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Fully nonlinear curvature flow of axially symmetric hypersurfaces

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Fully Nonlinear Curvature Flow of Axially Symmetric Hypersurfaces

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Abstract

In this thesis we consider axially symmetric evolving hypersurfaces mostly with boundary conditions between two parallel planes. The speed function is a fully nonlinear function of the principal curvatures of the hypersurface, homogeneous of degree one. We have results for several boundary conditions. Specifically, with a natural class of Neumann boundary conditions we show that evolving hypersurfaces exist for a finite maximal time. The maximal time is characterised by a curvature singularity at either boundary. Generally, the singularities of the flow are classified as Type I in the case of pure Neumann boundary conditions. In addition to the "curvature pinching estimate" that is obtained, Sturmian theory is applied to show the discreteness of singularities. Furthermore, some results carry over to higher degrees of homogeneity. Finally, we have some additional results including a gradient bound for the hight function in the volume preserve case.

Certification

I declare that this thesis, submitted in fulfilment of the requirements for the award of Doctor of Philosophy, in the School of Mathematics and Applied Statistics, University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. This document has not been submitted for qualification at any other academic institution.

Fatemah Mofarreh

2015

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List of Publications

The following publications have been emerged from this thesis:

[1] James A. McCoy, Fatemah Y. Y. Mofarreh and Graham H. Williams, Fully nonlinear curvature flow of axially symmetric hypersurfaces with boundary conditions. *Annali di Matematica Pura ed Applicata, V. 193, Issue 5, pp 1443-1455*, 2014, doi: 10.1007/s10231-013-0337-7 [54]

[2] James A. McCoy, Fatemah Y. Y. Mofarreh and Valentina M. Wheeler, Fully nonlinear curvature flow of axially symmetric hypersurfaces. *Nonlinear Differential Equations and Applications NoDEA, V. 22, Issue 2, pp 325-343*, 2015, doi: 10.1007/s00030-014-0287-9 [55]

[3] Additional publication under preparation

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Chapter 1

Introduction

1 Background

It is well known that heat equations are used physically to model the spread of temperature from hot to cold areas and this process always happens quickly and easily. Mathematically, this process is related to a family of geometric curvature flows. This method is considered important because it can smoothly deform complicated items into more easily understood ones. Recently, this area of research has been extensively studied by many mathematicians due to its many applications in physical models. There are many kinds of curvature flow of hypersurfaces, such as Gauss curvature flow (GCF), Ricci flow and mean curvature flow (MCF) and they are used in different areas such as: tumbling stones on the beach [30], crystal growth [66], image processing [60], annealing of metals [57] as well as in a proof of the Poincare conjecture [56]. MCF can be viewed as a nonlinear heat flow on manifolds because the partial differential equations (PDE) associated with the flow has a Laplacian as a leading term. In fact, they have both similar and different properties, such as singularities. In particular, the Laplacian is the quasilinear Laplace-Beltrami operator, not the standard Laplacian of the ordinary heat equation.

2 Literature review

The foundations for the topic of this thesis are found in the following selected works. Mean curvature flow was introduced in 1956 by Mullins [57] as a model of annealing metals. Then Firey, in 1974 [30], proposed the motion of a convex surface by its positive Gauss curvature as a model for the changing shape of a tumbling stone on a beach. He conjectured, that convex surfaces contract to spherical points. Firey's conjecture was resolved by Andrews [5] to show the roundness of the point.

Many studies have considered the contraction of hypersurfaces using curvature flow in a general sense where the speed function is positive and homogeneous and the initial hypersurface is convex. For example, the existence and regularity of solutions of Firey's conjecture is proved by Tso [67] without limiting the shape of the contracting surface. He also obtained the result for hypersurfaces. Additionally, for flows modelled by powers of Gaussian curvature some similar results were shown by Chow [24]. Another result for motion by square root of the scalar curvature was proved by Chow [25] but it requires stronger assumption for the initial hypersurface.

