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Strong restriction on inflationary vacua from the local *gauge* invariance III: Infrared regularity of graviton loops

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It has been claimed that the super-Hubble modes of the graviton generated during inflation can make loop corrections diverge. Even if we introduce an infrared (IR) cutoff at a comoving scale as an ad hoc but practical method of regularization, we encounter secular growth, which may lead to the breakdown of perturbative expansion for a sufficiently long-lasting inflation. In this paper, we show that the IR pathology concerning the graviton can be attributed to the presence of residual *gauge* degrees of freedom in the local observable universe, as in the case of the adiabatic curvature perturbation. We will show that choosing the Euclidean vacuum as the initial state ensures invariance under the above-mentioned residual *gauge* transformations. We will also show that, as long as we consider a *gauge* invariant quantity in the local universe, we encounter neither IR divergence nor secular growth. The argument in this paper applies to general single-field models of inflation up to a sufficiently high order in perturbation.

Subject Index E00, E81, E86

1. Introduction

The inflationary spacetime leads to the generation of gravitational waves. Even though the amplitude of gravitational waves is smaller than the amplitude of the adiabatic curvature perturbation, detection of the primordial gravitational waves generated during inflation is expected to be within our reach. Measurement of primordial gravitational waves is crucially important in uncovering the model of inflation, providing information that cannot be obtained through the measurement of scalar perturbations. In particular, the amplitude of the gravitational waves can directly measure the energy scale of inflation, unlike the amplitude of the adiabatic curvature perturbation, which is also sensitive to the detailed dynamics of the inflationary universe. We can find an example that highlights the importance of measuring the primordial gravitational waves for non-linear perturbations as well, say in Refs. [1–3], which elucidate the impact of parity violation in the gravity sector on the bi-spectrum of the primordial gravitational waves. Thus, it will be profitable to study the method of predicting primordial gravitational waves in the presence of the non-linear interaction too. In this paper, our focus is on the loop correction due to the primordial gravitational waves.

It is known that loop corrections of a massless scalar field generated during inflation can suffer infrared (IR) divergences [4–42]. Since a massless scalar field yields a scale-invariant spectrum in

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the IR limit as $\mathcal{P}(k) \propto 1/k^3$, a naive loop integral yields a factor $\int d^3\boldsymbol{k}/k^3 \sim \int dk/k$, which logarithmically diverges. As expected from the fact that the mode equation of the tensor perturbation, which we also refer to as the graviton, is nothing but the mode equation for a massless scalar field, graviton loop corrections also appear to yield IR divergences. To quantify the graviton loop corrections, we need to provide a way to regularize them. One may propose the introduction of an IR cutoff, say, at a comoving scale $k_{\rm IR}$ as a practical method of regularization. However, this will not provide a satisfactory solution, because the loop integral of the super-Hubble modes gives $\int_{k_{\rm IR}}^{aH} dk/k \sim \ln(aH/k_{\rm IR})$, which logarithmically increases in time. Here a and H are the scale factor and the Hubble parameter of the background spacetime, respectively. Compared with the tree-level contribution, the loop corrections are typically suppressed by the Planck scale as $(H/M_{\rm pl})^2$ with $M_{\rm pl}^2 \equiv (8\pi G)^{-1}$. However, this suppression may be compensated by the secular growth for a sufficiently long-lasting inflation. (A thorough overview of the possible origins of the IR divergences can be found in the review paper by Seery [43].)

The IR behavior of the graviton has also attracted attention as a possible origin of the running of coupling constants [10–12,24,44–53]. Tsamis and Woodard claimed that the logarithmically growing secular effect due to the graviton loops can lead to the screening of the cosmological constant, suggesting that the screening may provide a dynamical solution to the cosmological constant problem [24]. More recently, Kitamoto and Kitazawa studied the IR effect on gauge coupling and claimed that the secular growth from the graviton loops can screen the gauge coupling [34,35]. A related issue is discussed for the U(1) gauge field in Refs. [36,37]. If the secular growth due to the graviton loops is actually physical, it will have a phenomenological impact. However, these secular growths originate from the increasing IR contributions, and hence the predicted secular effects significantly depend on the regularization method of the IR contributions. Therefore, to trust the secular growth as a physical effect, its presence should be verified based on a rigorous method of regularization. The graviton loop corrections have mostly been discussed in the exact de Sitter background and their regularity is still under debate, even at the linear level. In Ref. [54], Higuchi et al. claimed the existence of a regular two-point function (see also Refs. [55,56]), while Miao et al. objected against it in Ref. [57]. This issue was also discussed in Refs. [58–60].

The IR pathology has been studied more intensively for the adiabatic curvature perturbation [61–76]. In the presence of gravity, we need to remove the influence of gauge degrees of freedom in calculating the observable fluctuations. In the cosmological perturbation theory, gauge artifacts are usually removed by employing gauge conditions that completely fix the coordinate choice. However, when we consider actual observations, the gauge conditions may not be fixed in a suitable way that preserves the gauge invariance for an actual observer. In fact, we can observe the fluctuations only within the region causally connected to us, which is limited to a finite portion of each time constant slice. Therefore, precisely speaking, the gauge conditions used in the conventional cosmological perturbation theory, which require all the information all over the time constant slice of the universe, does not match the actual process of observations. To regularize the IR contributions for the curvature perturbation, it is necessary to take into account this subtle issue [62,63,65–67]. Since in actual observations we can impose the gauge conditions only in the observable region, there inevitably appears an ambiguity associated with the degrees of freedom in choosing coordinates in the outside of the observable region. Such residual coordinate degrees of freedom can be attributed to the degrees of freedom in the boundary conditions of our observable local universe. It was shown that requesting the invariance under the change of such residual coordinate degrees of freedom in the local universe can ensure the IR regularity and the absence of the secular growth [62,63,65–67]. This is, so to speak, because we can absorb the singular IR contributions by the residual coordinate degrees of freedom (see also Refs. [77,78]). This issue is reviewed in Ref. [67]. It has been pointed out that the residual coordinate degrees of freedom can also affect the IR behavior of the graviton [63,67]. In the present paper, we will show that, when we require the invariance under the residual coordinate transformations, the IR regularity and the absence of secular growth are also guaranteed for the graviton loops. (The IR issues about the entropy perturbation were studied in Refs. [79,80] and those about a test scalar field in the de Sitter background were studied in Refs. [81–86].)

For the graviton, the relation between the IR divergence and the gauge artifact has been discussed during the past few decades. Allen showed that the IR divergence in the free graviton propagator that appears for some particular values of the gauge parameter can be understood from the fact that the gauge-fixing term does not select a unique gauge for these particular values of the gauge parameter [87]. (See also Refs. [88–91].) Even if we properly choose the gauge parameter, it is known that the transverse traceless mode can still suffer from IR divergence through the loop corrections, which is the target of this paper. The connection between the IR divergence and the gauge artifact was also pointed out in Ref. [92], where the secular growth predicted by Tsamis and Woodard in Ref. [24] was reexamined. It was shown that the spatially averaged Hubble expansion computed by Tsamis and Woodard is not invariant under the change of the time slice and hence the screening effect that shows up in their averaged Hubble parameter suffers from the gauge artifact [92,93]. Meanwhile, in Ref. [92], it was shown that the locally defined Hubble expansion, which may mimic the observable Hubble rate, becomes time-independent. This example suggests that computing observable quantities, unaffected by the influence from the outside of the observable region, will play an important role in quantifying the IR effects of the graviton [92] (see also Ref. [94]).

This paper is organized as follows. In Sect. 2, we will briefly show how the IR divergence and the secular growth can appear when we naively compute the loop corrections for the curvature and graviton perturbations. Then, in Sect. 2.2, we will point out the presence of the residual coordinate degrees of freedom in the local observable universe, which describes the influence from the outside of the observable region. One way to preserve the invariance under the residual coordinate transformations will be discussed in Sect. 2.3. Using the prescription that will be introduced in Sect. 3, we will show, in Sects. 4 and 5, that when we choose the Euclidean vacuum as the initial state, IR regularity and the absence of secular growth are ensured. In this paper, we will discuss the inflationary universe that contains a single scalar field and we will not directly discuss the pure gravity setup, although our argument may also provide some insight into the latter case.

2. Overview of IR issues

In this section, we will give an overview of the IR issues of the curvature perturbation and the graviton perturbation. In this paper, we consider a standard single-field inflation model whose action is given by

$$S = \frac{M_{\rm pl}^2}{2} \int \sqrt{-g}[R - g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 2V(\phi)]d^4x, \tag{2.1}$$

where $M_{\rm pl}$ is the Planck mass and ϕ is the dimensionless scalar field, which is an ordinary scalar field divided by $M_{\rm pl}$. We choose the time slicing by adopting the uniform field gauge

$$\delta\phi = 0. \tag{2.2}$$

Under the ADM metric decomposition, which is given by

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{2.3}$$

we further decompose the spatial metric g_{ij} as

$$g_{ij} = e^{2(\rho+\zeta)}\gamma_{ij} \equiv e^{2(\rho+\zeta)}\left[e^{\delta\gamma}\right]_{ij}, \tag{2.4}$$

where $a \equiv e^{\rho}$ is the scale factor, ζ is the so-called curvature perturbation, and $\delta\gamma_{ij}$ is the graviton perturbation that satisfies the transverse and traceless condition

$$\delta\gamma_{ii} = 0, \quad \partial_i \delta\gamma^i_j = 0. \tag{2.5}$$

Since $\delta\gamma_{ij}$ is traceless, we find $\det\gamma = 1$.

2.1. *Various divergences from the curvature and graviton perturbations*

In this subsection, after we briefly review the linear perturbation theory for the scalar and tensor perturbations, we will summarize several different origins of the divergences that possibly appear in the loop corrections of these two perturbations.

2.1.1. Scalar perturbation. The quadratic action for the scalar perturbation ζ, which describes the evolution of the interaction picture field ζ_I, is given by

$$S_0^s = M_{\rm pl}^2 \int dt \int d^3\boldsymbol{x} e^{3\rho}\varepsilon_1 \left[\dot{\zeta}_I^2 - e^{-2\rho}(\partial_i \zeta_I)^2\right], \tag{2.6}$$

and its equation of motion is given by

$$\left[\partial_t^2 + (3+\varepsilon_2)\dot{\rho}\partial_t - e^{-2\rho}\partial^2\right]\zeta_I = 0. \tag{2.7}$$

Here, the dot denotes the differentiation with respect to cosmological time t. Here, for notational convenience, we introduce the horizon flow functions as

$$\varepsilon_1 \equiv \dot{\rho}\frac{d}{d\rho}\frac{1}{\dot{\rho}}, \quad \varepsilon_n \equiv \frac{1}{\varepsilon_{n-1}}\frac{d}{d\rho}\varepsilon_{n-1}, \tag{2.8}$$

with $n \geq 2$. We do not assume that these functions are small, leaving the background inflation model unrestricted.

For quantization, we expand the curvature perturbation $\zeta_I(x)$ as

$$\zeta_I(x) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} v_k^s(t) a_{\boldsymbol{k}} + (\text{h.c.}), \tag{2.9}$$

where $a_{\boldsymbol{k}}$ is the annihilation operator, which satisfies

$$\left[a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}^{\dagger}\right] = \delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}'), \quad [a_{\boldsymbol{k}}, a_{\boldsymbol{k}}] = 0. \tag{2.10}$$

The mode function $v_k^s(t)$ should satisfy

$$\left[\partial_t^2 + (3+\varepsilon_2)\dot{\rho}\partial_t + (ke^{-\rho})^2\right]v_k^s(t) = 0, \tag{2.11}$$

and is normalized as

$$\left(v_k^s e^{i\boldsymbol{k}\cdot\boldsymbol{x}},\, v_p^s e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\right) = (2\pi)^3 \delta^{(3)}(\boldsymbol{k}-\boldsymbol{p}), \tag{2.12}$$

where the Klein–Gordon inner product is defined by

$$(f_1, f_2) \equiv -2iM_{\rm pl}^2 \int d^3\boldsymbol{x} e^{3\rho} \varepsilon_1 \{ f_1 \partial_t f_2^* - (\partial_t f_1) f_2^* \}. \tag{2.13}$$

Notice that Eq. (2.11) states that $v_k^s(t)$ becomes time-independent in the IR limit $k/(e^\rho \dot{\rho}) \ll 1$. When we choose the mode function for the adiabatic vacuum, which approaches the WKB solution in the UV limit $k/(e^\rho \dot{\rho}) \gg 1$, the power spectrum becomes almost scale invariant in the IR limit as

$$P^s(k) \equiv |v_k^s(t)|^2 = \frac{1}{4k^3} \frac{1}{\varepsilon_1(t_k)} \left(\frac{\dot{\rho}(t_k)}{M_{\rm pl}}\right)^2 \left[1 + \mathcal{O}\left((k/e^\rho \dot{\rho})^2\right)\right], \tag{2.14}$$

where we have evaluated $v_k^s(t)$ at the Hubble crossing time $t = t_k$ with $k = e^{\rho(t_k)} \dot{\rho}(t_k)$, since the curvature perturbation gets frozen rapidly after t_k.

When we assume that the corresponding free theory has an almost scale-invariant spectrum in the IR limit, a naive consideration can easily lead to the IR divergence due to loop corrections. For instance, one may expect that an interaction vertex that contains the curvature perturbation without the derivative operator yields the factor

$$\langle \zeta_I^2(x) \rangle = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} P^s(k), \tag{2.15}$$

whose momentum integral logarithmically diverges in the IR as $\int d^3\boldsymbol{k}/k^3$ for the scale-invariant spectrum. Even if the spectrum is not exactly scale invariant as in Eq. (2.14), the deep IR modes give a significant contribution to Eq. (2.15). Following Ref. [67], we refer to such unsuppressed contribution due to the integral over small k as the *IR divergence* (IRdiv).

When we introduce an IR cutoff for the regularization, say, at the Hubble scale at the initial time t_i, the super-Hubble (superH) modes in the variance of ζ_I give rise to secular growth that is logarithmic in the scale factor $a = e^\rho$ as

$$\langle \zeta_I^2(x) \rangle_{\rm superH} \propto \int_{e^{\rho(t_i)}\dot{\rho}(t_i)}^{e^{\rho(t)}\dot{\rho}(t)} \frac{dk}{k} = \ln\left(\frac{e^{\rho(t)}\dot{\rho}(t)}{e^{\rho(t_i)}\dot{\rho}(t_i)}\right). \tag{2.16}$$

Then, the loop corrections, which are suppressed by an extra power of the amplitude of the power spectrum $(\dot{\rho}/M_{\rm pl})^2$, may dominate if the inflationary epoch lasts sufficiently long, leading to the breakdown of the perturbative expansion. We refer to the modes with $e^{\rho(t_i)}\dot{\rho}(t_i) \lesssim k \lesssim e^{\rho(t)}\dot{\rho}(t)$ as the *transient IR* (tIR) modes and refer to the enhancement of the loop contributions due to the tIR modes as the *IR secular growth* (IRsec). From the definition, it is clear that the tIR modes were in the sub-Hubble (subH) range at the initial time t_i, but have been transmitted into the superH ones by the time t. The influence of the IRsec has been discussed by introducing an IR cutoff in Refs. [95–102]. We refer to the case when both the IRdiv and the IRsec are absent as IR regular.

So far, we have discussed the secular growth that originates from the momentum integration, keeping the time coordinates of the interaction vertices fixed. However, the time integration can also yield secular growth. If the contribution from the interaction vertex in the far past remains unsuppressed, it will diverge when we send the initial time to the infinite past. We refer to the secular growth due to the temporal integral as the SG, discriminating it from the previously discussed IRsec. (Regarding the SG, see also Refs. [103,104].)

2.1.2. Graviton perturbation. The quadratic action for the graviton perturbation $\delta\gamma_{ij}$, which describes the evolution of the interaction picture field $\delta\gamma_{ijI}$, is given by

$$S_0^t = \frac{M_{\rm pl}^2}{8}\int dt \int d^3\boldsymbol{x} e^{3\rho}\left[\delta\dot{\gamma}^i_{jI}\delta\dot{\gamma}^j_{iI} - e^{-2\rho}\partial^l\delta\gamma^i_{jI}\partial_l\delta\gamma^j_{iI}\right], \tag{2.17}$$

and its equation of motion is given by

$$\left[\partial_t^2 + 3\dot{\rho}\partial_t - e^{-2\rho}\partial^2\right]\delta\gamma_{ijI} = 0. \tag{2.18}$$

We quantize $\delta\gamma_{ijI}$ as

$$\delta\gamma^i_{jI}(x) = \sum_{\lambda=\pm}\int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} v_k^{t(\lambda)}(t) e^i_j(\boldsymbol{k},\lambda) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} a_{\boldsymbol{k}}^{(\lambda)} + (\text{h.c.}), \tag{2.19}$$

where λ labels the helicity, $e_{ij}(\boldsymbol{k},\lambda)$ is the transverse and traceless polarization tensor that satisfies

$$e^i_i(\boldsymbol{k},\lambda) = k^i e_{ij}(\boldsymbol{k},\lambda) = 0, \quad e_{ij}(\boldsymbol{k},\lambda)e^{ij}(\boldsymbol{k},\lambda') = \delta_{\lambda,\lambda'}, \tag{2.20}$$

and $a_{\boldsymbol{k}}^{(\lambda)}$ is the annihilation operator that satisfies

$$\left[a_{\boldsymbol{k}}^{(\lambda)}, a_{\boldsymbol{p}}^{(\lambda')\dagger}\right] = \delta_{\lambda\lambda'}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{p}). \tag{2.21}$$

When the graviton perturbation is isotropic, its variance (in the coincidence limit) is given by

$$\langle\delta\gamma_{ijI}(x)\delta\gamma_{klI}(x)\rangle = \frac{1}{4}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\left(\mathcal{P}_{ik}P_{jl} + \mathcal{P}_{il}\mathcal{P}_{jk} - \mathcal{P}_{ij}\mathcal{P}_{kl}\right)P^t(k), \tag{2.22}$$

where $\mathcal{P}_{ij}$ is the transverse traceless projection tensor:

$$\mathcal{P}_{ij} \equiv \delta_{ij} - \frac{k_i k_j}{k^2}, \tag{2.23}$$

and $P^t(k)$ is the power spectrum of the graviton perturbation:

$$P^t(k) = 2|v_k^t(t)|^2. \tag{2.24}$$

Here, the factor 2 shows the helicity number. Since the equation for $v_k^{t(\lambda)}$ is identical to that for a massless scalar field, the graviton spectrum in the adiabatic vacuum is almost scale invariant in the IR limit as

$$P^t(k) = \frac{4}{k^3}\left(\frac{\dot\rho(t_k)}{M_{\rm pl}}\right)^2\left[1+\mathcal{O}\left((k/e^\rho\dot\rho)^2\right)\right]. \tag{2.25}$$

Integrating over the angular coordinates in Eq. (2.22), we obtain

$$\langle\delta\gamma_{ij\,I}(x)\delta\gamma_{kl\,I}(x)\rangle = \frac{1}{20\pi^2}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\frac{2}{3}\delta_{ij}\delta_{kl}\right)\int\frac{dk}{k}k^3P^t(k). \tag{2.26}$$

Similarly to the curvature perturbation, we find that the IR and tIR modes in Eq. (2.26) yield the IRdiv and IRsec, respectively. The influence of the IRsec due to graviton loops is discussed in Ref. [98]. Meanwhile, the interaction vertices with the graviton can also yield the SG in the same way as the curvature perturbation.

2.2. *Residual gauge degrees of freedom and IR issues*

2.2.1. Solving the constraint equations. Eliminating the Lagrange multipliers N and N_i, we derive the action written solely in terms of the dynamical fields ζ and $\delta\gamma_{ij}$ [105]. In the gauge defined by Eqs. (2.2) and (2.5), the constraint equations are given by

$$^sR - 2V - (\kappa^{ij}\kappa_{ij}-\kappa^2) - N^{-2}\dot\phi^2 = 0, \tag{2.27}$$

$$D_j(\kappa^j_{\ i} - \delta^j_{\ i}\kappa) = 0, \tag{2.28}$$

where D_i is the covariant differentiation associated with the spatial metric g_{ij}, and

$$\kappa_{ij} \equiv \frac{1}{2N}(\dot g_{ij} - D_iN_j - D_jN_i) \quad \text{and} \quad \kappa \equiv g^{ij}\kappa_{ij} \tag{2.29}$$

are the extrinsic curvature and its trace, respectively. We expand the metric perturbations as

$$\zeta = \zeta_I + \zeta_2 + \cdots, \tag{2.30}$$

$$\delta\gamma_{ij} = \delta\gamma_{ijI} + \delta\gamma_{ij2} + \cdots, \tag{2.31}$$

$$N = 1 + N_1 + N_2 + \cdots, \tag{2.32}$$

$$N_i = N_{i1} + N_{i2} + \cdots, \tag{2.33}$$

where we use the subscript I to express the first-order curvature and graviton perturbations, since they correspond to the interaction picture fields. Then, the nth-order Hamiltonian constraint and the momentum constraints are expressed in the form

$$VN_n - 3\dot\rho\dot\zeta_n + e^{-2\rho}\partial^2\zeta_n + \dot\rho e^{-2\rho}\partial^iN_{in} = H_n, \tag{2.34}$$

$$4\partial_i\left(\dot\rho N_n - \dot\zeta_n\right) - e^{-2\rho}\partial^2N_{in} + e^{-2\rho}\partial_i\partial^jN_{jn} = M_{in}, \tag{2.35}$$

where ∂^2 denotes the Laplacian. For $n = 1$, H_1 and M_{i1} are 0 and for $n \geq 2$, H_n and M_{in} consist of n interaction picture fields (either ζ_I or $\delta\gamma_{ijI}$).

Since the constraint equations (2.34) and (2.35) are elliptic-type equations, we need to employ a (spatial) boundary condition to determine a solution for N_n and N_{in}. A general solution for Eqs. (2.34) and (2.35) in the absence of the graviton perturbation can be found in the Appendix of Ref. [66]. An extension to include the graviton perturbation proceeds in a straightforward manner and the general solution is given as

$$N_n = \frac{1}{\dot\rho}\dot\zeta_n + \frac{V}{4\dot\rho}e^{-2\rho}\left(e^{2\rho}\partial^{-2}\partial^i M_{in} - G_n\right), \tag{2.36}$$

$$\begin{aligned} N_{in} = \partial_i\partial^{-2}\left[\frac{\dot\phi^2}{2\dot\rho^2}e^{2\rho}\dot\zeta_n - \frac{1}{\dot\rho}\partial^2\zeta_n + \frac{e^{2\rho}}{\dot\rho}H_n - \frac{V}{4\dot\rho^2}\left\{e^{2\rho}\partial^{-2}\partial^j M_{jn} - G_n\right\}\right] \\ - \left(\delta_i^j - \partial_i\partial^{-2}\partial^j\right)\left\{e^{2\rho}\partial^{-2}\left(M_{jn} - \frac{4\dot\rho}{V}\partial_j H_n\right) - G_{jn}\right\}, \end{aligned} \tag{2.37}$$

where $G_n(x)$ and $G_{in}(x)$ are arbitrary solutions of the Laplace equations

$$\partial^2 G_n(x) = 0, \qquad \partial^2 G_{in}(x) = 0, \tag{2.38}$$

and are nth order in perturbation. Since the function $G_{in}(x)$ contributes only through its transverse part, the number of introduced independent functions is three. By employing the appropriate boundary conditions at spatial infinity, the solutions of elliptic-type equations are uniquely determined. Substituting the thus-obtained expression for N and N_i, the action $S = \int d^4x\mathcal{L}[\zeta, \delta\gamma_{ij}, N, N_i]$ can be, in principle, expressed only in terms of the dynamical fields ζ and $\delta\gamma_{ij}$. Then, the evolution of ζ and $\delta\gamma_{ij}$ is governed by the *non-local* action with the inverse Laplacian.

