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A decomposition approach via Fourier sine transform for valuing American knock-out options with rebates

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Keywords

transform, sine, rebates, fourier, options, via, approach, decomposition, knock-out, american, valuing

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Nhat-Tan Le Duy-Minh Dang Tran-Vu Khanh *

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Abstract

We present an innovative decomposition approach for computing the price and the hedging parameters of American knock-out options with a time-dependent rebate. Our approach is built upon: (i) the Fourier sine transform applied to the partial differential equation with a finite time-dependent spatial domain that governs the option price, and (ii) the decomposition technique that partitions the price of the option into that of the European counterpart and an early exercise premium. Our analytic representations can generalize a number of existing decomposition formulas for some European-style and American-style options. A complexity analysis of the method, together with numerical results, show that the proposed approach is significantly more efficient than the state-of-the-art adaptive finite difference methods, especially in dealing with spot prices near the barrier. Numerical results are also examined in order to provide new insight into the significant effects of the rebate on the option price, the hedging parameters, and the optimal exercise boundary.

Keywords. American barrier options, decomposition, Fourier sine transform, rebate, optimal exercise boundary, heat equation, time-dependent spatial domain.

1 Introduction

American vanilla options give the option holders the right to trade an underlying asset for a pre-determined strike price at anytime before and up to a pre-determined expiry date. American knock-out options are very similar to their vanilla counterparts, except that they are immediately terminated, i.e. knocked-out, as soon as the price of the underlying asset

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breaches a particular level, referred to as the barrier. In other words, the holder of an American knock-out option starts with a vanilla option, but will lose this, once the knock-out feature is activated. To compensate for this potential risk, the knock-out feature is usually accompanied by a rebate, which is cash paid out to the option holder at if the option is terminated early. In this paper, we assume the rebate be a decreasing function of time rather than be a constant over time because the rebate is usually set as a portion of the value of the embedded option, which decreases with time.

It is well-known that the pricing of an American option, even a vanilla one, is a challenging task, due to the "early exercise" feature of the option (Chen et al., 2008; Mitchell et al., 2014). Typically, at each time during the life of the option, there exists an unknown value of the underlying asset, referred to as the optimal exercise price, that divides the pricing domain into two subdomains: (i) the early exercise region, where the option should be exercised immediately, and (ii) the continuation region, where the option should be held. The existence of these time-dependent unknown optimal exercise prices prevents an explicit closed form solution for an American option in most cases. Consequently, numerical methods must be used.

For American knock-out options, the pricing and hedging is even more challenging, due to the existence of the barrier. There are two major approaches used to price American knock-out options without rebate. The first approach is essentially lattice/grid-based methods, such as binomial/trinomial tree methods (Boyle and Lau, 1994; Cheuk and Vorst, 1996; Figlewski and Gao, 1999; Ritchken, 1995), and numerical partial differential equation (PDE) methods, such as the finite difference method (Boyle and Tian, 1999; Zhu et al., 2013; Zvan et al., 2000). However, it is well-known that the lattice/grid-based methods cannot handle the knock-out feature very well, especially for asset prices near the barrier. This is because the option payoff is discontinuous at the barrier, and hence results in a high sensitivity of the option price and the hedging parameters in the region near the barrier. This issue has been dealt with, to some extent, in, for example, Cheuk and Vorst (1996); Figlewski and Gao (1999); Gao et al. (2000), and indeed forms the main motivation for the second approach, namely the decomposition approach. The work of Gao et al. (2000) is possibly the first published work that discusses the decomposition approach for American knock-out options. In this approach, using probabilistic techniques, the price of an American knock-out option without rebate can be decomposed into two components: (i) the price of the European counterpart and (ii) an exercise premium associated with the early exercise right, which involves the unknown "optimal exercise boundary". This optimal exercise boundary has been formulated as the solution to an integral equation, which needs to be solved before the option price and the hedging parameters can be obtained. This solution procedure, i.e. identifying the optimal early exercise boundary before setting the option price, is similar to those taken by Kallast and Kivinukk (2003); Mitchell et al. (2014). The decomposition approach developed in Gao et al. (2000) has been extended in a number of works, such as Detemple (2010); Farid et al. (2003); Kwok (2008).

While American knock-out options without rebate have been studied extensively, to the best of our knowledge, there has been no published work that comprehensively studies the rebate counterparts, despite the importance of the subject. It is also not clear whether the probabilistic-based decomposition approach pioneered by Gao et al. (2000) can be easily extended to price American-style knock-out options with time-dependent rebates. Therefore, there is a need for a new and efficient computational method that can examine carefully the effects of rebates on the options prices, the hedging parameters, and the optimal exercise boundaries. This is the main motivation for our work.

In this paper, we propose an innovative decomposition approach for valuing American knock-out options with time-dependent rebates. The continuous Fourier sine transform (FST) method, instead of a probabilistic method as adopted by Detemple (2010); Farid et al. (2003); Gao et al. (2000); Kwok (2008), is used in our decomposition approach. More specifically, the FST method is employed to solve the governing PDE on a finite time-dependent spatial domain, between the moving optimal exercise boundary and the fixed barrier. Applying FST to the PDE results in an ordinary differential equation (ODE), whose solution can be straightforwardly obtained (in the Fourier sine space) and analytically converted back to the original space. As a result, our decomposition technique can be used to partition the price of an American knock-out option with a time-dependent rebate into that of the European counterpart and an exercise premium. In our formulation, the optimal exercise boundary is governed by an integral equation. A striking feature of this integral equation is its independence from the current spot asset price. Therefore, the "near-barrier" issue faced by grid-based methods is eliminated from our formulation. Similar results can be obtained for the hedging parameters as well. Our decomposition results also include, as a special case, a number of existing decomposition formulas for some European-style and American-style options. In addition, our decomposition formulas allow us to compute both the option price and the hedging parameters significantly more efficiently than adaptive finite difference (FD) methods, which are among the most efficient FD methods currently available.

The remainder of this paper is organized as follows. In Section 2, we introduce the PDE system that governs the price of an American up-and-out put option with a time-dependent rebate. In Section 3, a decomposition method based on the FST technique is presented. We discuss a numerical implementation of the decomposition formula in Section 4. In Section 5, we present numerical results to illustrate the efficiency of this method and to provide insight into the significant effects of the rebate on the option price, the hedging parameters and the optimal exercise boundary.

2 Formulation

We assume that the underlying asset price, denoted by S, follows a geometric Brownian motion given by:

$$\frac{dS(t)}{S(t)} = (r-\delta)dt + \sigma dZ. \tag{2.1}$$

Here, r and σ denote the risk-free interest rate and the instantaneous volatility, respectively; δ is a constant continuous dividend yield; Z is a standard one-dimensional Brownian motion. We are interested in the valuation problem of American up-and-out put options with a time-dependent rebate written on S, with maturity T and strike E. The knockout barrier and the time-dependent rebate are respectively specified by the constant $\bar{S}$ and the deterministic time-dependent function R. We now make the usual assumption: $E < \bar{S}$ in the contract of an up-and-out put option because the holder often accepts the loss of his/her option only when the option is out-of-money.

For the rest of the paper, we will with the variable $\tau = T - t$ which represents the time to maturity. We denote by $V(S,\tau)$ the value of an American up-and-out put option with a time-dependent rebate $R(\tau)$. To derive the PDE system governing $V(S,\tau)$, we note the following. First, by definition, $V(S,\tau)$ is the associated value of the rebate when the asset price hits the barrier. As a result, we have:

$$V(\bar{S},\tau) = R(\tau). \tag{2.2}$$

It should be noted that after the asset price hits the barrier, the option expires. In addition, if S is below the unknown optimal exercise boundary, denoted by $S_b(\tau)$, the option should be exercised immediately. In this case, the option value is equal to the payoff of a put option. It is well-known that the two necessary conditions for determining $S_b(\tau)$ are (Chen et al., 2008):

$$V\left(S_b(\tau),\tau\right) = E - S_b(\tau), \qquad \frac{\partial V}{\partial S}\left(S_b(\tau),\tau\right) = -1. \tag{2.3}$$

It is also known that for an American put, $S_b(\tau) \le E$, and thus $S_b(\tau) < \bar{S}$, which follows from the natural assumption that $E < \bar{S}$. For $S_b(\tau) < S < \bar{S}$, $0 < \tau \le T$, under the

Black-Scholes framework, $V(S,\tau)$ satisfies the classical Black-Scholes PDE:

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV, \tag{2.4}$$

subject to the terminal condition:

$$V(S,0) = \max(E-S,0). \tag{2.5}$$

Putting everything together, the PDE system that governs $V(S,\tau)$ is given by:

$$\begin{cases} \dfrac{\partial V}{\partial \tau} = \dfrac{\sigma^2 S^2}{2}\dfrac{\partial^2 V}{\partial S^2} + (r-\delta)S\dfrac{\partial V}{\partial S} - rV, \\ V(S,0) = \max(E-S,0), \\ V(S_b(\tau),\tau) = E - S_b(\tau), \\ \dfrac{\partial V}{\partial S}(S_b(\tau),\tau) = -1, \\ V(\bar{S},\tau) = R(\tau)\ . \end{cases} \tag{2.6}$$

for any $(S,\tau) \in (S_b(\tau),\bar{S}) \times (0,T]$. It should be emphasized that in this paper, $R(\tau)$ is assumed to be a *smooth* and *monotonically increasing* function of τ, with the property $R(0) = 0$, for two main reasons. First, in finance practice, the purpose of providing rebates is to partly compensate for the loss of the option in the event that the knock-out feature is activated before expiry, but not at expiry. The earlier the knock-out feature is activated, the more loss the holder suffers, and thereby the more amount of rebate should be paid to the holder. As a result, the rebate function $R(\tau)$ should be chosen as a monotonically increasing function of τ, with the property $R(0) = 0$. Second, under the Black-Scholes model, $V(S,\tau)$ is assumed to be a smooth function with respect to τ, for all values of S. Therefore, from the condition (2.2), it is necessary to assume $R(\tau)$ be a smooth function with respect to τ in order to guarantee the existence and uniqueness of the solution of the PDE system (2.6).

3 A Fourier sine decomposition approach

To derive a decomposition for $V(S,\tau)$ amendable to computation, we solve the pricing system (2.6) by using the continuous FST. More specifically, the PDE system (2.6) is first reduced to a dimensionless heat equation in a finite time-dependent domain. Then by using FST, the resulting heat equation can be further reduced to an initial value ODE in the

Fourier sine space, the solution of which is readily obtainable.

We shall first non-dimensionalize by introducing variables:

$$x = \ln \frac{\bar{S}}{S}, \quad l = \tau \frac{\sigma^2}{2};$$

and constants:

$$L = T\frac{\sigma^2}{2}, \quad \gamma = \frac{2r}{\sigma^2}, \quad q = \frac{2\delta}{\sigma^2}, \quad \lambda = 1 + q - \gamma, \quad \alpha = -\frac{\lambda}{2}, \quad \beta = -\alpha^2 - \gamma;$$

and unknown functions $u(x,l)$, $x_b(l)$ defined by:

$$V(S,\tau) = \bar{S}e^{\alpha x + \beta l}u(x,l), \quad x_b(l) = \ln \frac{\bar{S}}{S_b(\tau)}.$$

Using this change of variable, the system (2.6) becomes dimensionless, and is given by:

$$\begin{cases} \dfrac{\partial u}{\partial l}(x,l) = \dfrac{\partial^2 u}{\partial x^2}(x,l), \\ u(x,0) = f(x), \\ u(0,l) = g_1(l), \\ u(x_b(l),l) = g_2(x_b(l),l), \\ \dfrac{\partial u}{\partial x}(x_b(l),l) = g_3(x_b(l),l), \end{cases} \tag{3.7}$$

for any $(x,l) \in [0, x_b(l)] \times [0, L]$. Here, the datum $f(x)$, $g_1(l)$, $g_2(x_b(l),l)$ and $g_3(x_b(l),l)$ are given by:

$$\begin{aligned} f(x) &= \max\left(\frac{E}{\bar{S}}\mathrm{e}^{-\alpha x} - \mathrm{e}^{-(\alpha+1)x}, 0\right), \\ g_1(l) &= \frac{1}{\bar{S}}\mathrm{e}^{-\beta l}R\left(\frac{2}{\sigma^2}l\right), \\ g_2(x_b(l),l) &= \frac{E}{\bar{S}}\mathrm{e}^{-\alpha x_b(l) - \beta l} - \mathrm{e}^{-(\alpha+1)x_b(l) - \beta l}, \\ g_3(x_b(l),l) &= (\alpha+1)\mathrm{e}^{-(\alpha+1)x_b(l) - \beta l} - \alpha\frac{E}{\bar{S}}\mathrm{e}^{-\alpha x_b(l) - \beta l}. \end{aligned} \tag{3.8}$$

Although the PDE system (3.7) is somewhat simpler than (2.6), it is still difficult to directly solve. In fact, it is a heat equation in a finite time-dependent domain — a non-classical PDE. The existence and uniqueness of the solution of the heat equation in time-dependent domains has been studied in (Burdzy et al., 2003, 2004a,b; Chiarella et al., 2004). Especially, Chiarella et al. (2004) have successfully solved a heat equation in a semi-infinite time-dependent domain by using the Fourier transform. However, their method would be difficult to be extended to solve (3.7) because the x-domain here is a finite time-dependent one. To the best of our knowledge, there has been no published work that uses a comprehensive process to simultaneously obtain the unknown pair $u(x, l)$ and $x_b(l)$ in (3.7). This is the focus of our work. In next subsection, we use the continuous FST to formulate $u(x, l)$ in terms of $x_b(l)$, where $x_b(l)$ is the solution of an explicit integral equation. A numerical method to approximate $V(S, \tau)$ and $S_b(\tau)$ (respectively equivalent to $u(x, l)$ and $x_b(l)$) is given in Section 4.

