2017

# The K-theoretic bulk-edge correspondence for topological insulators

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**Keywords**

correspondence, topological, bulk-edge, insulators, k-theoretic

**Disciplines**

Engineering | Science and Technology Studies

# The $K$-theoretic bulk-edge correspondence for topological insulators

C. Bourne, J. Kellendonk and A. Rennie

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## 1. Introduction

The bulk-edge correspondence is fundamental for topological insulators. Indeed, insulators in non-trivial topological phases are characterised by the existence of edge states at the Fermi energy which are robust against perturbations coming from disorder. This is a topological bulk-boundary effect, as explained in all details for complex topological insulators in [51] (see Corollary 6.5.5). The theory of [51] is based on the noncommutative topology ($K$-theory, cyclic cohomology and index theory) of complex $C^*$-algebras, an approach to solid state systems proposed and developed by Bellissard [5, 6].

Complex topological insulators are topological insulators which are not invariant under a symmetry implemented by anti-linear operators, such as time reversal symmetry. The class of insulators with

anti-linear symmetries require noncommutative topology for real algebras and here the bulk-edge correspondence is still in development.

The bulk-edge correspondence in its noncommutative formulation has two sides. One side is the $K$-theoretic correspondence, where the boundary map of $K$-theory yields an equality between $K$-group elements, one of the $K$-group of the observable (bulk) algebra, and the other of the $K$-group of the edge algebra. The other side is "dual to $K$-theory", namely a correspondence between elements of a theory which provides us with functionals on $K$-theory. The pairing of $K$-group elements with the dual theory can be used to obtain topological numbers from the $K$-theory elements and, possibly, with a physical explanation of these numbers as topologically quantised entities (non-dissipative transport coefficients).

In the complex case this has been achieved using cyclic cohomology [31, 33, 51]. The result of applying a functional coming from cyclic cohomology (one speaks of a pairing between cyclic cocycles and $K$-group elements) is a complex number and since the functional is additive the result is necessarily 0 on elements of finite order. The most exciting topological invariants, however, lead to $K$-group elements which have order 2, the Kane–Mele invariant being such an example. This is why we look into another theory dual to $K$-group theory, namely $K$-homology (its proper dual in the algebraic sense). It leads to functionals which applied to $K$-group elements are Clifford index valued and do not vanish on finite groups. *Our specific aim in this paper is to describe a computable version of the bulk-edge correspondence for $K$-homology.*

We actually work in the more general setting of Kasparov's $KK$-theory of which $K$-theory and $K$-homology are special cases. In doing so we generalise the approach using Kasparov's theory for the quantum Hall effect [8] to the case of all topological insulators.

Whereas the specific details of the insulator feed into the construction of $K$-group elements, the $K$-homology class appears to be stable for systems of the same dimension. We call it the fundamental $K$-cycle as it is constructed similarly to Kasparov's fundamental class for oriented manifolds [28, 40]. It may well be a key feature of these kinds of condensed matter models that the physics (alternatively the geometry and topology) is governed by classes of this type, which is effectively the fundamental class of the momentum space.

An important aspect of the $K$-cycle is that it involves the Dirac operator in momentum space $\sum_{j=1}^{d} X_j \gamma^j$ with $X_j$ the components of the position operator in $d$ dimensions, and $\gamma^j$ the gamma matrices acting in a physical representation of the relevant algebra on

$\ell^2(\mathbb{Z}^d, \mathbb{C}^\nu)$ or $\ell^2(\mathbb{Z}^{d-1}, \mathbb{C}^\nu)$ (we work in the tight binding approximation). This operator, or rather its phase, has played a fundamental role ever since it was employed in the integer quantum Hall effect (the phase induces a singular gauge transformation sending one charge to infinity, a process which is at the heart of Laughlin's Gedankenexperiment). The associated index formula has recently been generalised to all topological insulators [21].

Computing boundary maps in $K$-homology is notoriously difficult. It comes down to realising $K$-homology groups of $A$ as Kasparov $KKO_i(A, \mathbb{R})$-groups [27] and determining the Kasparov product with the so-called extension class. Until recently this seemed impossible in general. But recent progress on the formulation of $KK$-theory via unbounded Kasparov cycles [4, 10, 16, 18, 24, 37, 44, 46] make it feasible in our case. We determine an unbounded Kasparov cycle for the class of the Toeplitz extension of the observable algebra of the insulator, denoted ext. The task is then to compute the Kasparov product of the $KK$-class of this extension, denoted [ext], with the class of the fundamental $K$-cycle of the edge algebra, denoted $\lambda_e$. One of the important results of this paper is that the product is, up to sign, the $KK$-class of the bulk fundamental $K$-cycle $\lambda_b$,

$$[\text{ext}] \hat{\otimes}_{A_e} [\lambda_e] = (-1)^{d-1} [\lambda_b],$$

where $[\lambda_\bullet]$ denotes the class of the Kasparov module in $KK$-theory. This is the dual side to the bulk-boundary correspondence, showing how to relate the fundamental classes of the bulk and edge theories. We also emphasise that by working in the unbounded setting, our computations are explicit and have direct link to the underlying physics and geometry of the system.

We can relate our results about fundamental $K$-cycles to topological phases by identifying the (bulk) invariants of topological phases as a pairing/product of the $K$-cycle $\lambda_b$ with a $K$-theory class $[x_b]$. We do not prescribe what the $K$-group element $[x_b]$ of the bulk has to be. It can be the homotopy class of a symmetry compatible Hamiltonian (translated from van Daele $K$-theory to $KK$-theory) or the class of the symmetry of the insulator constructed in [9]. Leaving this flexible gives us the possibility to consider insulators systems of quite general symmetry type without affecting the central correspondence. It also allows our approach to be adapted to different experimental arrangements, an important feature given the difficulties of measuring $\mathbb{Z}_2$-labelled phases experimentally.

The $K$-theory side of the bulk-edge correspondence can be obtained by realising the $K$-groups of the bulk algebra $A$ as Kasparov

3

$KKO_j(\mathbb{R}, A)$-groups so that the boundary map applied to a $KK$-class $[x]$ of $A$ is given by the Kasparov product $[x_b]\hat{\otimes}_A[\text{ext}]$. The bulk-boundary correspondence between topological quantised numbers is then a direct consequence of the associativity of the Kasparov product

$$([x_b]\hat{\otimes}_A[\text{ext}])\hat{\otimes}_{A_e}[\lambda_e] = [x_b]\hat{\otimes}_A([\text{ext}]\hat{\otimes}_{A_e}[\lambda_e]).$$

The equation says that the two topological quantised entities, that for the bulk $[x_b]\hat{\otimes}_A[\lambda_b]$, and that for the edge $[x_e]\hat{\otimes}_A[\lambda_e]$, where $[x_e] = [x_b]\hat{\otimes}_A[\text{ext}]$ corresponds to the image of the boundary map on $[x_b]$, are equal. Having said that, the result of the pairing lies in $KKO_{i+j+1}(\mathbb{R}, \mathbb{R})$ which is a group generated by one element and not a number, and so still needs interpretation. While this number can sometimes be interpreted as an index (modulo 2 valued, in certain degrees) we lack a better understanding, which had been possible in the complex case due to the use of derivations which make up the cyclic cohomology classes.

Apart from [8], our work relies most substantially on [9] which used real Kasparov theory to derive the groups that appear in the 'periodic table of topological insulators and superconductors' as outlined by Kitaev [34]. Our work also builds on and complements that in [17] for the commutative case and [21, 29, 36, 56] for the noncommutative approach. We consider systems with weak disorder (that is, disordered Hamiltonians retaining a spectral gap). The substantial problem of strong disorder and localisation will not be treated here.

## 1.1. Relation to other work

There have been several mathematical papers detailing aspects of the bulk-edge correspondence for topological insulators with anti-linear symmetries. Graf and Porta prove a bulk-edge correspondence for two-dimensional Hamiltonians with odd time-reversal symmetry using Bloch bundles and without reference to $K$-theory [20]. Similar results are obtained by Avilla, Schulz-Baldes and Villegas-Blas using spin Chern numbers and an argument involving transfer matrices [3, 55]. Spin Chern numbers are related to the noncommutative approach to the quantum Hall effect and, as such, allow samples with disorder to be considered.

An alternative approach is taken by Loring, who derives the invariants of topological phases by considering almost commuting Hermitian matrices and their Clifford pseudospectrum [41]. Such a viewpoint gives expressions for the invariants of interest that are

4

amenable to numerical simulation. Loring also relates indices associated to $d$ and $d + 1$ dimensional systems, a bulk-edge correspondence [41, Section 7]. What is less clear is the link between Loring's results and earlier results on the bulk-edge correspondence for the quantum Hall effect, particularly [31, 33].

Papers by Mathai and Thiang establish a $K$-theoretic bulk-edge correspondence for time-reversal symmetric systems [42, 43], and more recently with Hannabuss [22] consider the general case of topological insulators. Mathai and Thiang show that the invariants of interest in $K$-theory pass from bulk to edge under the boundary map of the real or complex Pimsner–Voiculescu sequence. Mathai and Thiang also show that, under T-duality, the boundary map in $K$-theory of tori can be expressed as the conceptually simpler restriction map. Similar results also appear in the work of Li, Kaufmann and Wehefritz-Kaufmann, who consider time-reversal symmetric systems and their relation to the $KO$, $KR$ and $KQ$ groups of $\mathbb{T}^d$ [39]. A bulk-edge correspondence then links the topological $K$-groups associated to the bulk and boundary using the Baum–Connes isomorphism and Poincaré duality.

Recent work by Kubota establishes a bulk-edge correspondence in $K$-theory for topological phases of quite general type [36]. Kubota follows the general framework of [17, 56] and considers twisted equivariant $K$-groups of uniform Roe algebras, which can be computed using the coarse Mayer–Vietoris exact sequence and coarse Baum–Connes map. Classes in such groups are associated to gapped Hamiltonians compatible with a twisted symmetry group. An edge invariant is also defined and is shown to be isomorphic to the bulk class under the boundary map of the coarse Mayer–Vietoris exact sequence in $K$-theory [35, 36]. The use of Roe algebras and coarse geometry means that there is the potential for systems with impurities and uneven edges to be considered.

## 1.2. Outline of the paper

We begin with a short review of unbounded Kasparov theory in Section 2. We particularly focus on real Kasparov theory as it is relatively understudied in the literature.

The central content of this paper is in Section 3, where we construct fundamental $K$-cycles $\lambda^{(d)}$ for (possibly twisted) $\mathbb{Z}^d$-actions of unital $C^*$-algebras. We can then link actions of different order by the Pimsner–Voiculescu short exact sequence [49], which we represent with an unbounded Kasparov module. Finally we use the unbounded product to show that the fundamental $K$-cycle $\lambda^{(d)}$ can be factorised into the product of the extension Kasparov module and $\lambda^{(d-1)}$ (up

to a basis ordering of Clifford algebra elements, which may introduce a minus sign).

In Section 4 we relate our result to topological phases by pairing the $K$-cycle $\lambda^{(d)}$ with a $K$-theory class $[x_b]$ related to gapped Hamiltonians with time reversal and/or particle-hole and/or chiral symmetry. We also include a detailed discussion on the computation of the bulk and edge pairings, using both Clifford modules and the Atiyah–Bott–Shapiro construction as well as semifinite spectral triples and the semifinite local index formula. Limitations and open questions are also considered.

## 2. Preliminaries on real Kasparov theory

For the convenience of the reader and to establish notation we briefly summarise the results in real Kasparov theory of use to us for the bulk-edge correspondence. The reader may consult [27, 54] or [9, Appendix A] for more information.

### 2.1. $KKO$-groups

The use of Clifford algebras to define higher $KK$-groups makes it important to work with $\mathbb{Z}_2$-graded $C^*$-algebras and $\mathbb{Z}_2$-graded tensor products, $\hat{\otimes}$. A basic reference is [7, §14]. Given a real $C^*$-module $E_B$ (written $E$ when $B$ is clear) over a $\mathbb{Z}_2$-graded $C^*$-algebra $B$, we denote by $\mathrm{End}_B(E)$ the algebra of endomorphisms of $E$ which are adjointable with respect to the $B$-valued inner-product $(\cdot \mid \cdot)_B$ on $E_B$. The algebra of compact endomorphisms $\mathrm{End}_B^0(E)$ is generated by the operators $\Theta_{e_1,e_2}$ for $e_1, e_2 \in E$ such that for $e_3 \in E$

$$\Theta_{e_1,e_2}(e_3) = e_1 \cdot (e_2 \mid e_3)_B$$

with $e \cdot b$ the (graded) right-action of $B$ on $E$.