As was pointed out in Refs. [62,63], the inverse Laplacian ∂^{-2} may enhance the singular behavior of perturbation in the IR limit by introducing a term with the factor $1/k^2$ where k is a comoving momentum of the constituent fields. (A detailed explanation of this is given in the review article [67].) We will return to this issue in Sect. 4.2.2.

2.2.2. Observable local patch. To discuss the observable quantities, we introduce the observable region as the region causally connected to us. We express the observable region on the time slice at the end of inflation t_f and its comoving radius as $\mathcal{O}_{t_f}$ and L_{t_f}, respectively. The causality requires that L_{t_f} should satisfy

$$L_{t_f} \lesssim \int_{t_f}^{t_0} \frac{dt}{e^{\rho(t)}},$$

where t_0 is the present time. The cosmological observations can measure the n-point functions of the fluctuation with the arguments $(t_f, \boldsymbol{x})$ contained within the observable region $\mathcal{O}_{t_f}$. For later use, we refer to the causal past of $\mathcal{O}_{t_f}$ as the observable region $\mathcal{O}$ and refer to the intersection between $\mathcal{O}$ and a t-constant slice Σ_t as $\mathcal{O}_t$. We approximate the comoving radius of the region $\mathcal{O}_t$ as

$$L_t \equiv L_{t_f} + \int_t^{t_f} \frac{dt'}{e^{\rho(t')}} \simeq L_{t_f} + \frac{1}{e^{\rho(t)}\dot\rho(t)}. \tag{2.39}$$

As L_t is approximated by $L_t \simeq 1/(e^{\rho}\dot{\rho})$ in the distant past, the effects of the superH modes with $k \lesssim e^{\rho(t)}\dot{\rho}(t)$ can be understood as the influence from the outside of the observable region $\mathcal{O}$. These modes potentially affect the fluctuations in $\mathcal{O}_{t_f}$ in two ways. One is due to the non-local interaction through the inverse Laplacian ∂^{-2}, while the other is through the Wightman functions

$$G^{+s}(x_1, x_2) \equiv \langle \zeta_I(x_1)\zeta_I(x_2)\rangle \tag{2.40}$$

and

$$G^{+t}_{ijkl}(x_1, x_2) \equiv \langle \delta\gamma_{ijI}(x_1)\delta\gamma_{klI}(x_2)\rangle. \tag{2.41}$$

Even if the spatial distance $|\boldsymbol{x}_1 - \boldsymbol{x}_2|$ is bounded from above by confining both $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ within the observable region, the contribution to the Wightman functions from the superH modes with $k \le |\boldsymbol{x}_1 - \boldsymbol{x}_2|^{-1}$ is not suppressed. The superH modes make these Wightman functions divergent for a scale-invariant or red-tilted spectrum. (See Eqs. (2.16) and (2.26).) The regularity of the contribution from the superH modes can be verified only if their contribution is suppressed by an additional factor of k.

2.2.3. Residual gauge degrees of freedom. In Sect. 2.2.2, we introduced the observable region $\mathcal{O}$, which is a limited portion of the whole universe. We claim that the observable quantities must be composed of fluctuations within $\mathcal{O}$. Since only the fluctuations within $\mathcal{O}$ are relevant, there is no reason to request the regularity of the fluctuations at spatial infinity in solving the elliptic-type constraint equations (2.34) and (2.35) (at least, at the level of the Heisenberg equations of motion). Then, there arise degrees of freedom in choosing the boundary conditions, which are described by arbitrary homogeneous solutions of the Laplace equation, $G_n(x)$ and $G_{in}(x)$ in Eqs. (2.36) and (2.37). These arbitrary functions in N and N_i can be understood as the degrees of freedom in choosing the coordinates. Since the time slicing is fixed by the gauge condition (2.2), the residual gauge degrees of freedom can reside only in the spatial coordinates x^i.

Let us consider these residual coordinate transformations associated with $G_n(x)$ and $G_{in}(x)$. We add the subscript gl to the original global coordinate for the flat Friedmann–Lemaître–Robertson–Walker universe with metric perturbations in order to reserve the simple notation $\boldsymbol{x}$ for the coordinates after transformation. As we have shown in Refs. [62,63], the coordinate transformations $\boldsymbol{x}_{\rm gl} \to \boldsymbol{x}$ are specified as

$$x^i_{\rm gl} = x^i - \sum_{m=1}^{\infty} s^i{}_{j_1\ldots j_m}(t)x^{j_1}\ldots x^{j_m} + \cdots, \tag{2.42}$$

where $s^i{}_{j_1\ldots j_m}(t)$ is symmetric over j_s with $s = 1, \ldots, m$ and satisfies $\delta^{jj'}s^i{}_{j_1\ldots j\ldots j'\ldots j_m}(t) = 0$. Here, we abbreviated the non-linear terms in Eq. (2.42). These transformations diverge at spatial infinity, no matter how small the coefficients are. Nevertheless, within the local region $\mathcal{O}$, the magnitude of the coordinate transformations (2.42) can be kept perturbatively small. Since the transformations (2.42) are nothing but coordinate transformations, the Heisenberg equations for a diffeomorphism invariant theory remain unchanged under these transformations.

We should note that, once we substitute the expressions for N and N_i to obtain the equation of motion solely written in terms of the curvature perturbation ζ and the graviton perturbation $\delta\gamma_{ij}$, the symmetry under the residual coordinate transformations is lost, because N and N_i depend on the specified boundary conditions. In this sense the coordinate transformations (2.42) should be distinguished from the standard gauge transformations that leave the overall action invariant. Therefore, to avoid confusion, we distinguish the coordinate transformations (2.42), writing it in italics as the *gauge* transformation.

Among the residual *gauge* transformations, we focus on

$$x^i_{\rm gl} = e^{-s(t)} \left[e^{-S(t)/2}\right]^i_{\ j} x^j + \mathcal{O}\left(S^2\right), \tag{2.43}$$

which is concerned with the IR contributions of the curvature and graviton perturbations. Here, $s(t)$ is a time-dependent function and $S_{ij}(t)$ is a time-dependent traceless tensor. When we perform the time-dependent dilatation transformation, the homogeneous part of the curvature perturbation transforms as

$$\zeta \to \zeta - s(t). \tag{2.44}$$

(The precise meaning of this transformation will be explained later.) In Refs. [65,66], we showed that preserving the invariance under the dilatation transformation parametrized by $s(t)$ is crucial in proving the regularity of the loops of the curvature perturbation. Intriguingly, the transformation (2.43) shifts the graviton perturbation as

$$\delta\gamma_{ij} \to \delta\gamma_{ij} - S_{ij}(t) + \mathcal{O}\left(\delta\gamma S, S^2\right) \tag{2.45}$$

at the linear level, which is analogous to the shift for ζ. Although the non-linear extension of the above transformation is rather non-trivial, this observation suggests that an analogous proof of the IR regularity may also work for graviton loops. The relation between the IRdiv due to graviton loops and the homogeneous shift (2.45) has been pointed out several times. Gerstenlauer et al. [69] and Giddings and Sloth [70] showed that the leading IRdiv of the graviton loops can be attributed to the change of the spatial coordinates (2.43) with $s(t) = 0$ due to the accumulated effect of the IR graviton.

2.3. *Genuine gauge invariance and quantization*

The observable fluctuations should not be affected by the residual *gauge* degrees of freedom, which were discussed in the preceding subsection. In this subsection we discuss how to introduce a quantity that is invariant under the residual *gauge* transformations. We call such a quantity a genuinely *gauge* invariant quantity. One may think that the genuine *gauge* invariance will be preserved by fixing the residual *gauge* degrees of freedom completely. If we could perform a complete *gauge* fixing by employing appropriate boundary conditions for N and N_i at the boundary of the observable region $\mathcal{O}$, the IRdiv and IRsec will not appear, because the maximum wavelength of fluctuations in such a *gauge* would be bounded approximately by the size of $\mathcal{O}$. We pursued this possibility in Refs. [61,67]. When we perform the quantization after complete *gauge* fixing or, equivalently, perform the quantization within the local observable region $\mathcal{O}$, the global translation invariance is not easily preserved any longer, leading to technical complexities. To avoid the complexities, in Ref. [61], first we performed the quantization in the whole universe, and then we fixed the coordinates by carrying out the residual *gauge* transformation. In this manner we showed that the absence of the IRdiv and IRsec is guaranteed if the initial fluctuation does not suffer from these IR pathologies. However, it turned out that the IRdiv can arise even in the initial fluctuation after we perform the residual *gauge* transformation to employ complete *gauge* fixing [67].

Here, following Refs. [62,63], we perform the quantization, taking into account the whole universe without fixing the residual *gauge* degrees of freedom, which allows us to keep the global translation invariance manifestly. Then, we construct a genuine *gauge* invariant operator and choose an initial state that will be understood as the genuine *gauge* invariant state. Since the time slice is uniquely specified by the gauge condition (2.2), the genuine *gauge* invariance will be ensured when a quantity preserves the invariance under a change of spatial coordinates.

To construct a genuinely *gauge* invariant operator, we consider correlation functions for the scalar curvature of the induced metric on a $\phi =$ constant hypersurface, ${}^sR$. Since ${}^sR$ itself transforms as a scalar quantity under spatial coordinate transformations, the correlation functions of ${}^sR$ are not invariant. However, the n-point function of ${}^sR$ will become gauge invariant, if we specify its n arguments in a coordinate-independent manner. The distances measured by the spatial geodesics that connect all the pairs of n points characterize the configuration in a coordinate-independent manner. Here, we adopt a slightly easier way, specifying the n spatial points by the geodesic distances and the directional cosines that are measured from an arbitrarily chosen reference point. Although the choice of the reference point and the frame is a part of the residual *gauge*, this ambiguity will not matter as we will choose a quantum state that does not break the spatial homogeneity and isotropy of the universe.

Our geodesic normal coordinates are introduced by solving the spatial geodesic equation on each time slice:

$$\frac{d^2 x^i_{\rm gl}}{d\lambda^2} + {}^s\Gamma^i{}_{jk}\frac{dx^j_{\rm gl}}{d\lambda}\frac{dx^k_{\rm gl}}{d\lambda} = 0, \tag{2.46}$$

where ${}^s\Gamma^i{}_{jk}$ is the Christoffel symbol for the 3D spatial metric $e^{2\zeta}\gamma_{ij}$, and λ is the affine parameter. The initial "velocity" is given by

$$\left.\frac{dx^i_{\rm gl}(\boldsymbol{x},\lambda)}{d\lambda}\right|_{\lambda=0} = e^{-\zeta(\lambda=0)}\,[\gamma(\lambda=0)]^i{}_j\, x^j. \tag{2.47}$$

A point $\boldsymbol{x}$ in the geodesic normal coordinates is identified with the end point of the geodesic $x^i_{\rm gl}$ $(\boldsymbol{x}, \lambda = 1)$ in the original coordinates. Perturbatively expanding $x^i_{\rm gl}$ in terms of x^i, we obtain

$$x^i_{\rm gl} = x^i + \delta x^i(\boldsymbol{x}).$$

Notice that the relation between $\boldsymbol{x}_{\rm gl}$ and $\boldsymbol{x}$ depends on the metric perturbations, which become quantum operators after quantization. Finally, we find that, by means of the geodesic normal coordinates, the genuinely *gauge* invariant variable is given by

$${}^gR(x) \equiv {}^sR(t, x^i_{\rm gl}(\boldsymbol{x})) = {}^sR(t, x^i + \delta x^i(\boldsymbol{x})). \tag{2.48}$$

In order to calculate the n-point functions of ${}^gR$, we also need to specify the quantum state that is invariant under the residual *gauge* transformations. However, in the present approach, we cannot directly discuss this invariance as a condition for allowed quantum states. This is because the residual *gauge* degrees of freedom cease to exist when we quantize fields in the whole universe. Let us recall the discussion in the case of the curvature perturbation ζ [65–67]. By construction, the operator

${}^g R$ is not affected by the residual *gauge* degrees of freedom. However, the n-point functions of ${}^g R$ can be correlated to the fields in the causally disconnected region. In Sect. 2.2.3, we discussed two ways in which the outside of the observable region $\mathcal{O}$ can affect the fluctuation in $\mathcal{O}$. One is through the boundary conditions for the inverse Laplacian ∂^{-2}. Since changing these boundary conditions is nothing but performing the residual *gauge* transformation (see Sect. 4.2.2), the n-point functions of the genuine *gauge* invariant curvature perturbation ${}^g R$ are not affected even if we restrict the integration region of ∂^{-2} to the region $\mathcal{O}$. Therefore, as long as we consider a genuinely *gauge* invariant operator, the inverse Laplacian ∂^{-2} never gives the conjunction between the inside and the far outside of $\mathcal{O}$. The other leak of the influence from the outside of $\mathcal{O}$ is due to the long-range correlation through the Wightman functions, which can remain even if we consider genuinely *gauge* invariant variables. Therefore, this long-range correlation may give a possible origin for the IRdiv and IRsec. In the case with the curvature perturbation ζ, it is shown that requesting the IR regularity by suppressing the long-range correlation constrains the quantum state of the inflationary universe [65]. Interestingly, the IR regularity condition on quantum states can be interpreted as the condition that requests the quantum states to be unaffected by the time-dependent dilatation transformation, which is one of the residual *gauge* degrees of freedom [65]. In Appendix A, we show that a similar genuine *gauge* invariance condition on quantum states is derived from the IR regularity condition for the graviton perturbation as well.

3. Preliminaries

In this section, as preparation for analyzing the n-point functions of the genuine *gauge* invariant curvature perturbation, we introduce a family of canonical variables. First, in Sect. 3.1, we describe the basic formulation for the canonical quantization in terms of the original set of variables ζ, $\delta\gamma_{ij}$ and their conjugate momenta. In Sect. 3.2, we introduce a family of alternative sets of canonical variables, in terms of which the proof of the IR regularity becomes more transparent.

3.1. *Canonical quantization*

For notational simplicity, we suppress the subscript "gl" in this subsection. In the following discussion, we express the action for the curvature perturbation ζ and the graviton perturbation $\delta\gamma_{ij}$ derived by solving the Hamiltonian and momentum constraint equations as

$$S = \int dt \int d^3\boldsymbol{x}\mathcal{L}_{dyn}\left[\zeta(x), \partial_t\zeta(x), \delta\gamma_{ij}(x), \partial_t\delta\gamma_{ij}(x)\right], \tag{3.1}$$

which includes the non-local integration operator ∂^{-2}. Here, $\mathcal{L}_{dyn}$ denotes the functional form of the Lagrangian density obtained after we eliminate the Lagrange multipliers N and N_i. We also introduce the Hamiltonian H and the Hamiltonian density $\mathcal{H}$ as

$$\begin{aligned} H(t) &\equiv \int d^3\boldsymbol{x}\pi(x)\partial_t\zeta(x) + \int d^3\boldsymbol{x}\pi^{ij}(x)\partial_t\delta\gamma_{ij}(x) \\ &\quad - \int d^3\boldsymbol{x}\mathcal{L}_{dyn}\left[\zeta(x), \partial_t\zeta(x), \delta\gamma_{ij}(x), \partial_t\delta\gamma_{ij}(x)\right] \\ &\equiv \int d^3\boldsymbol{x}\mathcal{H}[\zeta(x), \pi(x), \delta\gamma_{ij}(x), \pi_{ij}(x)], \end{aligned} \tag{3.2}$$

where we have introduced the conjugate momenta as

$$\pi(x) \equiv \frac{\partial \mathcal{L}_{dyn}(x)}{\partial\left(\partial_t \zeta(x)\right)}, \quad \pi^{ij}(x) \equiv \frac{\partial \mathcal{L}_{dyn}(x)}{\partial\left(\partial_t \delta\gamma_{ij}(x)\right)}. \tag{3.3}$$

This set of canonical variables $\Phi \equiv \{\zeta, \pi, \delta\gamma_{ij}, \pi_{ij}\}$ should satisfy the standard commutation relations

$$[\zeta(t, \boldsymbol{x}), \pi(t, \boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \quad [\zeta(t, \boldsymbol{x}), \zeta(t, \boldsymbol{y})] = [\pi(t, \boldsymbol{x}), \pi(t, \boldsymbol{y})] = 0, \tag{3.4}$$

and

$$\left[\delta\gamma_{ij}(t, \boldsymbol{x}), \pi^{kl}(t, \boldsymbol{y})\right] = i\delta^{(3)kl}_{ij}(\boldsymbol{x}-\boldsymbol{y}), \quad \left[\delta\gamma_{ij}(t, \boldsymbol{x}), \delta\gamma_{kl}(t, \boldsymbol{y})\right] = \left[\pi^{ij}(t, \boldsymbol{x}), \pi^{kl}(t, \boldsymbol{y})\right] = 0, \tag{3.5}$$

where

$$\delta^{(3)kl}_{ij}(\boldsymbol{x}-\boldsymbol{y}) \equiv \frac{1}{2}\sum_{\lambda=\pm}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} e_{ij}(\boldsymbol{k}, \lambda) e^{kl}(\boldsymbol{k}, \lambda) \tag{3.6}$$

is the tensorial delta function with the transverse traceless projection.

3.2. *Canonical transformation associated with residual gauge transformations*

In this subsection, we introduce a family of alternative sets of canonical variables

$$\tilde{\Phi} \equiv \{\tilde{\zeta}, \tilde{\pi}, \delta\tilde{\gamma}_{ij}, \tilde{\pi}_{ij}\} \tag{3.7}$$

whose Hamiltonian $\tilde{H}(t)$ is written only in terms of

$$\tilde{\zeta}(x) - s(t), \quad \tilde{\pi}(x), \quad \delta\tilde{\gamma}_{ij}(x) - S_{ij}(t), \quad \tilde{\pi}^{ij}(x), \tag{3.8}$$

where $s(t)$ and $S_{ij}(t)$ are an arbitrary time-dependent function and a symmetric-traceless matrix, respectively. We treat both $s(t)$ and $S_{ij}(t)$ perturbatively, assuming that they are as small as metric perturbations. We also show that $s(t)$ and $S_{ij}(t)$ without time differentiation are only contained in the Hamiltonian $\tilde{H}(t)$ only in the combination described in Eq. (3.8).

3.2.1. Introducing new canonical variables. For illustrative purposes, we first consider a coordinate transformation with s and S_{ij} time independent, which induces constant shifts $\tilde{\zeta}(x) - s$ and $\delta\tilde{\gamma}_{ij}(t, \boldsymbol{x}) - S_{ij}$. To be concrete, we consider the coordinate transformation $\boldsymbol{x}_{\rm gl} \to \boldsymbol{x}$ with

$$x^i_{\rm gl} \equiv e^{-s}\Lambda^i_{Tj} x^j \equiv \Lambda^i_{\ j} x^j, \tag{3.9}$$

where Λ^i_{Tj} is a functional of S_{ij} that satisfies

$$\det \Lambda_T = 1 \tag{3.10}$$

and

$$\Lambda^{i}_{Tj} = \delta^{i}_{j} - \frac{1}{2}S^{i}_{j} + \mathcal{O}(S^2). \tag{3.11}$$

Notice that the coordinate transformation $\boldsymbol{x}_{\rm gl} \to \boldsymbol{x}$ does not change the boundary condition of N and N_i, and hence it is one of the gauge transformations that leave the action invariant. For the time being, we will not specify the terms of $\mathcal{O}(S^2)$ in Λ^{i}_{Tj}.

Next, we consider the change of the spatial metric g_{ij} under the gauge transformation (3.9). As addressed in Ref. [65], when we neglect the graviton perturbation, setting $\Lambda^{i}_{Tj} = 0$, the dilatation transformation changes the spatial metric as

$$e^{2(\rho+\zeta(x_{\rm gl}))}\delta_{ij}dx^i_{\rm gl}dx^j_{\rm gl} = e^{2(\rho+\tilde{\zeta}(x)-s)}\delta_{ij}dx^i dx^j,$$

where we have defined $\tilde{\zeta}(x) \equiv \zeta(x_{\rm gl})$. We find that the curvature perturbation $\zeta(x_{\rm gl})$ transforms to $\tilde{\zeta}(x) - s$, which suggests that this scaling transformation can be used to find the canonical variables $\tilde{\Phi}$ that are subjected to the necessary constant shift.

