this expression, the commutativity $[\![\mathbf{Q},\mathbf{L}^{\eta}]\!]=0$ is manifest. Differentiating (A.23) with respect to τ, we obtain

$$\partial_\tau\mathbf{L}^{\eta}(\tau) = -\mathbf{L}_2^{\mathrm{B}}-\int_0^{\tau}d\tau''[\![\mathbf{Q},\widehat{\mathbf{G}}(\tau'')\boldsymbol{\lambda}^{[1]}(\tau'')\widehat{\mathbf{G}}^{-1}(\tau'')]\!]\,, \tag{A.24}$$

$$\partial_\tau^2\mathbf{L}^{\eta}(\tau) = -[\![\mathbf{Q},\widehat{\mathbf{G}}(\tau)\boldsymbol{\lambda}^{[1]}(\tau)\widehat{\mathbf{G}}^{-1}(\tau)]\!]\,. \tag{A.25}$$

Therefore we can define the product $\boldsymbol{\rho}$ by two gauge products $\boldsymbol{\lambda}^{[0]}$ and $\boldsymbol{\lambda}^{[1]}$ as

$$\boldsymbol{\rho}(\tau) = -\mathbf{G}(\tau)\boldsymbol{\lambda}^{[1]}(\tau)\mathbf{G}^{-1}(\tau)\,. \tag{A.26}$$

The cyclicity of $\boldsymbol{\rho}$ follows from that of $\boldsymbol{\lambda}^{[0]}$ and $\boldsymbol{\lambda}^{[1]}$. Expanding (A.26) in powers of τ, we obtain the following expressions for the first few orders:

$$\boldsymbol{\rho}_3 = -\boldsymbol{\lambda}_3^{[1]}\,, \tag{A.27}$$

$$\boldsymbol{\rho}_4 = -\Big(\boldsymbol{\lambda}_4^{[1]}+[\![\boldsymbol{\lambda}_2^{[0]},\boldsymbol{\lambda}_3^{[1]}]\!]\Big)\,, \tag{A.28}$$

$$\boldsymbol{\rho}_5 = -\Big(\boldsymbol{\lambda}_5^{[1]}+[\![\boldsymbol{\lambda}_2^{[0]},\boldsymbol{\lambda}_4^{[1]}]\!]+\frac{1}{2}[\![\boldsymbol{\lambda}_3^{[0]},\boldsymbol{\lambda}_3^{[1]}]\!]+\frac{1}{2}[\![\boldsymbol{\lambda}_2^{[0]},[\![\boldsymbol{\lambda}_2^{[0]},\boldsymbol{\lambda}_3^{[1]}]\!]]\!]\Big)\,. \tag{A.29}$$

The explicit form of three-string dual gauge product is, for example, given by

$$\begin{aligned}(V_1,V_2,V_3)^{\eta} = -\frac{1}{4}\Big(&\xi[V_1,V_2,V_3]-[\xi V_1,V_2,V_3]\\ &-(-1)^{V_1}[V_1,\xi V_2,V_3]-(-1)^{V_1+V_2}[V_1,V_2,\xi V_3]\Big)\,. \end{aligned} \tag{A.30}$$

B Four-point amplitudes with external fermions

In this appendix, we illustrate how the on-shell physical amplitudes with external fermions are reproduced from the heterotic string field theory by concentrating on the case of four-point amplitudes which can be calculated only from the action up to $\mathcal{O}(\Psi^4)$ constructed in this paper. We follow the notations and conventions in [29].