**Definition 2.1.** Let $A$ and $B$ be $\mathbb{Z}_2$-graded $C^*$-algebras. A real unbounded Kasparov module $(\mathcal{A}, {}_\pi E_B, D)$ is a $\mathbb{Z}_2$-graded real $C^*$-module $E_B$, a dense $*$-subalgebra $\mathcal{A} \subset A$ with graded real homomorphism $\pi : \mathcal{A} \to \mathrm{End}_B(E)$, and an unbounded regular odd operator $D$ such that for all $a \in \mathcal{A}$

$$[D, \pi(a)]_\pm \in \mathrm{End}_B(E), \qquad \pi(a)(1 + D^2)^{-1/2} \in \mathrm{End}_B^0(E).$$

Where unambiguous, we will omit the representation $\pi$ and write unbounded Kasparov modules as $(\mathcal{A}, E_B, D)$. The results of Baaj and Julg [4] continue to hold for real Kasparov modules, so given an unbounded module $(\mathcal{A}, E_B, D)$ we apply the bounded transformation to obtain the real (bounded) Kasparov module $(A, E_B, D(1+D^2)^{-1/2})$, where $A$ is the $C^*$-closure of the dense subalgebra $\mathcal{A}$.

One can define notions of unitary equivalence, homotopy and degenerate Kasparov modules in the real setting (see [27, §4]). Hence we can define the group $KKO(A, B)$ as the equivalence classes of real (bounded) Kasparov modules modulo the equivalence relation generated by these relations.

Clifford algebras are used to define higher $KKO$-groups and encode periodicity. In the real setting, we define $C\ell_{p,q}$ to be the real span of the mutually anti-commuting generators $\gamma^1, \ldots, \gamma^p$ and $\rho^1, \ldots, \rho^q$ such that

$$(\gamma^i)^2 = 1, \quad (\gamma^i)^* = \gamma^i, \quad (\rho^i)^2 = -1, \quad (\rho^i)^* = -\rho^i.$$

We now recall the relation between real $KK$-groups and real $K$-theory.

**Proposition 2.2** ([27], §**6, Theorem 3**). *For trivially graded, $\sigma$-unital real algebras $A$, there is an isomorphism $KKO(C\ell_{n,0}, A) \cong KO_n(A)$.*

Each short exact sequence of real $C^*$-algebras $0 \to B \to C \to A \to 0$ with ideal $B$ and quotient algebra $A$ gives rise to an element of the extension group $\mathrm{Ext}_{\mathbb{R}}(A, B)$ and this group is related to the real Kasparov $KK$-groups.

**Proposition 2.3** ([27], §**7**). *If $A$ and $B$ are separable and nuclear real $C^*$-algebras, then*

$$\mathrm{Ext}_{\mathbb{R}}(A, B) \cong KKO(A\hat{\otimes}C\ell_{0,1}, B) \cong KKO(A, B\hat{\otimes}C\ell_{1,0}).$$

## 2.2. The product

The generality of the constructions and proofs in [27] mean that all the central results in complex $KK$-theory carry over into the real (and Real) setting. In particular, the intersection product

$$KKO(A, B) \times KKO(B, C) \to KKO(A, C)$$

is still a well-defined map and other important properties such as stability

$$KKO(A\hat{\otimes}\mathcal{K}(\mathcal{H}), B) \cong KKO(A, B)$$

continue to hold, where $\mathcal{K}(\mathcal{H})$ is the algebra of real compact operators on a separable real Hilbert space.

Let $(\mathcal{A}, E_B^1, D_1)$ and $(\mathcal{B}, E_C^2, D_2)$ be unbounded real $A$-$B$ and $B$-$C$ Kasparov modules. We would like to take the product at the unbounded level. Naively one would like to use the formula

$$(\mathcal{A}, (E^1 \hat{\otimes}_B E^2)_C, D_1 \hat{\otimes} 1 + 1 \hat{\otimes} D_2),$$

where $(E^1 \hat{\otimes}_B E^2)_C$ is the $B$-balanced $\mathbb{Z}_2$-graded tensor product. This does not make sense as $D_2$ is not $\mathcal{B}$-linear (nor $B$-linear) and so $1 \hat{\otimes} D_2$ does not descend to the balanced tensor product. Instead one needs to choose a connection on $E_B^1$ to correct the naive formula for $1 \hat{\otimes} \mathcal{D}_2$. First define the '$D_2$ one forms' by

$$\Omega_{D_2}^1(\mathcal{B}) = \Big\{ \sum_j b_j [D_2, c_j] \in \mathrm{End}_C(E^2) \, : \, b_j, \, c_j \in \mathcal{B} \Big\}.$$

Then a $D_2$ connection on $E^1$ is a choice of dense $\mathcal{B}$ submodule $\mathcal{E}^1 \subset E^1$ and a linear map

$$\nabla : \mathcal{E}^1 \to \mathcal{E}^1 \hat{\otimes}_{\mathcal{B}} \Omega_{D_2}^1(\mathcal{B}) \quad \text{such that} \quad \nabla(eb) = \nabla(e)b + e \hat{\otimes} [D_2, b].$$

Setting $m : \Omega_{D_2}^1(\mathcal{B}) \hat{\otimes} E^2 \to E^2$ to be $m(a[D_2, b] \hat{\otimes} f) = a[D_2, b]f$, we then define

$$(1 \hat{\otimes}_{\nabla} D_2)(e \hat{\otimes} f) = (-1)^{|e|} e \hat{\otimes} D_2 f + (1 \hat{\otimes} m)(\nabla(e) \hat{\otimes} f),$$

with $|e|$ the degree of $e$ in the $\mathbb{Z}_2$-graded module $E_B^1$. A short but illuminating calculation shows that $1 \hat{\otimes}_{\nabla} D_2$ is well-defined on the balanced tensor product, and it is reasonable to hope that the formula

$$D := D_1 \hat{\otimes} 1 + 1 \hat{\otimes}_{\nabla} D_2 \tag{1}$$

would define an operator on $E^1 \hat{\otimes}_B E^2$ such that $(\mathcal{A}, (E^1 \hat{\otimes}_B E^2)_C, D)$ represents the product class.

In fact this is true in very many cases as proved in [24, 40, 44, 46]. These papers provide very general settings where the formula (1) can be guaranteed to produce an unbounded Kasparov module representing the product. These proofs, all in the complex case, proceed by showing that the conditions of Kucerovsky's theorem are satisfied. We will not develop such a general framework, but concretely construct potential representatives according to the recipe in Equation (1), and check Kucerovsky's conditions directly. Importantly, Kucerovsky's theorem is valid in the real case.

To state Kucerovsky's conditions, we start by defining a creation operator. Given $e_1 \in E_B^1$ and a $*$-homomorphism $\psi : B \to \mathrm{End}_C(E^2)$, we let $T_{e_1} \in \mathrm{Hom}_C(E^2, E^1 \otimes_B E^2)$ be given by $T_{e_1} e_2 = e_1 \hat{\otimes} e_2$. One can check that $T_{e_1}$ is adjointable with $T_{e_1}^*(f_1 \hat{\otimes} e_2) = \psi((e_1 | f_1)_B) e_2$.

8

**Theorem 2.4 (Kucerovsky's criteria [37], Theorem 13).** *Let $(\mathcal{A},\, {}_{\pi_1}E^1_B, D_1)$ and $(\mathcal{B},\, {}_{\pi_2}E^2_C, D_2)$ be unbounded Kasparov modules. Write $E := E^1 \hat\otimes_B E^2$. Suppose that $(\mathcal{A},\, {}_{\pi_1}E_C, D)$ is an unbounded Kasparov module such that*

**Connection condition.** *For all $e_1$ in a dense subspace of $\pi_1(A)E^1$, the commutators*

$$\left[ \begin{pmatrix} D & 0 \\ 0 & D_2 \end{pmatrix}, \begin{pmatrix} 0 & T_{e_1} \\ T^*_{e_1} & 0 \end{pmatrix} \right]$$

*are bounded on $\mathrm{Dom}(D \oplus D_2) \subset E \oplus E^2$;*

**Domain condition.** $\mathrm{Dom}(D) \subset \mathrm{Dom}(D_1\hat\otimes 1)$;

**Positivity condition.** *For all $e \in \mathrm{Dom}(D)$,*

$$((D_1\hat\otimes 1)e | De) + (De | (D_1\hat\otimes 1)e) \geq K(e|e)$$

*for some $K \in \mathbb{R}$.*

*Then the class of $(\mathcal{A},\, {}_{\pi_1}E_C, D)$ in $KK(A,C)$ represents the Kasparov product.*

In fact, if $\nabla$ satisfies an extra Hermiticity condition (which can always be achieved), an operator of the form $D = D_1\hat\otimes 1 + 1\hat\otimes_\nabla D_2$ always satisfies the domain condition and connection condition. Under mild hypotheses, the operator $1\hat\otimes_\nabla D_2$ will be self-adjoint [46, Theorem 3.17] and the sum will have locally compact resolvent (that is, $\pi_1(a)(1 + D^2)^{-1/2}$ compact for $a \in \mathcal{A}$) [24, Theorem 6.7]. The self-adjointness of the sum and the boundedness of commutators $[D, \pi_1(a)]$ needs to be checked directly, as does the positivity condition. In our examples all these extra conditions are satisfied and so the task of checking that we have a spectral triple representing the product is relatively straightforward.

## 2.3. Semifinite theory

An unbounded $A$-$\mathbb{C}$ or $A$-$\mathbb{R}$ Kasparov module is precisely a complex or real spectral triple as defined by Connes. Complex spectral triples satisfying additional regularity properties have the advantage that the local index formula by Connes and Moscovici [14] gives computable expressions for the index pairing with $K$-theory, a special case of the Kasparov product

$$K_*(A) \times KK^*(A, \mathbb{C}) \to K_0(\mathbb{C}) \cong \mathbb{Z}.$$

We would like to find computable expressions for more general Kasparov products. To do this, we generalise the definition of spectral triple using more general semifinite von Neumann algebras in place of the bounded operators on Hilbert space.

Let $\tau$ be a fixed faithful, normal, semifinite trace on a von Neumann algebra $\mathcal{N}$. We let $\mathcal{K}_\mathcal{N}$ be the $\tau$-compact operators in $\mathcal{N}$ (that is, the norm closed ideal generated by the projections $P \in \mathcal{N}$ with $\tau(P) < \infty$).

**Definition 2.5 ([12]).** Let $\mathcal{N}$ be a graded semifinite von Neumann algebra with trace $\tau$. A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$, a graded $*$-algebra $\mathcal{A} \subset \mathcal{N}$ acting evenly on $\mathcal{H}$ and a densely defined odd unbounded self-adjoint operator $D$ affiliated to $\mathcal{N}$ such that

1. $[D, a]$ is well-defined on $\mathrm{Dom}(D)$ and extends to a bounded operator on $\mathcal{H}$ for all $a \in \mathcal{A}$,
2. $a(1 + D^2)^{-1/2} \in \mathcal{K}_\mathcal{N}$ for all $a \in \mathcal{A}$.

A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is $QC^\infty$ if for $\partial = [(1+D^2)^{1/2}, \cdot]$ and $b \in \mathcal{A} \cup [D, \mathcal{A}]$, $\partial^j(b) \in \mathcal{N}$ for all $j \in \mathbb{N}$.

A unital semifinite spectral triple is $p$-summable if $(1 + D^2)^{s/2}$ is $\tau$-trace-class for all $s > p$.

If we take $\mathcal{N} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{Tr}$, then we recover the usual definition of a spectral triple.

**Theorem 2.6 ([25]).** *Let $(\mathcal{A}, \mathcal{H}, D)$ be a complex semifinite spectral triple associated to $(\mathcal{N}, \tau)$ with $A$ the $C^*$-completion of $\mathcal{A}$. Then $(\mathcal{A}, \mathcal{H}, D)$ determines a class in $KK(A, C)$ with $C$ a separable $C^*$-subalgebra of $\mathcal{K}_\mathcal{N}$.*

Semifinite spectral triples can also be paired with $K$-theory elements by the following composition

$$K_0(A) \times KK(A, C) \to K_0(A) \xrightarrow{\tau_*} \mathbb{R}, \tag{2}$$

with the class in $KK(A, C)$ coming from Theorem 2.6. The semifinite index pairing is, in general, real-valued and as such can potentially detect finer invariants than the usual pairing. Importantly, the local index formula can be generalised to $QC^\infty$ and $p$-summable semifinite spectral triples for $p \geq 1$ [12, 13]. Hence the semifinite local index formula may be used to compute the map in Equation (2).

One may ask if the converse is true. That is, given an unbounded Kasparov $A$-$C$ module, can we construct a semifinite spectral triple? If the algebra $C$ possesses a faithful semifinite norm-lower semicontinuous trace, the answer is often yes (see for example [48]). If a sufficiently regular (complex) semifinite spectral triple can be constructed, then we may use the semifinite local index formula to compute the map given in Equation (2). Therefore semifinite spectral

10

triples and index theory can be employed in order to compute invariants of complex systems from pairings of $K$-theory classes with unbounded Kasparov modules.

*Remark* 2.7 (Real semifinite spectral triples and the local index formula). While the definition of a semifinite spectral triple and semifinite index pairing can be extended to real algebras, the semifinite local index formula can not be used to detect torsion invariants as it involves a mapping to cyclic cohomology. However, one may naturally ask whether the semifinite local index formula can be used to detect integer invariants in the real setting (e.g. arising from $K_0(\mathbb{R})$ or $K_4(\mathbb{R})$). We are of the opinion that a local formula can be employed to access real integer invariants as in [12, 13], though the details of the proof given for the complex case need to be checked for the real case.