Compared with the curvature perturbation, finding a transformation that shifts the graviton perturbation by $-S_{ij}$ is much more non-trivial, particularly at non-linear order. Therefore, introducing the transverse traceless tensor $\delta\tilde{\gamma}_{ij}$, we express the spatial metric obtained after the coordinate transformation (3.9) as

$$\tilde{g}_{ij}(x) \equiv e^{2\{\rho+\tilde{\zeta}(x)-s\}}\tilde{\gamma}_{ij}(x) \equiv e^{2\{\rho+\tilde{\zeta}(x)-s\}}\left[e^{\delta\tilde{\gamma}(x)-S}\right]_{ij}, \tag{3.12}$$

with the requested shift by $-S_{ij}$. In the following, we assume that $s(t)$ and $S_{ij}(t)$ are of the same order as $\tilde{\zeta}$ and $\delta\tilde{\gamma}_{ij}$. From $g_{ij}dx^i_{\rm gl}dx^j_{\rm gl} = \tilde{g}_{ij}dx^i dx^j$, we find that $\delta\tilde{\gamma}_{ij}(x)$ should be related to $\delta\gamma_{ij}(x_{\rm gl})$ as

$$\tilde{\gamma}_{ij}(x) = \gamma_{kl}(x_{\rm gl})(\Lambda_T)^{k}_{i}(\Lambda_T)^{l}_{j}. \tag{3.13}$$

Once the functional form of Λ^{i}_{Tj} is determined, Eq. (3.13) specifies $\delta\tilde{\gamma}_{ij}$ order by order in perturbation. By expanding the inverse matrix of Λ^{i}_{Tj} as

$$(\Lambda_T^{-1})_{ij} = \delta_{ij} + \frac{1}{2}S_{ij} + \mathcal{O}(S^2),$$

Eq. (3.13) leads to

$$\delta\gamma_{ij}(x_{\rm gl}) = \delta\tilde{\gamma}_{ij}(x_{\rm gl}) + \mathcal{O}\left(\delta\gamma S,\, S^2\right). \tag{3.14}$$

On the right-hand side, we have explicitly written only the linear order in perturbation. Since the left-hand side of Eq. (3.14) is independent of S_{ij}, the field $\delta\tilde{\gamma}_{ij}$ should be defined so that the S_{ij} dependence on the right-hand side is canceled. In particular, Eq. (3.14) states that $\delta\gamma_{ij}(x_{\rm gl})$ should agree with $\delta\tilde{\gamma}_{ij}(x_{\rm gl})$ at the linear order in perturbation. Since the diffeomorphism invariance of the action implies that the Lagrangian densities for g_{ij} and $\tilde{g}_{ij}$ should take the same functional form, using the Lagrangian density $\mathcal{L}_{dyn}$ in Eq. (3.1), we can express the action for $\tilde{g}_{ij}$ as

$$S = \int dt \int d^3\boldsymbol{x}\mathcal{L}_{dyn}\left[\tilde{\zeta}(x) - s,\, \partial_t\tilde{\zeta}(x),\, \delta\tilde{\gamma}_{ij}(x) - S_{ij},\, \partial_t\delta\tilde{\gamma}_{ij}(x)\right]. \tag{3.15}$$

Next, we extend the above argument to time-dependent transformations with

$$x^i_{\rm gl} \equiv e^{-s(t)}\Lambda^i_{T\,j}(t)x^j \equiv \Lambda^i_{\ j}(t)x^j\,, \tag{3.16}$$

where $\Lambda^i_{T\,j}(t)$ is a functional of $S_{ij}(t)$ whose explicit form will be specified later. Similarly to the case of constant $\Lambda^i_{\ j}$, we introduce a new set of canonical variables $\tilde{\Phi}$ by

$$\tilde{\zeta}(x) \equiv \zeta(x_{\rm gl}), \tag{3.17}$$

$$\tilde{\gamma}_{ij}(x) \equiv \left[e^{\delta\tilde{\gamma}(x)-S(t)}\right]_{ij} \equiv \gamma_{kl}(x_{\rm gl})(\Lambda_T)^k_{\ i}(\Lambda_T)^l_{\ j}, \tag{3.18}$$

with the formal definition of the conjugate momenta given by

$$\tilde{\pi}(x) \equiv \frac{\partial\mathcal{L}_{dyn}(x)}{\partial(\partial_t\tilde{\zeta}(x))}, \quad \tilde{\pi}^{ij}(x) \equiv \frac{\partial\mathcal{L}_{dyn}(x)}{\partial(\partial_t\delta\tilde{\gamma}_{ij}(x))}. \tag{3.19}$$

In general, the residual *gauge* transformation is not well defined in the whole universe, since the transformation can diverge at spatial infinity. However, the residual *gauge* transformations (3.16), exceptionally, keep the variables finite in the whole universe, as is manifest from the above relations. Therefore, we can consistently discuss quantum theory in terms of the set of canonical variables $\tilde{\Phi}$ as well.

3.2.2. Commutation relations. Next, we will show that the variables $\tilde{\Phi} = \{\tilde{\zeta}, \tilde{\pi}, \delta\tilde{\gamma}_{ij}, \tilde{\pi}_{ij}\}$, defined in Eqs. (3.17), (3.18), and (3.19), satisfy the standard commutation relation. Because of the time variation of $\Lambda^i_{\ j}$, the partial time derivative with the original global spatial coordinates $\boldsymbol{x}_{\rm gl}$ fixed differs from the one with the new coordinates $\boldsymbol{x}$ fixed. We choose the transformation matrix $\Lambda^i_{T\,j}$ such that the difference between the two partial time derivative operations does not give $S_{ij}(t)$ without the time derivative. Then, we find that $\Lambda^i_{T\,j}$ should satisfy

$$\frac{d}{dt}\Lambda_T[S(t)]^i_{\ j} = -\frac{1}{2}\Lambda_T[S(t)]^i_{\ k}\dot{S}(t)^k_{\ j}, \tag{3.20}$$

or equivalently

$$\frac{d}{dt}\Lambda_T^{-1}[S(t)]^i_{\ j} = \frac{1}{2}\dot{S}(t)^i_{\ k}\Lambda_T^{-1}[S(t)]^k_{\ j}. \tag{3.21}$$

In fact, with this choice of $\Lambda^i_{T\,j}$, we obtain

$$\partial_t\zeta(t,\boldsymbol{x}_{\rm gl}) = \partial_t\tilde{\zeta}(x) + \left[\dot{s}(t)\mathbf{1} + \frac{\dot{S}(t)}{2}\right]^n_{\ m} x^m\frac{\partial}{\partial x^n}\tilde{\zeta}(x), \tag{3.22}$$

$$\partial_t\gamma_{ij}(t,\boldsymbol{x}_{\rm gl}) = \left\{\partial_t\tilde{\gamma}_{kl}(x) + \dot{S}_{(k^m(t)}\tilde{\gamma}_{l)m}(x) + \left[\dot{s}(t)\mathbf{1} + \frac{\dot{S}(t)}{2}\right]^n_{\ m} x^m\frac{\partial}{\partial x^n}\tilde{\gamma}_{kl}(x)\right\}\left(\Lambda_T^{-1}\right)^k_{\ i}\left(\Lambda_T^{-1}\right)^l_{\ j}, \tag{3.23}$$

where $\mathbf{1}$ denotes the unit matrix and the round brackets on indices represent symmetrization. The terms with spatial derivatives on the right-hand sides of Eqs. (3.22) and (3.23) stem from the difference between the two partial time derivative operators. Since Eq. (3.20) implies

$$\frac{d}{dt}\det\Lambda_T[S(t)] = 0, \tag{3.24}$$

the condition (3.10) can be extended to the time-dependent case as

$$\det\Lambda_T[S(t)] = 1. \tag{3.25}$$

Using Eq. (3.22), we find that the conjugate momentum $\tilde{\pi}$ is related to π as

$$\tilde{\pi}(x) = \frac{\partial\mathcal{L}_{dyn}(x)}{\partial[\partial_t\tilde{\zeta}(x)]} = e^{-3s(t)}\frac{\partial\mathcal{L}_{dyn}(x_{\rm gl})}{\partial[\partial_t\zeta(x_{\rm gl})]} = e^{-3s(t)}\pi(x_{\rm gl}). \tag{3.26}$$

In the second equality we have used

$$[\det\Lambda^{-1}(t)]\mathcal{L}_{dyn}(x) = e^{3s(t)}\mathcal{L}_{dyn}(x) = \mathcal{L}_{dyn}(x_{\rm gl}), \tag{3.27}$$

which is derived by changing the spatial coordinates in the action from $\boldsymbol{x}$ to $\boldsymbol{x}_{\rm gl}$. Deriving the relation between $\tilde{\pi}^{ij}$ and π^{ij} is more non-trivial, but using

$$\frac{\partial(\partial_t\delta\gamma_{ij}(x_{\rm gl}))}{\partial(\partial_t\delta\tilde{\gamma}_{kl}(x))} = \frac{\partial(\partial_t\tilde{\gamma}_{mn}(x))}{\partial(\partial_t\delta\tilde{\gamma}_{kl}(x))}\frac{\partial(\partial_t\gamma_{pq}(x_{\rm gl}))}{\partial(\partial_t\tilde{\gamma}_{mn}(x))}\frac{\partial(\partial_t\delta\gamma_{ij}(x_{\rm gl}))}{\partial(\partial_t\gamma_{pq}(x_{\rm gl}))} = \frac{\partial\delta\gamma_{ij}(x_{\rm gl})}{\partial\delta\tilde{\gamma}_{kl}(x)}, \tag{3.28}$$

where in the second equality we have used

$$\frac{\partial(\partial_t\gamma_{ij}(x_{\rm gl}))}{\partial(\partial_t\delta\gamma_{kl}(x_{\rm gl}))} = \frac{\partial\gamma_{ij}(x_{\rm gl})}{\partial\delta\gamma_{kl}(x_{\rm gl})}, \qquad \frac{\partial(\partial_t\tilde{\gamma}_{ij}(x))}{\partial(\partial_t\delta\tilde{\gamma}_{kl}(x))} = \frac{\partial\tilde{\gamma}_{ij}(x)}{\partial\delta\tilde{\gamma}_{kl}(x)}, \tag{3.29}$$

and

$$\frac{\partial(\partial_t\gamma_{ij}(x_{\rm gl}))}{\partial(\partial_t\tilde{\gamma}_{kl}(x))} = \frac{\partial\gamma_{ij}(x_{\rm gl})}{\partial\tilde{\gamma}_{kl}(x)} = \left(\Lambda_T^{-1}\right)^k_{\left(i\left(\Lambda_T^{-1}\right)^l_j\right)}, \tag{3.30}$$

which can be derived by using Eq. (3.23), we obtain

$$\tilde{\pi}^{ij}(x) = \frac{\partial\mathcal{L}_{dyn}(x)}{\partial(\partial_t\delta\tilde{\gamma}_{ij}(x))} = e^{-3s(t)}\frac{\partial(\partial_t\delta\gamma_{kl}(x_{\rm gl}))}{\partial(\partial_t\delta\tilde{\gamma}_{ij}(x))}\frac{\partial\mathcal{L}_{dyn}(x_{\rm gl})}{\partial(\partial_t\delta\gamma_{kl}(x_{\rm gl}))} = e^{-3s(t)}\frac{\partial\delta\gamma_{kl}(x_{\rm gl})}{\partial\delta\tilde{\gamma}_{ij}(x)}\pi^{kl}(x_{\rm gl}). \tag{3.31}$$

Here, we simply assume that the operator ordering is properly chosen.

Once we establish the relations between the two sets of the canonical variables, Φ and $\tilde{\Phi}$, the commutation relations for Φ yield the commutation relations for $\tilde{\Phi}$. Using Eqs. (3.4), (3.17), and (3.26), we obtain

$$\left[\tilde{\zeta}(t,\boldsymbol{x}), \tilde{\pi}(t,\boldsymbol{y})\right] = ie^{-3s(t)}\delta^{(3)}(\boldsymbol{x}_{\rm gl} - \boldsymbol{y}_{\rm gl}) = i\delta^{(3)}\left(\boldsymbol{x} - \boldsymbol{y}\right). \tag{3.32}$$

Similarly, using Eqs. (3.5), (3.18), and (3.31), we find

$$\left[\delta\tilde{\gamma}_{ij}(t,\boldsymbol{x}),\tilde{\pi}^{kl}(t,\boldsymbol{y})\right]=ie^{-3s(t)}\frac{\partial\delta\tilde{\gamma}_{ij}(t,\boldsymbol{x})}{\partial\delta\gamma_{mn}(t,\boldsymbol{x}_{\rm gl})}\frac{\partial\delta\gamma_{pq}(t,\boldsymbol{y}_{\rm gl})}{\partial\delta\tilde{\gamma}_{kl}(t,\boldsymbol{y})}\delta^{(3)pq}_{{\rm gl}mn}(\boldsymbol{x}_{\rm gl}-\boldsymbol{y}_{\rm gl})$$
$$=i\delta^{(3)kl}_{ij}(\boldsymbol{x}-\boldsymbol{y})\,. \tag{3.33}$$

In the second equality we have noted that the tensorial delta function $\delta^{(3)\,kl}_{{\rm gl}ij}(\boldsymbol{x}_{\rm gl}-\boldsymbol{y}_{\rm gl})$, given in Eq. (3.6), can be expressed as

$$\delta^{(3)\,kl}_{{\rm gl}ij}(\boldsymbol{x}_{\rm gl}-\boldsymbol{y}_{\rm gl})=\frac{1}{2}\sum_{\lambda=\pm}\int d^3\boldsymbol{k}\frac{\partial\delta\gamma_{ij}(t,\boldsymbol{x}_{\rm gl})}{\partial\Gamma^{(\lambda)}(t,\boldsymbol{k})}\frac{\partial\Gamma^{(\lambda)}(t,\boldsymbol{k})}{\partial\delta\gamma_{kl}(t,\boldsymbol{y}_{\rm gl})}=\frac{1}{2}\frac{\delta(\delta\gamma_{ij}(t,\boldsymbol{x}_{\rm gl}))}{\delta(\gamma_{kl}(t,\boldsymbol{y}_{\rm gl}))},$$

by expanding $\delta\gamma_{ij}(x_{\rm gl})$ as

$$\delta\gamma_{ij}(t,\boldsymbol{x}_{\rm gl})=\sum_{\lambda=\pm}\int\frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}}e^{i\boldsymbol{k}\cdot\boldsymbol{x}_{\rm gl}}e_{ij}(\boldsymbol{k},\lambda)\Gamma^{(\lambda)}(t,\boldsymbol{k}),$$

and we have used the factor $e^{-3s(t)}$ to change the argument from $\boldsymbol{x}_{\rm gl}$ to $\boldsymbol{x}$. The remaining commutation relations can be shown in a similar way and hence we can verify that $\tilde{\Phi}$ actually qualifies as a set of canonical variables.

Solving Eq. (3.20), we can determine the transformation matrix $\Lambda^{i}_{Tj}(t)$. As a boundary condition to solve the first-order differential equation, we employ the condition

$$\Lambda^{i}_{Tj}(t_f)=\left[e^{-S(t_f)/2}\right]^{i}_{j}, \tag{3.34}$$

at the end of inflation t_f. Since we have chosen $\Lambda^{i}_{Tj}(t)$ so as to satisfy $\det\Lambda_T(t_f)=1$, Eq. (3.24) guarantees that the condition (3.25) holds for all t. Notice that we can formally solve Eq. (3.20) as

$$\Lambda^{i}_{Tj}(t)=\left[\Lambda_T(t_f)Te^{\frac{1}{2}\int_t^{t_f}dt'\dot{S}(t')}\right]^{i}_{j}, \tag{3.35}$$

using the time-ordered product denoted by the operator T. Perturbatively expanding $\Lambda^{i}_{Tj}(t)$ with respect to $S_{ij}(t)$ to the next to leading order, we obtain

$$\Lambda^{i}_{Tj}(t)=\delta^{i}_{j}-\frac{1}{2}S^{i}_{j}(t)+\mathcal{O}\left(S^2\right). \tag{3.36}$$

3.2.3. Hamiltonians. Next, we compute the Hamiltonian for $\tilde{\Phi}$ defined by

$$\tilde{H}(t)\equiv\int d^3\boldsymbol{x}\tilde{\pi}(x)\partial_t\tilde{\zeta}(x)+\int d^3\boldsymbol{x}\tilde{\pi}^{ij}(x)\partial_t\delta\tilde{\gamma}_{ij}(x)-\int d^3\boldsymbol{x}\mathcal{L}_{dyn}(x). \tag{3.37}$$

Using Eqs. (3.22), (3.23), (3.26), and (3.31), we can relate the Hamiltonian $\tilde{H}(t)$ to $H(t)$ as

$$\tilde{H}(t) = H(t) - \left[\dot{s}(t)\mathbf{1} + \frac{\dot{S}(t)}{2}\right]_k^l \int d^3\boldsymbol{x} \left[\tilde{\pi}(x)x^k \frac{\partial}{\partial x^l}\tilde{\zeta}(x) + \tilde{\pi}^{ij}(x)x^k \frac{\partial}{\partial x^l}\delta\tilde{\gamma}_{ij}(x)\right]$$
$$- \int d^3\boldsymbol{x}\tilde{\pi}^{ij}(x)\left[\dot{S}_k^m(t)\frac{\partial\delta\tilde{\gamma}_{ij}(x)}{\partial\tilde{\gamma}_{kl}(x)}\tilde{\gamma}_{ml}(x) - \dot{S}_{ij}(t)\right]. \tag{3.38}$$

Equation (3.38) reveals that, when $s(t)$ or $S_{ij}(t)$ is time-dependent, the Hamiltonian $\tilde{H}(t)$ differs from $H(t)$. However, this difference does not appear in the quadratic terms of the perturbed variables. In fact, the linear terms in the square brackets on the second line cancel each other. Using Eqs. (3.17), (3.18), (3.26), and (3.31), we can express the Hamiltonian $H(t)$ in terms of $\tilde{\Phi}$ as

$$H(t) = \int d^3\boldsymbol{x}\mathcal{H}\left[\tilde{\zeta}(x) - s(t), \tilde{\pi}(x), \delta\tilde{\gamma}_{ij}(x) - S_{ij}(t), \tilde{\pi}^{ij}(x)\right], \tag{3.39}$$

where $\mathcal{H}$ is the Hamiltonian density defined in Eq. (3.2). Rewriting the graviton part is slightly non-trivial, but this can be confirmed as follows. When we express $H(t)$ in terms of

$$\gamma_{ij}(x_{\rm gl}) \quad \text{and} \quad \frac{\partial\mathcal{L}_{dyn}(x_{\rm gl})}{\partial(\partial_t\gamma_{ij}(x_{\rm gl}))} = \frac{\partial\delta\gamma_{kl}(x_{\rm gl})}{\partial\gamma_{ij}(x_{\rm gl})}\pi^{kl}(x_{\rm gl}),$$

these two variables transform as standard tensors in three dimensions into

$$\tilde{\gamma}_{ij}(x) \quad \text{and} \quad \frac{\partial\mathcal{L}_{dyn}(x)}{\partial(\partial_t\tilde{\gamma}_{ij}(x))} = \frac{\partial\delta\tilde{\gamma}_{kl}(x)}{\partial\tilde{\gamma}_{ij}(x)}\tilde{\pi}^{kl}(x),$$

leaving aside the factor $e^{3s(t)}$, which will be absorbed to make the combination $\tilde{\zeta}(x) - s(t)$ (see Ref. [66]). Then, since γ_{ij} and $\tilde{\gamma}_{ij}$ are given by $[\exp(\delta\gamma)]_{ij}$ and $[\exp(\delta\tilde{\gamma} - S)]_{ij}$, respectively, we can verify Eq. (3.39).

Next, collecting the quadratic terms in $\tilde{\Phi}$ from the Hamiltonian, we identify the non-interacting Hamiltonian,

$$\tilde{H}_0(t) = \int d^3\boldsymbol{x}\mathcal{H}_0\left[\tilde{\zeta}(x), \tilde{\pi}(x), \delta\tilde{\gamma}_{ij}(x), \tilde{\pi}^{ij}(x)\right], \tag{3.40}$$

which is exactly the same form as the one for Φ. Since both $\tilde{\zeta}$ and $\delta\tilde{\gamma}_{ij}$, which are massless fields, appear with spatial derivative operators in the non-interacting Hamiltonian, the shifts by $-s(t)$ and $-S_{ij}(t)$, respectively, are eliminated. We also introduce the interaction Hamiltonian as

$$\tilde{H}_I(t) \equiv \tilde{H}(t) - \tilde{H}_0(t) \equiv \int d^3\boldsymbol{x}\tilde{\mathcal{H}}_I[\tilde{\zeta}(x) - s(t), \tilde{\pi}(x), \delta\tilde{\gamma}_{ij}(x) - S_{ij}(t), \tilde{\pi}^{ij}(x)], \tag{3.41}$$

with

$$\begin{aligned}
&\tilde{\mathcal{H}}_I\left[\tilde{\zeta}(x)-s(t),\tilde{\pi}(x),\delta\tilde{\gamma}_{ij}(x)-S_{ij}(t),\tilde{\pi}^{ij}(x)\right]\\
&\quad=\mathcal{H}_I\left[\tilde{\zeta}(x)-s(t),\tilde{\pi}(x),\delta\tilde{\gamma}_{ij}(x)-S_{ij}(t),\tilde{\pi}^{ij}(x)\right]-\left[\dot{s}(t)\mathbf{1}+\frac{\dot{S}(t)}{2}\right]^l_k\\
&\qquad\times\left[\tilde{\pi}(x)x^k\frac{\partial}{\partial x^l}\tilde{\zeta}(x)+\tilde{\pi}^{ij}(x)x^k\frac{\partial}{\partial x^l}\delta\tilde{\gamma}_{ij}(x)\right]\\
&\qquad-\tilde{\pi}^{ij}(x)\left[\dot{S}^m_k(t)\frac{\partial\delta\tilde{\gamma}_{ij}(x)}{\partial\tilde{\gamma}_{kl}(x)}\tilde{\gamma}_{ml}(x)-\dot{S}_{ij}(t)\right],
\end{aligned}\tag{3.42}$$

where

$$\mathcal{H}_I[\Phi]\equiv\mathcal{H}[\Phi]-\mathcal{H}_0[\Phi]\tag{3.43}$$

is the interaction Hamiltonian density for Φ.

3.2.4. Short summary of the strategy. It will be instructive to note the two important properties of the interaction Hamiltonian density $\tilde{\mathcal{H}}_I$, given in Eq. (3.42), which will become crucial in the discussion of the IR regularity: First, the fields $\tilde{\zeta}(x)$ and $\delta\tilde{\gamma}_{ij}(x)$ always accompany the time-dependent parameters $s(t)$ and $S_{ij}(t)$ as $\tilde{\zeta}(x)-s(t)$ and $\delta\tilde{\gamma}_{ij}(x)-S_{ij}(t)$. Second, $s(t)$ and $S_{ij}(t)$, which are not accompanied by $\tilde{\zeta}(x)$ and $\delta\tilde{\gamma}_{ij}(x)$, respectively, are always differentiated with respect to time. When we consider only the adiabatic perturbation, we can provide a new set of canonical variables that fulfills these two properties simply by considering the time-dependent dilatation transformation $\boldsymbol{x}_{\rm gl}\to\boldsymbol{x}$ with $\boldsymbol{x}_{\rm gl}=e^{-s(t)}\boldsymbol{x}$, which is one of the residual *gauge* transformations [65,66]. At the linear order, the residual *gauge* transformation with Eq. (2.43) shifts the spatially homogeneous part of the graviton perturbation. However, at non-linear order, due to the non-commutativity between matrices, the residual *gauge* transformation with (2.43) does not immediately introduce the shift of all the graviton perturbation in the action.

To provide a new set of canonical variables with their homogeneous parts shifted, we have introduced a more non-trivial transformation (3.16). By choosing $\delta\tilde{\gamma}_{ij}$ as in Eq. (3.18), the first property can be ensured. To guarantee the second property, we have determined (the time dependence of) $\Lambda^i_{Tj}(t)$, requesting Eq. (3.20). Then, $s(t)$ and $S_{ij}(t)$ without the time derivative appear neither on the right-hand sides of Eqs. (3.22) and (3.23) nor in the Hamiltonian $\tilde{H}(t)$. Using these properties, we later show the IR regularity of graviton loops in a parallel way to the case of the curvature perturbation.