3.1 Fourier sine transform

For reader's convenience, we recall that the continuous FST and its inversion are defined as:

$$F_s\left\{\Phi(x)\right\} = \hat{\Phi}(\omega) = \int_0^\infty \Phi(x)\sin(\omega x)dx, \quad \Phi(x) = \mathcal{F}_s^{-1}\left\{\hat{\Phi}(\omega)\right\} = \frac{2}{\pi}\int_0^\infty \hat{\Phi}(\omega)\sin(\omega x)d\omega,$$

respectively. As we will use the continuous Fourier cosine transform (FCT) in our solution procedure later, we also recall here the definition of FCT and its inversion as:

$$F_c\left\{\Phi(x)\right\} = \hat{\Phi}(\omega) = \int_0^\infty \Phi(x)\cos(\omega x)dx, \quad \Phi(x) = \mathcal{F}_s^{-1}\left\{\hat{\Phi}(\omega)\right\} = \frac{2}{\pi}\int_0^\infty \hat{\Phi}(\omega)\cos(\omega x)d\omega,$$

respectively. Here Φ is defined on $[0, \infty)$.

In order to apply the FST to (3.7), we first need to extend the finite x-domain, i.e. $[0, x_b(l)]$, to a semi-infinite one. This finite domain can be extended to $0 \leq x < \infty$ by multiplying the first equation of (3.7) with $H(x_b(l) - x)$, where $H(x)$ is the Heaviside function defined as:

$$H(x) \quad = \quad \begin{cases} 1 & \text{if} \quad x > 0, \\ 1/2 & \text{if} \quad x = 0, \\ 0 & \text{if} \quad x < 0. \end{cases} \tag{3.9}$$

We then can apply the FST, the PDE in (3.7) becomes:

$$\mathcal{F}_s\left\{H(x_b(l) - x)\frac{\partial u}{\partial l}(x, l)\right\} = \mathcal{F}_s\left\{H(x_b(l) - x)\frac{\partial^2 u}{\partial x^2}(x, l)\right\}. \tag{3.10}$$

Let $\hat{u}(\omega, l)$ denote the FST of the product $H(x_b(l) - x)u(x, l)$. We have:

$$\hat{u}(\omega, l) = \int_0^\infty H(x_b(l) - x)u(x, l)\sin(\omega x)dx = \int_0^{x_b(l)} u(x, l)\sin(\omega x)dx. \tag{3.11}$$

Direct calculation shows that:

$$\mathcal{F}_s\left\{H(x_b(l) - x)\frac{\partial u}{\partial l}(x, l)\right\} = \frac{\partial \hat{u}}{\partial l}(\omega, l) - x_b'(l)g_2(x_b(l), l)\sin(\omega x_b(l)), \tag{3.12}$$

and

$$\begin{aligned}\mathcal{F}_s\left\{H(x_b(l) - x)\frac{\partial^2 u}{\partial x^2}(x, l)\right\} &= \sin(\omega x)\frac{\partial u}{\partial x}(x, l)\bigg|_0^{x_b(l)} - \int_0^{x_b(l)} \omega\frac{\partial u}{\partial x}(x, l)\cos(\omega x)dx \\ &= \sin(\omega x_b(l))g_3(x_b(l), l) - \omega\cos(\omega x_b(l))g_2(x_b(l), l) + \omega g_1(l) - \omega^2\hat{u}(\omega, l),\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\mathcal{F}_c\left\{H(x_b(l) - x)\frac{\partial^2 u}{\partial x^2}(x, l)\right\} &= \cos(\omega x)\frac{\partial u}{\partial x}(x, l)\bigg|_0^{x_b(l)} + \int_0^{x_b(l)} \omega\frac{\partial u}{\partial x}(x, l)\sin(\omega x)dx \\ &= \cos(\omega x_b(l))g_3(x_b(l), l) - \frac{\partial u}{\partial x}(0, l) + \omega\sin(\omega x_b(l))g_2(x_b(l), l) - \omega^2\mathcal{F}_c\left\{H(x_b(l) - x)u(x, l)\right\}.\end{aligned} \tag{3.14}$$

We emphasize the importance of choosing FST over FCT in solving (3.7). It can be seen from the formulas (3.13) and (3.14) that while the term $\frac{\partial u}{\partial x}(0, l)$ vanishes from $\mathcal{F}_s\left\{H(x_b(l) - x)\frac{\partial^2 u}{\partial x^2}(x, l)\right\}$, it does appear in $\mathcal{F}_c\left\{H(x_b(l) - x)\frac{\partial^2 u}{\partial x^2}(x, l)\right\}$. Therefore, if the FCT is used to solve the PDE (3.7), the term $\frac{\partial u}{\partial x}(0, l)$ must be eliminated during the solution procedure, because it is also unknown. Since this complicates the solution procedure unnecessarily, to effectively solve the system (3.7), FST is a better choice than FCT.

Using (3.12) and (3.13), (3.10) can now be written as a linear first-order ODE:

$$\frac{\partial \hat{u}}{\partial l}(\omega, l) + \omega^2\hat{u}(\omega, l) = g(\omega, l) \tag{3.15}$$

with initial condition $\hat{u}(\omega, 0) = \mathcal{F}_s\{H(x_b(0) - x)f(x)\}$. Here,

$$g(\omega, l) = \sin(\omega x_b(l))\left[g_3(x_b(l), l) + x_b'(l)g_2(x_b(l), l)\right] - \omega\cos(\omega x_b(l))g_2(x_b(l), l) + \omega g_1(l).$$

Solving the ODE (3.15), we obtain:

$$\hat{u}(\omega, l) = \int_0^l e^{-\omega^2(l-\xi)} g(\omega, \xi) d\xi + \hat{u}(\omega, 0)\mathrm{e}^{-\omega^2 l}. \tag{3.16}$$

Our two next steps are to analytically solve the inverse FST of (3.16) and then convert the dimensionless variables to the original variables S and τ. As a result, we obtain important results, which will be presented in the next section.

3.2 Main results

Proposition 3.1. *The value $V(S,\tau)$ of the American up-and-out put satisfies the following integral equation:*

$$V(S,\tau) = -(E-S)\mathbb{I}_{S=S_b(\tau)}(S) + M(S,\tau,E) + \int_0^\tau Q(S,\tau,s,S_b(s))ds \tag{3.17}$$

for any $(S,\tau) \in [S_b(\tau), \bar{S}] \times [0,T]$, where:

$$M(x,y,z) = M_1(x,y,z) - \left(\frac{x}{\bar{S}}\right)^\lambda M_1\left(\frac{\bar{S}^2}{x}, y, z\right), \tag{3.18a}$$

$$Q(x,y,z,w) = Q_1(x,y,z,w) - \left(\frac{x}{\bar{S}}\right)^\lambda Q_1\left(\frac{\bar{S}^2}{x}, y, z, w\right) + \left(\frac{x}{\bar{S}}\right)^{\frac{\lambda}{2}} K(x,y,z). \tag{3.18b}$$

Here, M_1, Q_1 and K are defined by:

$$M_1(x,y,z) = E\mathrm{e}^{-ry}N(-d_2(x,y,z)) - x\mathrm{e}^{-\delta y}N(-d_1(x,y,z)), \tag{3.19a}$$

$$Q_1(x,y,z,w) = Er\mathrm{e}^{-r(y-z)}N(-d_2(x,y-z,w)) - x\delta\mathrm{e}^{-\delta(y-z)}N(-d_1(x,y-z,w)), \tag{3.19b}$$

$$K(x,y,z) = \frac{\ln\bar{S} - \ln x}{\sigma\sqrt{2\pi}\sqrt{(y-z)^3}}\mathrm{e}^{-\frac{(\ln x - \ln\bar{S})^2}{2\sigma^2(y-z)} + \beta\frac{\sigma^2}{2}(y-z)} R(z), \tag{3.19c}$$

with

$$d_1(x,y,z) = \frac{\ln x - \ln z + (r-\delta+\sigma^2/2)y}{\sigma\sqrt{y}}, \quad d_2(x,y,z) = \frac{\ln x - \ln z + (r-\delta-\sigma^2/2)y}{\sigma\sqrt{y}}.$$

Proof. See Appendix A. □

It is interesting to note that several existing decomposition formulas for some European-style options are special cases of formulas developed in Proposition 3.1. First, it should be noted that the quantities $M_1(S,\tau,E)$, defined in (3.19a), and $\left(S/\bar{S}\right)^\lambda M_1\left(\bar{S}^2/S,\tau,E\right)$ are the values of the E-strike and T-maturity European vanilla and European up-and-in put options written on S, respectively. Thus, the quantity $M(S,\tau,E)$, defined in (3.18a) is indeed the price of European up-and-out put options without rebate. In other words, the decomposition formula (3.18a) is the well-known formula given in Hull (2009)[Chapter 22] for the value of a European up-and-out put option without rebate. Second, formula (3.17) also covers, as a special case, the decomposition formula developed in Kwok (2008) for European up-and-out put options with a time-dependent rebate. More specifically, by substituting $S_b(t) = +\infty$ in (3.17), in which case the option is no longer of American-style and becomes a European-style option, we obtain the following decomposition formula of Kwok (2008):

$$U(S,\tau) = M(S,\tau,E) + \int_0^\tau \left(\frac{S}{\bar{S}}\right)^{\frac{\lambda}{2}} K(S,\tau,s)ds, \tag{3.20}$$

where $U(S,\tau)$ denotes the price of the European up-and-out put option with a time-dependent rebate $R(\tau)$. In (3.20), as noted above, the quantity $M(S,\tau,E)$ is the value of the no-rebate counterpart and the quantity $\int_0^\tau \left(\frac{S}{\bar{S}}\right)^{\frac{\lambda}{2}} K(S,\tau,s)ds$ represents the extra value due to the time-dependent rebate.

We now show that several well-known decomposition formulas for the price of American-style options are also special cases of Proposition 3.1. First, when there is no barrier, i.e. barrier tends to infinity, the formula (3.17) reduces to Kim (1990)'s well-known decomposition formula for American vanilla options: the value of a live American put can be expressed as a sum of its European counterpart and an early-exercise premium. More specifically, by using the L'Hospital rule, we can show that:

$$\begin{aligned}
&\lim_{\bar{S}\to+\infty} \left(\frac{S}{\bar{S}}\right)^{\lambda} M_1\left(\frac{\bar{S}^2}{S},\tau,E\right) = 0,\\
&\lim_{\bar{S}\to+\infty} \int_0^\tau \left(\frac{S}{\bar{S}}\right)^{\lambda} Q_1\left(\frac{\bar{S}^2}{S},\tau,s,S_b(s)\right) ds = 0,\\
&\lim_{\bar{S}\to+\infty} \int_0^\tau \left(\frac{S}{\bar{S}}\right)^{\frac{\lambda}{2}} K(S,\tau,s)ds = 0.
\end{aligned}$$

As a result, we obtain Kim (1990)'s formula:

$$\lim_{\bar{S}\to+\infty} V\left(S,\tau\right) = M_1\left(S,\tau,E\right) + \int_0^\tau Q_1\left(S,\tau,s,S_b(s)\right)ds,$$

where, as noted previously, $M_1\left(S,t,E\right)$, is the value of the European counterpart, and the integral quantity $\int_0^\tau Q_1\left(S,t,s,S_b(s)\right)ds$ represents the associated early-exercise premium.

Second, the well-known decomposition formula of Gao et al. (2000) for the price of an American up-and-out put options *without* rebate is also covered in (3.17). More precisely, when $R(\tau) = 0$ for all τ, we have:

$$\int_0^\tau \left(\frac{S}{\bar{S}}\right)^{\frac{\lambda}{2}} K(S,\tau,s)ds = 0,\ \forall \tau.$$

This implies that the decomposition formula (3.17) reduces to the formula of Gao et al. (2000):

$$V(S,\tau) = M(S,\tau,E) + X(S,\tau;S_b(\tau)),\ \forall S > S_b(\tau), \tag{3.21}$$

where $M(S,\tau,E)$, as previously mentioned, is the value of the European up-and-out put option without rebate counterpart and $X(S,\tau;S_b(\tau))$ is defined as:

$$X(S,\tau;S_b(\tau)) = \int_0^\tau \left[Q_1(S,\tau,s,S_b(s)) - \left(\frac{S}{\bar{S}}\right)^{\lambda} Q_1\left(\frac{\bar{S}^2}{S},\tau,s,S_b(s)\right)\right] ds, \tag{3.22}$$

which is the early exercise premium.