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B.1 Propagators and vertices

The kinetic terms of the NS string field V and the Ramond string field Ψ are obtained from (3.51) and (4.123) as

$$S_0 = \frac{1}{2}\langle \eta V, QV\rangle - \frac{1}{2}\langle \xi_0\Psi, YQ\Psi\rangle\,. \tag{B.1}$$

This is invariant under the gauge transformations

$$\delta V = Q\Lambda + \eta\Omega\,, \qquad \delta\Psi = Q\lambda\,, \tag{B.2}$$

which we fix here by gauge conditions

$$b_0^+ V = \xi_0 V = 0\,, \qquad b_0^+\Psi = 0\,. \tag{B.3}$$

Under these gauge conditions the propagators of the NS and the Ramond string fields can be found as

$$\begin{aligned} |\overline{V\rangle\langle V}| &= \xi_0 \frac{b_0^- b_0^+}{L_0^+}\delta(L_0^-) \\ &= \int d^2T\,(\xi_0 b_0^- b_0^+)\,e^{-TL_0^+ - i\theta L_0^-} \equiv \Pi_{NS}\,, \end{aligned} \tag{B.4}$$

$$\begin{aligned} |\overline{\Psi\rangle\langle \Psi}| &= -\frac{b_0^- b_0^+ X\eta}{L_0^+}\delta(L_0^-) \\ &= -\int d^2T\,(b_0^- b_0^+ X\eta)\,e^{-TL_0^+ - i\theta L_0^-} \equiv \Pi_{R}\,, \end{aligned} \tag{B.5}$$

respectively, where $\int d^2T = \int_0^\infty dT \int_0^{2\pi}\frac{d\theta}{2\pi}$. Notice that the Ramond propagator satisfies[8]

$$\eta\Pi_R = \Pi_R\eta = 0\,, \tag{B.6}$$

$$XY\Pi_R = \Pi_R(\xi_0 YX\eta) = \Pi_R\,, \tag{B.7}$$

which expresses that only the Ramond states satisfying the constraint (4.1) propagate.

The necessary interaction vertices can be read from (3.51), (4.123) and (4.170) by expanding them in the coupling constant κ:

$$S_1^{(0)} = \frac{\kappa}{3!}\langle \eta V, [V, QV]\rangle\,, \tag{B.8}$$

$$S_2^{(0)} = \frac{\kappa^2}{4!}\langle \eta V, [V, (QV)^2]\rangle + \frac{\kappa^2}{4!}\langle \eta V, [V, [V, QV]\rangle\,, \tag{B.9}$$

$$S_1^{(2)} = -\frac{\kappa}{2}\langle \Psi, [V, \Psi]\rangle\,, \tag{B.10}$$

$$\begin{aligned} S_2^{(2)} = -\frac{\kappa^2}{16}\Big(&\langle \xi_0\Psi, [QV, \eta V, \Psi]\rangle + \langle \Psi, [\xi_0 QV, \eta V, \Psi]\rangle \\ &+ \langle \Psi, [QV, \xi_0\eta V, \Psi]\rangle + \langle \Psi, [QV, \eta V, \xi_0\Psi]\rangle\Big) - \frac{\kappa^2}{2}\langle \Xi[\eta V, \Psi], [V, \Psi]\rangle\,, \end{aligned} \tag{B.11}$$

$$S_2^{(4)} = \frac{\kappa^2}{4!}\langle \xi_0\Psi, [\Psi^3]\rangle\,. \tag{B.12}$$

From these propagators and vertices, we can uniquely calculate the tree-level four-point amplitudes including external fermions.

[8] The condition for the BPZ conjugate, the latter equation in (B.7), can be understood from the inner product between restricted states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ in the large Hilbert space: $\langle\Psi_1|\xi_0 Y|\Psi_2\rangle = \langle\Psi_1|\xi_0 YXY|\Psi_2\rangle = \langle\Psi_1|(\xi_0 YX\eta)\xi_0 Y|\Psi_2\rangle$.