3.3. Coarse-grained gauge invariant operator

In Sect. 2.3, we introduced the genuine *gauge* invariant variable ${}^gR$, using the geodesic normal coordinates. Changing the spatial coordinates to the geodesic normal coordinates also modifies the UV contributions. Tsamis and Woodard [106] pointed out that using the geodesic normal coordinates can introduce an additional origin of UV divergence, yielding contributions that may not be renormalized by local counter terms [107]. This is presumably because specifying the spatial distance precisely in the presence of the gravitational perturbation requires taking account of all short-wavelength modes. In any realistic observations, what we actually observe is a smeared field with a finite resolution. However, it is not so trivial to describe a realistic smearing in a genuinely gauge invariant manner. Here, in order to remove the UV contribution in the measurement of the position, we replace the geodesic normal coordinates with approximate ones that are not affected by the UV contributions.

In this paper, we compute the n-point functions at the end of inflation $t = t_f$. Then, in place of the geodesic normal coordinates, we use the "smeared" coordinates x^i, which are related to the global coordinates $\hat{x}^i_{\rm gl}$ as[1]

$$\hat{x}^i_{\rm gl} \equiv e^{-{}^g\bar{\zeta}(t_f)} \left[e^{-\delta^g\bar{\gamma}(t_f)/2} \right]^i_{\ j} x^j , \tag{3.44}$$

where we have replaced $s(t_f)$ and $S_{ij}(t_f)$ in the transformation matrix $\Lambda^i_{\ j}(t_f)$ with the smeared metric perturbations:

$$ {}^g\bar{\zeta}(t_f) \equiv \frac{\int d^3\boldsymbol{x} W_{t_f}(\boldsymbol{x}) \zeta(t_f, \hat{\boldsymbol{x}}_{\rm gl})}{\int d^3\boldsymbol{x} W_{t_f}(\boldsymbol{x})}, \tag{3.45}$$

$$ \delta^g\bar{\gamma}_{ij}(t_f) \equiv \frac{\int d^3\boldsymbol{x} W_{t_f}(\boldsymbol{x}) \delta\gamma_{Sij}[t_f, \hat{\boldsymbol{x}}_{\rm gl}, \delta^g\bar{\gamma}(t_f)]}{\int d^3\boldsymbol{x} W_{t_f}(\boldsymbol{x})}. \tag{3.46}$$

Here, $W_t(\boldsymbol{x})$ is a window function that takes a non-vanishing value in the local region $\mathcal{O}_t$ and

$$\delta\gamma_{Sij}[t, \boldsymbol{x}_{\rm gl}; S] \equiv \left[\ln \left(\gamma(t, \boldsymbol{x}_{\rm gl}) \Lambda_T(t) \Lambda_T(t) \right) \right]_{ij} + S_{ij}, \tag{3.47}$$

which implicitly depends on the values of $S_{ij}(t')$ with $t \le t' \le t_f$ through $\Lambda^i_{Tj}(t)$. Notice that Λ^i_{Tj} at $t = t_f$ is exceptionally determined by the value of S_{ij} only at $t = t_f$ owing to the boundary condition (3.34). Although ${}^g\bar{\zeta}$ and $\delta^g\bar{\gamma}_{ij}$ appear on the right-hand sides of Eqs. (3.45) and (3.46), we can iteratively define ${}^g\bar{\zeta}$ and $\delta^g\bar{\gamma}_{ij}$ by these expressions. Using the quantities introduced above, we define ${}^g\zeta(t_f, \boldsymbol{x})$ and $\delta^g\gamma_{ij}(t_f, \boldsymbol{x})$ as

$$ {}^g\zeta(t_f, \boldsymbol{x}^i) \equiv \zeta(t_f, \hat{\boldsymbol{x}}_{\rm gl}), \tag{3.48}$$

$$ \delta^g\gamma_{ij}(t_f, \boldsymbol{x}^i) \equiv \delta\gamma_{Sij}[t_f, \hat{\boldsymbol{x}}_{\rm gl}; \delta^g\bar{\gamma}(t_f)]. \tag{3.49}$$

Notice that $\boldsymbol{x}_{\rm gl}$ includes ζ and $\delta\gamma_{ij}$ but does not include their canonical conjugate momenta. Hence, we can define ${}^g\bar{\zeta}(t_f)$, $\delta^g\bar{\gamma}_{ij}(t_f)$, ${}^g\zeta(t_f, \boldsymbol{x})$, and $\delta^g\gamma_{ij}(t_f, \boldsymbol{x})$ without ambiguity of the operator ordering.

The fields ${}^g\zeta(t_f, \boldsymbol{x})$ and $\delta^g\gamma_{ij}(t_f, \boldsymbol{x})$ introduced above are not genuinely *gauge* invariant. However, we can show that $\mathcal{R}_x {}^g\zeta(t_f, \boldsymbol{x})$ and $\mathcal{R}_x \delta^g\gamma_{ij}(t_f, \boldsymbol{x})$ preserve the invariance under the particular residual *gauge* transformation given in Eq. (3.44), where

$$\mathcal{R}_x \ni \frac{\partial_t}{\dot{\rho}}, \quad \frac{\partial_{\boldsymbol{x}}}{e^{\rho(t)}\dot{\rho}(t)}, \quad \left(1 - \frac{\int d^3\boldsymbol{x} W_t(\boldsymbol{x})}{\int d^3\boldsymbol{y} W_t(\boldsymbol{y})} \right), \quad \ldots \tag{3.50}$$

represents an operator that manifestly suppresses the IR contributions by acting on the fields ${}^g\zeta(t, \boldsymbol{x})$ and $\delta^g\gamma_{ij}(t, \boldsymbol{x})$. Here, the subscript associated with $\mathcal{R}_x$ specifies the argument of the fields on which the operator acts. In the following, we will address the IR regularity of the n-point functions of $\mathcal{R}_x {}^g\zeta(t_f, \boldsymbol{x})$ and $\mathcal{R}_x \delta^g\gamma_{ij}(t_f, \boldsymbol{x})$.

[1] Precisely speaking, the coordinates "$\boldsymbol{x}$", which will be used in the rest of this paper, are not the geodesic normal coordinates $\boldsymbol{x}$. However, for notational simplicity, we also use the same symbol $\boldsymbol{x}$ for the coarse-grained version of the geodesic coordinates.

4. Euclidean vacuum and regularization scheme

In order to compute genuinely *gauge* invariant quantities, we also need to specify the quantum state so as not to be affected by the residual *gauge* degrees of freedom. However, as we mentioned in Sect. 2.3, the genuine *gauge* invariance of the quantum state cannot be directly discussed in our current approach. Hence, focusing on the invariance under the restricted class of transformations (3.16), we discuss the equivalence among quantum states specified in terms of the set of variables $\tilde{\Phi}$ with various values of $s(t)$ and $S_{ij}(t)$. As discussed in Ref. [66], the boundary condition of the Euclidean vacuum selects the unique quantum state irrespective of the choice of canonical variables connected by the dilatation transformation. Namely, as long as we choose the Euclidean vacuum, the quantum state is unaltered by the dilatation scaling. In this section, we extend this argument to include the graviton perturbation, using different sets of the canonical variables Φ and $\tilde{\Phi}$, introduced in the previous section, by performing the residual *gauge* transformation from $x^i_{\rm gl}$ to x^i. We will show that, employing the boundary condition of the Euclidean vacuum, we can select the unique quantum state irrespective of the choice of the canonical variables. In Sect. 4.2, using this property of the Euclidean vacuum, we will reformulate the perturbative expansion.

4.1. Euclidean vacuum

In this subsection, we briefly summarize the basic properties of the Euclidean vacuum. In the case of a massive scalar field in de Sitter space, the boundary condition specified by rotating the time path on the complex plane can be understood as requesting the regularity of correlation functions on the Euclidean sphere that can be obtained by the analytic continuation from those on de Sitter space. The vacuum state defined in this way is called the Euclidean vacuum state. Here, we denote by the Euclidean vacuum the state specified by a similar boundary condition in more general spacetime.

To be more precise, we define the Euclidean vacuum by requesting the regularity of the n-point functions,

$$\langle T_c \delta\gamma_{i_1 j_1}(x_{\rm gl1}) \dots \delta\gamma_{i_m j_m}(x_{{\rm gl}m}) \zeta(x_{{\rm gl}m+1}) \dots \zeta(x_{{\rm gl}n}) \rangle_E < \infty \quad \text{for} \quad \eta(t_a) \to -\infty(1 \pm i\epsilon), \tag{4.1}$$

where $a = 1, \dots, n$ and T_c represents the path ordering along the closed time path, $-\infty(1 - i\epsilon) \to \eta(t_f) \to -\infty(1 + i\epsilon)$, in terms of conformal time

$$\eta(t) \equiv \int^t \frac{dt}{e^{\rho(t)}}. \tag{4.2}$$

For simplicity, here we assume that $e^{\rho(t)}\dot{\rho}(t)$ is rapidly increasing in time so that

$$|\eta(t)| = \mathcal{O}\left(1/e^{\rho(t)}\dot{\rho}(t)\right). \tag{4.3}$$

We add the subscript E to the expectation values for the Euclidean vacuum defined in terms of the canonical variables Φ.

For the canonical variables $\tilde{\Phi}$, the boundary condition of the Euclidean vacuum is similarly given by

$$\langle T_c \delta\tilde{\gamma}_{i_1 j_1}(x_1) \dots \delta\tilde{\gamma}_{i_m j_m}(x_m) \tilde{\zeta}(x_{m+1}) \dots \tilde{\zeta}(x_n) \rangle_{\tilde{E}} < \infty \quad \text{for} \quad \eta(t_a) \to -\infty(1 \pm i\epsilon). \tag{4.4}$$

The Euclidean vacuum is expected to be invariant under the residual *gauge* transformations, since the above boundary conditions of the Euclidean vacuum are formally independent of the choice of canonical variables. In fact, we can show the equivalence between the expectation values,

$$\langle T_c \mathcal{O} \rangle_E = \langle T_c \tilde{\mathcal{O}} \rangle_{\tilde{E}}, \tag{4.5}$$

where the operators $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are related to each other by the relations (3.17) and (3.18). A more detailed explanation regarding the uniqueness of the Euclidean vacuum can be found in Ref. [66] and the argument there can be extended to include the graviton modes in a straightforward manner. We will find that the distinctive property (4.5) is crucial in showing the IR regularity for the Euclidean vacuum.

4.2. *Rewriting the n-point functions*

In this subsection, we rewrite the expression for the n-point functions into a more suitable form to examine the regularity of the IR contributions. Namely, we perform the perturbative expansion of the n-point functions of ${}^g\zeta(t_f, \boldsymbol{x}_a)$ and $\delta^g\gamma_{ij}(t_f, \boldsymbol{x}_a)$ with $a = 1, \ldots, n$, using the canonical variables $\tilde{\Phi}$. In this subsection, we adopt the Schrödinger picture. Since all the operators will be in the Schrödinger picture, they do not have time dependence. Introducing the unitary operator of the time evolution

$$U(t, t') \equiv T_c \exp\left[-i \int_{t'}^{t} dt H(t)\right], \tag{4.6}$$

the n-point functions are expressed as

$$\langle 0| U(-\infty(1+i\epsilon), t_f)\, {}^g\zeta(\boldsymbol{x}_1) \ldots {}^g\zeta(\boldsymbol{x}_n) U(-\infty(1-i\epsilon), t_f)|0\rangle. \tag{4.7}$$

Here, we introduce the eigenstate of ζ and $\delta\gamma_{ij}$, $|\zeta^c, \delta\gamma^c\rangle$. For given values of s and S_{ij}, $|\zeta^c, \delta\gamma^c\rangle$ also becomes the eigenstate of ${}^g\bar{\zeta}$ and $\delta^g\bar{\gamma}$. That is,

$${}^g\bar{\zeta}(t) \left|\zeta^c, \delta\gamma^c\right\rangle = \frac{\int d^3\boldsymbol{x} W_t(\boldsymbol{x}) \zeta(\boldsymbol{x}_{\rm gl})}{\int d^3\boldsymbol{x} W_t(\boldsymbol{x})} \left|\zeta^c, \delta\gamma^c\right\rangle = s^{(\rm ev)}[t, \zeta^c, \delta\gamma^c, s; S] \left|\zeta^c, \delta\gamma^c\right\rangle, \tag{4.8}$$

$$\delta^g\bar{\gamma}_{ij}(t) \left|\zeta^c, \delta\gamma^c\right\rangle = \frac{\int d^3\boldsymbol{x} W_t(\boldsymbol{x}) \delta\gamma_{S ij}[t, \boldsymbol{x}_{\rm gl}; S]}{\int d^3\boldsymbol{x} W_t(\boldsymbol{x})} \left|\zeta^c, \delta\gamma^c\right\rangle = S^{(\rm ev)}_{ij}[t, \zeta^c, \delta\gamma^c, s; S] \left|\zeta^c, \delta\gamma^c\right\rangle, \tag{4.9}$$

where ζ^c and $\delta\gamma^c$ denote the eigenvalues of ζ and $\delta\gamma_{ij}$. Here the time dependence of the operators ${}^g\bar{\zeta}$ and $\delta^g\bar{\gamma}_{ij}$ appears through $W_t(\boldsymbol{x})$, $\boldsymbol{x}_{\rm gl}$, and S_{ij}, while $\zeta(\boldsymbol{x})$ and $\delta\gamma_{ij}(\boldsymbol{x})$ are time-independent Schrödinger operators. Since $\boldsymbol{x}_{\rm gl}$ and $\delta\gamma_{S,ij}$ depend on the value of $s(t)$ and the path for picked-up values of $S_{ij}(t')$ with $t \le t' \le t_f$, the eigenvalues of the operators ${}^g\bar{\zeta}$ and $\delta^g\bar{\gamma}_{ij}$, $s^{(\rm ev)}(t)$ and $S^{(\rm ev)}_{ij}(t)$, also depend on $s(t)$ and $S_{ij}(t')$.

Using the eigenstate $|\zeta^c, \delta\gamma^c\rangle$, we construct a decomposition of unity:

$$\mathbf{1} = \int \mathcal{D}\zeta^c \mathcal{D}\delta\gamma^c \Big|\zeta^c, \delta\gamma^c\Big\rangle\Big\langle\zeta^c, \delta\gamma^c\Big|. \tag{4.10}$$

Discretizing the time coordinate along the closed time path in Eq. (4.7), as is usually done in the path integral, we insert the unit operator (4.10) at each intermediate time step as

$$\text{Eq. (4.7)} = \left\langle 0 \middle| T_c \left(\prod_{a=0}^{\infty} \int \mathcal{D}\zeta^c \mathcal{D}\delta\gamma^c U(t_{a+1}, t_a) \middle| \zeta^c, \delta\gamma^c \right\rangle \left\langle \zeta^c, \delta\gamma^c \middle| \right) \\ \times \zeta(\hat{\boldsymbol{x}}_{\mathrm{gl}1}) \ldots \zeta(\hat{\boldsymbol{x}}_{\mathrm{gl}n}) \left(\prod_{b=-\infty}^{0} \int \mathcal{D}\zeta^c \mathcal{D}\delta\gamma^c \middle| \zeta^c, \delta\gamma^c \right\rangle \left\langle \zeta^c, \delta\gamma^c \middle| U(t_b, t_{b-1}) \right) \left| 0 \right\rangle, \quad (4.11)$$

where we have labeled the discretized time coordinate from the distant past to t_f by negative integers and that from t_f to the distant past by positive integers, with $t_0 = t_f$. For the time being, we focus on the n-point functions for a particular time path of ζ^c and $\delta\gamma^c$, picking up, at each time step, one representative state among the summed eigenstates in the unit operator (4.10) in Eq. (4.11). Namely, we consider the expectation value

$$\left\langle 0 \middle| T_c \left(\prod_{a=0}^{\infty} U(t_{a+1}, t_a) \middle| \zeta^c, \delta\gamma^c \right\rangle \left\langle \zeta^c, \delta\gamma^c \middle| \right) \zeta(\hat{\boldsymbol{x}}_{\mathrm{gl}1}) \ldots \zeta(\hat{\boldsymbol{x}}_{\mathrm{gl}n}) \\ \times \left(\prod_{b=-\infty}^{0} \middle| \zeta^c, \delta\gamma^c \right\rangle \left\langle \zeta^c, \delta\gamma^c \middle| U(t_b, t_{b-1}) \right) \left| 0 \right\rangle. \quad (4.12)$$

Once the path of ζ^c and $\delta\gamma^c$ is specified, we can choose self-consistent values of $s(t)$ and $S_{ij}(t)$ that satisfy

$$s(t) = s^{(\mathrm{ev})}[t, \zeta^c, \delta\gamma^c, s; S], \quad S_{ij}(t) = S_{ij}^{(\mathrm{ev})}[t, \zeta^c, \delta\gamma^c, s; S], \quad (4.13)$$

for all t order by order. Then, using the corresponding values of s and S_{ij}, we introduce the canonical variables $\tilde{\Phi}(x)$ as defined in the preceding section. Using $\tilde{\Phi}$, we can replace $\zeta(\hat{\boldsymbol{x}}_{\mathrm{gl}})$ in Eq. (4.12) with $\tilde{\zeta}(\boldsymbol{x})$. Notice that, in the canonical system with $\tilde{\Phi}$, we should use the unitary operator of time evolution defined by the Hamiltonian $\tilde{H}(t)$, which differs from $H(t)$, as

$$\text{Eq. (4.12)} = \left\langle 0 \middle| T_c \left(\prod_{a=0}^{\infty} \tilde{U}(t_{a+1}, t_a) \middle| \zeta^c, \delta\gamma^c \right\rangle \left\langle \zeta^c, \delta\gamma^c \middle| \right) \\ \times \tilde{\zeta}(\boldsymbol{x}_1) \ldots \tilde{\zeta}(\boldsymbol{x}_n) \left(\prod_{b=-\infty}^{0} \middle| \zeta^c, \delta\gamma^c \right\rangle \left\langle \zeta^c, \delta\gamma^c \middle| \tilde{U}(t_b, t_{b-1}) \right) \left| 0 \right\rangle, \quad (4.14)$$

with

$$\tilde{U}(t, t') \equiv T_c \exp\left[-i \int_{t'}^{t} dt \tilde{H}(t) \right]. \quad (4.15)$$

Here, s and S_{ij} are different between the forward and backward paths, and hence the new canonical variables $\tilde{\Phi}(x)$ will differ between them. Furthermore, Eqs. (4.8) and (4.9) imply

$$ {}^{g}\bar{\zeta}(t_a)\left|\zeta^c, \delta\gamma^c\right\rangle = s(t_a)\left|\zeta^c, \delta\gamma^c\right\rangle, \tag{4.16} $$

$$ \delta^{g}\bar{\gamma}_{ij}(t_a)\left|\zeta^c, \delta\gamma^c\right\rangle = S_{ij}(t_a)\left|\zeta^c, \delta\gamma^c\right\rangle, \tag{4.17} $$

with

$$ {}^{g}\bar{\zeta}(t) \equiv \frac{\int d^3\boldsymbol{x} W_t(\boldsymbol{x})\tilde{\zeta}(\boldsymbol{x})}{\int d^3\boldsymbol{x} W_t(\boldsymbol{x})}, \tag{4.18} $$

$$ \delta^{g}\bar{\gamma}_{ij}(t) \equiv \frac{\int d^3\boldsymbol{x} W_t(\boldsymbol{x})\delta\tilde{\gamma}_{ij}(\boldsymbol{x})}{\int d^3\boldsymbol{x} W_t(\boldsymbol{x})}. \tag{4.19} $$

Next, we write down the expression (4.12) in the interaction picture. Using the unitary operator

$$ \tilde{U}_0(t) \equiv T_c \exp\left[-i\int^t dt \int d^3\boldsymbol{x}\tilde{\mathcal{H}}_0\right], \tag{4.20} $$

with an appropriate choice of lower boundary for the t-integration, the Schrödinger picture fields $\tilde{\zeta}(\boldsymbol{x})$ and $\delta\tilde{\gamma}_{ij}(\boldsymbol{x})$ are related to the interaction picture fields $\tilde{\zeta}_I(t,\boldsymbol{x})$ and $\delta\tilde{\gamma}_{ijI}(t,\boldsymbol{x})$, respectively, as

$$ \tilde{\zeta}(\boldsymbol{x}) = \tilde{U}_0(t)\tilde{\zeta}_I(t,\boldsymbol{x})\tilde{U}_0^{\dagger}(t), \tag{4.21} $$

$$ \delta\tilde{\gamma}_{ij}(\boldsymbol{x}) = \tilde{U}_0(t)\delta\tilde{\gamma}_{ijI}(t,\boldsymbol{x})\tilde{U}_0^{\dagger}(t). \tag{4.22} $$

Similarly to Eqs. (4.16) and (4.17), we define the eigenstate for the interaction picture fields as

$$ \left|t;\zeta^c, \delta\gamma^c\right\rangle_I = \tilde{U}_0^{\dagger}(t)\left|\zeta^c, \delta\gamma^c\right\rangle. \tag{4.23} $$

In the interaction picture, we obtain

$$ \begin{aligned} \text{Eq. (4.14)} = &\left\langle 0\middle| T_c\left(\prod_{a=0}^{\infty}\tilde{U}_I(t_{a+1},t_a)\middle|t_a;\zeta^c,\delta\gamma^c\right\rangle_{II}\left\langle t_a;\zeta^c,\delta\gamma^c\middle|\right)\tilde{\zeta}_I(t_f,\boldsymbol{x}_1)\ldots\tilde{\zeta}_I(t_f,\boldsymbol{x}_n) \\ &\times\left(\prod_{b=-\infty}^{0}\middle|t_b;\zeta^c,\delta\gamma^c\right\rangle_{II}\left\langle t_b;\zeta^c,\delta\gamma^c\middle|\tilde{U}_I(t_b,t_{b-1})\right)\left|0\right\rangle, \end{aligned} \tag{4.24} $$

with

$$ \tilde{U}_I(t,t') \equiv T_c\exp\left[-i\int_{t'}^{t} dt\,\tilde{H}_I(t)\right]. \tag{4.25} $$

With ${}^{g}\bar{\zeta}_I(t)$ and $\delta^{g}\bar{\gamma}_{ijI}(t)$ defined as

$$ {}^{g}\bar{\zeta}_I(t) \equiv \frac{\int d^3\boldsymbol{x} W_t(\boldsymbol{x})\tilde{\zeta}_I(t,\boldsymbol{x})}{\int d^3\boldsymbol{x} W_t(\boldsymbol{x})}, \tag{4.26} $$

$$ \delta^{g}\bar{\gamma}_{ijI}(t) \equiv \frac{\int d^3\boldsymbol{x} W_t(\boldsymbol{x})\delta\tilde{\gamma}_{ijI}(t,\boldsymbol{x})}{\int d^3\boldsymbol{x} W_t(\boldsymbol{x})}, \tag{4.27} $$

Eqs. (4.16) and (4.17) indicate

$$ {}^g\bar{\zeta}_I(t_a)\left|t_a;\zeta^c,\delta\gamma^c\right\rangle_I = s(t_a)\left|t_a;\zeta^c,\delta\gamma^c\right\rangle_I, \tag{4.28} $$

$$ \delta^g\bar{\gamma}_{ijI}(t_a)\left|t_a;\zeta^c,\delta\gamma^c\right\rangle_I = S_{ij}(t_a)\left|t_a;\zeta^c,\delta\gamma^c\right\rangle_I. \tag{4.29} $$

Next, we will show that, when we choose the Euclidean vacuum as the initial state, the n-point functions for ${}^g\zeta(x)$ and $\delta^g\gamma_{ij}(x)$ can be expanded only in terms of the interaction picture fields $\tilde{\zeta}_I(x)$ and $\delta\tilde{\gamma}_{ijI}(x)$ with the IR-suppressing operators $\mathcal{R}_x$. While the interaction Hamiltonian density $\tilde{\mathcal{H}}_I$ is messy, the IR regularity can be shown just by using the fact that, as given in Eq. (3.42), $\tilde{\mathcal{H}}_I$ is expressed only in terms of

$$ \tilde{\zeta}_I(x)-s(t),\quad \delta\tilde{\gamma}_{ijI}(x)-S_{ij}(t), \tag{4.30} $$

and

$$ \tilde{\pi}_I(x)=2M_{\rm pl}^2e^{3\rho}\varepsilon_1\dot{\tilde{\zeta}}_I(x),\quad \tilde{\pi}_I^{ij}(x)=\frac{M_{\rm pl}^2}{4}e^{3\rho}\delta\dot{\tilde{\gamma}}_I^{ij}(x), \tag{4.31} $$

and also with the parameters

$$ \dot{s}(t),\quad \dot{S}_{ij}(t). \tag{4.32} $$

Notice that the terms in (4.30) are not suppressed by $\mathcal{R}_x$ and also that the inverse Laplacian ∂^{-2}, which arises from N and N_i, may decrease the power of k by $1/k^2$, depending on the choice of the boundary conditions.