## 3. The bulk-edge correspondence in $K$-homology

We consider the version of $K$-homology due to Kasparov in which the $K$-homology of the $C^*$-algebra $A$ arises as $KKO(A, \mathbb{R})$ or $(KK(A, \mathbb{C})$ in the complex case). It follows from Kasparov's work that any exact sequence of (trivially graded) $C^*$-algebras gives rise to a long exact sequence in $K$-homology whose boundary map is given by the internal Kasparov product with the extension class defined by the exact sequence. This is a very general statement which is in general not computable. In our case, however, we are able to compute the product and hence the boundary map on what we call fundamental classes. Our formula is very explicit and avoids homotopy arguments.

We apply this to the context of topological insulators. The fundamental classes do not depend on the symmetries. The Clifford algebra comes into play through the use of the Dirac operator and should be understood as a geometrical input. This will be different in Section 4 where Clifford algebras are used to encode the symmetries of an insulator.

### 3.1. The bulk algebra and its extension

The bulk algebra is the observable algebra of the solid seen as infinite and without boundary. It is also referred to as the noncommutative Brillouin zone and was developed in [5, 6]. We work in the tight binding approximation where the configuration space is $\mathbb{Z}^d$. In this case the bulk algebra is the crossed product algebra $C(\Omega, M_N) \rtimes_{\alpha.\theta} \mathbb{Z}^d$ of matrix-valued continuous functions over a compact space $\Omega$ by the group $\mathbb{Z}^d$.

The space $\Omega$ may be thought of as a space of disorder configurations. The disorder space is equipped with a topological structure and action $\alpha$ of $\mathbb{Z}^d$ by shift of the configuration. We also usually assume that $\Omega$ possesses a probability measure that is invariant and ergodic under the $\mathbb{Z}^d$-action to obtain expectation values. The space $\Omega$ could be taken to be a point, or contractible to a point. The latter point of view is taken in [51] but this excludes the interesting possibility of describing for instance quasicrystals. The algebra of $N \times N$ matrices, $M_N$, is used to incorporate internal degrees of freedom like spin. Whereas these internal components are important for the implementation of certain symmetries, for instance odd time reversal, they do not interfere with the main topological constructions which will follow and, in order not to overburden the notation we will suppress them.

An external magnetic field can be incorporated by a two cocycle $\theta : \mathbb{Z}^d \times \mathbb{Z}^d \to C(\Omega, S^1)$ to twist the action, but note that the existence of anti-linear symmetries puts restrictions on the form of such a magnetic field. Indeed, for systems with symmetries which are implemented by anti-linear operators we are bound to look at a real sub-algebra of the above algebra. In this case we consider therefore $C(\Omega)$ as the algebra of real valued continuous functions and exclude the possibility of complex-valued twisting cocycles. The presence of a boundary will impose further constraints on any possible cocycle, and we examine this below.

It will be useful to recall some details on crossed products. Let $B$ be a unital $C^*$-algebra with action $\alpha$ of $\mathbb{Z}^d$ and twisting cocycle $\theta : \mathbb{Z}^d \times \mathbb{Z}^d \to B$ (with values in the unitaries of the center of $B$). The twisted crossed product $A = B \rtimes_{\alpha,\theta} \mathbb{Z}^d$ is the universal $C^*$-completion of the algebraic crossed product $\mathcal{A} := B_{\alpha,\theta}\mathbb{Z}^d$ given by finite sums $\sum_{n \in \mathbb{Z}^d} b_n S^n$ where $b_n \in B$, $n \in \mathbb{Z}^d$ is a multi-index and $S^n = S_1^{n_1} \cdots S_d^{n_d}$ is a product of powers of $d$ abstract unitary elements $S_i$ subject to the multiplication extending that of $B$ by

$$S_i b = \alpha_i(b) S_i, \qquad S_i S_j = \theta_{ij} S_j S_i, \qquad S_i^* = S_i^{-1}.$$

The map $\alpha_i$ is the automorphism corresponding to the action of $e_i \in \mathbb{Z}^d$ for $e_i$ are the standard generators and the elements $\theta_{ij}$ belong to $B$ and can be obtained from the cocycle $\theta$.

Let us consider the case $d = 1$ in which $\theta = 1$. Given a left action $\rho$ of $B$ on a module $M$ one obtains a left action $\pi$ of $B_\alpha \mathbb{Z}$ on the module given by the algebraic tensor product $\ell^2(\mathbb{Z}) \otimes M$. The action $\pi$ is given on elementary tensors by

$$\pi(b)(\delta_j \otimes \xi) = \delta_j \otimes \rho(\alpha^{-j}(b))\xi, \qquad \pi(S)(\delta_n \otimes \xi) = \delta_{n+1} \otimes \xi \quad (3)$$

with $\{\delta_j\}_{j\in\mathbb{Z}}$ the standard basis of $\ell^2(\mathbb{Z})$ and $\xi \in M$. We will be interested in the case where $M$ is a Hilbert module so that we can expect the action to extend to an action of the $C^*$-crossed product $B \rtimes_\alpha \mathbb{Z}$ on the completion of the algebraic tensor product $\ell^2(\mathbb{Z}) \otimes M$ which we denote by $\ell^2(\mathbb{Z}, M)$.

The above procedure can be iterated and hence yields a representation of $C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d$, if it can be rewritten as an iterated crossed product. This is always that case if we allow $C(\Omega)$ to be replaced by its stabilisation [47] but the induction principle may also be modified to yield directly a representation of $C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d$ [47].

Of importance for our application to physics is the family of representations $\{\pi_\omega\}_{\omega\in\Omega}$ which are induced by the evaluation representations $\mathrm{ev}_\omega : C(\Omega) \to \mathbb{R}$ (or $\mathbb{C}$). These $\pi_\omega$ are thus representations of $C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d$ on $\ell^2(\mathbb{Z}^d)$. They look as follows

$$\pi_\omega(f)\delta_n = f(\alpha^n(\omega))\delta_n, \qquad \pi_\omega(S_i)\delta_n = \theta(n, e_i)(\omega)\delta_{n+e_i} \qquad (4)$$

with $f \in C(\Omega)$. A tight binding Hamiltonian is a self-adjoint element of $\mathcal{A}$,

$$h = \sum_{n\in\mathbb{Z}^d} v_n S^n \qquad (5)$$

where the functions $v_n \in C(\Omega)$ are chosen such that the expression is self-adjoint and the sum has only finitely many non zero terms (if we take internal degrees of freedom into account then the $v_n$ are matrix valued). The representations $\pi_\omega$ give rise to a family of Hamiltonians $H_\omega = \pi_\omega(h)$ which are unitarily equivalent for points $\omega$ in the same orbit.

It is important that the twisting cocycle $\theta$ can be arranged in such a way that $S_i S_d = S_d S_i$ for all $i \in \{1, \dots, d\}$. This implies that we can rewrite the bulk algebra as a crossed product by $\mathbb{Z}$ (where $\alpha^\parallel$ is the action of $\mathbb{Z}^{d-1}$ on $C(\Omega)$ given by restricting $\alpha$)

$$C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d \cong \left(C(\Omega) \rtimes_{\alpha^\parallel,\theta} \mathbb{Z}^{d-1}\right) \rtimes_{\alpha_d} \mathbb{Z}.$$

This allows us to view the bulk algebra as a quotient algebra of a Toeplitz extension.

We quickly recall the Toeplitz extension for a real or complex unital $C^*$-algebra $B$ with $\mathbb{Z}$-action $\alpha$. Very similar to the construction of the crossed product $B \rtimes_\alpha \mathbb{Z}$ we can consider $B_\alpha\mathbb{N}$ the algebra given by finite sums $\sum_{k\in\mathbb{N}} \tilde{S}^k b_k + (\tilde{S}^*)^k c_k$, where $b_k, c_k \in B$ and $\tilde{S}^k$ and $(\tilde{S}^*)^k$ are $k$-th powers of abstract elements and multiplication is determined by

$$\tilde{S}b = \alpha(b)\tilde{S}, \qquad \tilde{S}^*b = \alpha^{-1}(b)\tilde{S}, \qquad \tilde{S}^*\tilde{S} = 1, \qquad \tilde{S}\tilde{S}^* = 1 - p$$

with $p = p^* = p^2$ a projection. Thus $\tilde{S}$ is no longer unitary but an isometry. There is a unique $*$-algebra morphism $q : B_\alpha \mathbb{N} \to B_\alpha \mathbb{Z}$ determined by $q(\tilde{S}) = S$ which is the identity on the subalgebra $B$. Its kernel is the ideal generated by $p$ which can easily be seen to be isomorphic to $F \otimes B$ where $F$ is the algebra of the finite rank operators. The exact sequence

$$0 \to F \otimes B \to B_\alpha \mathbb{N} \xrightarrow{q} B_\alpha \mathbb{Z} \to 0$$

is the algebraic version of the Toeplitz extension, the $C^*$-version is obtained by taking the universal $C^*$-closures. The $C^*$-closure of $B_\alpha \mathbb{N}$, denoted by $\mathcal{T}(\alpha)$, is the Toeplitz algebra of the $\mathbb{Z}$-action $\alpha$ and the closure of $F \otimes B$ is the stabilisation of $B$, $\mathcal{K} \otimes B$ with $\mathcal{K}$ the algebra of complex operators on a separable (real or complex) Hilbert space.

As for crossed products, given a left action $\rho$ of $B$ on a module $M$ one obtains a left action $\tilde{\pi}$ of $B_\alpha \mathbb{N}$ on $\ell^2(\mathbb{N}) \otimes M$ by

$$\tilde{\pi}(b)(\delta_n \otimes \xi) = \delta_n \otimes \rho(\alpha^{-n}(b))\xi, \qquad \tilde{\pi}(\tilde{S})(\delta_n \otimes \xi) = \delta_{n+1} \otimes \xi,$$

where $\tilde{\pi}(\tilde{S}^*)(\delta_n \otimes \xi) = \delta_{n-1} \otimes \xi$ and we set $\delta_n = 0$ for $n < 0$.

## 3.2. The edge algebra

The Toeplitz algebra $\mathcal{T}(\alpha)$ has a physical interpretation which we now explain. Recall that the bulk algebra is of the form $\mathcal{A} = \mathcal{B}_{\alpha_d} \mathbb{Z}$ where $\mathcal{B} = C(\Omega)_{\alpha^{\parallel}, \theta} \mathbb{Z}^{d-1}$. Starting with the evaluation representation $\text{ev}_\omega : C(\Omega) \to \mathbb{R}$ (or $\mathbb{C}$) we obtain a representation of $\mathcal{B}$ on $\ell^2(\mathbb{Z}^{d-1})$. This representation, in turn, induces a representation of $\mathcal{B}_\alpha \mathbb{N}$ on $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z}^{d-1})$. The latter is dense in $\ell^2(\mathbb{Z}^{d-1} \times \mathbb{N})$ and $\mathbb{Z}^{d-1} \times \mathbb{N}$ is the configuration space for the insulator on half-space, that is, the insulator seen as infinite with a boundary, the so-called edge, which is at $\mathbb{Z}^{d-1} \times \{0\}$. The representation of $\mathcal{B}_{\alpha_d} \mathbb{N}$ looks away from the edge like that of $\mathcal{A} = \mathcal{B}_{\alpha_d} \mathbb{Z}$, but at the edge the translation $S_d$ in the perpendicular direction is truncated. The algebra $\mathcal{B}_{\alpha_d} \mathbb{N}$, or rather its closure $\mathcal{T}(\alpha)$ can therefore be seen as the observable algebra of the insulator with boundary.

Recall that the kernel of the quotient map is the ideal generated by the projection $p$. In the evaluation representation this projection becomes the projection onto the subspace $\mathbb{C} \otimes \ell^2(\mathbb{Z}^{d-1}) = \ell^2(\mathbb{Z}^{d-1} \times \{0\})$, and hence is naturally associated to the edge. Thus the elements of $F \otimes \mathcal{B}$ are represented as operators which are localised near the edge. The ideal of the Toeplitz extension $F \otimes \mathcal{B}$ (and its $C^*$-closure $\mathcal{K} \otimes B$) can therefore be referred to as the (stabilised) edge algebra.

14

### 3.3. Fundamental $K$-cycles for $\mathbb{Z}^d$-actions

Consider a real or complex (possibly twisted) crossed product $A = B \rtimes_{\alpha,\theta} \mathbb{Z}^d$ with its algebraic version $\mathcal{A} = B_{\alpha,\theta}\mathbb{Z}^d$, which is spanned by finite sums $\sum_n S^n b_n$. Our task is to construct an unbounded Kasparov module encoding the $\mathbb{Z}^d$-action. We remark that similar constructions appear in [23, 48, 50] in the complex setting.