More importantly, the result of Proposition 3.1 allows us to easily derive a composition for $V(S,\tau)$ amendable to computation. The main result of the paper is presented the following theorem.

Theorem 3.1. *The value $V(S,\tau)$ of an American up-and-out put with a time-dependent rebate can be decomposed into two components: the value $U(S,\tau)$ of its European counterpart and the early exercise premium, $X(S,\tau;S_b(\tau))$, as follows:*

$$V(S,\tau) = U(S,\tau) + X(S,\tau;S_b(\tau)),\ \forall S > S_b(\tau) \tag{3.23}$$

where $U(S,\tau)$ and $X(S,\tau;S_b(\tau))$ are defined in (3.20) *and* (3.22)*, respectively. Moreover, the optimal exercise boundary $S_b(\tau)$ satisfies the equation:*

$$E - S_b(\tau) = U(S_b(\tau),\tau) + X(S_b(\tau),\tau;S_b(\tau)). \tag{3.24}$$

We note that, intuitively, a positive rebate can be viewed as an insurance for possible adverse movements of the asset price which can lead to the loss of the embedded option. Therefore, we expect the price of an American up-and-out put option with a rebate to be an increasing function of the rebate. Formula (3.23) clearly verifies this expectation. More specifically, if the amount of rebate increases, the value of the quantity $\int_0^\tau \left(\frac{S}{\bar{S}}\right)^{\frac{\lambda}{2}} K(S,\tau,s)ds$, which is embedded in $U(S,\tau)$, defined in (3.20), is the only term in the formula (3.23) related to the amount of rebate, also increases. Consequently, the value of an American up-and-out put option is an increasing function of the amount of rebate.

3.3 Hedging parameters

It should also be stressed that the hedging parameters, or Greeks, such as Delta, Gamma, Theta, Vega and Rho, can also be readily obtained by differentiating the decomposition formula (3.23) with respect to the relevant parameter(s). As an illustrative example, we calculate explicitly Delta below. Other hedging parameters can be calculated in a similar manner.

Proposition 3.2. *The hedging parameter Delta (Δ) can be calculated as:*

$$\frac{\partial}{\partial S}V(S,\tau) = \tilde{U}(S,\tau) + \tilde{X}(S,\tau;S_b(\tau)), \quad \forall S > S_b(\tau). \tag{3.25}$$

Here, $\tilde{U}(S,\tau)$ and $\tilde{X}(S,\tau;S_b(\tau))$ are explicitly expressed as:

$$\begin{aligned}
\tilde{U}(S,\tau) &= \tilde{K_1}(S,\tau) + \tilde{M}(S,\tau,E), \\
\tilde{X}(S,\tau;S_b(\tau)) &= \int_0^{\tau} L(S,\tau,s,S_b(s))ds,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{K_1}(x,y) =& \frac{\lambda x^{\frac{\lambda}{2}-1}}{\sqrt{2\pi}\bar{S}^{\frac{\lambda}{2}}} \int_0^{+\infty} \mathrm{e}^{\left\{-\frac{1}{2}\left(u+\frac{\ln\bar{S}-\ln x}{\sigma\sqrt{y}}\right)^2+\frac{\beta}{2}\left(\frac{\ln\bar{S}-\ln x}{u+\frac{\ln\bar{S}-\ln x}{\sigma\sqrt{y}}}\right)^2\right\}} R\left(y-\left(\frac{\ln\bar{S}-\ln x}{\sigma(u+\frac{\ln\bar{S}-\ln x}{\sigma\sqrt{y}})}\right)^2\right) du \\
&+ \left(\frac{x}{\bar{S}}\right)^{\frac{\lambda}{2}} \frac{\sqrt{2}\sigma}{\sqrt{\pi}x} \int_0^{\sqrt{y}} \mathrm{e}^{-\frac{(\ln\bar{S}-\ln x)^2}{2\sigma^2 v^2}+\frac{\beta}{2}\sigma^2 v^2} \left[(-\beta)R(y-v^2) + \frac{2}{\sigma^2}R'(y-v^2)\right] dv, \\
\tilde{M}(x,y,z) =& \tilde{M_1}(x,y,z) - \frac{\lambda x^{\lambda-1}}{\bar{S}^{\lambda}} M_1\left(\frac{\bar{S}^2}{x},y,z\right) + \left(\frac{x}{\bar{S}}\right)^{\lambda-2} \tilde{M_1}\left(\frac{\bar{S}^2}{x},y,z\right), \\
\tilde{M_1}(x,y,z) =& -\mathrm{e}^{-\delta y} N(-d_1(x,y,z)), \qquad \tilde{N}(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}, \\
L(x,y,z,w) =& \tilde{Q_1}(x,y,z,w) - \frac{\lambda x^{\lambda-1}}{\bar{S}^{\lambda}} Q_1\left(\frac{\bar{S}^2}{x},y,z,w\right) + \left(\frac{x}{\bar{S}}\right)^{\lambda-2} \tilde{Q_1}\left(\frac{\bar{S}^2}{x},y,z,w\right), \\
\tilde{Q_1}(x,y,z,w) =& \mathrm{e}^{-\delta(y-z)}\left[-\delta N(-d_1(x,y-z,w)) + \frac{\tilde{N}\left(-d_1(x,y-z,w)\right)}{\sigma\sqrt{y-z}}\left(\delta - \frac{Er}{w}\right)\right].
\end{aligned}$$

Proof. See Appendix B. □

It should be noted that $\tilde{U}(S,\tau)$ is the Delta of the European counterpart.

4 Numerical implementation

In order to apply the results of Theorem 3.1 to compute the option value $V(S,\tau)$, when $\tau = T$, we need to find $U(S,T)$, and $X\left(S,T;S_b(T)\right)$, which both involves integrals from 0 to T, with the integrands being functions of the unknown optimal exercise boundary $S_b(\tau)$. As a result, we resort to numerical techniques to first approximate $S_b(\tau)$ at discrete points in time, via a Newton iteration, and then apply composite quadrature rules to compute these integrals.

4.1 A result on $S_b(0^+)$

As input to the numerical techniques, the value of the optimal exercise price just prior to expiry, i.e. at $\tau = 0^+$. is needed.

Corollary 4.1. *The value of the optimal exercise price of an American up-and-out put option with a time-dependent rebate just prior to expiry, i.e. $S_b(0^+)$, is given by:*

$$S_b(0^+) = min\left(E, \frac{rE}{\delta}\right). \tag{4.26}$$

Proof. See appendix C. □

4.2 Numerical procedure

We now describe a numerical procedure to approximate $S_b(\tau)$ from $\tau = 0^+$ to $\tau = T$, with $S_b(0^+)$ given by (4.26) (Corollary 4.1). From (3.24), we define:

$$F(S_b(\tau), \tau) = S_b(\tau) + U\left(S_b(\tau), \tau\right) + X\left(S_b(\tau), \tau; S_b(\tau)\right) - E. \tag{4.27}$$

Let $\{\tau_n\}_{n=0}^{p}$, $\tau_{n+1} - \tau_n = \Delta t = \frac{T}{p}$, be an uniform partition of the interval $[0, T]$. Denote by $S_b^{n,(k)}$ the approximation to $S_b(\tau_n)$ at the k-th Newton iteration. At each time τ_n, given $S_b^{n,(k)}$, the Newton iteration computes $S_b^{n,(k+1)}$ as follows:

$$S_b^{n,(k+1)} = S_b^{n,(k)} - \frac{F\left(S_b^{n,(k)}, \tau_n\right)}{F'\left(S_b^{n,(k)}, \tau_n\right)}, \tag{4.28}$$

where F' denotes the derivative of F with respect to $S_b(\tau)$. The stopping criteria, `tol`, is a user-defined tolerance.

$$\left|\frac{S_b^{n,(k+1)} - S_b^{n,(k)}}{S_b^{n,(k+1)}}\right| < \texttt{tol}, \tag{4.29}$$

In our numerical experiments, we observe that only a few iterations, namely 2-3 iterations, is needed to reach $\texttt{tol} = 10^{-7}$. We denote by S_b^n the numerical optimal exercise price to $S_b(\tau_n)$ produced by the above Newton iteration. The initial guess for the Newton scheme (4.28) is

$$S_b^{n,(0)} = \begin{cases} S_b(0^+) & \text{if} \quad \tau_n = \tau_{0^+}, \\ S_b^{n-1} & \text{if} \quad \tau_n = \tau_1, \\ 2S_b^{n-1} - S_b^{n-2} & \text{if} \quad \tau_n = \tau_i, \quad i = 2, \dots, p. \end{cases} \tag{4.30}$$

Here, $S_b(0^+)$ is defined in (4.26) (Corollary (4.1)). When $\tau \geq \tau_2$, the initial guess for each time step is based on linear extrapolation of the numerical optimal exercise prices from the two previous time steps.

In approximating $F\left(S_b^{n,(k)}, \tau_n\right)$ and $F'\left(S_b^{n,(k)}, \tau_n\right)$, we will need to evaluate several integrals from 0 to τ_n. For example, to approximate $F\left(S_b^{n,(k)}, \tau_n\right)$ we need to compute $U(S_b^{n,(k)}, \tau_n)$, defined in (3.20), and $X\left(S_b^{n,(k)}, \tau_n; S_b^{n,(k)}\right)$, defined in (3.22). Most of the integrals in these quantities are of smooth functions on finite domains and can be approximated by using the composite Gauss–Legendre rule (Kythe and Schaferkotter, 2014). The only one term that needs special attention is the second term of $U(S_b^{n,(k)}, \tau)$, which is

$$\int_0^{\tau_n} \left(\frac{S_b^{n,(k)}}{\bar{S}}\right)^{\frac{\lambda}{2}} K\left(S_b^{n,(k)}, \tau_n, s\right) ds. \tag{4.31}$$

Here, as defined in (3.18),

$$K(x, y, z) = \frac{\ln \bar{S} - \ln x}{\sigma\sqrt{2\pi}\sqrt{(y-z)^3}} \mathrm{e}^{\left\{-\frac{(\ln x - \ln \bar{S})^2}{2\sigma^2(y-z)} + \beta\frac{\sigma^2}{2}(y-z)\right\}} R(z).$$

It should be noted that the above integral has a singularity at $v = \tau_n$. To deal with the singularity, we first to transform (4.31) into an integral on a semi-infinite domain by using the following variable transformation:

$$w = \frac{\ln \bar{S} - \ln S_b^{n,(k)}}{\sigma\sqrt{2(\tau_n - v)}} - \frac{\ln \bar{S} - \ln S_b^{n,(k)}}{\sigma\sqrt{2\tau_n}}.$$

The integral (4.31) becomes:

$$\frac{2}{\sqrt{\pi}} \int_0^{+\infty} \left(\frac{S_b^{n,(k)}}{\bar{S}}\right)^{\frac{\lambda}{2}} \mathrm{e}^{\left\{-\left(w + \frac{\ln \bar{S} - \ln S_b^{n,(k)}}{\sigma\sqrt{2\tau_n}}\right)^2 + \frac{\beta(\ln \bar{S} - \ln S_b^{n,(k)})^2}{4\left(w + \frac{\ln \bar{S} - \ln S_b^{n,(k)}}{\sigma\sqrt{2\tau_n}}\right)^2}\right\}} R\left(\tau_n - \frac{(\ln \bar{S} - \ln S_b^{n,(k)})^2}{2\sigma^2\left(w + \frac{\ln \bar{S} - \ln S_b^{n,(k)}}{\sigma\sqrt{2\tau_n}}\right)^2}\right) dw.$$

The Gauss-Laguerre quadrature rule, which is an efficient way to evaluate integrals on semi-infinite domains, is then applied to handle the above integral (Kythe and Schaferkotter, 2014). In Algorithm 1, we present an algorithm to compute $V(S, \tau)$.

Algorithm 1 Algorithm to approximate V using p time steps, and the g-point Gauss quadrature rules.