4.2.1. Interaction picture fields without the IR-suppressing operator. We begin with discussing the first term in Eq. (3.42), i.e.,

$$ \mathcal{H}_I[\tilde{\zeta}_I(x)-s(t),\tilde{\pi}_I(x),\delta\tilde{\gamma}_{ijI}(x)-S_{ij}(t),\tilde{\pi}_I^{ij}(x)]. \tag{4.33} $$

If we can simply replace $s(t)$ and $S_{ij}(t)$ with ${}^g\bar{\zeta}_I(t)$ and $\delta^g\bar{\gamma}_{ijI}(t)$, respectively, in the above expression, $\tilde{\zeta}_I(x)-s(t)$ and $\delta\tilde{\gamma}_{ijI}(x)-S_{ij}(t)$ are reduced to $\tilde{\zeta}_I(x)-{}^g\bar{\zeta}_I(t)$ and $\delta\tilde{\gamma}_{ijI}(x)-\delta^g\bar{\gamma}_{ijI}(t)$, which are combinations suppressed by the IR-suppressing operator $\mathcal{R}_x$. We will show that the distinctive property of the Euclidean vacuum given in Eq. (4.5) allows us to perform this replacement just by adding terms that are composed only of $\mathcal{R}_x\tilde{\zeta}_I(x)$ and $\mathcal{R}_x\delta\tilde{\gamma}_{ijI}(x)$.

To perform the replacement, we notice that the operator $|t_a;\zeta^c,\delta\gamma^c\rangle_{II}\langle t_a;\zeta^c,\delta\gamma^c|$ is located next to the interaction Hamiltonian $\tilde{H}_I(t_a)$ as

$$ \ldots\tilde{H}_I(t_a)\left|t_a;\zeta^c,\delta\gamma^c\right\rangle_{II}\left\langle t_a;\zeta^c,\delta\gamma^c\right|\ldots, $$

where the abbreviation on the right-hand side of $\tilde{\mathcal{H}}_I$ denotes operators in the past of t_a along the closed time path and that on the left-hand side denotes operators in the future of t_a. For notational simplicity, we abbreviate the subscript a in the following discussion. Picking up a single $\tilde{\zeta}_I(x)-s(t)$ or $\delta\tilde{\gamma}_{ijI}(x)-S_{ij}(t)$ from the first term of $\tilde{\mathcal{H}}_I$, given in Eq. (4.33), and using Eqs. (4.28) and (4.29), we rewrite each term as

$$\Big(\tilde{\zeta}_I(x) - s(t)\Big)\,\mathcal{A}(x)\left|t;\zeta^c,\delta\gamma^c\right\rangle_I = \Big(\tilde{\zeta}_I(x) - {}^g\bar{\zeta}_I(t)\Big)\,\mathcal{A}(x)\left|t;\zeta^c,\delta\gamma^c\right\rangle_I + \left[{}^g\bar{\zeta}_I(t),\mathcal{A}(x)\right]\left|t;\zeta^c,\delta\gamma^c\right\rangle_I, \tag{4.34}$$

or

$$\Big(\delta\tilde{\gamma}_{ijI}(x) - S_{ij}(t)\Big)\mathcal{A}(x)\left|t;\zeta^c,\delta\gamma^c\right\rangle_I = \Big(\delta\tilde{\gamma}_{ijI}(x) - \delta^g\bar{\gamma}_{ijI}(t)\Big)\mathcal{A}(x)\left|t;\zeta^c,\delta\gamma^c\right\rangle_I + \left[\delta^g\bar{\gamma}_{ijI}(t),\mathcal{A}(x)\right]\left|t;\zeta^c,\delta\gamma^c\right\rangle_I, \tag{4.35}$$

where, using $\mathcal{A}(x)$, we have schematically expressed the operators sandwiched between $\tilde{\zeta}_I(x) - s(t)$ or $\delta\tilde{\gamma}_{ijI}(x) - S_{ij}(t)$ and $|t;\zeta^c,\delta\gamma^c\rangle_I$. Since $\mathcal{A}(x)$ is part of the Hamiltonian density in Eq. (4.33), it can be expressed solely in terms of the combinations in Eq. (4.30) and the conjugate momenta $\tilde{\pi}_I$ and $\tilde{\pi}_I^{ij}$. Since ${}^g\bar{\zeta}_I(t)$ commutes with $\tilde{\zeta}_I(x) - s(t)$, $\delta\tilde{\gamma}_{ijI}(x) - S_{ij}(t)$, and $\tilde{\pi}_I^{ij}(x)$, the non-vanishing contributions in $[{}^g\bar{\zeta}_I(t),\mathcal{A}(x)]$ arise only from the commutator

$$\left[{}^g\bar{\zeta}_I(t),\tilde{\pi}_I(t,\boldsymbol{x})\right] = \frac{1}{\int d^3\boldsymbol{x}\,W_t(\boldsymbol{x})}\int d^3\boldsymbol{y}\,W_t(\boldsymbol{y})\left[\tilde{\zeta}_I(t,\boldsymbol{y}),\tilde{\pi}_I(t,\boldsymbol{x})\right] = i\,\frac{W_t(\boldsymbol{x})}{\int d^3\boldsymbol{x}\,W_t(\boldsymbol{x})}, \tag{4.36}$$

which yields a local function whose Fourier mode is regular in the IR limit. Repeating this procedure, we can rewrite $(\tilde{\zeta}_I(x) - s(t))\mathcal{A}(x)$ solely in terms of

$$\tilde{\zeta}_I(x) - {}^g\bar{\zeta}_I(t), \quad \tilde{\pi}_I(x), \quad \delta\tilde{\gamma}_{ijI}(x) - \delta^g\bar{\gamma}_{ijI}(t), \quad \tilde{\pi}_I^{ij}(x). \tag{4.37}$$

The same argument can apply to $(\delta\tilde{\gamma}_{ijI}(x) - S_{ij}(t))\mathcal{A}(x)$. In this way all the interaction picture fields in the first term of $\tilde{\mathcal{H}}_I$ are now expressed by $\mathcal{R}_x\tilde{\zeta}_I$ and $\mathcal{R}_x\delta\tilde{\gamma}_{ijI}$.

Next, we consider the second term of the interaction Hamiltonian (3.42) with $\dot{s}$ and $\dot{S}_{ij}$. When we discretize the time coordinate, the time derivative should be regarded as the difference between the values at two adjacent time steps. We can express the second term in Eq. (3.42) sandwiched by ${}_I\langle t_{a+1};\zeta^c,\delta\gamma^c|$ and $|t_a;\zeta^c,\delta\gamma^c\rangle_I$ as

$$\begin{aligned}
&{}_I\!\left\langle t_{a+1};\zeta^c,\delta\gamma^c\right|\left[\tilde{\pi}_I(x_a)x^l\partial_m\tilde{\zeta}_I(x_a) + \tilde{\pi}_I^{ij}(x_a)x^l\partial_m\delta\tilde{\gamma}_{ijI}(x_a)\right]\left(\dot{s}(t_a)\delta_l^m + \dot{S}_l^m(t_a)/2\right)\left|t_a;\zeta^c,\delta\gamma^c\right\rangle_I \\
&\quad = {}_I\!\left\langle t_{a+1};\zeta^c,\delta\gamma^c\right|\left[\tilde{\pi}_I(x_a)x^l\partial_m\tilde{\zeta}_I(x_a) + \tilde{\pi}_I^{ij}(x_a)x^l\partial_m\delta\tilde{\gamma}_{ijI}(x_a)\right] \\
&\qquad \times \frac{\{s(t_{a+1}) - s(t_a)\}\delta_l^m + \{S_l^m(t_{a+1}) - S_l^m(t_a)\}/2}{t_{a+1} - t_a}\left|t_a;\zeta^c,\delta\gamma^c\right\rangle_I,
\end{aligned}$$

with $x_a = (t_a,\boldsymbol{x})$. Here, we neglect the terms irrelevant in the continuous limit. Using Eqs. (4.28) and (4.29), we can replace $s(t_a)$ and $S_{ij}(t_a)$ with ${}^g\bar{\zeta}_I(t_a)$ and $\delta^g\bar{\gamma}_{ijI}(t_a)$ placed next to $|t_a;\zeta^c,\delta\gamma^c\rangle_I$, and $s(t_{a+1})$ and $S_{ij}(t_{a+1})$ with ${}^g\bar{\zeta}_I(t_{a+1})$ and $\delta^g\bar{\gamma}_{ijI}(t_{a+1})$ next to ${}_I\langle t_{a+1};\zeta^c,\delta\gamma^c|$. For the same reason as in the previous case, the terms coming from the commutator between $[\tilde{\pi}_I(x_a)x^l\partial_m\tilde{\zeta}_I(x_a)+\cdots]$ and ${}^g\bar{\zeta}_I(t_a)$ or $\delta^g\bar{\gamma}_{ijI}(t_a)$ only give the IR-regular contributions, while the remaining part becomes

$$
\begin{aligned}
&{}_I\Big\langle t_{a+1};\zeta^c,\delta\gamma^c\Big|\left[{}^g\dot{\bar{\zeta}}_I(t_a)\delta_l^m+\delta^g\dot{\bar{\gamma}}_{lI}^m(t_a)/2\right]\\
&\quad\times\left[\tilde{\pi}_I(x_a)x^l\partial_m\tilde{\zeta}_I(x_a)+\tilde{\pi}_I^{ij}(x_a)x^l\partial_m\delta\tilde{\gamma}_{ijI}(x_a)\right]\Big|t_a;\zeta^c,\delta\gamma^c\Big\rangle_I.
\end{aligned}
$$

Similarly, we can replace $\dot{S}_{ij}(t)$ in the third term of the interaction Hamiltonian (3.42) with $\delta^g\dot{\bar{\gamma}}_{ijI}(t)$. Notice that ${}^g\dot{\bar{\zeta}}_I(t)$ is recast into

$$
\begin{aligned}
{}^g\dot{\bar{\zeta}}_I(t)&=\int d^3\boldsymbol{x}\partial_t\left\{\frac{W_t(\boldsymbol{x})}{\int d^3\boldsymbol{x}W_t(\boldsymbol{x})}\right\}\tilde{\zeta}_I(x)+\frac{\int d^3\boldsymbol{x}W_t(\boldsymbol{x})\partial_t\tilde{\zeta}_I(x)}{\int d^3\boldsymbol{x}W_t(\boldsymbol{x})}\\
&=\int d^3\boldsymbol{x}\partial_t\left\{\frac{W_t(\boldsymbol{x})}{\int d^3\boldsymbol{x}W_t(\boldsymbol{x})}\right\}\left\{\tilde{\zeta}_I(x)-{}^g\bar{\zeta}_I(t)\right\}+\frac{\int d^3\boldsymbol{x}W_t(\boldsymbol{x})\partial_t\tilde{\zeta}_I(x)}{\int d^3\boldsymbol{x}W_t(\boldsymbol{x})},
\end{aligned}
\tag{4.38}
$$

which is manifestly expressed in the IR-suppressed form, $\mathcal{R}_x\tilde{\zeta}_I(x)$. To make the IR regularity manifest, in the last equality we have added $0={}^g\bar{\zeta}_I(t)\partial_t\left\{\int d^3\boldsymbol{x}W_t(\boldsymbol{x})/\int d^3\boldsymbol{x}W_t(\boldsymbol{x})\right\}$. In a similar manner we can show that $\delta^g\dot{\bar{\gamma}}_{ijI}(t)$ is also in the IR-suppressed form.

In this way, we can show that all $\tilde{\zeta}_I$ and $\delta\tilde{\gamma}_{ijI}$ in the interaction vertices are multiplied by the IR-suppressing operator $\mathcal{R}_x$. The argument so far proceeds irrespective of the choice of the initial quantum states. Now, we focus on the distinctive property of the Euclidean vacuum given in Eq. (4.5), which states that the initial states chosen by the boundary condition of the Euclidean vacuum are specified uniquely and are independent of the canonical variables used for quantization. Therefore, requesting the Euclidean vacuum uniquely determines the initial state irrespective of the picked-up particular path of ζ^c and $\delta\gamma^c$. Therefore, after the above-mentioned replacements, the possible dependence of the n-point functions on the picked-up path remains only in $|t;\zeta^c,\delta\gamma^c\rangle_{I\,I}\langle t;\zeta^c,\delta\gamma^c|$, and hence we can remove the decomposition of unity.

4.2.2. Restricting the interaction vertices to the local region. Next, we will address the inverse Laplacian ∂^{-2}. If we choose the boundary conditions for ∂^{-2} in N and N_i appropriately, N and N_i with their argument $(t,\boldsymbol{x})$ in the region $\mathcal{O}_t$ can be specified by the fluctuations only within $\mathcal{O}_t$. In the general solutions of N and N_i given in Eqs. (2.36) and (2.37), the residual *gauge* degrees of freedom are expressed by arbitrary homogeneous solutions of the Laplace equation, $G_n(x)$ and $(\delta_i^j-\partial_i\partial^{-2}\partial^j)G_{jn}(x)$. We determine the homogeneous solution $G_n(x)$ such that the solution in the observable region $\mathcal{O}_t$ is given by the convolution between the Green function and the source restricted to the local region, i.e.,

$$
-\frac{1}{4\pi}\int\frac{d^3\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|}W_t(\boldsymbol{y})\partial^iM_{i,n}(t,\boldsymbol{y})=\partial^{-2}\partial^iM_{i,n}(x)-e^{-2\rho}G_n(x).
\tag{4.39}
$$

Similarly, using the transverse part of $G_{in}(x)$, we can determine the boundary conditions for the remaining ∂^{-2} so as to shut off the influence from the region far outside of $\mathcal{O}_t$. (For a detailed explanation, see Refs. [66,67].) Then, since all the interaction vertices are confined to the neighborhood of $\mathcal{O}$, the operation of the non-local operator ∂^{-2} no longer reduces the power law index with respect to k. Thus, when we choose the Euclidean vacuum as the initial states, we can expand the n-point functions for $\mathcal{R}_x{}^g\zeta(t_f,\boldsymbol{x})$ and $\mathcal{R}_x\delta^g\gamma_{ij}(t_f,\boldsymbol{x})$ only in terms of the interaction picture fields $\mathcal{R}_x\tilde{\zeta}_I(x)$ and $\mathcal{R}_x\delta\tilde{\gamma}_{ijI}(x)$.

Since $\mathcal{R}_x{}^g\zeta(x)$ and $\mathcal{R}_x\delta^g\gamma_{ij}(x)$ are not invariant under all the residual *gauge* transformations, their n-point functions can depend on the boundary conditions of N and N_i. However, if we calculate n-point functions for the genuinely *gauge* invariant operator ${}^gR$, changing the boundary conditions should not affect the result.

5. Regularity of loops

In this section, we will show that, when we choose the Euclidean vacuum as the initial state, the n-point functions of $\mathcal{R}_x{}^g\zeta$ and $\mathcal{R}_x\delta^g\gamma_{ij}$ no longer suffer from the IRdiv, IRsec, and SG. The discussion in this section goes almost in parallel with that in Sect. 3.2.4 of Ref. [66], where the regularity of the scalar loops is shown. Here, we briefly highlight the discussion, leaving the more detailed discussion to Ref. [66].

5.1. Euclidean vacuum from the $i\epsilon$ prescription

In the preceding section, we introduced the Euclidean vacuum, which satisfies the boundary conditions (4.1). Here, following Ref. [66], we show that these conditions lead to the $i\epsilon$ prescription in the ordinary perturbative description. For our current purpose, the explicit form of the interaction Hamiltonian density $\tilde{\mathcal{H}}_I$ is not necessary. We simply note that all the interaction vertices in $\tilde{\mathcal{H}}_I$ can be formally expressed as

$$M_{\rm pl}^2 e^{3\rho(t)}\dot{\rho}^2(t)\lambda(t)\prod_{m^s=1}^{N^s}\mathcal{R}_x^{(m^s)}\tilde{\zeta}_I(x)\prod_{m^t=1}^{N^t}\mathcal{R}_x^{(m^t)}\delta\tilde{\gamma}_{i_{m^t}j_{m^t}I}(x), \tag{5.1}$$

where N^s and N^t are non-negative integers with $N^s+N^t\geq 3$. Here, $\lambda(t)$ is a dimensionless time-dependent function that can be expressed only in terms of the horizon flow functions. To discriminate different IR-suppressing operators $\mathcal{R}_x$, we added a subscript (m^s) or (m^t) to $\mathcal{R}_x$. The spatial indices i_{m^t} and j_{m^t} will be contracted with other indices $i_{m^{t\prime}}$ and $j_{m^{t\prime}}$ or with indices in $\mathcal{R}_x$, which are abbreviated for notational simplicity. In the following, we use the formal expression (5.1) as the interaction vertices.

Since the boundary conditions for the Euclidean vacuum should also hold at tree level, the asymptotic form of the positive frequency mode function $v_k^\alpha(t)$ with $\alpha=s$ or t, in the limit $\eta\to-\infty$, should be proportional to $e^{-ik\eta(t)}$. Factoring out this time dependence, we express $v_k^\alpha(t)$ as

$$v_k^\alpha(t)=\frac{\mathcal{A}^\alpha(t)}{k^{3/2}}f_k^\alpha(t)e^{-ik\eta(t)}, \tag{5.2}$$

where we have introduced

$$\mathcal{A}^s(t)\equiv\frac{\dot{\rho}(t)}{\sqrt{\varepsilon_1(t)}M_{\rm pl}}, \tag{5.3}$$

$$\mathcal{A}^t(t)\equiv\frac{\dot{\rho}(t)}{M_{\rm pl}} \tag{5.4}$$

as approximate amplitudes of the curvature perturbation and the graviton perturbation. The function $f_k^\alpha(t)$ satisfies the regular second-order differential equation with the boundary condition

$$f_k^\alpha(t)\propto\frac{k}{e^\rho\dot{\rho}}\quad\text{for}\quad -k\eta(t)\to\infty. \tag{5.5}$$

Since both the differential equation and the boundary condition of $f_k^\alpha(t)$ are analytic in k for any t, the resulting function should be analytic as well. In fact, $f_k^\alpha(t)$ does not have any singularity such as a pole on the complex k-plane as a consequence of the boundary conditions of the Euclidean vacuum.

On the other hand, in the limit $-k\eta(t_k) \ll 1$, the function $f_k^\alpha(t)$ is proportional to $\mathcal{A}^\alpha(t_k)/\mathcal{A}^\alpha(t)$, where t_k is the Hubble crossing time defined by $-k\eta(t_k) = 1$, because the curvature and graviton perturbations should be constant in this limit. Hence, the expansion for small k is in general given by

$$\mathcal{A}^\alpha(t) f_k^\alpha(t) = \mathcal{A}^\alpha(t_k)\left[1 + \mathcal{O}(k^2|\eta(t)|^2)\right]. \tag{5.6}$$

By using Eq. (5.2), the Wightman function for the curvature perturbation is given by

$$\begin{aligned} G^{+s}(x, x') &= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} v_k^s(t) v_k^{s*}(t') \\ &= \mathcal{A}^s(t)\mathcal{A}^s(t') \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{k^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} f_k^s(t) f_k^{s*}(t') e^{ik(\eta(t')-\eta(t))}, \end{aligned} \tag{5.7}$$

and the Wightman function for the graviton is given by

$$\begin{aligned} G^{+t}_{ijkl}(x, x') &= \sum_{\lambda=\pm} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} e^{(\lambda)}_{ij}(\boldsymbol{k}) e^{(\lambda)}_{kl}(\boldsymbol{k}) v_k^t(t) v_k^{t*}(t') \\ &= \frac{\mathcal{A}^t(t)\mathcal{A}^t(t')}{2} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \left(\mathcal{P}_{ik}\mathcal{P}_{jl} + \mathcal{P}_{il}\mathcal{P}_{jk} - \mathcal{P}_{ij}\mathcal{P}_{kl}\right) \\ &\quad \times \frac{1}{k^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} f_k^t(t) f_k^{t*}(t') e^{ik(\eta(t')-\eta(t))}, \end{aligned} \tag{5.8}$$

where in the second equality we have assumed that the quantum state is isotropic.