We consider the representation of the crossed product $B \rtimes_{\alpha,\theta} \mathbb{Z}^d$ on the Hilbert module $\ell^2(\mathbb{Z}^d) \otimes B =: \ell^2(\mathbb{Z}^d, B)$ obtained by iteration of (3) associated with the left regular representation of $B$ on $B$ (by left multiplication). More precisely, the map $\imath : B_{\alpha,\theta}\mathbb{Z}^d \to \ell^2(\mathbb{Z}^d) \otimes B$, $\imath(S^n b) = \delta_n \otimes b$ is an inclusion of the dense subalgebra $\mathcal{A}$ into the Hilbert module such that the representation of $\mathcal{A}$ on the image of $\imath$ is given by left multiplication:

$$S^n b_1 \cdot (\delta_m \otimes b_2) = \imath(S^n b_1 S^m b_2) = \delta_{n+m} \otimes \alpha^{-m-n}(\theta(m,n))\alpha^{-m}(b_1)b_2$$

The Hilbert module is the completion of $\imath(\mathcal{A})$ w.r.t. the $B$-valued inner product

$$(\lambda_1 \otimes b_1 \mid \lambda_2 \otimes b_2)_B = \langle \lambda_1, \lambda_2 \rangle_{\ell^2(\mathbb{Z}^d)} b_1^* b_2,$$

and norm $\|\xi\|^2 = \|(\xi \mid \xi)_B\|_B$. The $C^*$-module also carries a right-action of $B$ by right-multiplication. Note that on $\imath(\mathcal{A})$ the inner product may be written as

$$(\imath(a_1) \mid \imath(a_2))_B = \Phi_0(a_1^* a_2)$$

Where $\Phi_0 : A \to B$, $\Phi_0(S^n b) = \delta_{n0} b$ is the usual conditional expectation. This expectation being positive we have $\Phi_0(b^* ab) \leq \|a\|_A \Phi_0(b^* b)$ for any positive $a \in A$ and $b \in B$.

**Proposition 3.1.** *The left-action of $\mathcal{A}$ on $\ell^2(\mathbb{Z}^d, B)$ extends to an adjointable representation of $B \rtimes_{\alpha,\theta} \mathbb{Z}^d$.*

*Proof.* We first check adjointability of the elements of $\mathcal{A}$ acting on $\imath(\mathcal{A})$ by computing

$$(a \cdot \imath(a_1) \mid \imath(a_2))_B = \Phi_0((aa_1)^* a_2)$$
$$= \Phi_0(a_1 a^* a_2) = (\imath(a_1) \mid a^* \cdot \imath(a_2))_B$$

Furthermore, the action of $\mathcal{A}$ on $\imath(\mathcal{A})$ is uniformly bounded:

$$\|a\|_{\mathrm{End}}^2 = \| \sup_{\substack{a' \in \mathcal{A} \\ \|\imath(a')\|=1}} (a \cdot \imath(a') \mid a \cdot \imath(a'))_B\|$$
$$\leq \sup_{\substack{a' \in \mathcal{A} \\ \|\imath(a')\|=1}} \|a^* a\|_A \|\Phi_0(a'^* a')\| = \|a^* a\|_A.$$

15

Hence the action extends by continuity, first from $\mathcal{A}$ on $\iota(\mathcal{A})$ to a bounded action of $\mathcal{A}$ on $\ell^2(\mathbb{Z}^d, B)$ and then to a bounded action of the whole $C^*$-algebra $A$ on $\ell^2(\mathbb{Z}^d, B)$. It then follows that all elements of $A$ are adjointable. $\qquad\square$

Inside $\ell^2(\mathbb{Z}^d, B)$ we consider the elements $e_m = \delta_m \otimes 1_B$ for $m \in \mathbb{Z}^d$. We compute that for $b \in B$

$$\Theta_{e_l, e_m} e_n b = \Theta_{e_l, e_m}(\delta_n \otimes b) = (\delta_l \otimes 1_B) \cdot (\delta_m \otimes 1_B \mid \delta_n \otimes b)_B$$
$$= \delta_{m,n} \delta_l \otimes b = \delta_{m,n} e_l \cdot b.$$

In particular we note that

$$\sum_{m \in \mathbb{Z}^d} \Theta_{e_m, e_m} = \mathrm{Id}_{\ell^2(\mathbb{Z}^d, B)}$$

and so $\{e_m\}_{m \in \mathbb{Z}^d}$ is a frame for the module $\ell^2(\mathbb{Z}^d, B)$.

The last ingredient we need for a Kasparov module is a Dirac-like operator, which we construct using the position operators, $X_j :$ $\mathrm{Dom}(X_j) \to \ell^2(\mathbb{Z}^d, B)$ for $j \in \{1, \ldots, d\}$ such that $X_j(\delta_m \otimes b) = m_j(\delta_m \otimes b)$ for any $m \in \mathbb{Z}^d$. We construct the Dirac-like operator by employing an irreducible Clifford representation. On the graded vector space $\bigwedge^* \mathbb{R}^d$ (we denote the grading by $\gamma_{\bigwedge^* \mathbb{R}^d}$) there is a representation of $C\ell_{d,0}$ and a representation of $C\ell_{0,d}$. The generators $\gamma^j$ of $C\ell_{d,0}$ and the generators $\rho^j$ of $C\ell_{0,d}$ act by

$$\gamma^j(w) = e_j \wedge w + \iota(e_j)w, \qquad \rho^j(w) = e_j \wedge w - \iota(e_j)w,$$

for $\{e_j\}_{j=1}^d$ the standard basis of $\mathbb{R}^d$, $w \in \bigwedge^* \mathbb{R}^d$ and $\iota(v)w$ the contraction of $w$ along $v$. These two actions graded-commute. On the tensor product space $\ell^2(\mathbb{Z}^d, B) \otimes \bigwedge^* \mathbb{R}^d$ we define

$$D := \sum_{j=1}^d X_j \otimes \gamma^j.$$

**Proposition 3.2.** *Consider a possibly twisted $\mathbb{Z}^d$-action $\alpha, \theta$ on a $C^*$-algebra $B$. Let $A$ be the associated crossed product with dense subalgebra $\mathcal{A} = B_{\alpha,\theta}\mathbb{Z}^d$. The data*

$$\lambda^{(d)} = \left( \mathcal{A}\hat{\otimes}C\ell_{0,d}, \ell^2(\mathbb{Z}^d, B)_B \otimes \bigwedge^* \mathbb{R}^d, \sum_{j=1}^d X_j \otimes \gamma^j, \gamma_{\bigwedge^* \mathbb{R}^d} \right)$$

*defines an unbounded $\mathcal{A}\hat{\otimes}C\ell_{0,d}$-$B$ Kasparov module. The $C\ell_{0,d}$-action is generated by the operators $\rho^j$. In the complex case we have $\mathbb{C}$ in place of $\mathbb{R}$ in the above formula.*

*Proof.* The left-action of $\mathcal{A}$ is adjointable by Proposition 3.1. By construction $D$ graded-commutes with the left Clifford representation. What remains to be checked is that $[D, a] = \sum_j [X_j, a] \otimes \gamma^j$ is adjointable for $a \in \mathcal{A}$ and $(1+D^2)^{-1/2}$ is compact. It is directly verified that

$$[X_j, S^n b] = n_j S^n b$$

and so we see that $[X_j, a] \in \mathcal{A}$ for all $a \in \mathcal{A}$. In particular the commutator is an adjointable operator. Furthermore, $D^2 = \sum_j X_j^2 \otimes 1$ and hence $(1+D^2)^{-1/2} = (1+|X|^2)^{-1/2} \otimes 1$, where $|X|^2 = \sum_j X_j^2$. Using the frame $\{e_m\}_{m \in \mathbb{Z}^d}$, we note that $(1+|X|^2)e_m = (1+|m|^2)e_m$ and so

$$(1 + D^2)^{-\frac{1}{2}} = \sum_{m \in \mathbb{Z}^d} (1 + |m|^2)^{-\frac{1}{2}} \Theta_{e_m, e_m} \otimes 1,$$

which is a norm-convergent sum of finite-rank operators and so is compact. $\qquad\square$

### 3.4. The fundamental $K$-cycles for the bulk and the edge

We apply the construction of Proposition 3.2 to $A_b = C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d$ and to $A_e = C(\Omega) \rtimes_{\alpha^\|,\theta} \mathbb{Z}^{d-1}$ to obtain the fundamental class for the bulk and for the edge, respectively. We let $\mathcal{A}_b = C(\Omega)_{\alpha,\theta} \mathbb{Z}^d$ and $\mathcal{A}_e = C(\Omega)_{\alpha^\|,\theta} \mathbb{Z}^{d-1}$ and obtain

$$\lambda_b = \left( \mathcal{A}_b \hat{\otimes} C\ell_{0,d}, \ell^2(\mathbb{Z}^d, C(\Omega))_{C(\Omega)} \otimes \bigwedge\nolimits^* \mathbb{R}^d, D_b, \gamma_{\bigwedge^* \mathbb{R}^d} \right), \quad (6)$$

with $D_b = \sum_{j=1}^k X_j \otimes \gamma^k$. We call $\lambda_b$ the fundamental $K$-cycle of the bulk. Similarly we call $\lambda_e$ the fundamental $K$-cycle of the edge given by

$$\left( \mathcal{A}_e \hat{\otimes} C\ell_{0,d-1}, \ell^2(\mathbb{Z}^{d-1}, C(\Omega))_{C(\Omega)} \otimes \bigwedge\nolimits^* \mathbb{R}^{d-1}, D_e, \gamma_{\bigwedge^* \mathbb{R}^{d-1}} \right), \quad (7)$$

with $D_e = \sum_{j=1}^{d-1} X_j \otimes \gamma^j$. These are unbounded representatives for elements in $KKO^d(A, C(\Omega))$ and $KKO^{d-1}(B, C(\Omega))$ (in the real case, otherwise it's $KK$).

### 3.5. The extension module

We recall that given a twisted crossed product $A = B \rtimes_{\alpha,\theta} \mathbb{Z}^d$, we can unwind the $\mathbb{Z}^d$ action to write $A = \left( B \rtimes_{\alpha^\|,\theta} \mathbb{Z}^{d-1} \right) \rtimes_{\alpha_d} \mathbb{Z}$. We can then construct the Toeplitz algebra $\mathcal{T}(\alpha_d)$ and short exact sequence

$$0 \to \mathcal{K} \otimes B \rtimes_{\alpha^\|,\theta} \mathbb{Z}^{d-1} \to \mathcal{T}(\alpha_d) \to \left( B \rtimes_{\alpha^\|,\theta} \mathbb{Z}^{d-1} \right) \rtimes_{\alpha_d} \mathbb{Z} \to 0. \quad (8)$$

We can associate the algebras $B \rtimes_{\alpha,\theta} \mathbb{Z}^d$ and $\mathcal{K} \otimes B \rtimes_{\alpha^\|,\theta} \mathbb{Z}^{d-1}$ to bulk and edge systems respectively. Hence we denote $A_e = B \rtimes_{\alpha^\|,\theta} \mathbb{Z}^{d-1}$ and $A_b = B \rtimes_{\alpha,\theta} \mathbb{Z}^d \cong A_e \rtimes_{\alpha_d} \mathbb{Z}$.

17

Equation (8) gives rise to a class [ext] in the extension group $\mathrm{Ext}(A_b, A_e)$, which by Proposition 2.3 is the same as the group $KKO(A_b \hat{\otimes} C\ell_{0,1}, A_e)$ (or $KK(A_b \hat{\otimes} \mathbb{C}\ell_1, A_b)$ if complex).

We compute boundary maps in $K$-theory and $K$-homology by taking the Kasparov product with the class [ext] representing the short exact sequence of Equation (8) and identified as an element of $KKO(A_b \hat{\otimes} C\ell_{0,1}, A_e)$. In order to make boundary maps computable, we require an unbounded representative of this class.

**Proposition 3.3.** *Let $B$ be a unital $C^*$-algebra and $A = B \rtimes_\alpha \mathbb{Z}$. The extension class of the Toeplitz extension of the $\mathbb{Z}$-action is represented by the fundamental $K$-cycle of the $\mathbb{Z}$-action,*

$$\left( B_\alpha \mathbb{Z} \hat{\otimes} C\ell_{0,1}, \, \ell^2(\mathbb{Z}, B)_B \otimes \bigwedge\nolimits^* \mathbb{R}, \, D = X_1 \otimes \gamma^1, \gamma_{\bigwedge^* \mathbb{R}} \right). \quad (9)$$

*There is an analogous result for complex algebras.*

It serves the clarity of arguments further down to denote $D = N \otimes \gamma_{\mathrm{ext}}$ for the extension module.