1: set $\mathcal{E} = \varnothing$;
2: compute $S_b(0^+)$ using Corollary 4.1; set $\mathcal{E} = \mathcal{E} \bigcup S_b(0^+)$;
3: compute abscissae and weights for the g-point Gauss–Legendre and Laguerre rules;
4: **for** $n = 1, 2, \ldots, p$ **do**
5: set $S_b^{n,(0)}$ according to (4.30);
6: **for** $k = 0, 1, \ldots,$ until convergence **do**
7: apply g-point Gauss quadrature rules to compute $U(S_b^{n,(k)}, \tau_n)$, $U'(S_b^{n,(k)}, \tau_n)$, $X\left(S_b^{n,(k)}, \tau_n; S_b^{n,(k)}\right)$, and $X'\left(S_b^{n,(k)}, \tau_n; S_b^{n,(k)}\right)$;
8: compute $F\left(S_b^{n,(k)} \tau_n\right)$ and $F'\left(S_b^{n,(k)}, \tau_n\right)$;
9: apply (4.28) to compute $S_b^{n,(k+1)}$;
10: **if** $k > 0$ and $\left|\frac{S_b^{n,(k+1)} - S_b^{n,(k)}}{S_b^{n,(k+1)}}\right| <$ `tol`, **then**
11: $\mathcal{E} = \mathcal{E} \bigcup S_b^{n,(k+1)}$;
12: break from the iteration;
13: **end if**
14: **end for**
15: **end for**
16: compute $U(S, T)$ and $X(S, T; S_b(T))$ using g-point Gauss rules with $\mathcal{E}$ and $\{\tau_n\}_{n=0}^{p}$;
17: return $V(S, T) = U(S, T) + X(S, T; S_b(T))$;

4.3 Computational complexity

For use later in comparison of the efficiency of numerical PDE methods, we discuss the computational complexity of Algorithm 1. A breakdown of the major costs required by the algorithm is as follows:

1. Step 3: A construction of abscissae and weights for the g-point Gauss–Legendre and Laguerre rules entails a cost of approximately $2g^2$ (flops)[1].

2. Steps 6-14 (Construction of $\mathcal{E}$): These are for computing numerical approximation to each $S_b(\tau_n)$, $\tau = 1, \ldots, p$ via (4.28). At each time τ_n, these steps involves a cost of:

$$\text{cost-per-iteration} \times \text{total number of iterations (flops)}.$$

At each iteration (4.28), to compute $F\left(S_b^{n,(k)} \tau_n\right)$, 3 integrals need to be evaluated using the g-points Gauss rules. For each integral evaluation, there are approximately g additions, and

[1] A flop is one addition, or one multiplication, or one division of two floating-point numbers.

g multiplications between the Gauss weights and the values of the integrand evaluated at the Gauss points. By examining the integrands in (3.18), the average cost for evaluating an integrand at a Gauss point at each time τ_n is about 8 (flops). Thus the cost for evaluating $F\left(S_b^{n,(k)}\tau_n\right)$ is about $3 \times (10g) = 30g$ (flops). Approximately, the same cost is needed for evaluating $F'\left(S_b^{n,(k)}\tau_n\right)$. Thus the cost-per-iteration is approximately $60g$ (flops).
3. Step 16: approximately requires $30pg$ (flops), taking into account that there are p time steps, and the cost per time step is $30g$ (flops) (3 integrals to evaluate at the cost $10g$ (flops) each).

Thus, the total cost can be approximated by

$$\text{total cost} \approx 2g^2 + (60g)(\text{total number of iterations}) + 30pg \text{ (flops)}.$$

We conclude by highlighting that, as illustrated later in Section 5, the average number of iterations per time step required for achieving the stopping criterion (4.29) is relatively small, only 2-3 iterations, and is independent of the time step size used, which is a desirable property. In addition, we emphasize that no computational grids are required for S as in finite difference.

5 Numerical results

In this section, we provide selected examples to validate our proposed approach. We then also compare our numerical method with an adaptive method in term of efficiency. In addition, the significant effects of rebates on the price, the Delta and the optimal exercise boundary of American up-and-out put options with time-dependent rebates are also examined through numerical examples.

5.1 Validation examples on no-rebate case

We now provide selected examples to validate our proposed approach. Since the valuation of American up-and-out put options with a time-dependent rebate has not previously been studied, we only consider validation examples on American up-and-out put options without rebate, which have been studied extensively in the literature. For this test, we compare the results obtained by our Fourier Sine decomposition (FSD) method, with those obtained by the trinomial tree method developed by Ritchken (1995), and employed in in Gao et al. (2000).

As a further check, these results are also compared with those obtained by an adaptive finite difference ("adaptiveFD") method. This adaptiveFD method is built upon the highly efficient adaptive techniques developed in Christara and Dang (2011) for American vanilla options. In the adaptiveFD, the penalty method of Forsyth and Vetzal (2002) is employed to handle the non-linear PDE that arises.[2] To control the space error given a fixed number of spatial grid points, an adaptive grid point distribution based on an error equidistribution principle is employed. Essentially, more points are automatically distributed to regions that the option price lacks regularity, such as those around the optimal exercise prices and the barrier, to minimize the error. As shown in Christara and Dang (2011), the adaptive FD technique is significantly more efficient than both the uniform and pre-determined non-uniform FD methods.

For validation tests, we consider three different volatility values, namely $\sigma = \{0.2, 0.3, 0.4\}$, and two different maturities, namely $T = \{0.25, 1\}$ (years), along with other parameters $E = \$45$, $\bar{S} = 50$, $r = 4.88\%$, and the dividend $\delta = 0\%$. These are the parameters used in Gao et al. (2000). Table 1 presents selected prices and Deltas for the American up-and-out put options without rebate. The results of the trinomial tree method, reported in Gao et al. (2000), were obtained by using 10^4 time steps. We implemented the adaptiveFD method using 640 spatial grid points (in the S-direction) and 320 time steps (in the τ-direction). We emphasize that the FSD method only uses 40 time steps and 57-points for Gauss–Legendre and Laguerre rules. Here, uniform timestep sizes are used. All of our experiments were performed using Matlab R2014b on an Intel Core i7, 3.40 GHZ machine. For both FSD and adaptiveFD, a tolerance $tol = 10^{-6}$ is used.

From the result in Table 1, it is clear that our analytic results agree well with those reported in Gao et al. (2000) as well as with those obtained by using the adaptiveFD method. It should be mentioned that the point-wise relative errors are less than 0.2% for both prices and Deltas.

5.2 Time-dependent rebate case

In this section, we first compare the FSD with the adaptiveFD method in term of efficiency, i.e. accuracy per unit of cost. We then study the effects of rebates on the price, the Delta and the optimal exercise boundary of American up-and-out put options with time-dependent rebates.

We use the following parameters $E = \$45, T = 1, \sigma = 0.4, r = 0.0488$, $\delta = 0$, which also come from Gao et al. (2000). In addition, we choose three different time-dependent

[2]The penalty iteration described in Forsyth and Vetzal (2002) is essentially a Newton iteration, but, to be consistent with Forsyth and Vetzal (2002), we use the term "penalty iteration".

Table 1: Validation tests. Prices and Deltas at time $\tau = T$ (i.e. $t = 0$) for American up-and-out put options without rebate. $E = \$45$, $\bar{S} = \$50$, $r = 4.88\%$, $\delta = 0\%$. Method of Gao et al. (2000): 10^4 time steps are used. adaptiveFD: 640 spatial grid points and 320 time steps. FSD: 40 time steps and 57-point Gauss–Legendre and Laguerre rules. For both adaptiveFD and FSD, uniform timestep sizes are used.

		$T = 0.25$			$T = 1$		
σ	S	Gao et al. (2000)	adaptiveFD	FSD	Gao et al. (2000)	adaptiveFD	FSD
price (V)							
0.2	40	5.0357	5.0357	5.0359	5.3861	5.3862	5.3863
	45	1.5445	1.5445	1.5446	2.2151	2.2153	2.2153
	49.5	0.1103	0.1103	0.1103	0.1936	0.1936	0.1936
0.3	40	5.4639	5.4640	5.4642	6.1455	6.1460	6.1458
	45	2.2250	2.2252	2.2252	2.8399	2.8402	2.8401
	49.5	0.1990	0.1990	0.1990	0.2684	0.2684	0.2684
0.4	40	5.9773	5.9774	5.9776	6.7054	6.7060	6.7058
	45	2.7007	2.7009	2.7009	3.2145	3.2148	3.2147
	49.5	0.2563	0.2563	0.2563	0.3117	0.3117	0.3117
Delta (Δ)							
0.2	40	-0.9253	-0.9255	-0.9252	-0.7696	-0.7695	-0.7695
	45	-0.4657	-0.4657	-0.4657	-0.5197	-0.5197	-0.5197
	49.5	-0.2244	-0.2245	-0.2245	-0.3920	-0.3921	-0.3921
0.3	40	-0.7828	-0.7829	-0.7827	-0.7223	-0.7223	-0.7223
	45	-0.5233	-0.5233	-0.5234	-0.6078	-0.6078	-0.6078
	49.5	-0.4006	-0.4007	-0.4007	-0.5398	-0.5399	-0.5399
0.4	40	-0.7372	-0.7372	-0.7372	-0.7331	-0.7331	-0.7331
	45	-0.5839	-0.5840	-0.5840	-0.6672	-0.6672	-0.6672
	49.5	-0.5144	-0.5144	-0.5144	-0.6253	-0.6254	-0.6253

rebate functions $R_i(\tau)$, $i = 1, \ldots, 3$, that are: (i) smoothly increasing functions of τ, and (ii) satisfy the property that $R_i(0) = 0$, $i = 1, \ldots, 3$. Note that any smoothly increasing function of τ satisfying (i) and (ii) can be a legitimate rebate function, and the choice is typically depends on the contract specification. For illustration purposes, we choose linear functions of time: $R_1(\tau) = 100\sigma^3\tau$, $R_2(\tau) = 50\sigma^3\tau$, and $R_3(\tau) = 0$, where σ is the volatility rate. It should be noted that there is no particular reason for the presence of σ in our choices for $R_1(\tau)$ and $R_2(\tau)$.

5.2.1 Comparison with adaptive finite difference

In this subsection, we compare the FSD with the adaptiveFD method in term of efficiency, i.e. accuracy per unit of cost. Since we compare the efficiency of two methods applied to option pricing, it is important to determine the computational cost of each method. The computational cost of FSD method can be computed as detailed in Subsection 4.3.

Like the adaptive methods developed in Christara and Dang (2011), the complexity of adaptiveFD, at each penalty iteration, consists of solving a tridiagonal linear system of size "# S points" $\times$ "# S points", where "# S points" denotes the number of grid points in the S-direction. Hence, the total computation cost of adaptiveFD can be modeled by the formula

$$\text{total cost} = \ \# \text{ penalty iter.} \times \# \ S \text{ points (flops)},$$

where "# penalty iter. " is the total number of penalty iterations required by adaptiveFD.

Table 2: Efficiency comparison to finite difference. Prices and Deltas at time $\tau = T$ (i.e. $t = 0$) for American up-and-out put option with a rebate function $R_1(\tau) = 100\sigma^3\tau$. $E = \$45, T = 1, r = 0.0488$, $\delta = 0$, $\bar{S} = \$50$. Uniform timestep sizes are used.

adaptiveFD						
S	# τ points	# S points	price (V)	Delta (Δ)	penalty iter. #	total cost (flops)
	80	200	7.21408	-0.24944	226	4.5×10^4
	160	400	7.21429	-0.24947	458	1.9×10^5
45	320	800	7.21438	-0.24949	995	8.0×10^5
	640	1600	7.21444	-0.24950	2088	3.3×10^6
	1280	3200	7.21448	-0.24951	4182	1.4×10^7
	80	200	6.46278	-5.53819	226	4.5×10^4
	400	320	6.46508	-4.36801	458	1.9×10^5
49.5	320	800	6.46527	-0.21101	995	8.0×10^5
	640	1600	6.46533	-0.09188	2088	3.3×10^6
	1280	3200	6.46537	-0.09176	4183	1.4×10^7
FSD						
	# τ points	# Gauss points	price (V)	Delta (Δ)	Newton iter. #	total cost (flops)
	40	57	7.21460	-0.24951	105	3.9×10^5
45	80	57	7.21450	-0.24951	186	7.1×10^5
	160	57	7.21450	-0.24951	345	1.3×10^6
	40	57	6.46539	-0.09171	105	3.9×10^5
49.5	80	57	6.46538	-0.09170	186	7.1×10^5
	160	57	6.46538	-0.09170	345	1.3×10^6

In Table 2, we present selected results when the rebate function is $R_1(\tau) = 100\sigma^3\tau$ obtained by the FSD and adaptiveFD methods. We make the following observations. First, the prices (V) and Delta (Δ) obtained by the two methods appear to converge to the same values,with the point-wise relative errors at the finest levels being less than 0.1% for both prices and Deltas. These results could serve as a validation test for accuracy of FSD in case of non-zero rebate. More importantly, FSD appears to be significantly more efficient than adaptiveFD. More specifically, when $S = E = \$45$, to achieve 5 decimal digits of accuracy, the adaptiveFD cost seems to be 18 times more than the cost of the

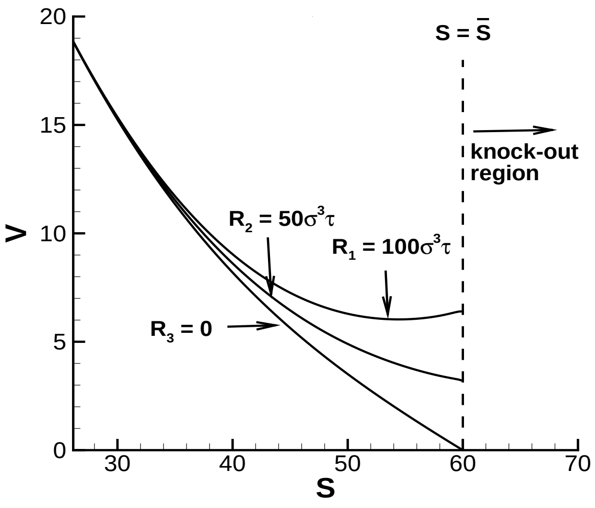
FSD (1.4×10^7 (flops) v.s. 7.1×10^5 (flops)). Particularly, when $S = \$49.5$, which is near the barrier, the adaptiveFD becomes even more costly than the FSD. For example, it can be seen that the Delta values computed by adaptiveFD are quite erratic, which is an expected result from a grid-based method when dealing with knock-out options with the spot asset prices being near the barrier. The accuracy and efficiency of adaptiveFD deteriorates significantly in computing Delta in this case. To achieve 4 decimal digits of accuracy in Delta the FSD needs less than 3.9×10^5 (flops), while the adaptiveFD needs more than 35 times that cost ($> 1.4 \times 10^7$ (flops)).