Using the in–in formalism, the n-point functions can be expanded by the Wightman functions $G^{+s}(x, x')$, $G^{+t}_{ijkl}(x, x')$, and their complex conjugates. When we impose the boundary conditions of the Euclidean vacuum, we need to start the vertex integrals at $\eta = -\infty$. Although the vertex integrals are infinitely oscillating in the limit $\eta \to -\infty$, the time integration can be made convergent by adding a small imaginary part to the time coordinate, which is nothing but the ordinary $i\epsilon$ prescription. To see the convergence of the time integration more explicitly, using the formal expression for the interaction vertex (5.1), we first consider the integral for the vertex that is closest to the past infinity $\eta \to -\infty(1 - i\epsilon)$. The interaction picture field $\tilde{\zeta}_I(x)$ included in this vertex is contracted with $\tilde{\zeta}_I(x_{m^s})$ in vertices labeled by $m^s = 1, 2, \ldots, N^s$, and gives the Wightman function $G^{+s}(x_{m^s}, x)$. Similarly, the interaction picture field $\delta\tilde{\gamma}_{I\,i_{m^t} j_{m^t}}(x)$ included in the vertex is contracted with $\delta\gamma_{k_{m^t} l_{m^t} I}(x_{m^t})$ in vertices labeled by $m^t = 1, 2, \ldots, N^t$, and gives the Wightman function $G^{+t}_{k_{m^t} l_{m^t} i_{m^t} j_{m^t}}(x_{m^t}, x)$. Then, the vertex integration with N^s $\tilde{\zeta}_I$ and N^t $\delta\tilde{\gamma}_{ijI}$ gives

$$\begin{aligned} V^{(1)}(t', \{x_{m^\alpha}\}) \equiv M_{\rm pl}^2 \int_{t_i}^{t'} dt \int d^3\boldsymbol{x} e^{3\rho(t)} \dot{\rho}(t)^2 \lambda(t) \prod_{m^s=1}^{N^s} \mathcal{R}_{x_{m^s}} \mathcal{R}_x^{(m^s)} G^{+s}(x_{m^s}, x) \\ \times \prod_{m^t=1}^{N^t} \mathcal{R}_{x_{m^t}} \mathcal{R}_x^{(m^t)} G^{+t}_{k_{m^t} l_{m^t} i_{m^t} j_{m^t}}(x_{m^t}, x), \end{aligned} \tag{5.9}$$

where x_{m^α} denotes either x_{m^s} or x_{m^t}. The Euclidean vacuum condition requires the convergence of this integral in the limit $\eta(t_i) \to -\infty$. Since the Wightman functions contain the exponential factor

$$e^{i\eta(t)\left(\sum_{m^s} k_{m^s} + \sum_{m^t} k_{m^t}\right)},$$

the integral can be made convergent by adding $+i\epsilon$ to $\eta(t)$, which is again exactly what is known as the $i\epsilon$ prescription. Here, k_{m^s} denotes the momentum of $G^{+s}(x_{m^s}, x)$ and k_{m^t} denotes the momentum of $G^{+t}_{k_{m^t} l_{m^t} i_{m^t} j_{m^t}}(x_{m^t}, x)$.

The vertex integration second-closest to the past infinity

$$V^{(2)}(t'', \{x_{m^\alpha}\}, \{x_{m^{\alpha\prime}}\}) \equiv M_{\rm pl}^2 \int_{t_i}^{t''} dt' \int d^3\boldsymbol{x}' e^{3\rho(t')} \dot{\rho}(t')^2 \lambda(t') \prod_{m^{s\prime}=1}^{N^{s\prime}} \mathcal{R}_{x_{m^{s\prime}}} \mathcal{R}_{x'}^{(m^{s\prime})} G^{+s}(x_{m^{s\prime}}, x')$$
$$\times \prod_{m^{t\prime}=1}^{N^{t\prime}} \mathcal{R}_{x_{m^{t\prime}}} \mathcal{R}_{x'}^{(m^{t\prime})} G^{+t}_{k_{m^{t\prime}} l_{m^{t\prime}} i_{m^{t\prime}} j_{m^{t\prime}}}(x_{m^{t\prime}}, x') V^{(1)}(t', \{x_{m^{\alpha\prime}}\}) \tag{5.10}$$

can be done in a similar manner, where $N^{s\prime}$ and $N^{t\prime}$ are the numbers of scalar and graviton propagators that connect between this second vertex and vertices other than the first one. If we perform the integration over the time coordinate of the first vertex t up to t', the exponential factor in $G^{+s}(x_{m^s}, x)$ or $G^{+t}_{k_{m^t} l_{m^t} i_{m^t} j_{m^t}}(x_{m^t}, x)$ can be replaced as

$$e^{ik_{m^\alpha}(\eta(t)-\eta(t_m))} \to e^{ik_{m^\alpha}(\eta(t')-\eta(t_m))}. \tag{5.11}$$

Therefore, all the Wightman functions connecting the vertices at t' or in the past of t' with the vertices in the future of t' give an exponential factor that is suppressed by adding $+i\epsilon$ to $\eta(t')$. (Here, we mean the future and past in the chronological sense, and not those in the sense of the Closed Time Path.) This is again consistent with the boundary condition of the Euclidean vacuum. The same argument can be made for the other vertices as well.

In this subsection, considering the time integration at vertices with fixed momenta of the Wightman propagators, we have shown that the boundary condition of the Euclidean vacuum can be imposed in a perturbative expansion by employing the $i\epsilon$ prescription. However, as we will describe in the next subsection, in our proof of the IR regularity, we will perform the momentum integration of the propagator ahead of the vertex integration.

5.2. IR/UV-suppressed Wightman function

Since all $\tilde{\zeta}_I(x)$ and $\delta\tilde{\gamma}_{ijI}(x)$ in the interaction Hamiltonian are multiplied by the IR-suppressing operators $\mathcal{R}_x$, the n-point functions of $\mathcal{R}_x{}^g\zeta(x)$ and $\mathcal{R}_x\delta^g\gamma_{ij}(x)$ can be expanded by the Wightman functions $\mathcal{R}_x\mathcal{R}_{x'}G^{+s}(x, x')$ and $\mathcal{R}_x\mathcal{R}_{x'}G^{+t}_{ijkl}(x, x')$ and their complex conjugates. In this subsection, we calculate the Wightman functions multiplied by the IR-suppressing operator $\mathcal{R}_x\mathcal{R}_{x'}G^{+}(x, x')$ and $\mathcal{R}_x\mathcal{R}_{x'}G^{+t}_{ijkl}(x, x')$ for $t > t'$. After integration over the angular part of the momentum, the Wightman function $\mathcal{R}_x\mathcal{R}_{x'}G^{+s}(x, x')$ is given as

$$\mathcal{R}_x\mathcal{R}_{x'}G^{+s}(x, x') = \frac{1}{2\pi^2}\int_0^\infty \frac{dk}{k} \mathcal{R}_x\mathcal{R}_{x'}\mathcal{A}^s(t) f_k^s(t) \mathcal{A}^s(t') f_k^{s*}(t') \left[\frac{e^{ik\sigma_+(x,x')} - e^{ik\sigma_-(x,x')}}{ik(\sigma_+(x,x') - \sigma_-(x,x'))}\right], \tag{5.12}$$

where we have introduced

$$\sigma_\pm(x, x') \equiv \eta(t') - \eta(t) \pm |\boldsymbol{x} - \boldsymbol{x}'|.$$

The Wightman function $\mathcal{R}_x \mathcal{R}_{x'} G^{+t}_{ijkl}(x, x')$ is given by a similar expression as

$$\mathcal{R}_x \mathcal{R}_{x'} G^{+t}_{ijkl}(x, x') = \frac{1}{4\pi^2} \int_0^\infty \frac{dk}{k} \mathcal{R}_{x'} \left(\mathcal{P}^{(d)}_{ik} \mathcal{P}^{(d)}_{jl} + \mathcal{P}^{(d)}_{il} \mathcal{P}^{(d)}_{jk} - \mathcal{P}^{(d)}_{ij} \mathcal{P}^{(d)}_{kl} \right) \times \mathcal{R}_x \mathcal{A}^t(t) f^t_k(t) \mathcal{A}^t(t') f^{t*}_k(t') \left[\frac{e^{ik\sigma_+(x,x')} - e^{ik\sigma_-(x,x')}}{ik(\sigma_+(x, x') - \sigma_-(x, x'))} \right]. \tag{5.13}$$

Here, before we integrate over the angular coordinates, we replace the projection tensor $\mathcal{P}_{ij}$ with the derivative form:

$$\mathcal{P}^{(d)}_{ij} = \delta_{ij} - \partial_{x'^i} \partial_{x'^j} \partial^{-2}_{\boldsymbol{x}'}, \tag{5.14}$$

which commutes with $\mathcal{R}_x$.

We first show the regularity of the k integration in Eqs. (5.12) and (5.13). The regularity of the Wightman function $G^{+s}(x, x')$ is shown in Ref. [66]. We will see that the same argument also leads to the regularity of $G^{+t}_{ijkl}(x, x')$. Since the functions $f^\alpha_k(t)$ with $\alpha = s, t$ are not singular, the regularity can be verified if the integration converges both in the IR and UV limits. The regularity in the IR limit is guaranteed by the presence of the IR-suppressing operator. The IR-suppressing operators $\mathcal{R}_x$ add at least one extra factor of $k|\eta(t)|$ or eliminate the leading t-independent term in the IR limit, and yield

$$\mathcal{R}_x \mathcal{A}^s(t) f^s_k(t) \left[\frac{e^{ik\sigma_+(x,x')} - e^{ik\sigma_-(x,x')}}{ik(\sigma_+(x, x') - \sigma_-(x, x'))} \right] = \mathcal{A}^s(t_k) e^{ik\eta(t')} \mathcal{O}\left(k|\eta(t)|\right) = \mathcal{A}^s(t) e^{ik\eta(t')} \mathcal{O}\left(\{k|\eta(t)|\}^{(n_s+1)/2} \right), \tag{5.15}$$

and

$$\mathcal{R}_x \mathcal{A}^t(t) f^t_k(t) \left[\frac{e^{ik\sigma_+(x,x')} - e^{ik\sigma_-(x,x')}}{ik(\sigma_+(x, x') - \sigma_-(x, x'))} \right] = \mathcal{A}^t(t) e^{ik\eta(t')} \mathcal{O}\left(\{k|\eta(t)|\}^{(n_t+2)/2} \right), \tag{5.16}$$

where we have introduced the spectral indices n_s and n_t as

$$n_s - 1 \equiv d \ln(|\mathcal{A}^s(t_k)|^2)/d \ln k, \tag{5.17}$$

$$n_t \equiv d \ln(|\mathcal{A}^t(t_k)|^2)/d \ln k. \tag{5.18}$$

Thus, the operation of $\mathcal{R}_x$ makes the k integration in Eqs. (5.12) and (5.13) regular in the IR limit, ensuring the IR regularity. Next, we consider the convergence in the UV limit. When we choose the Euclidean vacuum, the $i\epsilon$ prescription facilitates the regularization of the UV modes in Eqs. (5.12) and (5.13), because adding a small imaginary part to all the time coordinates as $\eta \to \eta \times (1 - i\epsilon)$ leads to the replacement

$$\eta(t') - \eta(t) \to \eta(t') - \eta(t) + i\epsilon|\eta(t') - \eta(t)|$$

with $\eta(t') - \eta(t) < 0$. Then, the manifest exponential suppression factor is introduced for large k. This UV regulator makes the integral finite for the large k contribution, except for the case $\sigma_\pm(x, x') = 0$, where x and x' are mutually light-like. Since the expression of the Wightman functions obtained after the k integration is independent of the value of ϵ, this regulator makes the UV contributions convergent even after ϵ is sent to zero. For $\sigma_\pm(x, x') = 0$, the integral becomes divergent in the limit $\epsilon \to 0$, but the divergence related to the behavior of the Wightman functions in this limit is to be interpreted as ordinary UV divergences, whose contribution to the vertex integrals must be renormalized by introducing local counter terms. Thus, the Wightman functions $\mathcal{R}_x\mathcal{R}_{x'}G^{\pm s}(x, x')$ and $\mathcal{R}_x\mathcal{R}_{x'}G^{\pm t}_{ijkl}(x, x')$ are now shown to be regular functions.

Since the amplitudes of the Wightman functions with the IR-suppressing operator are bounded from above, we can show the regularity of the n-point functions, if the non-vanishing support of the integrands of the vertex integrals is effectively restricted to a finite spacetime region. Since the causality has been established with the aid of the residual *gauge* degrees of freedom (see Sect. 4.2.2), the question to address is whether contributions from the distant past are shut off or not. In short, this question can be rephrased as whether the SG due to the time integral exists or not. To address such a long-term correlation, we discuss the asymptotic behavior of the Wightman functions $\mathcal{R}_x\mathcal{R}_{x'}G^{\pm s}(x, x')$ and $\mathcal{R}_x\mathcal{R}_{x'}G^{\pm t}_{ijkl}(x, x')$, sending t' to a distant past. Recall that when $\sigma_\pm(x, x') \neq 0$, we can rotate the integration contour with respect to k even toward the direction parallel to the imaginary axis, making ϵ finite. Rotating the direction of the path appropriately depending on the sign of $\sigma_\pm(x, x')$, the integrand shows an exponential decay for $k \gtrsim 1/|\sigma_\pm(x, x')| \simeq 1/|\eta(t')|$. Since we send t' to the past infinity, where $|\eta(t')| \gg |\eta(t)|$, $\sigma_\pm(x, x')$ becomes $\mathcal{O}(|\eta(t')|)$, except for the region where the two points are mutually light-like (see Ref. [66] regarding the estimation of the contribution from this region). The rotation of the k integration contour can be done without hitting any singularity in the complex k-plane, because the functions $f^\alpha_k(t)$ are guaranteed to be analytic by construction. If we choose other vacua, this operation yields extra contributions from singularities. After the rotation, the integrations of k on the right-hand sides of Eqs. (5.12) and (5.13) are totally dominated by wavenumbers with $k \lesssim 1/|\eta(t')| \ll 1/|\eta(t)|$. Using Eq. (5.15), which gives the asymptotic expansion in the limit $k|\eta(t)| \ll 1$, we obtain

$$\begin{aligned}\mathcal{R}_x\mathcal{R}_{x'}G^{+s}(x, x') &= \mathcal{A}^s(t) \times \mathcal{O}\left[\int_0^\infty \frac{dk}{k}\,\{k|\eta(t)|\}^{(n_s+1)/2}\,\mathcal{R}_{x'}\mathcal{A}^s(t')f^{s*}_k(t')e^{ik\eta(t')}\right] \\ &= \mathcal{A}^s(t)\mathcal{A}^s(t')\mathcal{O}\left(\left(\frac{|\eta(t)|}{|\eta(t')|}\right)^{\frac{n_s+1}{2}}\right),\end{aligned} \tag{5.19}$$

where in the second equality we have performed the k integration, rotating the integration contour. Similarly, using Eq. (5.16), we obtain

$$\mathcal{R}_x\mathcal{R}_{x'}G^{+t}_{ijkl}(x, x') = \mathcal{A}^t(t)\mathcal{A}^t(t')\mathcal{O}\left(\left(\frac{|\eta(t)|}{|\eta(t')|}\right)^{\frac{n_t+2}{2}}\right). \tag{5.20}$$

We should emphasize that we do not employ the long-wavelength approximation regarding the Hubble scale at t' to properly evaluate the modes k of $\mathcal{O}(1/|\eta(t')|)$ as well.

5.3. *Secular growth (SG) due to the time integral*

In this subsection, focusing on the long-term correlation, we discuss the convergence of the vertex integrals of the n-point functions for the Euclidean vacuum. We start with the integration of the n-point interaction vertex that is closest to $\eta = -\infty(1 - i\epsilon)$. By inserting the expression of the Wightman functions $\mathcal{R}_x\mathcal{R}_{x'}G^{+}(x, x')$ and $\mathcal{R}_x\mathcal{R}_{x'}G^{+t}_{ijkl}(x, x')$ with $t \gg t'$, given in Eqs. (5.19) and

(5.20), into Eq. (5.9), the vertex integral $V^{(1)}$ can be estimated as

$$V^{(1)}(t', \{x_{m^\alpha}\}) = \mathcal{O}\Bigg[M_{\rm pl}^2 \int_{t_i}^{t'} dt \int d^3\boldsymbol{x} e^{3\rho(t)} \dot{\rho}(t)^2 \lambda(t) \{\mathcal{A}^s(t)\}^{N^s} \prod_{m^s=1}^{N^s} \mathcal{A}^s(t_{m^s}) \left(\frac{\eta(t_{m^s})}{\eta(t)}\right)^{\frac{n_s+1}{2}} \times \{\mathcal{A}^t(t)\}^{N^t} \prod_{m^t=1}^{N^t} \mathcal{A}^t(t_{m^t}) \left(\frac{\eta(t_{m^t})}{\eta(t)}\right)^{\frac{n_t+2}{2}}\Bigg]. \tag{5.21}$$

As we explained in Sect. 4.2.2, the interaction vertices are confined within the observable region, i.e., the non-vanishing support of the integrand is bounded by $|\boldsymbol{x}| \lesssim L_t$, where L_t can be approximated by $|\eta(t)|$ in the distant past. Thus, we obtain

$$V^{(1)}(t', \{x_{m^\alpha}\}) = \mathcal{O}\Bigg[\int_{-\infty}^{\eta(t')} \frac{d\eta}{\eta} \lambda(\eta) \{\mathcal{A}^s(\eta)\}^{N^s} \{\mathcal{A}^t(\eta)\}^{N^t-2} \times \prod_{m^s=1}^{N^s} \mathcal{A}^s(t_{m^s}) \left(\frac{\eta(t_{m^s})}{\eta}\right)^{\frac{n_s+1}{2}} \prod_{m^t=1}^{N^t} \mathcal{A}^t(t_{m^t}) \left(\frac{\eta(t_{m^t})}{\eta}\right)^{\frac{n_t+2}{2}}\Bigg]. \tag{5.22}$$

Since we have performed the momentum integral first, the exponential suppression for large $|\eta|$, required for the Euclidean vacuum, no longer remains. However, picking up the η dependence of the integrand of Eq. (5.22), we still find that the contribution from the distant past is suppressed if

$$\left|\lambda(\eta) \left\{\mathcal{A}^s(\eta)\right\}^{N^s} \left\{\mathcal{A}^t(\eta)\right\}^{N^t-2} \eta^{-\frac{N^s(n_s+1)+N^t(n_t+2)}{2}}\right| \to 0 \quad \text{as} \quad \eta \to -\infty. \tag{5.23}$$

When this condition is satisfied, the time integral converges, and the amplitude of $V_n^{(1)}(\eta', \{x_m\})$ is estimated by the value of the integrand at the upper end of the integration as

$$V^{(1)}(t', \{x_{m^\alpha}\}) = \mathcal{O}\left[\lambda(t')\{\mathcal{A}^s(t')\}^{N^s}\{\mathcal{A}^t(t')\}^{N^t-2} \prod_{m^s=1}^{N^s} \mathcal{A}^s(t_{m^s}) \left(\frac{\eta(t_{m^s})}{\eta(t')}\right)^{\frac{n_s+1}{2}} \prod_{m^t=1}^{N^t} \mathcal{A}^t(t_{m^t}) \left(\frac{\eta(t_{m^t})}{\eta(t')}\right)^{\frac{n_t+2}{2}}\right]. \tag{5.24}$$

Then, when one of the Wightman propagators is connected to a vertex located in the future of x', i.e., $t_m > t'$, the t-integration yields the suppression factor $\{\eta(t_{m^s})/\eta(t')\}^{\frac{n_s+1}{2}}$ or $\{\eta(t_{m^t})/\eta(t')\}^{\frac{n_t+2}{2}}$. We denote the numbers of such scalar and graviton propagators by $\tilde{N}^s$ and $\tilde{N}^t$, respectively.

Similarly, we can evaluate the amplitude of $V^{(2)}$ as

$$V^{(2)}(t'', \{x_{m^\alpha}\}, \{x_{m^{\alpha'}}\}) = \mathcal{O}\Bigg[\int_{-\infty}^{\eta(t'')} \frac{d\eta'}{\eta'} \lambda'(\eta') \{\mathcal{A}^s(\eta')\}^{N^{s\prime}} \{\mathcal{A}^t(\eta')\}^{N^{t\prime}-2} \times \prod_{m^{s\prime}=1}^{N^{s\prime}} \mathcal{A}(t_{m^{s\prime}}) \left(\frac{\eta(t_{m^{s\prime}})}{\eta'}\right)^{\frac{n_s+1}{2}} \prod_{m^{t\prime}=1}^{N^{t\prime}} \mathcal{A}(t_{m^{t\prime}}) \left(\frac{\eta(t_{m^{t\prime}})}{\eta'}\right)^{\frac{n_t+2}{2}} V^{(1)}(t(\eta'), \{x_m\})\Bigg]. \tag{5.25}$$

Extracting the η'-dependent part in the above expression, we obtain

$$\int_{-\infty}^{\eta(t'')} \frac{d\eta'}{\eta'} \lambda(\eta')\lambda'(\eta') \{\mathcal{A}^s(\eta')\}^{N^s+N^{s\prime}} \{\mathcal{A}^t(\eta')\}^{N^t+N^{t\prime}-4} |\eta'|^{-\frac{n_s+1}{2}(N^{s\prime}+\tilde{N}^s)-\frac{n_s+2}{2}(N^{t\prime}+\tilde{N}^t)}. \tag{5.26}$$

Now the generalization proceeds in a straightforward way. For the N_vth vertex, the temporal integration becomes

$$\int \frac{d\eta_{N_v}}{\eta_{N_v}} \hat{\lambda}(\eta_{N_v}) \{\mathcal{A}^s(\eta_{N_v})\}^{N_f^s} \{\mathcal{A}^t(\eta_{N_v})\}^{N_f^t - 2N_v} |\eta_{N_v}|^{-\frac{(n_s+1)M^s + (n_t+2)M^t}{2}}, \tag{5.27}$$

where N_f^s and N_f^t, respectively, denote the numbers of $\tilde{\zeta}_I$ and $\delta\tilde{\gamma}_{ijI}$ contained in the vertices that have been integrated before the N_vth vertex, M^s and M^t denote the numbers of the Wightman propagators connected to a vertex with $\eta > \eta_{N_v}$, and $\hat{\lambda}$ is the product of all the interaction coefficients contained in the integrated vertices. Thus, the convergence condition is derived as

$$\left| \hat{\lambda}(\eta) \{\mathcal{A}^s(\eta)\}^{N_f^s} \{\mathcal{A}^t(\eta)\}^{N_f^t - 2N_v} \eta^{-\frac{(n_s+1)M^s + (n_t+2)M^t}{2}} \right| \to 0 \quad \text{as} \quad \eta \to -\infty. \tag{5.28}$$

As a simple example, we consider the case where ε_1 is constant. In this case, $\hat{\lambda}$ is expressed only in terms of ε_1 and takes a constant value. By assuming $M = 1$ and using $n_s - 1 = -2\varepsilon_1$, the convergence condition yields

$$(1 - \varepsilon_1)^2 M - \varepsilon_1 N > 0, \tag{5.29}$$

where $N \equiv N_f^s + N_f^t - 2N_v$ and $M \equiv M^s + M^t$. In the slow-roll limit $\varepsilon_1 \ll 1$, the above condition is recast into

$$N < \mathcal{O}\left(\frac{M}{\varepsilon_1}\right). \tag{5.30}$$

Since all interaction vertices contain at least one propagator that is connected to a vertex in their future, M should be $M \geq 1$. Therefore, unless an extremely high order in perturbation with $N > \mathcal{O}(1/\varepsilon_1)$ is concerned, the contributions from the distant past are suppressed and hence the time integrals at the interaction vertices do not yield the SG.