*Proof.* The above module defines the same $KK$-class as the bounded cycle

$$\left( \mathcal{A}, \, \ell^2(\mathbb{Z}, B)_B, \, 2P - 1 \right)$$

where $P = \chi_{[0,\infty)}(N)$. From [53, Proposition 3.14] we know that this class is the extension class of the short exact sequence

$$0 \to \mathrm{End}_B^0[P(\ell^2(\mathbb{Z}, B))] \to C^*(PB_\alpha \mathbb{Z}P, \, \mathrm{End}_B^0[P(\ell^2(\mathbb{Z}, B))])$$
$$\to Q \to 0, \quad (10)$$

where $C^*(PB_\alpha \mathbb{Z}P, \, \mathrm{End}_B^0[P(\ell^2(\mathbb{Z}, B))])$ is a closed subalgebra of the adjointable operators, $\mathrm{End}_B[P(\ell^2(\mathbb{Z}, B))]$, and $Q$ the quotient. Now $P$ is the projection onto $\ell^2(\mathbb{N}, B)$ and hence $PB_\alpha \mathbb{Z}P = B_\alpha \mathbb{N}$. Moreover, using [7, §13]

$$\mathrm{End}_B^0[P(\ell^2(\mathbb{Z}, B))] \cong \mathcal{K}(\ell^2(\mathbb{N})) \otimes B.$$

Hence $C^*(PB_\alpha \mathbb{Z}P, \, \mathrm{End}_B^0[P(\ell^2(\mathbb{Z}, B))])$ is the completion of $B_\alpha \mathbb{N}$ and $Q$ is the completion of $B_\alpha \mathbb{Z}$ as required. $\qquad \square$

Applying Proposition 3.3 to Equation (8), we have the unbounded representative of the extension class,

$$\left( \mathcal{A}_b \hat{\otimes} C\ell_{0,1}, \, \ell^2(\mathbb{Z}, A_e)_{A_e} \otimes \bigwedge\nolimits^* \mathbb{R}, \, N \otimes \gamma_{\mathrm{ext}}, \gamma_{\bigwedge^* \mathbb{R}} \right)$$

with $A_e = B \rtimes_{\alpha^\parallel, \theta} \mathbb{Z}^{d-1}$ and $A_b = A_e \rtimes_{\alpha_d} \mathbb{Z}$. The Clifford actions of $C\ell_{0,1}$ and $C\ell_{1,0}$ on $\bigwedge^* \mathbb{R}$ are generated by the operators

$$\rho_{\mathrm{ext}}(w) = e \wedge w - \iota(e)w, \qquad \gamma_{\mathrm{ext}}(w) = e \wedge w + \iota(e)w,$$

18

respectively, where $e$ is the basis element of $\mathbb{R}$.

### 3.6. From Kasparov module to spectral triple

Recall the map $\mathrm{ev}_\omega : C(\Omega) \to \mathbb{F}$ (for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), which gives a family of representations $\{\pi_\omega\}_{\omega \in \Omega}$ of the crossed product $C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d$ on $\ell^2(\mathbb{Z}^d)$ subject to the covariance condition, $S^n \pi_\omega(a)(S^*)^n = \pi_{\alpha^n(\omega)}(a)$. The representations $\pi_\omega$ were used in [9] to define a real or complex spectral triple

$$\lambda_b(\omega) = \left( \mathcal{A} \hat{\otimes} C\ell_{0,d}, \ _{\pi_\omega} \ell^2(\mathbb{Z}^d) \otimes \bigwedge{}^* \mathbb{R}^d, \sum_{j=1}^d X_j \otimes \gamma^j, \gamma_{\bigwedge^* \mathbb{R}^d} \right). \quad (11)$$

The spectral triple gives a $K$-homology class for the crossed-product algebra, which by the covariance relation is independent under a fixed $\alpha$-orbit of $\Omega$.

Let us link the bulk spectral triple of Equation (11) to the fundamental $K$-cycle studied in Proposition 3.2. We take the trivially graded Kasparov module

$$\mathrm{ev}_\omega = (C(\Omega), \ _{\mathrm{ev}_\omega} \mathbb{F}, 0)$$

with $\mathrm{ev}_\omega$ the evaluation representation of $C(\Omega)$. The Kasparov module $\mathrm{ev}_\omega$ represents a class in $KKO(C(\Omega), \mathbb{R})$ (or $KK(C(\Omega), \mathbb{C})$) that can be paired with our fundamental $K$-cycle.

**Proposition 3.4.** *The spectral triple $\lambda_b(\omega)$ is unitarily equivalent to the internal product of the unbounded bulk Kasparov module $\lambda_b$ with $\mathrm{ev}_\omega$.*

*Proof.* Because the evaluation Kasparov module is very simple, we can easily compute the internal product

$$\left( \mathcal{A} \hat{\otimes} C\ell_{0,d}, \ \ell^2(\mathbb{Z}^d, C(\Omega)) \otimes \bigwedge{}^* \mathbb{R}^d, \sum_{j=1}^d X_j \otimes \gamma^j, \gamma_{\bigwedge^* \mathbb{R}^d} \right)$$

$$\hat{\otimes}_{C(\Omega)} (C(\Omega), \ _{\mathrm{ev}_\omega} \mathbb{F}, 0)$$

$$\cong \left( \mathcal{A} \hat{\otimes} C\ell_{0,d}, \ \mathcal{H} \otimes \bigwedge{}^* \mathbb{R}^d, \sum_{j=1}^d X_j \otimes 1 \otimes \gamma^j, \gamma_{\bigwedge^* \mathbb{R}^d} \right)$$

with $\mathcal{H} = \ell^2(\mathbb{Z}^d, C(\Omega)) \otimes_{\mathrm{ev}_\omega} \mathbb{F}$. We identify $\ell^2(\mathbb{Z}^d, C(\Omega)) \otimes_{\mathrm{ev}_\omega} \mathbb{F}$ with $\ell^2(\mathbb{Z}^d)$ under which $X_j \otimes 1 \mapsto X_j$ and the left-action of $\mathcal{A}$ takes the form

$$S^m g \cdot \delta_n = \theta(n, m) g(\alpha^n(\omega)) \delta_{n+m} = \pi_\omega(S^m g) \delta_n.$$

Hence we recover $\lambda_b(\omega)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

### 3.7. The product

Recall that for the case $B = C(\Omega)$, the bulk and boundary algebra are related by a Toeplitz extension.

$$0 \to \mathcal{K} \otimes C(\Omega) \rtimes_{\alpha^{\parallel},\theta} \mathbb{Z}^{d-1} \to \mathcal{T}(\alpha_d) \to C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d \to 0.$$

Proposition 3.3 gives us an unbounded Kasparov module that represents this short exact sequence. We now state the most important result of the paper, namely that the Kasparov product of the extension module with the fundamental edge module $\lambda_e$ gives, up to a cyclic permutation of the Clifford generators, the fundamental bulk module. The product with the extension class is the boundary map arising from the bulk-edge short exact sequence and the corresponding long exact sequence in $KK$-theory. Our result is actually a special case of the following theorem.

**Theorem 3.5.** *Let $B$ be a unital real or complex $C^*$-algebra with fundamental $K$-cycles $\lambda^{(d)}$ and $\lambda^{(d-1)}$ for (possibly twisted) $\mathbb{Z}^d$ and $\mathbb{Z}^{d-1}$-actions. Then the unbounded Kasparov product of the extension Kasparov module from Proposition 3.3 with $\lambda^{(d-1)}$ gives, up to a cyclic permutation of the Clifford generators, the fundamental module $\lambda^{(d)}$. On the level of $KK$-classes this means*

$$[\text{ext}]\hat{\otimes}[\lambda^{(d-1)}] = (-1)^{d-1}[\lambda^{(d)}],$$

*where $-[x]$ denotes the inverse of the $KK$-class.*

*Proof.* We will focus on the real setting though note that the case of complex algebras and modules follows the same argument. We denote by $A_e = B \rtimes_{\alpha^{\parallel},\theta} \mathbb{Z}^{d-1}$ and $A_b = B \rtimes_{\alpha,\theta} \mathbb{Z}^d \cong A_e \rtimes_{\alpha_d} \mathbb{Z}$ with dense subalgebras $\mathcal{A}_e$ and $\mathcal{A}_b$. We are taking the product of an $A_b \hat{\otimes} C\ell_{0,1}$-$A_e$ module with an $A_e \hat{\otimes} C\ell_{0,d-1}$-$B$ module. Our first step is an external product of the $A_b \hat{\otimes} C\ell_{0,1}$-$A_e$ module with the identity class in $KKO(C\ell_{0,d-1}, C\ell_{0,d-1})$. This class can be represented by the Kasparov module

$$\left( C\ell_{0,d-1}, (C\ell_{0,d-1})_{C\ell_{0,d-1}}, 0, \gamma_{C\ell_{0,d-1}} \right)$$

with right and left actions given by right and left Clifford multiplication. The external product results in the $A_b \hat{\otimes} C\ell_{0,d}$-$A_e \hat{\otimes} C\ell_{0,d-1}$ module represented by

$$\left( \mathcal{A}_b \hat{\otimes} C\ell_{0,d}, \left( \ell^2(\mathbb{Z}, A_e) \otimes \bigwedge\nolimits^* \mathbb{R} \hat{\otimes} C\ell_{0,d-1} \right)_{A_e \hat{\otimes} C\ell_{0,d-1}}, \right.$$

$$\left. N \otimes \gamma_{\text{ext}} \hat{\otimes} 1, \gamma_{\bigwedge^* \mathbb{R}} \hat{\otimes} \gamma_{C\ell_{0,d-1}} \right).$$

We now take the internal product of this module with the $A_e \hat\otimes C\ell_{0,d-1}$-$B$-module representing the edge. We start with the $C^*$-modules, where

$$\left(\ell^2(\mathbb{Z}, A_e) \otimes_\mathbb{R} \bigwedge\nolimits^* \mathbb{R} \hat\otimes_\mathbb{R} C\ell_{0,d-1}\right) \hat\otimes_{A_e \hat\otimes C\ell_{0,d-1}}$$

$$\left(\ell^2(\mathbb{Z}^{d-1}, B) \otimes_\mathbb{R} \bigwedge\nolimits^* \mathbb{R}^{d-1}\right)$$

$$\cong \left(\ell^2(\mathbb{Z}, A_e) \otimes_{A_e} \ell^2(\mathbb{Z}^{d-1}, B)\right) \otimes_\mathbb{R} \bigwedge\nolimits^* \mathbb{R} \hat\otimes_\mathbb{R} \left(C\ell_{0,d-1} \cdot \bigwedge\nolimits^* \mathbb{R}^{d-1}\right)$$

$$\cong \left(\ell^2(\mathbb{Z}, A_e) \otimes_{A_e} \ell^2(\mathbb{Z}^{d-1}, B)\right) \otimes_\mathbb{R} \bigwedge\nolimits^* \mathbb{R} \hat\otimes_\mathbb{R} \bigwedge\nolimits^* \mathbb{R}^{d-1}$$

as the action of $C\ell_{0,d-1}$ on $\bigwedge^* \mathbb{R}^{d-1}$ by left-multiplication is nondegenerate.

Next we define $1 \otimes_\nabla X_j$ on the dense submodule $\ell^2(\mathbb{Z}, \mathcal{A}_e) \otimes_{\mathcal{A}_e} \ell^2(\mathbb{Z}^{d-1}, B)$ for all $j \in \{1, \ldots, d-1\}$, where

$$(1 \otimes_\nabla X_j)(\delta_k \otimes a_e \otimes \delta_n \otimes b)$$
$$= \delta_k \otimes a_e \otimes X_j \delta_n \otimes b + \delta_k \otimes 1 \otimes [X_j, a_e]\delta_n \otimes b \quad (12)$$

with $\{\delta_k\}_{k\in\mathbb{Z}}$ an orthonormal basis of $\ell^2(\mathbb{Z})$, $a_e \in \mathcal{A}_e$, $\{\delta_n\}_{n\in\mathbb{Z}^{d-1}}$ an orthonormal basis for $\ell^2(\mathbb{Z}^{d-1})$ and $b \in B$. The operator is well-defined as $[X_j, a_e] \in \mathcal{A}_e$ for $a_e \in \mathcal{A}_e$. Then

$$\left(\mathcal{A}_b \hat\otimes C\ell_{0,1} \hat\otimes C\ell_{0,d-1}, \left(\ell^2(\mathbb{Z}, A_e) \otimes_{A_e} \ell^2(\mathbb{Z}^{d-1}, B)\right)\right. \quad (13)$$

$$\left. \otimes \bigwedge\nolimits^* \mathbb{R} \hat\otimes_\mathbb{R} \bigwedge\nolimits^* \mathbb{R}^{d-1}, \ N \otimes 1 \otimes \gamma_{\text{ext}} \hat\otimes 1 + \sum_{j=1}^{d-1}(1 \otimes_\nabla X_j) \otimes 1 \hat\otimes \gamma^j\right)$$

is a candidate for the unbounded product module, where the Clifford actions take the form

$$\rho_{\text{ext}} \hat\otimes 1(\omega_1 \hat\otimes \omega_2) = (e_1 \wedge \omega_1 - \iota(e_1)\omega_1) \hat\otimes \omega_2$$
$$1 \hat\otimes \rho^j(\omega_1 \hat\otimes \omega_2) = (-1)^{|\omega_1|}\omega_1 \hat\otimes (e_j \wedge \omega_2 - \iota(e_j)\omega_2),$$

for $j \in \{1, \ldots, d-1\}$ with $|\omega_1|$ is the degree of the form $\omega_1$. Similar equations hold for $\gamma_{\text{ext}} \hat\otimes 1$ and $1 \hat\otimes \gamma^j$. Arguments very similar to the proof of Proposition 3.2 show that Equation (13) is a real or complex Kasparov module depending on what setting we are in. We now briefly check Kucerovsky's criterion (Theorem 2.4). The connection criterion holds precisely because we have used a connection $\nabla$ to construct $1 \otimes_\nabla X_j$. The domain condition is a simple check and the positivity condition is explicitly checkable as the operators of interest act as number operators. Therefore the unbounded Kasparov module of Equation (13) is an unbounded representative of the product $[\text{ext}] \hat\otimes_{A_e} [\lambda^{(d-1)}]$ at the level of $KK$-classes.