B.2 Four-fermion amplitude

Let us first consider the amplitude with four external fermions in the Ramond sector. The contributions to the four-fermion amplitude come from the four Feynman diagrams, the s-, t-, u-channel diagrams and a contact type diagram containing a four-string vertex. Following the convention in [29] we label each external string with A, B, C, and D, and denote the s-, t-, and u-channel contributions as $(AB|CD)$, $(AC|BD)$, and $(AD|BC)$, respectively. Then the s-channel contribution can be calculated from the NS propagator (B.4) and the three-string interaction (B.10) as

$$\begin{aligned}
\mathcal{A}_{F^4}^{(AB|CD)} &= (-\kappa)^2 \langle \Psi_A \Psi_B \, \overbrace{V\rangle\langle V} \, \Psi_C \Psi_D \rangle \\
&= \kappa^2 \int d^2T_s \langle \Psi_A \Psi_B (\xi_0 b_0^- b_0^+) \Psi_C \Psi_D \rangle_{W_s} \\
&= \kappa^2 \int d^2T_s \langle\!\langle \Psi_A \Psi_B (b_0^- b_0^+) \Psi_C \Psi_D \rangle\!\rangle_{W_s} \,,
\end{aligned} \tag{B.13}$$

where we denoted the moduli of the s-channel propagator as (T_s, θ_s) and the correlation $\langle \cdots \rangle_{W_s}$, or $\langle\!\langle \cdots \rangle\!\rangle_{W_s}$, is evaluated as the conformal field theory on the s-channel string diagram depicted in [14]. The inserted ξ_0 and $b_0^{\pm}$ are the zero mode with respect to the local coordinate on the NS propagator.[9] The symbols $\Psi_A, \cdots, \Psi_D$ represent the wave functions of the Ramond external states, which satisfy the constraints (4.1), gauge condition (B.3) and the on-shell condition $Q\Psi = 0$. They can be given by the specific vertex operators if necessary.

In this notation the t- and u-channel contributions can similarly be calculated as

$$\mathcal{A}_{F^4}^{(AC|BD)} = \kappa^2 \int d^2T_t \langle\!\langle \Psi_A \Psi_C (b_0^- b_0^+) \Psi_B \Psi_D \rangle\!\rangle_{W_t} \,, \tag{B.14}$$

$$\mathcal{A}_{F^4}^{(AD|BC)} = \kappa^2 \int d^2T_u \langle\!\langle \Psi_A \Psi_D (b_0^- b_0^+) \Psi_B \Psi_C \rangle\!\rangle_{W_u} \,. \tag{B.15}$$

A contact type interaction (B.12) gives the other contribution, denoted $(ABCD)$, which fills the gap in the moduli integration [18]:

$$\mathcal{A}_{F^4}^{(ABCD)} = \kappa^2 \int d\theta_1 d\theta_2 \langle\!\langle (b_{C_1} b_{C_2}) \Psi_A \Psi_B \Psi_C \Psi_D \rangle\!\rangle_{W_4} \,, \tag{B.16}$$

where the integration parameters θ_1 and θ_2 determine the shape of a tetrahedron along which four strings in the vertex (B.12) are glued, and $b_{C_{1,2}}$ denote the corresponding anti-ghost insertions, the details of which are given in [20].

Summing up all these contributions the on-shell four-fermion amplitude becomes

$$\begin{aligned}
\mathcal{A}_{F^4} &= \mathcal{A}_{F^4}^{(AB|CD)} + \mathcal{A}_{F^4}^{(AC|BD)} + \mathcal{A}_{F^4}^{(AD|BC)} + \mathcal{A}_{F^4}^{(ABCD)} \\
&= \kappa^2 \int d^2T_s \, \langle\!\langle \Psi_A \Psi_B (b_0^- b_0^+) \Psi_C \Psi_D \rangle\!\rangle_{W_s} + \kappa^2 \int d^2T_t \, \langle\!\langle \Psi_A \Psi_C (b_0^- b_0^+) \Psi_B \Psi_D \rangle\!\rangle_{W_t} \\
&\quad + \kappa^2 \int d^2T_u \, \langle\!\langle \Psi_A \Psi_D (b_0^- b_0^+) \Psi_B \Psi_C \rangle\!\rangle_{W_u} + \kappa^2 \int d^2\theta \, \langle\!\langle (b_{C_1} b_{C_2}) \Psi_A \Psi_B \Psi_C \Psi_D \rangle\!\rangle_{W_4} \,.
\end{aligned} \tag{B.17}$$

[9] These are denoted ξ_c and $b_c^{\pm}$ in [14].