The presence of the above suppression can be intuitively understood in the same way as in the discussion for the loops of the curvature perturbation [66]. When we choose the Euclidean vacuum as the initial state, both the IR and UV modes in the Wightman functions are suppressed and then only the contributions around the Hubble scale at each time are left unsuppressed. Being affected only by the modes around the Hubble scale, i.e., $k|\eta| \simeq k/e^\rho \dot{\rho} = \mathcal{O}(1)$, the Wightman functions $\mathcal{R}_x \mathcal{R}_{x'} G^{\pm s}(x, x')$ and $\mathcal{R}_x \mathcal{R}_{x'} G^{\pm t}_{ijkl}(x, x')$ are necessarily suppressed when $\eta(t)/\eta(t') \ll 1$. This is because, if the spacetime points x and x' are largely separated in time, any Fourier mode in the Wightman function cannot be of the order of the Hubble scale simultaneously at t and t'. When we consider the contribution of vertices located far in the past, at least one Wightman function should satisfy $\eta(t)/\eta(t') \ll 1$, and therefore it is suppressed. Equation (5.28) shows such suppression by M^s scalar propagators and M^t graviton propagators. As shown in Eq. (5.28), as we increase the number of operators included in or connected to the interaction vertex, denoted by N_f^s and N_f^t, the contributions from the distant past become less suppressed. On the other hand, as we increase the number of propagators connected to the vertices around the observation time, labeled by M^s and M^t, the contributions from the distant past are more suppressed. When N is sufficiently large, i.e., $N > \mathcal{O}(M/\varepsilon_1)$, the suppression due to M propagators can be overwhelmed by the large amplitude of the fluctuation, which increases when the energy scale of inflation increases, as in the far past. However, we should also stress that the SG never appears in slow-roll inflation, unless the order of perturbative expansion N takes an extremely large value, such as $1/\varepsilon_1 \simeq \mathcal{O}(10^2)$.

6. Concluding remarks

In this paper, we have addressed the regularity of the graviton loops. We have shown that, when we choose the Euclidean vacuum as the initial state, similar to the curvature perturbation, the graviton perturbation does not cause the IRdiv and IRsec in the n-point functions of genuine *gauge* invariant operators. In the absence of the graviton, simply performing the dilatation transformation provides a new set of canonical variables in which all ζ in the Hamiltonian are shifted by the free parameter s. The presence of this new set of canonical variables is important to show the IR regularity for the Euclidean vacuum. Extending this previous result to the graviton perturbation, we have provided a new set of canonical variables whose Hamiltonian includes the curvature perturbation and the graviton perturbation with the shifts by arbitrary time-dependent parameters s and S_{ij}, respectively. Then, following a similar argument to the one in Ref. [66], we established the IR regularity, i.e., the absence of the IRdiv and IRsec for the Euclidean vacuum to any order of perturbation. We also showed the absence of the SG in slow-roll inflation, at least, unless extremely high orders in perturbation are involved.

As is also argued in Ref. [66], when we evaluate the SG, considering only the superH modes is not sufficient, because all modes are subH modes when we send the initial time t_i to the past infinity. In Sect. 5.3, to evaluate the SG, we used the Wightman functions obtained in Sect. 5.2. These Wightman functions $\mathcal{R}_x \mathcal{R}_{x'} G^{\pm s}(x, x')$ and $\mathcal{R}_x \mathcal{R}_{x'} G^{\pm t}_{ijkl}(x, x')$ are shown to take finite values, as long as the two arguments x and x' are not mutually light-like. In this paper and also in Ref. [66], assuming that these UV divergences will be renormalized by local counter terms, we have not explicitly examined the contributions from the singular UV modes. We leave a detailed discussion about the UV renormalization for future study. (See Refs. [76,108], where the UV regularization is discussed.)

In this paper, we considered the inflationary universe as the background spacetime. When we take an exact de Sitter space as the background spacetime without introducing a scalar field, the curvature perturbation will disappear, while the graviton perturbation can still exist. For the pure gravity in the de Sitter limit, the accumulation of residual *gauge* degrees of freedom is still an issue of debate. It has been claimed that the IR graviton can become a trigger for the running of the coupling constant. For instance, in Ref. [10], Tsamis and Woodard claimed that the IR graviton can screen the cosmological constant, suggesting the possibility that the cosmological constant problem might be dynamically solved. In our forthcoming publication, we will address the IR issues of the graviton in the exact de Sitter background and discuss whether the screening of the cosmological constant can still exist even if we request the genuine *gauge* invariance.

Finally, we make several comments on the quantum states allowed from the IR regularity conditions. We have seen that, when we choose the Euclidean vacuum as the initial state, the n-point functions of the genuine *gauge* invariant operator become IR regular. Then, the question is whether the regularity can be maintained for other initial states or not. In the simple setup adopted in Appendix A, which immediately ensures the standard commutation relations for the interaction picture fields, we found that requesting the IR regularity of the graviton loops yields the same condition on the mode function v_k^s that was requested from the IR regularity of the loop corrections due to the curvature perturbation. (In Ref. [63], we claimed that the IR regularity of the graviton loops does not yield any condition on v_k^s. However, as mentioned in Appendix A, in Ref. [63], we chose an alternative heuristic iteration scheme that does not immediately guarantee the standard commutation relations for the interaction picture fields. Therefore there is no contradiction with the current result.) It will be intriguing to elaborate how strictly the IR regularity condition constrains the quantum state in the inflationary universe. We will also leave this issue for future study. (See also the studies on the scalar field by Einhorn and Larsen in Refs. [109,110] and by Marolf et al. in Ref. [111].)

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Appendix A: Constraining the initial states from the IR regularity

When we set the initial state to the Euclidean vacuum, as we mentioned in Sect. 4.1, the equivalence between the two sets of canonical variables that are connected by the residual *gauge* transformations is ensured. Making use of the privileged property of the Euclidean vacuum, we can write the perturbative expansion in a way that all the interaction picture fields are associated with the IR-suppressing operator $\mathcal{R}_x$, which plays a crucial role in showing the IR regularity. This consideration suggests that the IR regularity will not be guaranteed for an arbitrary choice of the initial state. In this section, we will show that the requirement of IR regularity actually yields a non-trivial restriction on the quantum state, choosing a simple setup where the interaction is turned on at a finite initial time t_i. In this appendix, all field variables without the subscript I are supposed to be those in the Heisenberg picture.

A.1. Solving the equations of motion

In this subsection, we compute the two-point function of $\mathcal{R}_x{}^g\zeta(x)$ up to one-loop order to derive the IR regularity condition on the initial state. Assuming that the interaction is turned on at the initial time t_i, we set the relation between the Heisenberg fields and the interaction picture fields as

$$\zeta(t_i, \boldsymbol{x}) = \zeta_I(t_i, \boldsymbol{x}), \qquad \pi(t_i, \boldsymbol{x}) = \pi_I(t_i, \boldsymbol{x}), \tag{A1}$$

and

$$\delta\gamma_{ij}(t_i, \boldsymbol{x}) = \delta\gamma_{ij\,I}(t_i, \boldsymbol{x}), \qquad \pi_{ij}(t_i, \boldsymbol{x}) = \pi_{ij\,I}(t_i, \boldsymbol{x}), \tag{A2}$$

where π_I and $\pi_{ij\,I}$ are the conjugate momenta of the interaction picture fields ζ_I and $\delta\gamma_{ij\,I}$, respectively. The advantage of choosing this initial condition is that the commutation relations for the Heisenberg field Φ, given in Eqs. (3.4) and (3.5), also immediately guarantee the standard commutation relations for the interaction picture fields, i.e.,

$$[\zeta_I(t, \boldsymbol{x}), \pi_I(t, \boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \quad [\zeta_I(t, \boldsymbol{x}), \zeta_I(t, \boldsymbol{y})] = [\pi_I(t, \boldsymbol{x}), \pi_I(t, \boldsymbol{y})] = 0, \tag{A3}$$

and

$$\left[\delta\gamma_{ij\,I}(t, \boldsymbol{x}), \pi_I^{kl}(t, \boldsymbol{y})\right] = i\delta_{ij}^{(3)\,kl}(\boldsymbol{x}-\boldsymbol{y}), \quad \left[\delta\gamma_{ij\,I}(t, \boldsymbol{x}), \delta\gamma_{kl\,I}(t, \boldsymbol{y})\right] = \left[\pi_I^{ij}(t, \boldsymbol{x}), \pi_I^{kl}(t, \boldsymbol{y})\right] = 0. \tag{A4}$$

Here, we compute the two-point function of $\mathcal{R}_x{}^g\zeta(x)$ by solving the Heisenberg operator equations of motion for ζ and $\delta\gamma_{ij}$. Using the retarded Green functions $G_R(x,x')$ and $G_{Rijkl}(x,x')$ given by

$$G_R(x,x') = -i\theta(t-t')\left[\zeta_I(x), \zeta_I(x')\right], \tag{A5}$$

$$G_{Rijkl}(x,x') = -i\theta(t-t')\left[\delta\gamma_{ijI}(x), \delta\gamma_{klI}(x')\right], \tag{A6}$$

we can solve the equations of motion for ζ and $\delta\gamma_{ij}$, employing the initial conditions (A1) and (A2) as

$$\zeta(x) = \zeta_I(x) + \mathcal{L}_{R,s}^{-1}\mathcal{S}_{\rm NL}(x), \tag{A7}$$

$$\delta\gamma_{ij}(x) = \delta\gamma_{ijI}(x) + \mathcal{L}_{R,t}^{-1}\mathcal{S}_{{\rm NL}ij}(x), \tag{A8}$$

with

$$\mathcal{L}_{R,s}^{-1}\mathcal{S}_{\rm NL}(t,\boldsymbol{x}) \equiv -2M_{\rm pl}^2 \int d^4x'\varepsilon_1(t')e^{3\rho(t')}G_R(x,x')\mathcal{S}_{\rm NL}(x'), \tag{A9}$$

$$\mathcal{L}_{R,t}^{-1}\mathcal{S}_{{\rm NL}ij}(t,\boldsymbol{x}) \equiv -\frac{M_{\rm pl}^2}{4}\int d^4x' e^{3\rho(t')}G_{Rkl}^{kl}(x,x')\mathcal{S}_{{\rm NL}ij}(x'), \tag{A10}$$

where the explicit form of the non-linear source terms $\mathcal{S}_{\rm NL}(x)$ and $\mathcal{S}_{{\rm NL}ij}(x')$ will be derived later. Evaluating Eqs. (A9) and (A10) iteratively, we can obtain expressions for ζ and $\delta\gamma_{ij}$, respectively.

Inserting the thus-obtained solution ζ and $\delta\gamma_{ij}$ into Eq. (3.45), we can perturbatively compute ${}^g\zeta(x)$ as

$${}^g\zeta(x) = \zeta_I(x) + {}^g\zeta_2(x) + {}^g\zeta_3(x) + \cdots, \tag{A11}$$

where ${}^g\zeta_n(x)$ represents the term that consists of n interaction picture fields. Expanding the interaction picture fields ζ_I and $\delta\gamma_{ijI}$ as in Eqs. (2.9) and (2.19), the initial vacuum state is defined by

$$a_{\boldsymbol{k}}|0\rangle = a_{\boldsymbol{k}}^{(\lambda)}|0\rangle = 0. \tag{A12}$$

Notice that the n-point functions computed by using the Heisenberg operator solved with the retarded Green function can be formally shown to agree with those calculated in the in–in formalism (see, for instance, the Appendix of Ref. [65]).

Using Eq. (A11), the one-loop contributions to the two-point function of $\mathcal{R}_x{}^g\zeta(x)$ are given by

$$\begin{aligned}&\langle\mathcal{R}_{x_1}{}^g\zeta(x_1)\mathcal{R}_{x_2}{}^g\zeta(x_2)\rangle_{\rm 1loop}\\ &\quad = \langle\mathcal{R}_{x_1}{}^g\zeta_2(x_1)\mathcal{R}_{x_2}{}^g\zeta_2(x_2)\rangle + \langle\mathcal{R}_{x_1}\zeta_I(x_1)\mathcal{R}_{x_2}{}^g\zeta_3(x_2)\rangle + \langle\mathcal{R}_{x_1}{}^g\zeta_3(x_1)\mathcal{R}_{x_2}\zeta_I(x_2)\rangle.\end{aligned} \tag{A13}$$

As discussed in Sect. 4.2.2, after we choose the boundary conditions for ∂^{-2} appropriately, the inverse Laplacian does not enhance the singular behavior of the superH modes, and hence the IRdiv and IRsec can appear only from the variances of $\zeta_I(x)$ and $\delta\gamma_{ijI}(x)$, whose superH contributions give

$$\langle\bar{\zeta}_I^2(t)\rangle = \int_{k\le 1/L_t}\frac{d^3\boldsymbol{k}}{(2\pi)^3}P^s(k) \propto \int_{k\le 1/L_t}\frac{dk}{k}, \tag{A14}$$

and

$$\langle \delta\bar{\gamma}_{ijI}(t)\delta\bar{\gamma}_{klI}(t)\rangle = \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right)\langle \delta\bar{\gamma}_I^2(t)\rangle, \tag{A15}$$

with

$$\langle \delta\bar{\gamma}_I^2(t)\rangle \equiv \frac{1}{20\pi^2}\int_{k\le 1/L_t}\frac{dk}{k}k^3 P^t(k) \propto \int_{k\le 1/L_t}\frac{dk}{k}. \tag{A16}$$

Here $\bar{\zeta}_I(t)$ and $\delta\bar{\gamma}_{ijI}(t)$, respectively, denote ζ_I and $\delta\gamma_{ijI}$ with only the superH modes, which mimic their spatially averaged values in $\mathcal{O}_t$.

When ${}^g\zeta_2$ includes terms with ζ_I or $\delta\gamma_{ijI}$ without differentiation, the first term in the second line of Eq. (A13) can give $\langle\bar{\zeta}_I^2\rangle$ or $\langle\delta\bar{\gamma}_I^2\rangle$. These variances can also appear from the second and third terms, when ${}^g\zeta_3$ includes terms with two ζ_I or two $\delta\gamma_{ijI}$ without differentiation. To make our discussion compact and transparent, here, we pick up only the potentially divergent contributions, which yield $\langle\bar{\zeta}_I^2\rangle$ or $\langle\delta\bar{\gamma}_I^2\rangle$. We introduce the symbol

$$\overset{\mathrm{IR}}{\approx}$$

to denote the approximate equality neglecting the terms that yield neither $\langle\bar{\zeta}_I^2\rangle$ nor $\langle\delta\bar{\gamma}_I^2\rangle$ at the one-loop level [62,63].

Now, we derive approximate equations of motion for ζ and $\delta\gamma_{ij}$. In the following, we will use

$$\mathcal{L}_{R,\alpha}^{-1}Q_I\mathcal{R}_x Q_I' \overset{\mathrm{IR}}{\approx} Q_I\mathcal{L}_{R,\alpha}^{-1}\mathcal{R}_x Q_I', \tag{A17}$$

with $\alpha = s, t$. Here, Q_I and Q_I' are either ζ_I or $\delta\gamma_{ijI}$ and $\mathcal{R}_x$ is a derivative operator that suppresses the IR modes. Equation (A17) can be proved as follows. The Fourier transformation of $\mathcal{L}_{R,s}^{-1}Q_I\mathcal{R}_x Q_I'$ is proportional to

$$\int d^3\boldsymbol{p}\int_{t_i}^{t} dt'\varepsilon_1(t')e^{3\rho(t')}\dot{\rho}^2(t')\{v_k(t)v_k^*(t') - v_k^*(t)v_k(t')\}Q_{I\boldsymbol{p}}(t')\left(\mathcal{R}Q_I'\right)_{\boldsymbol{k}-\boldsymbol{p}}(t'),$$

where $Q_{I\boldsymbol{k}}$ and $(\mathcal{R}Q_I')_{\boldsymbol{k}}$ denote the Fourier modes of Q_I and $\mathcal{R}Q_I'$. Since $(\mathcal{R}Q_I)_{\boldsymbol{k}-\boldsymbol{p}}(t') - \left(\mathcal{R}Q_I'\right)_{\boldsymbol{k}}(t')$ is suppressed and $Q_{I,\boldsymbol{p}}$ becomes time-independent in the limit $\boldsymbol{p}\to 0$, the IR-relevant piece of the integrand of the momentum integral can be recast into

$$Q_{I\boldsymbol{p}}\int_{t_i}^{t} dt'\varepsilon_1(t')e^{3\rho(t')}\dot{\rho}^2(t')\{v_k(t)v_k^*(t') - v_k^*(t)v_k(t')\}\left(\mathcal{R}Q_I'\right)_{\boldsymbol{k}}(t'). \tag{A18}$$

Similarly, we can also prove Eq. (A17) for $\mathcal{L}_{R,t}^{-1}$. In the following discussion, we will also use the approximate identities

$$\mathcal{L}_{R,\alpha}^{-1}f(x) \overset{\mathrm{IR}}{\approx} 0, \qquad \text{for } f(x) \overset{\mathrm{IR}}{\approx} 0. \tag{A19}$$

In the one-loop corrections to $\mathcal{R}_x{}^g\zeta$, $\delta\gamma_{ij2}$ contributes only through ${}^g\zeta_3$, and the $\delta\gamma_{ijn}$ with $n\ge 3$ do not contribute. Since at least one of the two interaction picture fields included in $\delta\gamma_{ij2}$ is suppressed by $\mathcal{R}_x$, we find $\delta\gamma_{ij2} \overset{\mathrm{IR}}{\approx} 0$. Then, we find

$$\delta\gamma_{ij}(x) \overset{\mathrm{IR}}{\approx} \delta\gamma_{ijI}(x), \tag{A20}$$

and hence the one-loop corrections can be given without computing the non-linear contributions in $\delta\gamma_{ij}$.

Next, we derive an approximate equation of motion for ζ. Under the equality $\overset{\text{IR}}{\approx}$, the non-linear action is reduced to

$$S \overset{\text{IR}}{\approx} M_{\text{pl}}^2 \int dt d^3\boldsymbol{x} e^{3(\rho+\zeta)} \varepsilon_1 \left[(\partial_t \zeta)^2 - e^{-2(\rho+\zeta)} \left[e^{-\delta\gamma} \right]^{ij} \partial_i \zeta \partial_j \zeta \right], \tag{A21}$$

where the terms with more than two fields with differentiation, which give neither $\langle \bar{\zeta}_I^2 \rangle$ nor $\langle \delta\bar{\gamma}_{ijI} \delta\bar{\gamma}_{klI} \rangle$, are abbreviated. The variation of the above action gives the equation of motion as

$$\mathcal{L}_s \zeta(x) = \mathcal{S}_{\text{NL}}(x), \tag{A22}$$

with

$$\mathcal{L}_s \equiv \partial_t^2 + (3 + \varepsilon_2)\, \dot{\rho} \partial_t - e^{-2\rho} \partial^2, \tag{A23}$$

and

$$\mathcal{S}_{\text{NL}}(x) \overset{\text{IR}}{\approx} e^{-2\rho} \left(e^{-2\zeta} [e^{-\delta\gamma}]^{ij} - \delta^{ij} \right) \partial_i \partial_j \zeta(x) - \delta(t - t_i)(e^{3\zeta} - 1) \partial_t \zeta(x), \tag{A24}$$

where the last term is added so that the solution satisfies the second condition in Eq. (A1) [65].

A.2. Computation of ${}^g\zeta$

Here, we solve the equation of motion (A22), employing the initial conditions (A1) and (A2). Expanding ζ as in Eq. (2.30), the equation of motion (A22) is recast into

$$\mathcal{L}_s \zeta_I = 0, \tag{A25}$$

$$\mathcal{L}_s \zeta_2 \overset{\text{IR}}{\approx} -(2\zeta_I \delta^{ij} + \delta\gamma_I^{ij}) \nabla_i \nabla_j \delta\zeta_I - 3\delta(t - t_i) \zeta_I \partial_t \zeta_I, \tag{A26}$$

$$\begin{aligned} \mathcal{L}_s \zeta_3 \overset{\text{IR}}{\approx} & -2 \left(\zeta_2 \Delta \zeta_I + \zeta_I \Delta \zeta_2 - \zeta_I^2 \Delta \zeta_I \right) \\ & + \frac{9}{2} \delta(t - t_i) \zeta_I^2 \partial_t \zeta_I - \left\{ \delta\gamma_I^{ij} \nabla_i \nabla_j \zeta_2 - \frac{1}{2} (\delta\gamma_I^2)^{ij} \nabla_i \nabla_j \zeta_I \right\}, \end{aligned} \tag{A27}$$

where we have introduced

$$\nabla_i \equiv e^{-\rho} \partial_i, \qquad \Delta \equiv \delta^{ij} \nabla_i \nabla_j. \tag{A28}$$

In deriving Eq. (A27), we used

$$\partial_t \zeta(t_i, \boldsymbol{x}) \overset{\text{IR}}{\approx} e^{-3\zeta_I(t_i, \boldsymbol{x})} \partial_t \zeta_I(t_i, \boldsymbol{x}), \tag{A29}$$

which is derived from the initial conditions (A2). Solving Eqs. (A26) and (A27) formally, we obtain

$$\zeta_2 \overset{\text{IR}}{\approx} -\bar{\zeta}_I \mathcal{L}_{R,s}^{-1} \left[2\Delta + 3\delta(t - t_i) \partial_t \right] \zeta_I - \delta\bar{\gamma}_I^{ij} \mathcal{L}_{R,s}^{-1} \nabla_i \nabla_j \zeta_I, \tag{A30}$$

$$\begin{aligned} \zeta_3 \overset{\text{IR}}{\approx} & \frac{1}{2} \bar{\zeta}_I^2 \left[4\mathcal{L}_{R,s}^{-1} \Delta \mathcal{L}_{R,s}^{-1} (2\Delta + 3\delta(t - t_i)\partial_t) + 4\mathcal{L}_{R,s}^{-1} \Delta + 9\mathcal{L}_{R,s}^{-1} \delta(t - t_i) \partial_t \right] \zeta_I \\ & + \delta\bar{\gamma}_I^{ij} \delta\bar{\gamma}_I^{kl} \mathcal{L}_{R,s}^{-1} \nabla_i \nabla_j \mathcal{L}_{R,s}^{-1} \nabla_k \nabla_l \zeta_I + \frac{1}{2} (\delta\bar{\gamma}_I^2)^{ij} \mathcal{L}_{R,s}^{-1} \nabla_i \nabla_j \zeta_I, \end{aligned} \tag{A31}$$

using the properties of the retarded integration given in Eqs. (A17) and (A19). Here, we have replaced ζ_I and $\delta\gamma_{ijI}$ with their superH contributions $\bar{\zeta}_I$ and $\delta\bar{\gamma}_{ijI}$, which contribute to the IRdiv and IRsec.