5.2.2 Effects on option price and Delta

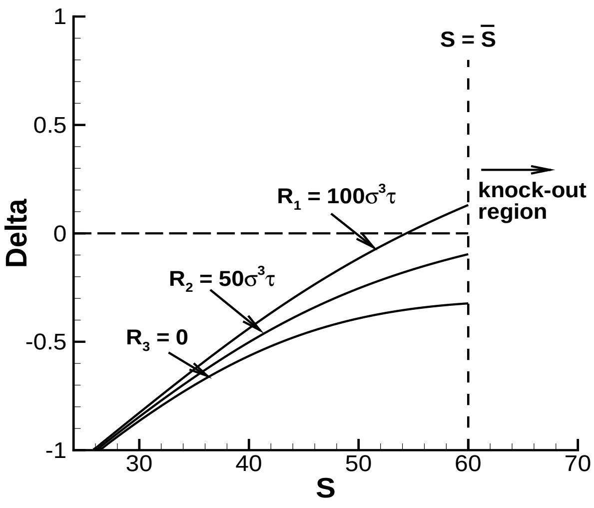
In this subsection, we study the effects of rebates on the option price and Delta. We consider all three rebate functions $R_i(\tau)$, $i = 1, 2, 3$. It is clear from our choice that, for a given τ, $R_1(\tau) > R_2(\tau) > R_3(\tau)$. In Figure 1, we plot the resulting option price and Delta for two barrier values $\bar{S} = \{60, 80\}$. We make the following interesting observations. For the case of no rebate, i.e. $R_3(\tau) = 0$ in Figures 1(a) and 1(c), V is a monotonically decreasing function of S for both $\bar{S} = \{60, 80\}$. This observation is consistent with the plots of Delta on Figures 1(b) and 1(d), where both the corresponding Delta curves are below zero. This behavior of V is expected since the closer S is to (the left of) the barrier $\bar{S}$, the more likely the option will be out-of-money and be knocked out, in which case the holder receives nothing. In other words, the holder of an American-style up-and-out put option without rebate does not benefit from increasing asset price.

However, with the presence of a rebate, the holder of an American-style up-and-out put option receives some compensation when the knock-out feature is activated. Therefore, with sufficiently large rebate, the holder can even benefit from the increase in the asset price towards the barrier. This may consequently change the monotonicity of the option price, i.e., the option price may not always be a monotonically decreasing function of S. For instance, for the case: $R_2 = 50\sigma^3\tau$ and $\bar{S} = 60$, V is always a decreasing function of S (see Figure 1(a)) and the associated Delta is always negative (see Figure 1(b)). However, for the same rebate, but with $\bar{S} = 80$, V in fact increases as S tends towards the barrier $\bar{S}$ (see Figure 1(c)) and the associated Delta changes sign, from negative to positive (see Figure 1(d)). More interestingly, for the case: $R_1 = 100\sigma^3\tau$, V is always an increasing function of S as S tends towards $\bar{S}$ (see both Figures 1(a)-1(c)), and the associated Delta changes sign when S sufficiently close to $\bar{S}$ (see Figures 1(b)-1(d)).

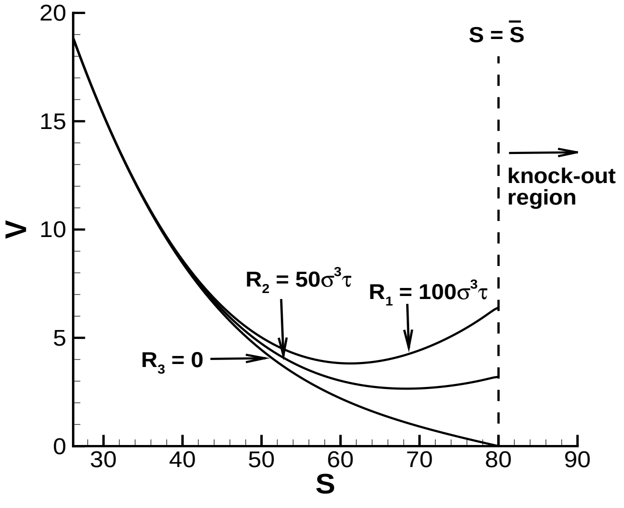
The source of this interesting phenomenon is the combined effect of the magnitudes of the barrier and the rebate. More specifically, when S tends towards $\bar{S}$, the put option



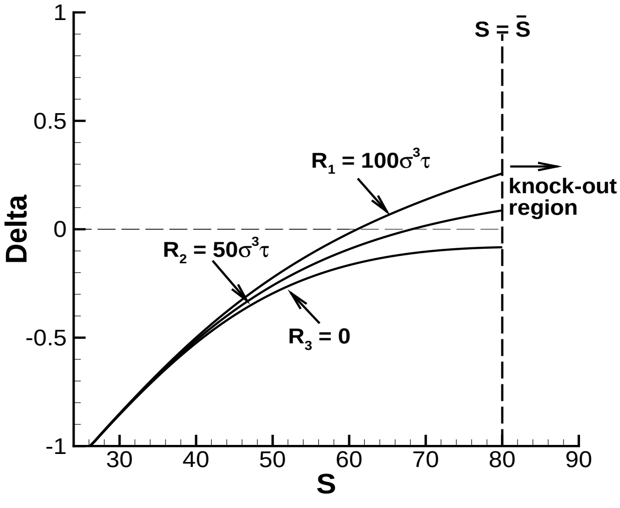
(a) price ($\bar{S} = 60$)



(b) Delta ($\bar{S} = 60$)



(c) price ($\bar{S} = 80$)



(d) Delta ($\bar{S} = 80$)

Figure 1: Prices and Deltas versus asset price at time $\tau = T$, i.e. ($t = 0$) of the American up-and-out put for different rebate functions.

becomes gradually out-of-money (recall that $E < \bar{S}$). As a result, the effect of the rebate on the option price becomes more pronounced as S approaches $\bar{S}$. Thus, if $\bar{S}$ is large enough compared to E (e.g. $\bar{S} = 80$), the option becomes deeply out-of-money, and hence, its value mainly comes from the rebate. That is why we observe the change in the monotonicity of V for both R_1 and R_2 in Figure 1(c). However, if $\bar{S}$ is not large enough compared to E (e.g. $\bar{S} = 60$), then the rebate must be sufficiently large to affect the monotonicity of the option price. This is what we observe in Figure 1(a), where V changes it monotonicity with R_1, but not with $R_2 < R_1$.

To further study the effects of the rebates across different barrier values, we plot the option prices when $\bar{S} = \{60, 80\}$ for both $R_1(\tau)$ and $R_3(\tau)$ on the same figure, Figure 2. We observe from Figure 2 that for the case $(\bar{S} = 80, R_3(\tau) = 0)$, the option prices are always strictly greater than the those obtained with $(\bar{S} = 60, R_3(\tau) = 0)$. This observation is consistent with the fact that the price of an American up-and-out put without rebate is a monotonically increasing function of the barrier. However, with the presence of a fixed rebate, the price of an American up-and-out put associated with a lower asset barrier might be greater than that associated with a higher asset barrier, at some asset prices. In fact, we observe from Figure 2 that option prices obtained with $(\bar{S} = 60, R_1 = 100\sigma^3\tau)$ are indeed above those obtained with $(\bar{S} = 80, R_1 = 100\sigma^3\tau)$. The presence of the rebate, which changes the monotonicity of the option, results in this interesting phenomenon.

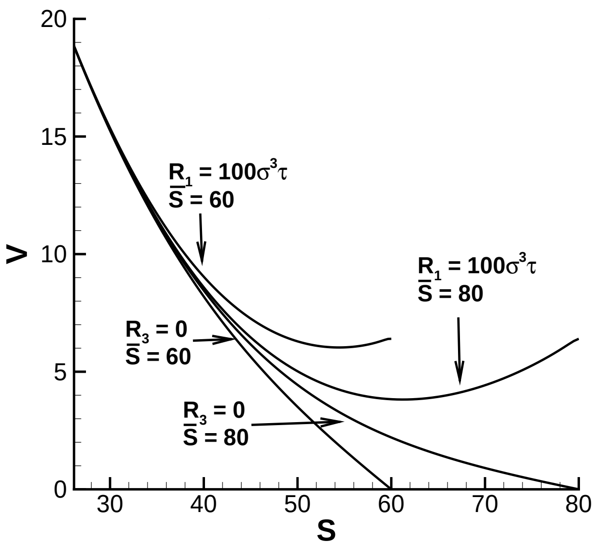
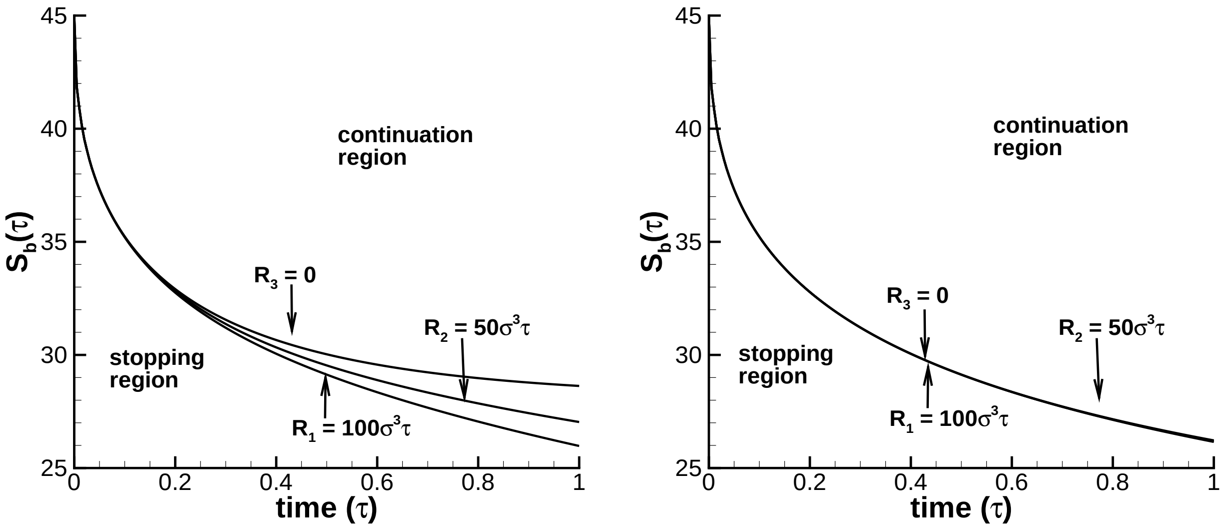


Figure 2: Price at time $\tau = T$, i.e. $(t = 0)$ of the American up-and-out put versus different rebate functions and barriers. Compiled plot from Figures 1(a)-1(c).

5.2.3 Effects on optimal exercise boundary

Rebates also have pronounced effects on the optimal exercise boundary $S_b(\tau)$. Figure 3 compares the optimal exercise boundaries of the American up-and-out put obtained with rebate functions $R_i(\tau)$, $i = 1, 2, 3$. We show plots for two different barriers $\bar{S} = \{60, 80\}$.

First, from Figure 3(a), it is clear that the optimal exercise boundary $S_b(\tau)$ associated with



(a) $\bar{S} = 50$

(b) $\bar{S} = 80$

Figure 3: Optimal exercise boundary of an American-style up-and-out put for different rebate functions.

a larger rebate is lower than those associated with smaller rebates. This demonstrates the fact that for a fixed barrier $\bar{S}$, at a given time τ, $S_b(\tau)$ is a decreasing function of the rebate amount $R(\tau)$. This can be financially explained by the fact that a larger rebate would increase the value of the barrier option, and thus the put option holder would prefer to choose a lower asset price to optimally exercise the option.