21

Our next task is to relate the module (13) to $\lambda^{(d)}$. We first identify $\bigwedge^* \mathbb{R} \hat{\otimes}_{\mathbb{R}} \bigwedge^* \mathbb{R}^{d-1} \cong \bigwedge^* \mathbb{R}^d$ and use the graded isomorphism $C\ell_{p,q} \hat{\otimes} C\ell_{r,s} \cong C\ell_{p+r,q+s}$ from [27, §2.16] on the left and right Clifford generators by the mapping

$$\rho_{\text{ext}} \hat{\otimes} 1 \mapsto \rho^1, \qquad\qquad 1 \hat{\otimes} \rho^j \mapsto \rho^{j+1},$$
$$\gamma_{\text{ext}} \hat{\otimes} 1 \mapsto \gamma^1, \qquad\qquad 1 \hat{\otimes} \gamma^j \mapsto \gamma^{j+1}.$$

Therefore applying this isomorphism gives the unbounded Kasparov module

$$\left( \mathcal{A}_b \hat{\otimes} C\ell_{0,d}, \ (\ell^2(\mathbb{Z}, A_e) \otimes_{A_e} \ell^2(\mathbb{Z}^{d-1}, B)) \otimes \bigwedge{}^* \mathbb{R}^d, \right.$$
$$\left. N \otimes 1 \otimes \gamma^1 + \sum_{j=1}^{d-1} (1 \otimes_\nabla X_j) \otimes \gamma^{j+1} \right)$$

with $C\ell_{0,d}$-action generated by $\rho^j(\omega) = e_j \wedge \omega - \iota(e_j)\omega$ and $C\ell_{d,0}$-action generated by $\gamma^j(\omega) = e_j \wedge \omega + \iota(e_j)\omega$ for $\omega \in \bigwedge^* \mathbb{R}^d$ and $\{e_j\}_{j=1}^d$ the standard basis of $\mathbb{R}^d$.

Next we define the map $\ell^2(\mathbb{Z}, A_e) \otimes_{A_e} \ell^2(\mathbb{Z}^{d-1}, B) \to \ell^2(\mathbb{Z}^d, B)$ on generators as

$$\delta_k \otimes a_e \otimes \delta_n \otimes b \mapsto \alpha_d^k(a_e) \cdot \delta_{(n,k)} \otimes b$$

for $k \in \mathbb{Z}$, $n \in \mathbb{Z}^{d-1}$ and $\alpha_d$ the automorphism on $A_e$ such that $A_b = A_e \rtimes_{\alpha_d} \mathbb{Z}$. One easily checks that this map is compatible with the right-action of $B$ and $B$-valued inner product and, thus, is a unitary map on $C^*$-modules. We check compatibility with the left action by $A_b \cong C^*(A_e, S_d)$, where for $c_e \in A_e$,

$$c_e(\delta_k \otimes a_e \otimes \delta_n \otimes b) = \delta_k \otimes \alpha_d^{-k}(c_e)a_e \otimes \delta_n \otimes b$$
$$\mapsto \alpha_d^k\big(\alpha_d^{-k}(c_e)a_e\big) \cdot \delta_{(n,k)} \otimes b$$
$$= c_e\left(\alpha_d^k(a_e) \cdot \delta_{(n,k)} \otimes b\right).$$

Furthermore,

$$S_d(\delta_k \otimes A_e \otimes \delta_n \otimes b) = \delta_{k+1} \otimes a_e \otimes \delta_n \otimes b$$
$$\mapsto \alpha_d^{k+1}(a_e) \cdot \delta_{(n,k+1)} \otimes b$$
$$= S_d \alpha_d^k(a_e) S_d^* \cdot \delta_{(n,k+1)} \otimes b$$
$$= S_d\left(\alpha_d^k(a_e) \cdot \delta_{(n,k)} \otimes b\right)$$

and so the generators of the left-action are compatible with the isomorphism. It is then straightforward to check that the compatibility

22

of the left-action extends to the $C^*$-completion. Next we compute that

$$N(\delta_k \otimes a_e \otimes \delta_n \otimes b) = k\delta_k \otimes a_e \otimes \delta_n \otimes b$$
$$\mapsto k\alpha_d^k(a_e) \cdot \delta_{(n,k)} \otimes b = X_d \left(\alpha_d^k(a_e) \cdot \delta_{(n,k)} \otimes b\right)$$

as $\alpha_d^k(a_e)$ commutes with $X_d$. Lastly we use Equation (12) to check that

$$(1 \otimes_\nabla X_j)(\delta_k \otimes a_e \otimes \delta_n \otimes b) = \delta_k \otimes (a_e + [X_j, a_e]) \otimes \delta_n \otimes b$$
$$\mapsto \alpha_d^k(a_e + [X_j, a_e]) \cdot \delta_{(n,k)} \otimes b$$
$$= \alpha_d^k(X_j a_e) \cdot \delta_{(n,k)} \otimes b$$
$$= X_j \left(\alpha_d^k(a_e) \cdot \delta_{(n,k)} \otimes b\right)$$

as $[X_j, a_e] \in \mathcal{A}_e$ for $a_e \in \mathcal{A}_e$ and $\alpha_d^k(X_j) = X_j$ for $j \in \{1, \ldots, d-1\}$. We note that the identification $1 \otimes_\nabla X_j \mapsto X_j$ requires us to use the smooth subalgebra $\mathcal{A}_e \subset A_e$ in order for $[X_j, a_e]$ to be a well-defined element of $A_e$. This is typical of the unbounded product and why we need to have the equality $\overline{\mathcal{A}_e \cdot \ell^2(\mathbb{Z}^{d-1}, B)} = \ell^2(\mathbb{Z}^{d-1}, B)$.

To summarise, the unbounded Kasparov module representing the product is unitarily equivalent to

$$\left(A_b \hat{\otimes} C\ell_{0,d}, \ell^2(\mathbb{Z}^d, B)_B \otimes \bigwedge{}^* \mathbb{R}^d, X_d \otimes \gamma^1 + \sum_{j=1}^{d-1} X_j \otimes \gamma^{j+1}, \gamma_{\bigwedge^* \mathbb{R}^d}\right) \tag{14}$$

with left and right Clifford actions as previously. This is almost the same as the bulk module $\lambda^{(d)}$, the only difference being the labelling of the Clifford basis. The map $\xi(\gamma^j) = \gamma^{\sigma(j)}$ and $\xi(\rho^j) = \rho^{\sigma(j)}$ for $\sigma(j) = (j-1) \bmod d$ is potentially an orientation reversing map on Clifford algebras. Taking the canonical orientation $\omega_{C\ell_{0,d}} = \rho^1 \cdots \rho^d$ on $C\ell_{0,d}$,

$$\xi(\omega_{C\ell_{0,d}}) = \rho^d \rho^1 \cdots \rho^{d-1} = (-1)^{d-1} \rho^1 \cdots \rho^d = (-1)^{d-1} \omega_{C\ell_{0,d}},$$

and similarly for the $\gamma^j$ and $C\ell_{d,0}$. Using [27, §5, Theorem 3], such a map on Clifford algebras will send the $KK$-class of the Kasparov module of Equation (14) to its inverse if $d$ is even or leaves the $KK$-class invariant if $d$ is odd. Therefore at the level of $KK$-classes, $[\text{ext}] \hat{\otimes}_{A_e} [\lambda^{(d-1)}] = (-1)^{d-1} [\lambda^{(d)}]$ as required. $\square$

*Remark* 3.6. It would be preferable to have an explicit unitary equivalence implementing the coordinate permutation of Clifford indices as existing proofs demonstrating the signed equality arising from a permutation require (explicit) homotopies. The difficulty of defining reasonable equivalence relations on unbounded Kasparov modules

23

stronger than 'unitary equivalence modulo bounded perturbation' means that such homotopies compromise the computability and any interpretation stronger than mere (signed) equality of $KK$-classes.

*Remark* 3.7. Applying Theorem 3.5 to the case of $B = C(\Omega)$, we can decompose the bulk module $\lambda_b$ into the product of the extension class linking bulk and edge algebras with the edge module (up to the sign $(-1)^{d-1}$).

We also obtain a bulk-edge correspondence for the bulk spectral triple $\lambda_b(\omega)$ using the decomposition

$$[\lambda_b(\omega)] = [\lambda_b]\hat{\otimes}_{C(\Omega)}[\mathrm{ev}_\omega] = (-1)^{d-1}[\mathrm{ext}]\hat{\otimes}_{A_e}[\lambda_e]\hat{\otimes}_{C(\Omega)}[\mathrm{ev}_\omega],$$

where $[\lambda_e]\hat{\otimes}_{C(\Omega)}[\mathrm{ev}_\omega]$ is represented by the spectral triple

$$\lambda_e(\omega) = \left( \mathcal{A}_e\hat{\otimes}C\ell_{0,d-1}, \pi_\omega\,\ell^2(\mathbb{Z}^{d-1})\otimes{\bigwedge}^*\mathbb{R}^{d-1}, \sum_{j=1}^{d-1}X_j\otimes\gamma^j, \gamma_{\bigwedge^*\mathbb{R}^{d-1}} \right)$$

and is associated to an edge system.

# 4. $K$-theory, pairings and computational challenges

In this section we outline how our rather general results about Kasparov modules and twisted $\mathbb{Z}^d$-actions can be related to topological phases and the bulk-edge correspondence.

As mentioned in the introduction, the measured quantities in insulator models involve the pairing of a $K$-theory class with a dual theory, be it cyclic cohomology, $K$-homology or the more general Kasparov product.

Computing these pairings gives rise to analytic index formulas, which are in general a Clifford module valued index in the sense of Atiyah–Bott–Shapiro [1]. In the case of integer invariants, cyclic formulas may be used to obtain more computable expressions for disorder-averaged quantities. The case of torsion indices is more complicated and we finish with some brief remarks about computing such invariants.

## 4.1. Symmetries and $K$-theory
Recall that our Hamiltonian of interest is a self-adjoint element $h$ in the crossed-product algebra $C(\Omega)\rtimes_{\alpha,\theta}\mathbb{Z}^d$. A Hamiltonian $h$ with spectral gap at 0 represents an extended topological phase if $h$ is

compatible with certain symmetry involutions. The symmetries of interest to us are time-reversal symmetry (TRS), particle-hole (charge-conjugation) symmetry (PHS) and chiral (sublattice) symmetry, although we emphasise that other symmetries such as spatial involution may be considered. The following result, due in various forms to numerous authors, associates a $K$-theory class to a symmetry compatible Hamiltonian.

**Proposition 4.1** ([9, 17, 29, 36, 56]). *Let $h \in C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}$ be self-adjoint with a spectral gap at $0$. If $h$ is compatible with time-reversal symmetry and/or particle-hole symmetry and/or chiral symmetry then we may associate a class in $K_j(C(\Omega) \rtimes \mathbb{Z}^d)$ or $KO_j(C(\Omega) \rtimes \mathbb{Z}^d)$, where $j$ is determined by the symmetries present. The details are summarised in Table 1.*

The specific class associated to a symmetry compatible Hamiltonian in Proposition 4.1 may arise in several (largely equivalent) ways.

One may consider a subgroup $G$ of the $CT$-symmetry group $\{1, T, C, CT\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with $C$ denoting charge-conjugation (particle-hole), $T$ time-reversal and $CT$ sublattice/chiral symmetry. A topological phase can then be considered as a Hamiltonian with spectral gap (assumed to be at $0$), whose phase $h|h|^{-1}$ acts as a grading for a projective unitary/anti-unitary (PUA) representation of $G \subset \{1, T, C, CT\}$ on a complex Hilbert space $\mathcal{H}$. This is the perspective first developed in [17] in the commutative setting and then extended to noncommutative algebras in [9, 36, 56]. The even/odd nature of the time-reversal and charge-conjugation symmetries is encoded in the cocycle that arises in the projective representation. The class in $K_j(A)$ or $KO_j(A)$ for $A \subset C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d$ from Proposition 4.1 and Table 1 is determined by the subgroup $G$ of the $CT$-symmetry group and its PUA representation.

One may also consider the gapped Hamiltonian $h$ to be oddly graded by the presence of a chiral/sublattice symmetry, in which case the phase $h|h|^{-1}$ determines a class in the van Daele $K$-theory of the algebra, which can then be related to real or complex $K$-theory.[1] Other symmetries can be incorporated by considering Real structures on the observable algebra. By considering the various types of symmetries and whether they are even and odd, one can derive Table 1. See [29] for further information.