Next, using Eqs. (A30) and (A31), we express ${}^g\zeta(t_f, \boldsymbol{x})$, defined in Eq. (3.48), as

$$ {}^g\zeta(t_f, \boldsymbol{x}) = \zeta\left(t_f, e^{-{}^g\bar{\zeta}(t_f)}[e^{-\delta^g\bar{\gamma}(t_f)}]^i{}_j x^j\right). \tag{A32} $$

Inserting Eq. (A30) into Eq. (A32), we can easily obtain

$$ {}^g\zeta_2(t_f, \boldsymbol{x}) \overset{\text{IR}}{\approx} -\bar{\zeta}_I \mathcal{D}^s_x \zeta_I - \frac{1}{2}\delta\bar{\gamma}^{ij}_I \mathcal{D}^t_{x\,ij}\zeta_I, \tag{A33} $$

with

$$ \mathcal{D}^s_x \equiv 2\mathcal{L}^{-1}_{R,s}\Delta + 3\mathcal{L}^{-1}_{R,s}\delta(t-t_i)\partial_t + \boldsymbol{x}\cdot\partial_{\boldsymbol{x}}, \tag{A34} $$

$$ \mathcal{D}^t_{x\,ij} \equiv 2\mathcal{L}^{-1}_{R,s}\nabla_i\nabla_j + x_i\partial_j. \tag{A35} $$

The computation of ${}^g\zeta_3$ is slightly lengthy but straightforward. Using Eqs. (A30) and (A32), we find

$$ \begin{aligned} {}^g\zeta_3(t_f, \boldsymbol{x}) \overset{\text{IR}}{\approx}\ & \zeta_3 + \bar{\zeta}^2_I \boldsymbol{x}\cdot\partial_{\boldsymbol{x}}\mathcal{L}^{-1}_{R,s}(2\Delta + 3\delta(t-t_i)\partial_t)\zeta_I + \frac{1}{2}\bar{\zeta}^2_I(\boldsymbol{x}\cdot\partial_{\boldsymbol{x}})^2\zeta_I \\ & + \frac{1}{2}\delta\bar{\gamma}^i_{jI}\delta\bar{\gamma}^{kl}_I x^j\partial_i\mathcal{L}^{-1}_{R,s}\nabla_k\nabla_l\zeta_I + \frac{1}{8}\delta\bar{\gamma}^i_{jI}\delta\bar{\gamma}^k_{lI}x^j\partial_i x^l\partial_k\zeta_I. \end{aligned} \tag{A36} $$

To rewrite the terms with $x^i\partial_j\mathcal{L}^{-1}_{R,s}$ in ${}^g\zeta_3$ into a more tractable form, we use the identity

$$ x^i\partial_j\mathcal{L}^{-1}_{R,s} = \frac{1}{2}\left(x^i\partial_j\mathcal{L}^{-1}_{R,s} + \mathcal{L}^{-1}_{R,s}\mathcal{L}_s x^i\partial_j\mathcal{L}^{-1}_{R,s}\right), \tag{A37} $$

which obviously holds if $\mathcal{L}^{-1}_{R,s}\mathcal{L}_s$ can be replaced with unity. In general, for

$$ \delta_R \equiv (1 - \mathcal{L}^{-1}_{R,s}\mathcal{L}_s)x^i\partial_j\mathcal{L}^{-1}_{R,s}(\cdots), $$

we have $\mathcal{L}_s\delta_R = 0$, and hence δ_R is a homogeneous solution of the second-order differential equation, i.e., $\mathcal{L}_s\delta_R = 0$. Since δ_R and $\partial_t\delta_R$ are both zero at the initial time, which is automatically satisfied by the definition of the retarded integral $\mathcal{L}^{-1}_{R,s}$, we can confirm that δ_R vanishes for all $t \geq t_i$. Using

$$ \left[\mathcal{L}_s, x^i\partial_j\right] = -2\nabla^i\nabla_j, $$

the right-hand side of Eq. (A37) is further rewritten as

$$ x^i\partial_j\mathcal{L}^{-1}_{R,s} = \frac{1}{2}(x^i\partial_j\mathcal{L}^{-1}_{R,s} + \mathcal{L}^{-1}_{R,s}x^i\partial_j) - \mathcal{L}^{-1}_{R,s}\nabla^i\nabla_j\mathcal{L}^{-1}_{R,s}. \tag{A38} $$

Using Eqs. (A31), (A36), and (A38), we obtain

$$ \begin{aligned} {}^g\zeta_3(t_f, \boldsymbol{x}) \overset{\text{IR}}{\approx}\ & \frac{1}{2}\bar{\zeta}^2_I(\mathcal{D}^s_x)^2\zeta_I + \frac{1}{8}\delta\bar{\gamma}^{ij}_I\delta\bar{\gamma}^{kl}_I\mathcal{D}^t_{x\,ij}\mathcal{D}^t_{x\,kl}\zeta_I \\ & - 3\bar{\zeta}^2_I\mathcal{L}^{-1}_{R,s}\delta(t-t_i)\partial_t\mathcal{L}^{-1}_{R,s}\Delta\zeta_I + \frac{9}{2}\bar{\zeta}^2_I\{\mathcal{L}^{-1}_{R,s}\delta(t-t_i)\partial_t - (\mathcal{L}^{-1}_{R,s}\delta(t-t_i)\partial_t)^2\}\zeta_I. \end{aligned} \tag{A39} $$

Noticing that the definition of $\mathcal{L}_R^{-1}$ implies

$$\mathcal{L}_{R,s}^{-1}\delta(t-t_i)\partial_t\mathcal{L}_{R,s}^{-1}\Delta\zeta_I=0,\quad \{\mathcal{L}_{R,s}^{-1}\delta(t-t_i)\partial_t-(\mathcal{L}_{R,s}^{-1}\delta(t-t_i)\partial_t)^2\}\zeta_I=0, \tag{A40}$$

we obtain

$${}^g\zeta_3(t_f,\boldsymbol{x})\overset{\mathrm{IR}}{\approx}\frac{1}{2}\bar{\zeta}_I^2\mathcal{D}_x^{s\,2}\zeta_I+\frac{1}{8}\delta\bar{\gamma}_I^{ij}\delta\bar{\gamma}_I^{kl}\mathcal{D}_{x\,ij}^t\mathcal{D}_{x\,kl}^t\zeta_I. \tag{A41}$$

In the above expressions (A33) and (A41), $\bar{\zeta}_I$ multiplied by the delta function $\delta(t-t_i)$ in $\mathcal{D}_x^s$ should be understood as $\bar{\zeta}_I(t_i)$.

A.3. One-loop corrections

Using Eqs. (A33) and (A41) into Eq. (A13), we obtain the one-loop corrections to $\mathcal{R}_x{}^g\zeta(t_f,\boldsymbol{x})$ as

$$\begin{aligned}&\langle\mathcal{R}_{x_1}{}^g\zeta(t_f,\boldsymbol{x}_1)\mathcal{R}_{x_2}{}^g\zeta(t_f,\boldsymbol{x}_2)\rangle_{1\mathrm{loop}}\\&\overset{\mathrm{IR}}{\approx}\frac{1}{2}\langle\bar{\zeta}_I^2(t_i)\rangle\mathcal{F}_{\mathrm{IRdiv}}^s(x_1,x_2)+\frac{1}{2}\langle\{\bar{\zeta}_I(t_f)-\bar{\zeta}_I(t_i)\}^2\rangle\mathcal{F}_{\mathrm{IRsec}}^s(x_1,x_2)\\&\quad+\frac{1}{8}\langle\delta\bar{\gamma}_I^{ij}(t_f)\delta\bar{\gamma}_I^{kl}(t_f)\rangle\mathcal{F}_{ijkl}^t(x_1,x_2),\end{aligned} \tag{A42}$$

with

$$\mathcal{F}_{\mathrm{IRdiv}}^s(x_1,x_2)\equiv\mathcal{R}_{x_1}\mathcal{R}_{x_2}\Big\langle 2\mathcal{D}_{x_1}^s\zeta_I(x_1)\mathcal{D}_{x_2}^s\zeta_I(x_2)+\mathcal{D}_{x_1}^{s\,2}\zeta_I(x_1)\zeta_I(x_2)+\zeta_I(x_1)\mathcal{D}_{x_2}^{s\,2}\zeta_I(x_2)\Big\rangle, \tag{A43}$$

$$\mathcal{F}_{\mathrm{IRsec}}^s(x_1,x_2)\equiv\mathcal{R}_{x_1}\mathcal{R}_{x_2}\Big\langle 2\mathcal{D}_{x_1}^{s\prime}\zeta_I(x_1)\mathcal{D}_{x_2}^{s\prime}\zeta_I(x_2)+\mathcal{D}_{x_1}^{s\prime\,2}\zeta_I(x_1)\zeta_I(x_2)+\zeta_I(x_1)\mathcal{D}_{x_2}^{s\prime\,2}\zeta_I(x_2)\Big\rangle, \tag{A44}$$

$$\begin{aligned}\mathcal{F}_{ijkl}^t(x_1,x_2)\equiv\mathcal{R}_{x_1}\mathcal{R}_{x_2}\Big\langle&2\mathcal{D}_{x_1\,ij}^t\zeta_I(x_1)\mathcal{D}_{x_2\,kl}^t\zeta_I(x_2)\\&+\mathcal{D}_{x_1\,ij}^t\mathcal{D}_{x_1\,kl}^t\zeta_I(x_1)\zeta_I(x_2)+\zeta_I(x_1)\mathcal{D}_{x_2\,ij}^t\mathcal{D}_{x_2\,kl}^t\zeta_I(x_2)\Big\rangle,\end{aligned} \tag{A45}$$

where we have introduced $x_a\equiv(t_f,\boldsymbol{x}_a)$ for $a=1,2$ and

$$\mathcal{D}_x^{s\prime}\equiv 2\mathcal{L}_{R,s}^{-1}\Delta+\boldsymbol{x}\cdot\partial_{\boldsymbol{x}}, \tag{A46}$$

which agrees with the trace of $\mathcal{D}_{x\,ij}^t$. The first term in Eq. (A42) can yield the IRdiv of the curvature perturbation, which can be removed only if $\mathcal{F}_{\mathrm{IRdiv}}^s(x_1,x_2)$ vanishes. The second term, which accompanies

$$\langle\{\bar{\zeta}_I(t_f)-\bar{\zeta}_I(t_i)\}^2\rangle\simeq\int_{1/L_{t_i}\le k\le 1/L_{t_f}}\frac{d^3\boldsymbol{k}}{(2\pi)^3}P^s(k)\propto\ln\left\{\frac{e^{\rho(t_f)}\dot{\rho}(t_f)}{e^{\rho(t_i)}\dot{\rho}(t_i)}\right\},$$

appears to yield the IRsec due to the curvature perturbation. This term can be removed only if $\mathcal{F}_{\mathrm{IRsec}}^s(x_1,x_2)$ vanishes. The third term appears to yield the IRdiv and IRsec due to the graviton perturbation, which can be removed only if $\mathcal{F}_{ijkl}^t(x_1,x_2)$ vanishes.

A.4. IR regularity condition on the mode function

Next, we discuss a condition that eliminates the IRdiv and IRsec due to the curvature perturbation and the graviton perturbation. One may think that, if the conditions

$$\mathcal{D}_x^s \zeta_I(x) = 0, \tag{A47}$$

$$\mathcal{D}_x^{s\prime} \zeta_I(x) = 0, \tag{A48}$$

$$\mathcal{D}^t_{x\,ij} \zeta_I(x) = 0 \tag{A49}$$

were fulfilled, $\mathcal{F}^s_{\rm IRdiv}(x_1, x_2)$, $\mathcal{F}^s_{\rm IRsec}(x_1, x_2)$, and $\mathcal{F}^t_{ijkl}(x_1, x_2)$ would vanish, and hence the IR regularity could be guaranteed without imposing any further conditions. However, these conditions are immediately contradicted when we insert the mode expansion of ζ_I, given in (2.9), into Eq. (A47). Operating $\boldsymbol{x} \cdot \partial_{\boldsymbol{x}}$ on a Fourier mode $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ yields the factor $(\boldsymbol{x} \cdot \boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$, which cannot be canceled by the remaining two terms in Eq. (A47), since the retarded integral $\mathcal{L}^{-1}_{R,s}$ [65] acting on $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ leaves it proportional to $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$. Similarly, Eqs. (A48) and (A49) cannot be compatible with the Fourier mode decomposition, as long as we use the solution with the retarded Green function $\mathcal{L}^{-1}_{R,s}$, fixed by the initial condition (A1) and (A2).

Here, following Ref. [65], we look for a simple alternative way to remove the IRdiv and IRsec of the curvature and graviton perturbations. In Ref. [65], we pointed out that, when

$$\mathcal{D}_x^s \zeta_I(x) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} \left(a_{\boldsymbol{k}} D e^{i\boldsymbol{k}\cdot\boldsymbol{x}} v_k^s + (\text{h.c.}) \right) \tag{A50}$$

is satisfied, where D is defined as

$$D \equiv k^{-3/2} e^{-i\phi(k)} \boldsymbol{k} \cdot \partial_{\boldsymbol{k}} k^{3/2} e^{i\phi(k)}, \tag{A51}$$

and $\phi(k)$ is an arbitrary phase function $\phi(k)$, $\mathcal{F}^s_{\rm IRdiv}(x_1, x_2)$ can be summarized in the total derivative form as

$$\mathcal{F}^s_{\rm IRdiv}(x_1, x_2) = \mathcal{R}_{x_1} \mathcal{R}_{x_2} \int \frac{d(\ln k) d\Omega_{\boldsymbol{k}}}{(2\pi)^3} \partial^2_{\ln k} \left\{ k^3 |v_k^s|^2 e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)} \right\}, \tag{A52}$$

where $\int d\Omega_{\boldsymbol{k}}$ denotes the integration over the angular directions of $\boldsymbol{k}$. Then, since the integral of a total derivative vanishes, the IRdiv can be eliminated. Using the mode expansion (2.9), the condition (A50) can be recast into a condition on mode functions as

$$\mathcal{L}^{-1}_{R,\boldsymbol{k}} \left(-2(ke^{-\rho})^2 + 3\delta(t - t_i)\partial_t \right) v_k^s = D v_k^s, \tag{A53}$$

where $\mathcal{L}^{-1}_{R,\boldsymbol{k}}$ is the Fourier mode of $\mathcal{L}^{-1}_{R,s}$. Similarly, we can also eliminate the IRsec of the curvature perturbation, by requesting

$$\mathcal{D}_x^{s\prime} \zeta_I(x) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} \left(a_{\boldsymbol{k}} D e^{i\boldsymbol{k}\cdot\boldsymbol{x}} v_k^s + (\text{h.c.}) \right), \tag{A54}$$

which leads to a slightly different condition from Eq. (A53) as

$$-2\mathcal{L}^{-1}_{R,\boldsymbol{k}} (ke^{-\rho})^2 v_k^s = D v_k^s. \tag{A55}$$

Next, we will derive the IR regularity condition for the graviton loop. To compute $\mathcal{F}^t_{ijkl}(x_1, x_2)$, we first rewrite $\delta\bar{\gamma}^{ij}_I \mathcal{D}^t_{x\,ij}\zeta_I(x)$ as

$$\begin{aligned}
&\delta\bar{\gamma}^{ij}_I \mathcal{D}^t_{x\,ij}\zeta_I(x)\\
&\quad = \delta\bar{\gamma}^{ij}_I \int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} a_{\boldsymbol{k}} e^{-i\phi(k)}\\
&\qquad \left[\frac{\partial}{\partial k^i} k_j e^{i\phi(k)} v^s_k e^{i\boldsymbol{k}\cdot\boldsymbol{x}} - e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{\boldsymbol{k}_i \boldsymbol{k}_j}{k^2} (\mathcal{L}^{-1}_{R,\boldsymbol{k}} 2(ke^{-\rho})^2 + \boldsymbol{k}\cdot\partial_{\boldsymbol{k}}) e^{i\phi(k)} v^s_k\right] + (\text{h.c.}),
\end{aligned} \tag{A56}$$

where the terms multiplied by δ_{ij} in the square bracket vanish, being contracted with $\delta\bar{\gamma}^{ij}$. Noticing that $\partial/\partial k^i v^s_k = (k^i/k)\partial/\partial k v^s_k$, since v^s_k does not depend on the direction of $\boldsymbol{k}$, we find that the terms that potentially yield IRdiv and IRsec due to the graviton vanish as

$$\begin{aligned}
&\langle \delta\bar{\gamma}^{ij}_I(t_f)\delta\bar{\gamma}^{kl}_I(t_f)\rangle \mathcal{F}^t_{ijkl}(x_1, x_2)\\
&\quad = \langle \delta\bar{\gamma}^{ij}_I(t_f)\delta\bar{\gamma}^{kl}_I(t_f)\rangle \mathcal{R}_{x_1}\mathcal{R}_{x_2} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{\partial}{\partial k^i} k_j \frac{\partial}{\partial k^k} k_l \{|v^s_k(t_f)|^2 e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}\} = 0,
\end{aligned} \tag{A57}$$

if the mode function satisfies

$$-2\mathcal{L}^{-1}_{R,\boldsymbol{k}}(ke^{-\rho})^2 v^s_k = e^{-i\phi(k)} \boldsymbol{k}\cdot\partial_{\boldsymbol{k}} e^{i\phi(k)} v^s_k. \tag{A58}$$

Thus, if we require Eq. (A58), we can eliminate the IRdiv and IRsec due to the graviton loops.

In the case with the isotropic graviton spectrum, the IR regularity can be guaranteed if the mode function satisfies Eq. (A55). In fact, when we request the condition (A55), we find

$$\delta\bar{\gamma}^{ij}_I \mathcal{D}^t_{x\,ij}\zeta_I(x) = \delta\bar{\gamma}^{ij}_I \int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} a_{\boldsymbol{k}} e^{-i\phi(k)} \left[k^{-3/2}\frac{\partial}{\partial k^i} k^{3/2} k_j e^{i\phi(k)} v^s_k e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right] + (\text{h.c.}), \tag{A59}$$

and then the one-loop contribution from the graviton is given by

$$\begin{aligned}
&\langle \delta\bar{\gamma}^{ij}_I(t_f)\delta\bar{\gamma}^{kl}_I(t_f)\rangle \mathcal{F}^t_{ijkl}(x_1, x_2)\\
&\quad = \langle \delta\bar{\gamma}^{ij}_I(t_f)\delta\bar{\gamma}^{kl}_I(t_f)\rangle \mathcal{R}_{x_1}\mathcal{R}_{x_2} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} k^{-3} \frac{\partial}{\partial k^i} k_j \frac{\partial}{\partial k^k} k_l \{k^3 |v^s_k(t_f)|^2 e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}\}.
\end{aligned} \tag{A60}$$

Using the following relations:

$$\left[k^{-3}, \frac{\partial}{\partial k^i}\right] = 3\frac{k_i}{k^5}, \quad \left[\frac{k_i k_j}{k^5}, \frac{\partial}{\partial k^l}\right] = 5\frac{k_i k_j k_l}{k^7} - \frac{\delta_{il}k_j + \delta_{jl}k_i}{k^5}, \tag{A61}$$

we can rewrite Eq. (A60) as

$$\begin{aligned}
&\langle \delta\bar{\gamma}^{ij}_I(t_f)\delta\bar{\gamma}^{kl}_I(t_f)\rangle \mathcal{F}^t_{ijkl}(x_1, x_2)\\
&\quad = \langle \delta\bar{\gamma}^{ij}_I(t_f)\delta\bar{\gamma}^{kl}_I(t_f)\rangle \mathcal{R}_{x_1}\mathcal{R}_{x_2} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \left\{\frac{\partial}{\partial k^j}\frac{k_i}{k^3}\frac{\partial}{\partial k^l} k_k + 3\frac{\partial}{\partial k^l}\frac{k_i k_j k_k}{k^5}\right\} \{k^3 |v^s_k(t_f)|^2 e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}\}\\
&\qquad + 3\langle \delta\bar{\gamma}^{ij}_I(t_f)\delta\bar{\gamma}^{kl}_I(t_f)\rangle \mathcal{R}_{x_1}\mathcal{R}_{x_2} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{k_k}{k^4} \{5k_i k_j k_l - k^2(\delta_{il}k_j + \delta_{jl}k_i)\} |v^s_k(t_f)|^2 e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}.
\end{aligned} \tag{A62}$$

Using Eq. (A15), we can show that the terms in the last line cancel among them. Then, since the terms in the second line, which are total derivatives, vanish, we find that the condition (A55) can ensure the IR regularity of graviton loops as well.

As is pointed out in Ref. [65], no mode function can consistently satisfy the IR regularity conditions (A53) and (A55), suggesting the necessity of modifying the initial condition (A1) and (A2). Apart from that, it is shown that the same conditions as Eqs. (A53) and (A55) are derived from the requirement that the quantum states, selected operationally in the same way in terms of two different canonical variables related by the dilatation transformation, should agree with each other. This is in harmony with our claim that choosing the Euclidean vacuum that guarantees Eq. (4.5) is crucial for the IR regularity.

In our previous work [63], we computed the one-loop contribution of the graviton in the two-point function of ${}^gR(x)$, which can be expressed in the form $\mathcal{R}_x {}^g\zeta(x)$ by neglecting the terms that do not contribute to the IRdiv or IRsec. Then, we claimed that the one-loop contribution in the two-point function of ${}^gR(x)$ becomes IR regular without restricting the mode function v_k^s. However, in Ref. [63], to compute the graviton loop, we adopted

$$\zeta_2 \overset{\text{IR}}{\approx} \cdots - \mathcal{L}_s^{-1} \delta\bar{\gamma}_I^{ij} \nabla_i \nabla_j \zeta_I \overset{\text{IR}}{\approx} \cdots + \frac{1}{2} \delta\bar{\gamma}_I^{ij} x_i \partial_j \zeta_I \tag{A63}$$

as the solution for Eq. (A22), where ellipses represent the terms that do not include $\delta\gamma_{ijI}$. Notice that, in Eq. (A63), the solution that satisfies

$$2\mathcal{L}_s^{-1} \delta\bar{\gamma}_I^{ij} \nabla_i \nabla_j \zeta_I = -\delta\bar{\gamma}_I^{ij} x_i \partial_j \zeta_I$$

is selected. Based on the discussion after Eq. (A49), we find that this solution cannot be obtained by using the retarded Green function, $\mathcal{L}_{R,s}^{-1}$, with the initial conditions (A1) and (A2). Therefore, in order to eliminate the IRdiv and IRsec from the graviton loops for an arbitrary mode function v_k^s, we need to abandon the initial conditions (A1) and (A2). Then, however, there is no longer any guarantee that the standard commutation relations also hold for the interaction picture fields.

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