However, as we increase $\bar{S}$, the effects appear to diminish very quickly. This of course also depends on how large the rebate is. More precisely, for larger rebates, it might take a larger $\bar{S}$ to diminish the effect. As shown in Figure 3(b), when $\bar{S} = 80$, all three optimal exercise boundaries merge into one. This phenomenon can be financially explained as follows. As $\bar{S}$ increases, the chance that the option will be knocked out is smaller, and hence the effect of the rebate on the optimal exercise boundary is also smaller. In particular, when $\bar{S} \to \infty$, the behavior of S_b is the same with that of the optimal exercise boundary of the vanilla counterpart, and therefore S_b does not depend on the rebate at all.

6 Conclusion

This paper presents an innovative decomposition approach to price American up-and-out put options with a time-dependent rebate. A key step of our solution approach is to use the continuous FST to transform the PDE that governs the option price on a finite time-dependent domain into a simple ODE. The solution of this ODE can be easily obtained in the Fourier space and can be analytically converted back to the real space coordinate. As a result, we obtain an analytic representation that decomposes the price of an American up-and-out put with a time-dependent rebate into two components, namely the price of its European counterpart with the given rebate and the early exercise premium associated with the American-style early exercise right. Our decomposition results cover a number of existing decomposition formulas for some European-style and American-style options. Moreover, our proposed numerical procedure is very efficient in computing the option price and hedging parameters, even more efficient the adaptive FD method built upon Christara and Dang (2011), which are among the most efficient FD methods currently available. Furthermore, our numerical results also show that a rebate can have substantial effects on the price, the hedging parameters and the optimal exercise boundary. The numerical analysis reveal several interesting properties of the option price, hedging parameters, such as Delta, and the optimal exercise, which were not explored previously in the literature.

Appendix A Proof of Proposition 3.1.

Taking the inverse FST of (3.16) and using Fubini's theorem, we obtain:

$$H(x_b(l)-x)u(x,l) = \underbrace{\int_0^l \mathcal{F}_s^{-1}\left\{g(\xi,\omega)\mathrm{e}^{-\omega^2(l-\xi)}\right\}d\xi}_{\textbf{(I)}} + \underbrace{\mathcal{F}_s^{-1}\left\{\hat{u}(\omega,0)\mathrm{e}^{-\omega^2 l}\right\}}_{\textbf{(II)}}. \tag{A.32}$$

Compute the term (I) of (A.32)**.** By the basic integral formulas, the inverse FST of $g(\omega,\xi)\mathrm{e}^{-\omega^2(l-\xi)}$ can be calculated as:

$$
\begin{aligned}
&\mathcal{F}_s^{-1}\left\{g(\xi,\omega)\,\mathrm{e}^{-\omega^2(l-\xi)}\right\} = \frac{2}{\pi}\int_0^\infty g(\omega,\xi)\mathrm{e}^{-\omega^2(l-\xi)}\sin(x\omega)d\omega\\
=&\frac{2}{\pi}\left(g_3(x_b(\xi),\xi)+x_b'(\xi)g_2(x_b(\xi),\xi)\right)\int_0^\infty \mathrm{e}^{-\omega^2(l-\xi)}\sin(x_b(\xi)\omega)\sin(x\omega)d\omega\\
&-\frac{2}{\pi}g_2(x_b(\xi),\xi)\int_0^\infty \omega\mathrm{e}^{-\omega^2(l-\xi)}\cos(\omega x_b(\xi))\sin(x\omega)d\omega\\
&+\frac{2}{\pi}g_1(\xi)\int_0^\infty \omega\mathrm{e}^{-\omega^2(l-\xi)}\sin(x\omega)d\omega\\
=&\frac{g_3(x_b(\xi),\xi)+x_b'(\xi)g_2(x_b(\xi),\xi)}{2\sqrt{\pi(l-\xi)}}\left(\mathrm{e}^{-\frac{(x-x_b(\xi))^2}{4(l-\xi)}}-\mathrm{e}^{-\frac{(x+x_b(\xi))^2}{4(l-\xi)}}\right)+\frac{g_1(\xi)x\mathrm{e}^{-\frac{x^2}{4(l-\xi)}}}{2\sqrt{\pi(l-\xi)^3}}\\
&-\frac{g_2(x_b(\xi),\xi)}{4\sqrt{\pi(l-\xi)^3}}\left((x+x_b(\xi))\mathrm{e}^{-\frac{(x+x_b(\xi))^2}{4(l-\xi)}}+(x-x_b(\xi))\mathrm{e}^{-\frac{(x-x_b(\xi))^2}{4(l-\xi)}}\right)\\
=&\underbrace{\left(\frac{g_3(x_b(\xi),\xi)+x_b'(\xi)g_2(x_b(\xi),\xi)}{2\sqrt{\pi(l-\xi)}}-\frac{g_2(x_b(\xi),\xi)(x-x_b(\xi))}{4\sqrt{\pi(l-\xi)^3}}\right)\mathrm{e}^{-\frac{(x-x_b(\xi))^2}{4(l-\xi)}}}_{J^-}\\
&-\underbrace{\left(\frac{g_3(x_b(\xi),\xi)+x_b'(\xi)g_2(x_b(\xi),\xi)}{2\sqrt{\pi(l-\xi)}}+\frac{g_2(x_b(\xi),\xi)(x+x_b(\xi))}{4\sqrt{\pi(l-\xi)^3}}\right)\mathrm{e}^{-\frac{(x+x_b(\xi))^2}{4(l-\xi)}}}_{J^+}\\
&+\underbrace{\frac{g_1(\xi)x\mathrm{e}^{-\frac{x^2}{4(l-\xi)}}}{2\sqrt{\pi(l-\xi)^3}}}_{J^0}.
\end{aligned}
\tag{A.33}
$$

We now calculate J^-. Substituting g_2 and g_3 given in (3.8) into J^-, we can split J^- into:

$$
J^- = \frac{E}{\bar{S}}J_\alpha^- - J_{\alpha+1}^-,
$$

where

$$
J_\alpha^- = \left(\frac{-\alpha+x_b'(\xi)}{2\sqrt{\pi(l-\xi)}}-\frac{x-x_b(\xi)}{4\sqrt{\pi(l-\xi)^3}}\right)\mathrm{e}^{-\frac{(x-x_b(\xi))^2}{4(l-\xi)}-\alpha x_b(\xi)-\beta\xi}. \tag{A.34}
$$

Note that the factor $\left(\frac{-\alpha+x_b'(\xi)}{2\sqrt{\pi(l-\xi)}}-\frac{x-x_b(\xi)}{4\sqrt{\pi(l-\xi)^3}}\right)$ and the exponent in (A.34) can be written as:

$$
\frac{-\alpha+x_b'(\xi)}{2\sqrt{\pi(l-\xi)}}-\frac{x-x_b(\xi)}{4\sqrt{\pi(l-\xi)^3}} = -\frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial\xi}\left(\frac{x-x_b(\xi)-2\alpha(l-\xi)}{\sqrt{2(l-\xi)}}\right)
$$

and

$$-\frac{(x-x_b(\xi))^2}{4(l-\xi)} - \alpha x_b(\xi) - \beta\xi = -\alpha x + \alpha^2 l - (\alpha^2+\beta)\xi - \frac{1}{2}\left(\frac{x-x_b(\xi)-2\alpha(l-\xi)}{\sqrt{2(l-\xi)}}\right)^2,$$

respectively. Hence,

$$J_\alpha^- = -\mathrm{e}^{-\alpha x+\alpha^2 l-(\alpha^2+\beta)\xi}\frac{\partial}{\partial\xi}N\left(\frac{x-x_b(\xi)-2\alpha(l-\xi)}{\sqrt{2(l-\xi)}}\right),$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-a^2/2}\,da,$$

which is the cumulative distribution function of the standard normal distribution. Integrating J_α^- with respect to ξ from 0 to l and using integral by parts, we obtain:

$$\begin{aligned}\int_0^l J_\alpha^- d\xi =& -e^{-\alpha x+\alpha^2 l}\lim_{\xi\to l} e^{-(\alpha^2+\beta)\xi}N\left(\frac{x-x_b(\xi)-2\alpha(l-\xi)}{\sqrt{2(l-\xi)}}\right)\\ &+e^{-\alpha x+\alpha^2 l}N\left(\frac{x-x_b(0)-2\alpha l}{\sqrt{2l}}\right)\\ &-(\alpha^2+\beta)e^{-\alpha x+\alpha^2 l}\int_0^l e^{-(\alpha^2+\beta)\xi}N\left(\frac{x-x_b(\xi)-2\alpha(l-\xi)}{\sqrt{2(l-\xi)}}\right)d\xi.\end{aligned} \tag{A.35}$$

Since $x_b(\xi)$ is a C^1-smooth function and $N(0)=\frac{1}{2}$, the limit term is given by:

$$\lim_{\xi\to l}\left(\mathrm{e}^{-(\alpha^2+\beta)\xi}N\left(\frac{x-x_b(\xi)-2\alpha(l-\xi)}{\sqrt{2(l-\xi)}}\right)\right) = \mathrm{e}^{-(\alpha^2+\beta)l}\mathbb{I}_{x=x_b(l)}(x),$$

where

$$\mathbb{I}_{x=x_b(l)}(x) = \begin{cases}\frac{1}{2} & \text{if} \quad x = x_b(l),\\ 0 & \text{if} \quad x\neq x_b(l).\end{cases} \tag{A.36}$$

Therefore, (A.35) can be written as:

$$\begin{aligned}\int_0^l J_\alpha^- d\xi =& -\mathrm{e}^{-\alpha x-\beta l}\mathbb{I}_{x=x_b(l)}(x) + \mathrm{e}^{-\alpha x+\alpha^2 l}N\left(\frac{x-x_b(0)-2\alpha l}{\sqrt{2l}}\right)\\ &+\gamma\mathrm{e}^{-\alpha x+\alpha^2 l}\int_0^l \mathrm{e}^{\gamma\xi}N\left(\frac{x-x_b(\xi)-2\alpha(l-\xi)}{\sqrt{2(l-\xi)}}\right)d\xi.\end{aligned}$$

For the term $J^-_{\alpha+1}$, we just replace α by $\alpha+1$ in J^-_{α}. Therefore, the integral of J^- is expressed by:

$$\begin{aligned}\int_0^l J^- d\xi =& -\left(\frac{E}{\bar{S}}\mathrm{e}^{-\alpha x-\beta l} - \mathrm{e}^{-(\alpha+1)x-\beta l}\right)\mathbb{I}_{x=x_b(l)}(x) + \frac{E}{\bar{S}}\mathrm{e}^{-\alpha x+\alpha^2 l}N\left(\frac{x-x_b(0)-2\alpha l}{\sqrt{2l}}\right)\\ &- \mathrm{e}^{-(\alpha+1)x+(\alpha+1)^2 l}N\left(\frac{x-x_b(0)-2(\alpha+1)l}{\sqrt{2l}}\right)\\ &+ \frac{E}{\bar{S}}\gamma\mathrm{e}^{-\alpha x+\alpha^2 l}\int_0^l \mathrm{e}^{\gamma\xi}N\left(\frac{x-x_b(\xi)-2\alpha(l-\xi)}{\sqrt{2(l-\xi)}}\right)d\xi\\ &- q\mathrm{e}^{-(\alpha+1)x+(\alpha+1)^2 l}\int_0^l \mathrm{e}^{q\xi}N\left(\frac{x-x_b(\xi)-2(\alpha+1)(l-\xi)}{\sqrt{2(l-\xi)}}\right)d\xi.\end{aligned} \tag{A.37}$$

For the term J^+, we just replace x by $-x$ in J^- with notice that the limit term is always zero since $x \neq -x_b(l)$, then obtain:

$$\begin{aligned}\int_0^l J^+ d\xi =& \frac{E}{\bar{S}}\mathrm{e}^{\alpha x+\alpha^2 l}N\left(\frac{-x-x_b(0)-2\alpha l}{\sqrt{2l}}\right) - \mathrm{e}^{(\alpha+1)x+(\alpha+1)^2 l}N\left(\frac{-x-x_b(0)-2(\alpha+1)l}{\sqrt{2l}}\right)\\ &+ \frac{E}{\bar{S}}\gamma\mathrm{e}^{\alpha x+\alpha^2 l}\int_0^l \mathrm{e}^{\gamma\xi}N\left(\frac{-x-x_b(\xi)-2\alpha(l-\xi)}{\sqrt{2(l-\xi)}}\right)d\xi\\ &- q\mathrm{e}^{(\alpha+1)x+(\alpha+1)^2 l}\int_0^l \mathrm{e}^{q\xi}N\left(\frac{-x-x_b(\xi)-2(\alpha+1)(l-\xi)}{\sqrt{2(l-\xi)}}\right)d\xi.\end{aligned} \tag{A.38}$$