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[1] We have assumed $A = C(\Omega) \rtimes_{\alpha,\theta} \mathbb{R}^d$ to be trivially graded. Hence for $h$ to have odd grading then we must first consider a larger graded algebra which we can then reduce to the trivially graded $A$. See [29].

| Symmetry type | $K$-theory group |
|---|---|
| even TRS | $KO_0(A)$ |
| even TRS, even PHS | $KO_1(A)$ |
| even PHS | $KO_2(A)$ |
| even PHS, odd TRS | $KO_3(A)$ |
| odd TRS | $KO_4(A)$ |
| odd TRS, odd PHS | $KO_5(A)$ |
| odd PHS | $KO_6(A)$ |
| even TRS, odd PHS | $KO_7(A)$ |
| N/A | $K_0(A)$ |
| chiral | $K_1(A)$ |

TABLE 1. Symmetry types and the corresponding $K$-theory group of the ungraded algebra $A = C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d$ (real or complex). See Proposition 4.1.

The generality and flexibility of $KK$-theory mean that we can use either the symmetry class or the van Daele class in terms of pairing or the bulk-edge correspondence. The choice of $K$-theory class can therefore be determined in order to model the specifics of an experimental set-up. This is important as the techniques being employed to define/measure $\mathbb{Z}_2$-invariants of topological phases are still in development.

### 4.2. Pairings, the Clifford index and the bulk-edge correspondence

As briefly explained, a symmetry compatible gapped Hamiltonian gives rise to a class in $K_j(A)$ or $KO_j(A)$ for $A$ the real or complex ungraded crossed product $C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d$. By Theorem 2.2, we can relate the $K$-theory groups to $KK(\mathbb{C}\ell_j, A)$ or $KKO(C\ell_{j,0}, A)$. Hence we can consider the map

$$KKO(C\ell_{j,0}, C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d) \times KKO(C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d \hat{\otimes} C\ell_{0,d}, C(\Omega))$$
$$\to KKO(C\ell_{j,d}, C(\Omega)) \quad (15)$$

given by the internal Kasparov product, where the class in the group $KKO(C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d, C(\Omega))$ is represented by the fundamental $K$-cycle for the $\mathbb{Z}^d$-action, $\lambda_b$ (an analogous map occurs in the complex case).

Like the case of the quantum Hall effect, where the Hall conductance is related to a pairing of the Fermi projection with a cyclic

cocycle or $K$-homology class, we claim that the quantities of interest in topological insulator systems arise as pairings/products of this type.

Without specifying a particular $K$-theory class, we can make some general comments about the pairing with the fundamental $K$-cycle. The unbounded product with the fundamental cycle has the general form

$$\left( C\ell_{j,d},\, E_{C(\Omega)},\, \widetilde{X},\, \Gamma \right),$$

with $E_{C(\Omega)}$ a countably generated $C^*$-module with a left-action of $C\ell_{j,0}$. The construction of the product is done in such a way that the left-action of $C\ell_{j,d}$ graded-commutes with the unbounded operator $\widetilde{X}$. This implies that the topological information of interest is contained in the kernel $\mathrm{Ker}(\widetilde{X})$ as a $C^*$-submodule of $E_{C(\Omega)}$, see [9, Appendix B].

Let us now associate an analytic index to the product in Equation (15).

**Definition 4.2.** We let $_{r,s}\mathfrak{M}_{C(\Omega)}$ be the Grothendieck group of equivalence classes of real $\mathbb{Z}_2$-graded right $C(\Omega)$-$C^*$-modules carrying a graded left-representation of $C\ell_{r,s}$.

Using the notation of Clifford modules, $\mathrm{Ker}(\widetilde{X})$ determines a class in the quotient group $_{j,d}\mathfrak{M}_{C(\Omega)}/i^*(_{j+1,d}\mathfrak{M}_{C(\Omega)})$, where $i^*$ comes from restricting a Clifford action of $C\ell_{j+1,d}$ to $C\ell_{j,d}$. Next, we use an elementary extension of the Atiyah–Bott–Shapiro isomorphism, [1] and [54, §2.3], to make the identification

$$_{j,d}\mathfrak{M}_{C(\Omega)}/i^*{}_{j+1,d}\mathfrak{M}_{C(\Omega)} \cong KO_{j-d}(C(\Omega)).$$

**Definition 4.3.** The Clifford index, $\mathrm{Index}_{j-d}(\widetilde{X})$, of $\widetilde{X}$ is given by the class

$$[\mathrm{Ker}(\widetilde{X})] \in {}_{j,d}\mathfrak{M}_{C(\Omega)}/i^*{}_{j+1,d}\mathfrak{M}_{C(\Omega)} \cong KO_{j-d}(C(\Omega))$$

We remark that $\mathrm{Index}_k$ reproduces the usual (real) $C^*$-module Fredholm index as studied in [19, Chapter 4] if $k = 0$, see [54].

Of course we may instead wish to use an invariant probability measure $\mathbf{P}$ on $\Omega$ to obtain invariants averaged over the disorder space rather than $K$-theory classes of the disorder. We will return to this question in Section 4.3.

**4.2.1. The bulk-edge correspondence.** Let us now apply Theorem 3.5 to our study on pairings. Denoting by $[x_b]$ the $K$-theory class represented by the symmetry compatible Hamiltonian (using either

van Daele $K$-theory or the graded PUA representation), we have that
$$[x_b]\hat{\otimes}_{A_b}[\lambda_b] = (-1)^{d-1}[x_b]\hat{\otimes}_{A_b}[\text{ext}]\hat{\otimes}_{A_e}[\lambda_e].$$
Using associativity of the Kasparov product to group the terms on the right in two different ways, the real index pairing will either be a pairing
$$KKO(C\ell_{j,0}, C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d) \times KKO(C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d\hat{\otimes}C\ell_{0,d}, \mathbb{R})$$
$$\to KO_{j-d}(C(\Omega)),$$
the bulk invariant, or a new (but equivalent) pairing
$$KKO(C\ell_{j,0}\hat{\otimes}C\ell_{0,1}, C(\Omega) \rtimes_{\alpha^{\parallel},\theta} \mathbb{Z}^{d-1})$$
$$\times KKO(C(\Omega) \rtimes_{\alpha^{\parallel},\theta} \mathbb{Z}^{d-1}\hat{\otimes}C\ell_{0,d-1}, \mathbb{R}) \to KO_{j-d}(C(\Omega)).$$
The second pairing yields an invariant that comes from the edge algebra $C(\Omega) \rtimes_{\alpha^{\parallel},\theta} \mathbb{Z}^{d-1}$ of a system with boundary. Theorem 3.5 ensures that regardless of our choice of pairing, the result is the same and so we obtain the bulk-edge correspondence for real and complex algebras. In particular non-trivial bulk invariants imply non-trivial edge invariants and vice versa. Furthermore, we see that the bulk-edge correspondence continues to hold under the addition of weak disorder. In complex examples, the value of the edge pairing can be interpreted as a response coefficient of the edge system like, for instance, the conductance of a current concentrated at the boundary of the sample [30, 31, 33, 51].

*Remark* 4.4 (Wider applications of Theorem 3.5). The bulk-edge correspondence and Theorem 3.5 are largely independent of the symmetry considerations of topological phases. Instead, it is a general property of the (real or complex) unbounded Kasparov module representing the short exact sequence
$$0 \to \mathcal{K} \otimes C(\Omega) \rtimes_{\alpha^{\parallel},\theta} \mathbb{Z}^{d-1} \to \mathcal{T}(\alpha_d) \to C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d \to 0$$
and the fundamental $K$-cycles on the ideal and quotient algebras we have constructed.

In particular, the fact that the factorisation occurs on the $K$-homological part of the index pairing means other $K$-theory classes and symmetry types can be considered without changing the result and is independent of the symmetries present. For example, if we were to consider symmetry compatible Hamiltonians of a group $\tilde{G}$ that included spatial involution or other symmetries, then provided that the symmetry data can be associated to a class in $KKO(C^*(\tilde{G}), A)$ (or complex), the pairing would still display the bulk-edge correspondence.

*Separating the topological information arising from the internal symmetries of the Hamiltonian from the (non-commutative) geometry of the Brillouin zone highlights an advantage of using Kasparov theory to study topological systems with internal symmetries. We obtain the flexibility to change the K-theoretic data without affecting the geometric information that is used to obtain the topological invariants of interest and vice versa.*

### 4.3. Semifinite spectral triples and integer invariants

Given a unital $C^*$-algebra (real or complex) $B$ with possibly twisted $\mathbb{Z}^d$-action, we have given a general procedure to obtain an unbounded $B \rtimes_{\alpha,\theta} \mathbb{Z}^d$-$B$ module and class in $KKO(B \rtimes_{\alpha,\theta} \mathbb{Z}^d \hat{\otimes} C\ell_{0,d}, B)$. If the algebra $B$ has a faithful, semifinite and norm-lower semicontinuous trace $\tau_B$, there is a general method by which we can obtain a semifinite spectral triple as studied in [25, 38, 48]. Such a condition on $B$ is satisfied in the physically interesting case of $B = C(\Omega, M_N)$, where the disorder space of configurations is equipped with a probability measure $\mathbf{P}$ such that $\mathrm{supp}(\mathbf{P}) = \Omega$. The measure is usually assumed to also be invariant and ergodic under the $\mathbb{Z}^d$-action.

Given the $C^*$-module $\ell^2(\mathbb{Z}^d, B)$ and trace $\tau_B$, we consider the inner-product

$$\langle \lambda_1 \otimes b_1, \lambda_2 \otimes b_2 \rangle = \tau_B((\lambda_1 \otimes b_1 \mid \lambda_2 \otimes b_2)_B) = \langle \lambda_1, \lambda_2 \rangle_{\ell^2(\mathbb{Z}^d)} \, \tau_B(b_1^* b_2),$$

which defines the Hilbert space $\ell^2(\mathbb{Z}^d) \otimes L^2(B, \tau_B)$ where $L^2(B, \tau_B)$ is the GNS space.

**Lemma 4.5.** *The algebra $A = B \rtimes_{\alpha,\theta} \mathbb{Z}^d$ acts on $\ell^2(\mathbb{Z}^d) \otimes L^2(B, \tau_B)$.*

*Proof.* This follows from the identification of $\ell^2(\mathbb{Z}) \otimes L^2(B, \tau_B)$ with $\ell^2(\mathbb{Z}, B) \otimes_B L^2(B, \tau_B)$. $\square$

**Proposition 4.6** ([38], Theorem 1.1). *Given $T \in \mathrm{End}_B(\ell^2(\mathbb{Z}^d, B))$ with $T \geq 0$, define*

$$\mathrm{Tr}_\tau(T) = \sup_I \sum_{\xi \in I} \tau_B[(\xi \mid T\xi)_B],$$

*where the supremum is taken over all finite subsets $I$ of $\ell^2(\mathbb{Z}, B)$ such that $\sum_{\xi \in I} \Theta_{\xi,\xi} \leq 1$. Then $\mathrm{Tr}_\tau$ is a semifinite norm-lower semicontinuous trace with the property $\mathrm{Tr}_\tau(\Theta_{\xi_1,\xi_2}) = \tau_B[(\xi_2 \mid \xi_1)_B]$.*

**Lemma 4.7.** *Let $\mathrm{End}_B^{00}(\ell^2(\mathbb{Z}^d, B))$ be the algebra of the span of rank-1 operators, $\Theta_{\xi_1,\xi_2}$ with $\xi_1, \xi_2 \in \ell^2(\mathbb{Z}^d, B)$, and $\mathcal{N}$ be the von Neumann algebra $\mathrm{End}_B^{00}(\ell^2(\mathbb{Z}^d, B))''$ with weak-closure taken in the bounded operators on $\ell^2(\mathbb{Z}^d) \otimes L^2(B, \tau_B)$. Then the trace $\mathrm{Tr}_\tau$ extends to a trace on the positive cone $\mathcal{N}_+$.*

*Proof.* This is just the dual trace construction, or an explicit check can be made as in [48, Proposition 5.11]. □

**Proposition 4.8.** *The tuple*

$$\left( \mathcal{A} \hat{\otimes} C\ell_{0,d}, \, \ell^2(\mathbb{Z}^d) \otimes L^2(B, \tau_B) \otimes \bigwedge^* \mathbb{R}^d, \, \sum_{j=1}^{d} X_j \otimes \gamma^j, \, (\mathcal{N}, \mathrm{Tr}_\tau) \right)$$

*is a $QC^\infty$ and $d$-summable semifinite spectral triple.*

*Proof.* The commutators $[X_j, a]$ are bounded by analogous arguments to the proof of Proposition 3.2. We use the frame $\{e_m\}_{m \in \mathbb{Z}^d} \subset \ell^2(\mathbb{Z}^d, B)$ with $e_m = \delta_m \otimes 1_B$ to compute that

$$\mathrm{Tr}_\tau((1 + |X|^2)^{-s/2}) = \mathrm{Tr}_\tau \left( \sum_{m \in \mathbb{Z}^d} (1 + |m|^2)^{-s/2} \Theta_{e_m, e_m} \right)$$

$$= \sum_{m \in \mathbb{Z}^d} (1 + |m|^2)^{-s/2} \tau_B((e_m \mid e_m)_B)$$

$$= \sum_{m \in \mathbb{Z}^d} (1 + |m|^2)^{-s/2} \tau_B(1_B)$$

by the properties of $\mathrm{Tr}_\tau$. The sum of $(1 + |m|^2)^{-s/2}$ for $m \in \mathbb{Z}^d$ will be finite for $s > d$ and $\tau_B(1_B) = 1$. Hence $(1 + |X|^2)^{-s/2}$ is $\mathrm{Tr}_\tau$-trace-class for $s > d$ as required.