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From the fact that the bosonic closed string field theory reproduces the correct perturbative amplitudes, we can conclude that the amplitude (B.17) agrees with that obtained in the first quantized formulation.

B.3 Two-fermion-two-boson amplitude

We can similarly calculate the two-fermion-two-boson amplitude. Suppose strings A and B are fermions in the Ramond sector and strings C and D are bosons in the NS sector. The s-channel contributions can be calculated using the NS propagator (B.4) and the vertices (B.8) and (B.10):

$$\begin{aligned} \mathcal{A}^{(AB|CD)}_{F^2B^2} &= -\frac{\kappa^2}{2}\langle \Psi_A \Psi_B\, \overbrace{V\rangle\langle V}\, \big((QV_C)\,(\eta V_D) + (\eta V_C)\,(QV_D)\big)\rangle_{W_s} \\ &= -\frac{\kappa^2}{2}\int d^2T_s\, \langle \Psi_A \Psi_B (\xi_0 b_0^- b_0^+)\big((QV_C)\,(\eta V_D) + (\eta V_C)\,(QV_D)\big)\rangle_{W_s} \\ &= -\frac{\kappa^2}{2}\int d^2T_s\, \langle\!\langle \Psi_A \Psi_B (b_0^- b_0^+)\big((X_0\eta V_C)\;(\eta V_D) + (\eta V_C)\;(X_0\eta V_D)\big)\rangle\!\rangle_{W_s}\,, \end{aligned} \tag{B.18}$$

where V_C and V_D represent the wave functions of the NS external states, which satisfy the gauge conditions (B.3) and the on-shell condition $Q\eta V = 0$. In the last equation, we used $QV = X_0\eta V$ for this wave function V.

The t-channel contribution in this case is calculated using the Ramond propagator (B.5) and the vertex (B.10):

$$\begin{aligned} \mathcal{A}^{(AC|BD)}_{F^2B^2} &= -\kappa^2 \langle \Psi_A V_C\, \overbrace{\Psi\rangle\langle\Psi}\, \Psi_B V_D\rangle \\ &= -\kappa^2 \int d^2T_t\, \langle \Psi_A V_C (b_0^- b_0^+ X\eta)\Psi_B V_D\rangle_{W_t}\,. \end{aligned} \tag{B.19}$$

In order to smoothly connect the contributions of the four diagrams at their boundaries and sum up them to the integration over the whole moduli space, we rearrange the integrand so as to be the correlation function of the same external vertices, Ψ_A, Ψ_B, $((X_0\eta V_C)\,(\eta V_D) + (\eta V_C)\,(X_0\eta V_D))/2$, and the operator insertion $(b_0^- b_0^+)$ as those in (B.18). In particular we move the picture changing operator X in the Ramond propagator, whose form is highly depend on the coordinate system of the propagator, to an external state by using the relation $X = \{Q, \Xi\}$. After a little manipulation, we obtain

$$\begin{aligned} \mathcal{A}^{(AC|BD)}_{F^2B^2} &= -\frac{\kappa^2}{2}\int d^2T_t\, \Big(\langle \Psi_A V_C (b_0^- b_0^+ X)\Psi_B\,(\eta V_D)\rangle_{W_t} + \langle \Psi_A\,(\eta V_C)(b_0^- b_0^+ X)\Psi_B V_D\rangle_{W_t}\Big) \\ &= -\frac{\kappa^2}{2}\int d^2T_t\, \Big(\langle\!\langle \Psi_A\,(X_0\eta V_C)\,(b_0^- b_0^+)\Psi_B\,(\eta V_D)\rangle\!\rangle_{W_t} \\ &\qquad\qquad + \langle\!\langle \Psi_A\,(\eta V_C)\,(b_0^- b_0^+)\,\Psi_B\,(X_0\eta V_D)\rangle\!\rangle_{W_t}\Big) \\ &\quad - \frac{\kappa^2}{2}\oint d\theta_t\, \Big(\langle \Psi_A\, V_C (b_0^- \Xi)\Psi_B\,(\eta V_D)\rangle_{W_t} + \langle \Psi_A\,(\eta V_C)(b_0^- \Xi)\Psi_B\, V_D\rangle_{W_t}\Big)\Big|_{T_t=0}\,, \end{aligned} \tag{B.20}$$