Finally, the integral of J^0 with respect to ξ from 0 to l is:

$$\int_0^l J^0 d\xi = \int_0^l \frac{x\mathrm{e}^{-\beta\xi}\mathrm{e}^{-\frac{x^2}{4(l-\xi)}}R(\frac{2\xi}{\sigma^2})}{2\bar{S}\sqrt{\pi(l-\xi)^3}}d\xi.$$

Compute the term (II) of (A.32)**.** Applying the convolution theorem for the FST:

$$\mathcal{F}_s^{-1}\left\{\mathcal{F}_s(f)\mathcal{F}_c(g)\right\} = \frac{1}{2}\int_0^\infty f(\zeta)\left[g(|x-\zeta|) - g(|x+\zeta|)\right]d\zeta,$$

it follows:

$$
\begin{aligned}
\mathcal{F}_s^{-1}\left\{\hat{u}(\omega,0)\mathrm{e}^{-\omega^2 l}\right\} = \mathcal{F}_s^{-1}&\left\{\mathcal{F}_s\left\{H(x_b(0)-x)f(x)\right\}\mathcal{F}_c\left\{\frac{\mathrm{e}^{-\frac{x^2}{4l}}}{\sqrt{\pi l}}\right\}\right\}\\
&=\int_0^{+\infty} H(x_b(0)-\zeta)f(\zeta)\frac{1}{2\sqrt{\pi l}}\left(\mathrm{e}^{-\frac{(x-\zeta)^2}{4l}}-\mathrm{e}^{-\frac{(x+\zeta)^2}{4l}}\right)d\zeta\\
&=\int_0^{x_b(0)} \max\left\{\frac{E}{\bar{S}}\mathrm{e}^{-\alpha\zeta}-\mathrm{e}^{-(\alpha+1)\zeta},0\right\}\frac{1}{2\sqrt{\pi l}}\left(\mathrm{e}^{-\frac{(x-\zeta)^2}{4l}}-\mathrm{e}^{-\frac{(x+\zeta)^2}{4l}}\right)d\zeta\\
&=\frac{1}{2\sqrt{\pi l}}\int_{\ln\frac{\bar{S}}{E}}^{x_b(0)}\left(\frac{E}{\bar{S}}\mathrm{e}^{-\alpha\zeta}-\mathrm{e}^{-(\alpha+1)\zeta}\right)\left(\mathrm{e}^{-\frac{(x-\zeta)^2}{4l}}-\mathrm{e}^{-\frac{(x+\zeta)^2}{4l}}\right)d\zeta\\
&=\frac{E}{\bar{S}}K_\alpha^- - K_{\alpha+1}^- - \frac{E}{\bar{S}}K_\alpha^+ + K_{\alpha+1}^+.
\end{aligned}
\tag{A.39}
$$

Here,

$$
\begin{aligned}
K_\alpha^\pm =&\frac{1}{2\sqrt{\pi l}}\int_{\ln\frac{\bar{S}}{E}}^{x_b(0)} \mathrm{e}^{-\alpha\zeta-\frac{(\zeta\pm x)^2}{4l}}d\zeta\\
=&\frac{1}{2\sqrt{\pi l}}\mathrm{e}^{\pm\alpha x+\alpha^2 l}\int_{\ln\frac{\bar{S}}{E}}^{x_b(0)} \mathrm{e}^{-\frac{1}{2}(\frac{\zeta\pm x+2\alpha l}{\sqrt{2l}})^2}d\zeta\\
=&\mathrm{e}^{\pm\alpha x+\alpha^2 l}\left[N\left(\frac{x_b(0)\pm x+2\alpha l}{\sqrt{2l}}\right)-N\left(\frac{\ln\frac{\bar{S}}{E}\pm x+2\alpha l}{\sqrt{2l}}\right)\right]\\
=&\mathrm{e}^{\pm\alpha x+\alpha^2 l}\left[N\left(\frac{\mp x-\ln\frac{\bar{S}}{E}-2\alpha l}{\sqrt{2l}}\right)-N\left(\frac{\mp x-x_b(0)-2\alpha l}{\sqrt{2l}}\right)\right].
\end{aligned}
\tag{A.40}
$$

Substituting (A.33), (A.37), (A.38) and (A.39) into (A.32), and multiplying both sides of the resulting equation with $S\mathrm{e}^{\alpha x+\beta l}$, we obtain the following relation between $u(x,l)$ and $x_b(l)$:

$$
\bar{S}e^{\alpha x+\beta l}H(x_b(l)-x)u(x,l)=-(E-\bar{S}e^{-x})\mathbb{I}_{x=x_b(l)}(x)+\mathcal{M}(x,l)+\int_0^l \mathcal{Q}(x,l,\xi,x_b(\xi))d\xi
\tag{A.41}
$$

where

$$
\begin{aligned}
\mathcal{M}(x,l) = &Ee^{-\gamma l}N\left(\frac{x-\ln(\frac{\bar{S}}{E})-2\alpha l}{\sqrt{2l}}\right)-\bar{S}e^{-ql-x}N\left(\frac{x-\ln(\frac{\bar{S}}{E})-2(\alpha+1)l}{\sqrt{2l}}\right)\\
&-e^{2\alpha x}\left[Ee^{-\gamma l}N\left(\frac{-x-\ln(\frac{\bar{S}}{E})-2\alpha l}{\sqrt{2l}}\right)-\bar{S}e^{-ql+x}N\left(\frac{-x-\ln(\frac{\bar{S}}{E})-2(\alpha+1)l}{\sqrt{2l}}\right)\right],
\end{aligned}
\tag{A.42}
$$

and

$$
\begin{aligned}
\mathcal{Q}(x,l,\xi,x_b(\xi)) =& E\gamma \mathrm{e}^{-\gamma(l-\xi)} N\left(\frac{x-x_b(\xi)-2\alpha(l-\xi)}{\sqrt{2(l-\xi)}}\right) \\
& - q\bar{S}\mathrm{e}^{-q(l-\xi)-x} N\left(\frac{x-x_b(\xi)-2(\alpha+1)(l-\xi)}{\sqrt{2(l-\xi)}}\right) \\
& - e^{2\alpha x}\left[E\gamma \mathrm{e}^{-\gamma(l-\xi)} N\left(\frac{-x-x_b(\xi)-2\alpha(l-\xi)}{\sqrt{2(l-\xi)}}\right)\right. \\
& \left. - q\bar{S}e^{-q(l-\xi)+x} N\left(\frac{-x-x_b(\xi)-2(\alpha+1)(l-\xi)}{\sqrt{2(l-\xi)}}\right)\right] \\
& + \frac{xR(\frac{2\xi}{\sigma^2})}{2\sqrt{\pi(l-\xi)^3}} e^{\beta(l-\xi)+\alpha x-\frac{x^2}{4(l-\xi)}}.
\end{aligned} \tag{A.43}
$$

Converting the dimensionless variables to the original variables S and τ, one can easily obtain Proposition 3.1.

Appendix B Proof of Proposition 3.2.

From the formulae (3.23–3.22), we can derive that, $\forall S > S_b(\tau)$,

$$
\begin{aligned}
&\frac{\partial V}{\partial S}(S,\tau) = \frac{\partial U}{\partial S}(S,\tau) + \frac{\partial X}{\partial S}(S,\tau;S_b(\tau)) \\
&= \frac{\partial M_1}{\partial S}(S,\tau,E) - \frac{\lambda S^{\lambda-1}}{\bar{S}^\lambda} M_1\left(\frac{\bar{S}^2}{S},\tau,E\right) + \left(\frac{S}{\bar{S}}\right)^{\lambda-2} \frac{\partial M_1}{\partial \frac{\bar{S}^2}{S}}\left(\frac{\bar{S}^2}{S},\tau,E\right) + \frac{\partial K_1}{\partial S}(S,\tau) \\
&+ \int_0^\tau \left[\frac{\partial Q_1}{\partial S}(S,\tau,s,S_b(s)) - \frac{\lambda S^{\lambda-1}}{\bar{S}^\lambda} Q_1\left(\frac{\bar{S}^2}{S},\tau,s,S_b(s)\right) + \left(\frac{S}{\bar{S}}\right)^{\lambda-2} \frac{\partial Q_1}{\partial \frac{\bar{S}^2}{S}}\left(\frac{\bar{S}^2}{S},\tau,s,S_b(s)\right)\right] ds,
\end{aligned}
$$

where $K_1(x,y) = \int_0^y \left(\frac{x}{\bar{S}}\right)^{\frac{\lambda}{2}} K(x,y,z)dz$, and $K(x,y,z), M(x,y,z), Q_1(x,y,z,t)$ are defined in (3.18). Therefore, in order to prove the formula (3.25), we only need to show that

$$
\frac{\partial}{\partial x} M_1(x,y,z) = \tilde{M}_1(x,y,z), \quad \frac{\partial}{\partial x} Q_1(x,y,z,w) = \tilde{Q}_1(x,y,z,w), \quad \text{and} \quad \frac{\partial}{\partial x} K_1(x,y) = \tilde{K}_1(x,y).
$$

where $\tilde{K}_1, \tilde{M}_1, \tilde{Q}_1$ are defined as above. Before going to the proof of these equalities, we notice that $d_1(x,y,z) - d_2(x,y,z) = \sigma\sqrt{y}$. This implies

$$
x\mathrm{e}^{-\delta y}\tilde{N}(-d_1(x,y,z)) = z\mathrm{e}^{-ry}\tilde{N}(-d_2(x,y,z)), \tag{B.44}
$$

where $\tilde{N}(x) = \frac{1}{2\pi}\mathrm{e}^{-\frac{x^2}{2}}$.

Proof of $\frac{\partial}{\partial x} M_1(x,y,z) = \tilde{M}_1(x,y,z)$. We have

$$\begin{aligned}\frac{\partial}{\partial x} M_1(x,y,z) &= \frac{\partial}{\partial x}\left[z\mathrm{e}^{-ry}N(-d_2(x,y,z)) - x\mathrm{e}^{-\delta y}N(-d_1(x,y,z))\right]\\ &= -\frac{z\mathrm{e}^{-ry}}{x\sigma\sqrt{y}}\tilde{N}(-d_2(x,y,z)) - \mathrm{e}^{-\delta y}N(-d_1(x,y,z)) + \frac{\mathrm{e}^{-\delta y}}{\sigma\sqrt{y}}\tilde{N}(-d_1(x,y,z)).\end{aligned}$$

Using the formula (B.44), it follows

$$-\frac{z\mathrm{e}^{-ry}}{x\sigma\sqrt{y}}\tilde{N}(-d_2(x,y,z)) + \frac{\mathrm{e}^{-\delta y}}{\sigma\sqrt{y}}\tilde{N}(-d_1(x,y,z)) = 0.$$

Therefore,

$$\frac{\partial}{\partial x} M_1(x,y,z) = -\mathrm{e}^{-\delta y}N(-d_1(x,y,z)) = \tilde{M}_1(x,y,z).$$

Proof of $\frac{\partial}{\partial x} Q_1(x,y,z) = \tilde{Q}_1(x,y,z)$. We have

$$\begin{aligned}\frac{\partial}{\partial x} Q_1(x,y,z) &= \frac{\partial}{\partial x}\left[Er\mathrm{e}^{-r(y-z)}N(-d_2(x,y-z,w)) - x\delta\mathrm{e}^{-\delta(y-z)}N(-d_1(x,y-z,w))\right]\\ &= -\frac{Er\mathrm{e}^{-r(y-z)}}{x\sigma\sqrt{y-z}}\tilde{N}(-d_2(x,y-z,w)) - \delta\mathrm{e}^{-\delta(y-z)}N(-d_1(x,y-z,w))\\ &\quad +\frac{\delta\mathrm{e}^{-\delta(y-z)}}{\sigma\sqrt{y-z}}\tilde{N}(-d_1(x,y-z,w)).\end{aligned}$$

Using the formula (B.44), it follows

$$\mathrm{e}^{-r(y-z)}\tilde{N}(-d_2(x,y-z,w)) = \frac{x\mathrm{e}^{-\delta(y-z)}}{w}\tilde{N}(-d_1(x,y-z,w)).$$

Therefore,

$$\frac{\partial}{\partial x} Q_1(x,y,z) = \mathrm{e}^{-\delta(y-z)}\left[-\delta N(-d_1(x,y-z,w)) + \frac{\tilde{N}\left(-d_1(x,y-z,w)\right)}{\sigma\sqrt{y-z}}\left(\delta - \frac{Er}{w}\right)\right] = \tilde{Q}_1(x,y,z).$$