  Our spectral triple is $QC^\infty$ if $\mathcal{A} \hat{\otimes} C\ell_{0,d}$ preserves the domain of $(1 + D^2)^{k/2}$ for any $k \in \mathbb{N}$, which reduces to checking that $\mathcal{A}$ preserves the domain of $(1 + |X|^2)^{k/2} \otimes 1$ on $\ell^2(\mathbb{Z}^d) \otimes L^2(B, \tau_B)$. We recall that elements of $\mathcal{A}$ are of the form $\sum_n S^n b_n$ with $b_n \in B$ and $n \in \mathbb{Z}^d$ a multi-index such that $|n| < R$ for some $R$. We compute that

$$(1 + |X|^2)^{k/2} \sum_{|n| < R} S^n b_n \cdot (\psi(x) \otimes b)$$

$$= \sum_{|n| < R} (1 + |x - n|^2)^{k/2} \psi(x - n) \otimes \alpha^{-x+n}(\theta(x, n)) \alpha^{-x}(b_n) b$$

for $x \in \mathbb{Z}^d$. Because $n$ is strictly bounded and $\theta(x, n)$ is unitary for all $n, x \in \mathbb{Z}^d$, the Hilbert space norm of the sum over finite $n$ of $(1 + |x - n|^2)^{k/2} \psi(x - n) \otimes \alpha^{-x}(\alpha^n(\theta(x, n)) b_n) b$ is well-defined and norm-convergent for $\psi \otimes b$ in the domain of $(1 + |X|^2)^{k/2}$. □

Let us return to the case of $B = C(\Omega)$, where $C(\Omega)$ is endowed with the invariant probability measure $\mathcal{T}_{\mathbf{P}}$. The semifinite index pairing with the symmetry $K$-theory class is given by the composition

$$KO_j(C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d) \times KKO(C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d \hat{\otimes} C\ell_{0,d}, C(\Omega))$$

$$\to KO_{j-d}(C(\Omega)) \xrightarrow{(\mathcal{T}_{\mathbf{P}})_*} \mathbb{R}$$

or a complex analogue. Hence the semifinite index pairing of the spectral triple of Proposition 4.8 measures disorder-averaged topological invariants. Furthermore in the case of complex algebras and modules, the semifinite local index formula [12, 13] gives us computable expressions for the index pairing. We expect similar results to hold for the case of real integer invariants (e.g. those that arise from $KO_0(\mathbb{R})$ or $KO_4(\mathbb{R})$), though delay a more detailed investigation to another place.

### 4.4. The Kane–Mele model and torsion invariants

As an example, we consider 2-dimensional systems with odd time reversal symmetry from [15, 26] and considered in [9, Section 4]. Recall the bulk Hamiltonian $H_{KM}^\omega = \begin{pmatrix} h_\omega & g \\ g^* & \mathcal{C}h_\omega\mathcal{C} \end{pmatrix}$ acting on $\mathcal{H}_b = \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^{2N}$, with $h_\omega$ a Haldane Hamiltonian, $g$ the Rashba coupling and $\mathcal{C}$ component-wise complex conjugation such that $g^* = -\mathcal{C}g\mathcal{C}$. We take the symmetry group $G = \{1, T\}$ whose time-reversal involution is implemented by the operator $R_T = \begin{pmatrix} 0 & \mathcal{C} \\ -\mathcal{C} & 0 \end{pmatrix}$ on $\mathcal{H}_b$. We follow [9, Section 4] and construct the projective symmetry class $[P_\mu^G] \in KKO(C\ell_{4,0}, C(\Omega) \rtimes \mathbb{Z}^2)$. The class $[P_\mu^G]$ is represented by the Kasparov module

$$\left( C\ell_{4,0}, P_\mu(A \rtimes G)_A^{\oplus 2}, 0, \Gamma \right),$$

where $P_\mu$ is the Fermi projection, $(A \rtimes G)_A$ is a $C^*$-module given by the completion of $A \rtimes G$ under the conditional expectation of the $G$-action on $A$ and $\Gamma$ is the self-adjoint unitary given by the phase $H_{KM}^\omega |H_{KM}^\omega|^{-1}$.

For simplicity, we will pair $[P_\mu^G]$ with the spectral triple coming from the evaluation map $\mathrm{ev}_\omega$. Namely we have the real spectral triple

$$\lambda_b(\omega) = \left( \mathcal{A}_b \hat{\otimes} C\ell_{0,2}, \ _{\pi_\omega}(\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^{2N}) \otimes \bigwedge{}^* \mathbb{R}^2, X_b, \gamma_{\bigwedge^* \mathbb{R}^2} \right)$$

with $X_b = \sum_{j=1}^2 X_j \otimes 1_{2N} \otimes \gamma^j$ and $\mathcal{A}_b$ a dense subalgebra of $C(\Omega) \rtimes \mathbb{Z}^2$ given by matrices of elements in $C(\Omega)_\alpha \mathbb{Z}^2$.

31

Let us now consider a bulk-edge system. We may link bulk and edge algebras by a short exact sequence, which gives rise to a class $[\text{ext}] \in KKO(C(\Omega) \rtimes \mathbb{Z}^2 \hat{\otimes} C\ell_{0,1}, C(\Omega) \rtimes \mathbb{Z})$ by the procedure in Section 3.5. We will omit the details and instead focus on the bulk and edge pairings, where the edge spectral triple is given by

$$\lambda_e(\omega) = \left( \mathcal{A}_e \hat{\otimes} C\ell_{0,1}, \pi_\omega(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^{2N}) \otimes \bigwedge{}^* \mathbb{R}, X_1 \otimes \gamma_{\text{edge}}, \gamma_{\bigwedge^* \mathbb{R}} \right)$$

with $\mathcal{A}_e$ a dense subalgebra of $C(\Omega) \rtimes \mathbb{Z}$. Our bulk pairing is the product

$$[P_\mu^G] \, \hat{\otimes}_A \left[ \left( \mathcal{A} \hat{\otimes} C\ell_{0,2}, \pi_\omega(\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^{2N}) \otimes \bigwedge{}^* \mathbb{R}^2, X_b, \gamma_{\bigwedge^* \mathbb{R}^2} \right) \right]$$

$$KKO(C\ell_{4,0}, A) \times KKO(A \hat{\otimes} C\ell_{2,0}, \mathbb{R}) \to KO_2(\mathbb{R}) \cong \mathbb{Z}_2,$$

which can be expressed concretely as the Clifford module valued index,

$$\langle [P_\mu^G], [\lambda_b(\omega)] \rangle = [\text{Ker}(P_\mu X_b P_\mu)]$$
$$\cong \text{Dim}_{\mathbb{C}} \text{Ker}[P_\mu((X_1 + iX_2) \otimes 1)P_\mu] \bmod 2$$

using a particular choice of Clifford generators. By Theorem 3.5 and the associativity of the Kasparov product, this is the same as the pairing

$$- \left( [P_\mu^G] \hat{\otimes}_A [\text{ext}] \right) \hat{\otimes}_{A_e}$$
$$\left[ \left( \mathcal{A}_e \hat{\otimes} C\ell_{0,1}, \pi_\omega(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^{2N}) \otimes \bigwedge{}^* \mathbb{R}, X_1 \otimes \gamma_{\text{edge}}, \gamma_{\bigwedge^* \mathbb{R}} \right) \right],$$

a map

$$KKO(C\ell_{4,0} \hat{\otimes} C\ell_{0,1}, C(\Omega) \rtimes \mathbb{Z}) \times KKO((C(\Omega) \rtimes \mathbb{Z}) \hat{\otimes} C\ell_{0,1}, \mathbb{R})$$
$$\to KO_{4-1-1}(\mathbb{R}) \cong \mathbb{Z}_2,$$

which is now an invariant of the edge algebra $C(\Omega) \rtimes \mathbb{Z}$. Analytic expressions can be derived for the edge invariant by taking the Clifford index of the (edge) product module.

We would like to examine the edge pairing more closely. We first review what occurs in the complex setting as developed in [30, 31]. Let $\Delta \subset \mathbb{R}$ be an open interval of $\mathbb{R}$ such that $\mu \in \Delta$ and $\Delta$ is in the complement of the spectrum of $H_{KM}^\omega$. By considering states in the image of the spectral projection $P_\Delta = \chi_\Delta(\Pi_s H_{KM}^\omega \Pi_s)$ for $\Pi_s : \ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z} \otimes \{\ldots, s-1, s\})$ the projection, we are focusing precisely on eigenstates of the Hamiltonian with edge that do not exist in the bulk system, namely edge states. We use the projection $P_\Delta$ to define the unitary

$$U(\Delta) = \exp\left( -2\pi i \frac{\Pi_s H_{KM}^\omega \Pi_s - \inf(\Delta)}{\text{Vol}(\Delta)} P_\Delta \right).$$

It is a key result of [30, 31] that $U(\Delta)$ is a unitary in $(C(\Omega) \rtimes \mathbb{Z})_{\mathbb{C}}$ and, furthermore, represents the image of the Fermi projection under the (complex) $K$-theory boundary map. That is, the unitary $[U(\Delta)] \in K_1((C(\Omega) \rtimes \mathbb{Z})_{\mathbb{C}})$ represents the complex Kasparov product $[P_\mu]\hat{\otimes}_{A_{\mathbb{C}}}[\text{ext}]$ for trivially graded algebras. The authors of [31] show that the pairing of $[U(\Delta)]$ with the boundary spectral triple can be expressed as

$$\sigma_e = -\frac{e^2}{h}\hat{\mathcal{T}}\left(U(\Delta)^*i[X_1, U(\Delta)]\right)$$

$$= -\lim_{\Delta \to \mu} \frac{1}{\text{Vol}(\Delta)}\hat{\mathcal{T}}(P_\Delta i[X_1, \Pi_s H^\omega_{KM}\Pi_s]), \qquad (16)$$

where $\hat{\mathcal{T}} = \mathcal{T} \otimes \text{Tr}$ is the trace per unit volume along the boundary and operator trace normal to the boundary. One recognises Equation (16) as measuring the conductance of an edge current (as $P_\Delta$ projects onto edge states). Unfortunately, in the Kane–Mele example, the expression $\hat{\mathcal{T}}(P_\Delta i[X_1, \Pi_s H^\omega_{KM}\Pi_s])$ is zero as there is no net current and the cyclic cocycle cannot detect the $\mathbb{Z}_2$-index we associate to the edge channels. This leads us to summarise the remaining difficulties in our approach.

## 5. Open problems

**Interpreting torsion-valued invariants.** A concrete representation of the torsion-valued index pairings that give rise to both bulk and edge pairings is a much more difficult task than in the complex case, where invariants can be expressed as the Fredholm index of the operators of interest. This is because we have to consider Kasparov products, which give rise to a 'Clifford module' index.

One advantage of unbounded Kasparov theory is that the operators we deal with and the modules we build have geometric or physical motivation and so can be linked to the underlying system. In particular it would be desirable to link the Clifford module index with a more physical expression for the edge pairing as the edge invariant is meant to be directly linked to the existence of edge channels on a system with boundary. This remains an open problem in the field and is related to the difficulty of measuring torsion-labelled states.

A further complication is that there are two copies of $\mathbb{Z}_2$ in the real $K$-theory of a point: in degrees 1 and 2. These different indices arise in different ways, and have different interpretations: see [11].

**Disorder, localisation, and spectral gaps.** Throughout this article we have required that the Hamiltonian retains a spectral gap. This is so that there is an unambiguous method of constructing $K$-theory classes associated to the Hamiltonian. It is well-known, however, that the complex pairing with the fundamental class continues to make sense whenever the spectrum of the Hamiltonian has a *gap of extended states*.

For the quantum Hall effect Bellissard et al. showed that this phenomenon arises physically from localisation arising from disorder [6]. Mathematically this was reflected in the properties of commutators between the position operator $X$ and the projector $\chi_{(-\infty,a]}(H)$ for $a$ a point in such a gap of extended states of $H$. In turn this behaviour is seen in the expression for the Chern character, which makes sense for non-torsion invariants, but seemingly has no analogue for torsion invariants. Importantly, disorder does not affect the construction of the $K$-homology fundamental class.

**The real local index formula.** For non-torsion invariants of topological insulators, it seems reasonable to expect that the entirety of the machinery developed for complex topological insulators in [51] has an exact analogue. The use of a Chern character, the analysis of disorder and localisation, the Kubo formula relating topological invariants to linear response coefficients all make perfect sense.

While all this seems quite reasonable, there is a quite a lot of detail to check. There are many steps in the production of, for instance, the local index formula, and the heavy reliance on spectral theory means that numerous details of the proof require a careful check.

**The $K$-theory class.** Relating experiments on topological phases to details of single particle Hamiltonians with symmetries is not a simple task. The measured quantities should correspond to pairings of the fundamental class with $K$-theory classes of the observable algebra arising from the Hamiltonian.

Having the $K$-homology data (the fundamental class) fixed allows us to compare the results of future experiments with the predictions arising from pairing with different $K$-theory classes.

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