where we denoted $\oint d\theta \equiv \int_0^{2\pi} d\theta/2\pi$ for simplicity. Here the first line of the final expression has the same form of the external states as those in (B.18), but the extra contribution in the second line appears from the boundary $T_t = 0$ by exchanging Q and b_0^+ using

$$\{Q, \int_0^\infty dT\ b_0^+ e^{-TL_0^+}\} = \int_0^\infty dT\ L_0^+ e^{-TL_0^+} = 1\,.$$

The contribution from the u-channel has the same structure and is calculated as

$$\begin{aligned}
\mathcal{A}_{F^2B^2}^{(AD|BC)} &= -\kappa^2 \int d^2T_u\, \langle \Psi_A V_D (b_0^- b_0^+ X\eta) \Psi_B V_C \rangle_{W_u} \qquad \text{(B.21)}\\
&= -\frac{\kappa^2}{2} \int d^2T_u \Big(\langle \Psi_A\, (QV_D)(b_0^- b_0^+ \Xi) \Psi_B\, (\eta V_C) \rangle_{W_u} \\
&\qquad\qquad + \langle \Psi_A\, (\eta V_D)(b_0^- b_0^+ \Xi) \Psi_B\, (QV_C) \rangle_{W_u} \Big) \\
&\quad - \frac{\kappa^2}{2} \oint d\theta_u\, \big(\langle \Psi_A V_D (b_0^- \Xi) \Psi_B\, (\eta V_C) \rangle_{W_u} + \langle \Psi_A\, (\eta V_D)(b_0^- \Xi) \Psi_B V_C \rangle\!\rangle_{W_u} \big) \Big|_{T_u=0} \\
&= -\frac{\kappa^2}{2} \int d^2T_u \Big(\langle\!\langle \Psi_A\, (X_0 \eta V_D)(b_0^- b_0^+) \Psi_B\, (\eta V_C) \rangle\!\rangle_{W_u} \\
&\qquad\qquad + \langle\!\langle \Psi_A\, (\eta V_D)(b_0^- b_0^+) \Psi_B\, (X_0 \eta V_C) \rangle\!\rangle_{W_u} \Big) \\
&\quad - \frac{\kappa^2}{2} \oint d\theta_u\, \big(\langle \Psi_A V_D (b_0^- \Xi) \Psi_B\, (\eta V_C) \rangle_{W_u} + \langle \Psi_A\, (\eta V_D)(b_0^- \Xi) \Psi_B V_C \rangle_{W_u} \big) \Big|_{T_u=0}.
\end{aligned}$$

The contribution from the contact type diagram can be obtained using the vertices (B.11):