Proof of $\frac{\partial}{\partial x} K_1(x,y) = \tilde{K}_1(x,y)$. Note that $K_1(x,y) = \int_0^y \left(\frac{x}{\bar{S}}\right)^{\frac{\lambda}{2}} K(x,y,z)dz$ has removable singularities at $(\bar{S},y)$. In this case, in order to calculate $\frac{\partial K_1}{\partial x}(x,y)$ when x closes to $\bar{S}$, we first need to remove these singularities by using the following variable transformation:

$$\xi = \frac{\ln\bar{S} - \ln x}{\sigma\sqrt{y-z}}.$$

As a result,

$$K_1(x,y) = \int_{\frac{\ln\bar{S}-\ln x}{\sigma\sqrt{y}}}^{+\infty} \left(\frac{x}{\bar{S}}\right)^{\frac{\lambda}{2}} \frac{\sqrt{2}}{\sqrt{\pi}} \mathrm{e}^{-\frac{\xi^2}{2}+\beta\frac{\sigma^2}{2}\left(\frac{\ln\bar{S}-\ln x}{\sigma\xi}\right)^2} R\left(y - \frac{(\ln\bar{S}-\ln x)^2}{\sigma^2\xi^2}\right) d\xi.$$

By using the Leibniz integral rule, we can calculate the derivative of the above integral. We have

$$\begin{aligned}
\frac{\partial}{\partial x}K_1(x,y) &= \frac{\lambda x^{\frac{\lambda}{2}-1}}{\sqrt{2\pi}\bar{S}^{\frac{\lambda}{2}}} \int_{\frac{\ln\bar{S}-\ln x}{\sigma\sqrt{y}}}^{+\infty} \mathrm{e}^{\left\{-\frac{\xi^2}{2}+\beta\frac{\sigma^2}{2}\frac{(\ln\bar{S}-\ln x)^2}{\sigma^2\xi^2}\right\}} R\left(y - \frac{(\ln\bar{S}-\ln x)^2}{\sigma^2\xi^2}\right) d\xi \\
&\quad + \left(\frac{x}{\bar{S}}\right)^{\frac{\lambda}{2}} \frac{\sqrt{2}}{\sqrt{\pi}} \int_{\frac{\ln\bar{S}-\ln x}{\sigma\sqrt{y}}}^{+\infty} \mathrm{e}^{\left\{-\frac{\xi^2}{2}+\beta\frac{\sigma^2}{2}\frac{(\ln\bar{S}-\ln x)^2}{\sigma^2\xi^2}\right\}} \frac{\ln\bar{S}-\ln x}{x\xi^2} \\
&\quad \cdot \left[(-\beta)R\left(y - \frac{(\ln\bar{S}-\ln x)^2}{\sigma^2\xi^2}\right) + \frac{2}{\sigma^2}R'\left(y - \frac{(\ln\bar{S}-\ln x)^2}{\sigma^2\xi^2}\right)\right] d\xi.
\end{aligned}$$

By using variable transformations $u = \xi - \dfrac{\ln\bar{S}-\ln x}{\sigma\sqrt{y}}$ and $v = \dfrac{\ln\bar{S}-\ln x}{\sigma\xi}$ for the first and second integral in the above formula of $\dfrac{\partial}{\partial x}K_1(x,y)$, we obtain the following expression for $\dfrac{\partial}{\partial x}K_1(x,y)$:

$$\begin{aligned}
\frac{\partial}{\partial x}K_1(x,y) &= \frac{\lambda x^{\frac{\lambda}{2}-1}}{\sqrt{2\pi}\bar{S}^{\frac{\lambda}{2}}} \int_{\frac{\ln\bar{S}-\ln x}{\sigma\sqrt{y}}}^{+\infty} \mathrm{e}^{\left\{-\frac{(u+\frac{\ln\bar{S}-\ln x}{\sigma\sqrt{y}})^2}{2}+\frac{\beta}{2}\left(\frac{\ln\bar{S}-\ln x}{(u+\frac{\ln\bar{S}-\ln x}{\sigma\sqrt{y}})}\right)^2\right\}} R\left(y - \left(\frac{\ln\bar{S}-\ln x}{\sigma(u+\frac{\ln\bar{S}-\ln x}{\sigma\sqrt{y}})}\right)^2\right) d\xi \\
&\quad + \left(\frac{x}{\bar{S}}\right)^{\frac{\lambda}{2}} \frac{\sqrt{2}\sigma}{\sqrt{\pi}x} \int_0^{\sqrt{y}} \mathrm{e}^{\left\{-\frac{(\ln\bar{S}-\ln x)^2}{2\sigma^2v^2}+\frac{\beta}{2}\sigma^2v^2\right\}} \left[(-\beta)R(y-v^2) + \frac{2}{\sigma^2}R'(y-v^2)\right] ds \\
&= \tilde{K}_1(x,y).
\end{aligned}$$

This completes the proof of Proposition (3.2).

Appendix C Proof of Corollary 4.1.

By switching terms, the integral equation (3.24) can be rewritten as

$$\frac{S_b(\tau)}{E} = \frac{T_1(\tau)}{T_2(\tau)}, \tag{C.45}$$

where

$$
\begin{aligned}
T_1(\tau) &= 1-\mathrm{e}^{-r\tau}N(-d_2\left(S_b(\tau),\tau,E\right))+\left(\frac{S_b(\tau)}{\bar S}\right)^{\lambda}\mathrm{e}^{-r\tau}N\left(-d_2\left(\frac{\bar S^2}{S_b(\tau)},\tau,E\right)\right)\\
&\quad-\int_0^{\tau} r\mathrm{e}^{-r(\tau-u)}N\left(-d_2(S_b(\tau),\tau-u,S_b(u))\right)du\\
&\quad+\left(\frac{S_b(\tau)}{\bar S}\right)^{\lambda}\int_0^{\tau} r\mathrm{e}^{-r(\tau-u)}N\left(-d_2\left(\frac{\bar S^2}{S_b(\tau)},\tau-u,S_b(u)\right)\right)du,
\end{aligned}
$$

and

$$
\begin{aligned}
T_2(\tau) &= 1-\mathrm{e}^{-\delta\tau}N(-d_1(S_b(\tau),\tau,E))+\left(\frac{S_b(\tau)}{\bar S}\right)^{\lambda-2}\mathrm{e}^{-\delta\tau}N\left(-d_1\left(\frac{\bar S^2}{S_b(\tau)},\tau,E\right)\right)\\
&\quad-\int_0^{\tau}\delta\mathrm{e}^{-\delta(\tau-u)}N(-d_1(S_b(\tau),\tau-u,S_b(u)))du\\
&\quad+\left(\frac{S_b(\tau)}{\bar S}\right)^{\lambda-2}\int_0^{\tau}\delta\mathrm{e}^{-\delta(\tau-u)}N\left(-d_1\left(\frac{\bar S^2}{S_b(\tau)},\tau-u,S_b(u)\right)\right)du\\
&\quad-\frac{S_b(\tau)^{\alpha-1}}{\bar S^{\alpha}}\frac{1}{\sigma\sqrt{2\pi}}\int_0^{\tau}\frac{\ln S_b(\tau)-\ln\bar S}{\sqrt{(\tau-u)^3}}\mathrm{e}^{-\frac{(\ln S_b(\tau)-\ln\bar S)^2}{2\sigma^2(\tau-u)}+\beta\frac{\sigma^2}{2}(\tau-u)}R(u)du.
\end{aligned}
$$

Before proceeding further, we note that $S_b(\tau)\le E$ as the put option should be exercised only when it is in-the-money or at-the-money.

Consider the first case where $S_b(0^+)=E$. Taking the limit of equation (C.45) as τ tends to 0^+, we obtain $\lim_{\tau\to0^+}\dfrac{S_b(\tau)}{E}=1$ and thus $S_b(0^+)=E$ is one possible solution for $S_b(0^+)$.

Now we consider the second case where $S_b(0^+)<E$. As $\lim_{\tau\to0^+}T_1(\tau)=\lim_{\tau\to0^+}T_2(\tau)=0$, the limit of equation (C.45) is an indeterminate form which can be resolved by using L'Hospital's rule. However, before applying L'Hospital's rule, we should eliminate "redundant terms" in T_1 and T_2. For T_1, we have the following claim.

<u>Claim.</u> *When $\tau\to0^+$, we have*

(a) $\left(\frac{S_b(\tau)}{\bar S}\right)^{\lambda}\mathrm{e}^{-r\tau}N\left(-d_2\left(\frac{\bar S^2}{S_b(\tau)},\tau,E\right)\right)$ *is eliminated by* $1-e^{-r\tau}N\left(-d_2\left(S_b(\tau),\tau,E\right)\right)$,

(b) $\left(\frac{S_b(\tau)}{\bar S}\right)^{\lambda}\int_0^{\tau}r\mathrm{e}^{-r(\tau-u)}N\left(-d_2\left(\frac{\bar S^2}{S_b(\tau)},\tau-u,S_b(u)\right)\right)du$ *is eliminated by* $\int_0^{\tau}r\mathrm{e}^{-r(\tau-u)}N\left(-d_2\left(S_b(\tau),\tau-u,S_b(u)\right)\right)du$.

<u>*Proof of Claim (a)*</u>. It is straightforward to see that

$$
\lim_{\tau\to0^+}\frac{1-\mathrm{e}^{-r\tau}N(-d_2(S_b(\tau),\tau,E))}{\tau}=\lim_{\tau\to0^+}\frac{1-\mathrm{e}^{-r\tau}}{\tau}=r.
$$

Thus $1 - \mathrm{e}^{-r\tau}N(-d_2(S_b(\tau),\tau,E)) \backsim r\tau$ as $\tau \to 0$, where the notation $\backsim$ denotes the equivalence of two infinitesimal functions of τ. Moreover, as $\tau \to 0$, we have

$$N\left(-d_2\left(\frac{\bar{S}^2}{S_b(\tau)},\tau,E\right)\right) \backsim N\left(\frac{1}{\sqrt{\tau}}\right) \backsim \int_{-\infty}^{\frac{1}{\sqrt{\tau}}} \mathrm{e}^{-\frac{t^2}{2}}dt$$

$$\lim_{\tau\to 0^+} \frac{\int_{-\infty}^{\frac{1}{\sqrt{\tau}}} \mathrm{e}^{-\frac{t^2}{2}}dt}{\tau} = \lim_{\tau\to 0^+} \frac{\mathrm{e}^{-\frac{1}{2\tau}}}{\tau} = 0.$$

Therefore, the term $\left(\frac{S_b(\tau)}{\bar{S}}\right)^\lambda \mathrm{e}^{-r\tau}N\left(-d_2\left(\frac{\bar{S}^2}{S_b(\tau)},\tau,E\right)\right)$ decays to 0, as $\tau \to 0$, at a faster rate than the term $1 - \mathrm{e}^{-r\tau}N\left(-d_2\left(S_b(\tau),\tau,E\right)\right)$.

Proof of Claim (b). We have

$$\lim_{u\to\tau}\left(\frac{S_b(\tau)}{\bar{S}}\right)^\lambda N\left(-d_2\left(\frac{\bar{S}^2}{S_b(\tau)},\tau-u,S_b(u)\right)\right) = 0,$$

$$\lim_{u\to\tau} N\left(-d_2\left(S_b(\tau),\tau-u,S_b(u)\right)\right) = \frac{1}{2}.$$

Therefore, the terms $\left(\frac{S_b(\tau)}{\bar{S}}\right)^\lambda \int_0^\tau r\mathrm{e}^{-r(\tau-u)}N\left(-d_2\left(\frac{\bar{S}^2}{S_b(\tau)},\tau-u,S_b(u)\right)\right)du$ decays to 0, as $\tau \to 0$, at a faster rate than the term $\int_0^\tau r\mathrm{e}^{-r(\tau-u)}N(-d_2(S_b(\tau),\tau-u,S_b(u)))du$. The proof of the claim is complete.

From the claim, we conclude that as $\tau \to 0$

$$T_1 \backsim T_3 = 1 - \mathrm{e}^{-r\tau}N(-d_2(S_b(\tau),\tau,E)) - \int_0^\tau r\mathrm{e}^{-r(\tau-u)}N(-d_2(S_b(\tau),\tau-u,S_b(u)))du.$$

Similarly, we have

$$T_2 \backsim T_4 = 1 - \mathrm{e}^{-\delta\tau}N(-d_1(S_b(\tau),\tau,E)) - \int_0^\tau \delta\mathrm{e}^{-\delta(\tau-u)}N(-d_1(S_b(\tau),\tau-u,S_b(u)))du.$$

Chiarella et al. (2004) shows that $\lim_{\tau\to 0}\frac{T_3(\tau)}{T_4(\tau)} = \frac{r}{\delta}$. Therefore,

$$\lim_{\tau\to 0^+}\frac{S_b(\tau)}{E} = \lim_{\tau\to 0^+}\frac{T_1(\tau)}{T_2(\tau)} = \lim_{\tau\to 0^+}\frac{T_3(\tau)}{T_4(\tau)} = \frac{r}{\delta}. \tag{C.46}$$

Combining the results of the two cases, we obtain

$$S_b(0^+) = \min\left(E, \frac{rE}{\delta}\right).$$

This completes the proof of Corollary 4.1.

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