$$\begin{aligned}
\mathcal{A}_{F^2B^2}^{(ABCD)} &= -\frac{\kappa^2}{2} \int d\theta_1 d\theta_2 \langle (\xi_0 b_{C_1} b_{C_2}) \Psi_A \Psi_B \big((QV_C)\,(\eta V_D) + (\eta V_C)\,(QV_D) \big) \rangle_{W_4} \\
&\quad - \frac{\kappa^2}{2} \oint d\theta_t \Big(\langle \Xi b_0^- \big(\Psi_A\, (\eta V_C) \big)\, \Psi_B\, V_D \rangle_{W_t} \langle \Xi b_0^- \big(\Psi_B\, (\eta V_C) \big)\, \Psi_A\, V_D \rangle_{W_t} \Big) \Big|_{T_t=0} \\
&\quad - \frac{\kappa^2}{2} \oint d\theta_u \Big(\langle \Xi b_0^- \big(\Psi_A (\eta V_D) \big)\, \Psi_B\, V_C \rangle_{W_u} + \langle \Xi b_0^- \big(\Psi_B (\eta V_D) \big)\, \Psi_A\, V_C \rangle_{W_u} \Big) \Big|_{T_u=0} \\
&= -\frac{\kappa^2}{2} \int d\theta_1 d\theta_2 \langle\!\langle (b_{C_1} b_{C_2}) \Psi_A \Psi_B \big((X_0 \eta V_C)\,(\eta V_D) + (\eta V_C)\,(X_0 \eta V_D) \big) \rangle\!\rangle_{W_4} \\
&\quad + \frac{\kappa^2}{2} \oint d\theta_t \Big(\langle \Psi_A\, (\eta V_C)(b_0^- \Xi) \Psi_B V_D \rangle_{W_t} + \langle \Psi_A V_C (b_0^- \Xi) \Psi_B\, (\eta V_D) \rangle_{W_t} \Big) \Big|_{T_t=0} \\
&\quad + \frac{\kappa^2}{2} \oint d\theta_u \Big(\langle \Psi_A\, (\eta V_D)(b_0^- \Xi) \Psi_B V_C \rangle_{W_u} + \langle \Psi_A V_D (b_0^- \Xi) \Psi_B\, (\eta V_C) \rangle_{W_u} \Big) \Big|_{T_u=0}.
\end{aligned} \qquad \text{(B.22)}$$

The first line is the similar contribution to the one in the four-fermion amplitude (B.16), coming from the proper four-string interaction defined by the three-string product $[\,\cdot\,,\,\cdot\,,\,\cdot]$. Extra contribution in the second line comes from the vertices represented by the nested two string products $[\,\cdot\,,[\,\cdot\,,\,\cdot\,]]$, represented by the diagram with the collapsed-propagator, the propagator with $T = 0$, integrated only over its angle. For the more details, see [29].

Summing up all these contributions, one can find that all the extra contributions integrated only one parameter are cancelled out. The two-fermion-two-boson amplitude finally becomes

$$
\begin{aligned}
\mathcal{A}_{F^2B^2} &= \mathcal{A}_{F^2B^2}^{(AB|CD)} + \mathcal{A}_{F^2B^2}^{(AC|BD)} + \mathcal{A}_{F^2B^2}^{(AD|BC)} + \mathcal{A}_{F^2B^2}^{(ABCD)} \\
&= -\frac{\kappa^2}{2}\int d^2T_s\Big(\langle\!\langle \Psi_A\Psi_B(b_0^-b_0^+)\big((X_0\eta V_C)\,(\eta V_D) + (\eta V_C)\,(X_0\eta V_D)\big)\rangle\!\rangle_{W_s}\Big) \\
&\quad -\frac{\kappa^2}{2}\int d^2T_t\Big(\langle\!\langle \Psi_A\,(X_0\eta V_C)\,(b_0^-b_0^+)\Psi_B\,(\eta V_D)\rangle\!\rangle_{W_t} \\
&\qquad\qquad\qquad + \langle\!\langle \Psi_A\,(\eta V_C)\,(b_0^-b_0^+)\,\Psi_B\,(X_0\eta V_D)\rangle\!\rangle_{W_t}\Big) \\
&\quad -\frac{\kappa^2}{2}\int d^2T_u\Big(\langle\!\langle \Psi_A\,(X_0\eta V_D)(b_0^-b_0^+)\Psi_B\,(\eta V_C)\rangle\!\rangle_{W_u} \\
&\qquad\qquad\qquad + \langle\!\langle \Psi_A\,(\eta V_D)(b_0^-b_0^+)\Psi_B\,(X_0\eta V_C)\rangle\!\rangle_{W_u}\Big) \\
&\quad -\frac{\kappa^2}{2}\int d\theta_1 d\theta_2\ \langle\!\langle (b_{C_1}b_{C_2})\Psi_A\Psi_B\big((X_0\eta V_C)\,(\eta V_D) + (\eta V_C)\,(X_0\eta V_D)\big)\rangle\!\rangle_{W_4}\,.
\end{aligned}
\tag{B.23}
$$

For the same reason as the four-fermion amplitude, this agrees with the result in the first-quantized formulation.

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