Andrews in [3] introduced a class of fully nonlinear speed functions that are homogeneous of degree one and proved that convex surfaces contract to spherical points for all flows in the class. Later in [8] he made this class bigger using a generalisation of maximum principles of Hamilton for tensors and he further expanded this study for surfaces in [9]. This class is extended further in Andrews, McCoy and Zheng [16].

McCoy in [53] considered similar classes of speed of curvature flow and he showed that, for closed hypersurfaces, not necessarily convex, under certain conditions the only self-similar solutions are spheres. More related work appears in Han [36] and Andrews, Langford and McCoy [12].

While Mullins [57] gave some solutions, Huisken [37] used partial differential equations and differential geometry techniques to study mean curvature flow of compact hypersurfaces in $\mathbb{R}^{n+1}$ with $n \geq 2$ without boundary. Huisken's result was about strictly convex hypersurfaces that evolve under the mean curvature flow and contract smoothly to a spherical point in finite time. Later, in 1990 the same author

[39] observed neckpinch singularities developing in finite time under mean curvature flow for rotationally symmetric two-dimensional surfaces of positive mean curvature. Asymptotically the behaviour of such singularities was like a cylinder.

The rotationally symmetric mean curvature flows generated more interest by researchers Dziuk and Kawohl [26], who in 1991 show that, for a compact rotationally symmetric surface without boundary, "the solution degenerates in the sense that its curvature develops a singularity at exactly one point". In 1995, Altschuler, Angenent and Giga [2] generalised this, showing regularity of the solution of MCF except at isolated points for compact, smooth, rotationally symmetric hypersurfaces given by rotating a graph around an axis. Also [26] is generalised by Matioc [50] for more general boundary conditions to enforce the formation of a singularity in finite time using parabolic maximum principles.

Another interesting type of curvature flow is started by adding a positive "global term" geometric property of the evolving hypersurface that will be preserved under the flow. Surface volume or area are some of these properties. Huisken proved that volume preserved mean curvature flow for a uniformly convex, compact manifold without boundary converges to a sphere [38]. A similar result was obtained by McCoy [51] for the surface area preserving mean curvature flow. For mixed volumes the result was generalised by the same author [52]. Athanassenas [19] shows that for rotationally symmetric, volume-preserving mean curvature flow, singularities form a finite, discrete set along the axis of rotation. Later for axially symmetric surfaces with Neumann boundary conditions a similar result was verified by Athanassenas and Kandanaarachchi [21]. In addition, they classified first singularity as Type I under an additional lower height bound on the boundary of a specific region.

Cabezas-Rivas and Sinestrari [22] prove that if the initial closed convex hypersurface satisfies a suitable pinching condition flow with speed it is given by a power of the mth mean curvature plus a volume preserving term, the solution exists for all times and converges to a round sphere. A similar result was proven [65] in Euclidean space where speed is given by a positive power of the mean curvature without assuming the curvature pinching properties or restrictions on the dimension.

3 Mean curvature flow

Several authors with different points of view have studied the motion of surfaces by their mean curvature. It is particularly important to note that under mean curvature flow the solution is evolved considering time. Therefore, there is a relation between the normal velocity at each point and the mean curvature vector. Surfaces of positive mean curvature shrink under the mean curvature flow. Generally, the more highly curved regions will shrink faster, and a singularity may happen at some point before the hypersurface disappears. The natural question here will be about the possible limiting shapes of an evolving hypersurface when the first singular time is approached. More specific, singularity is usually developed near the neck. A simple example of an explicit solution to MCF is the homothetically shrinking sphere.

For the mean curvature flow, the position vector $X(x,t)$ of the evolving hypersurface $M_t = X(M,t)$ satisfies evolution equation

$$\frac{\partial X}{\partial t}(x,t) = -H(x,t)\,\nu(x,t),$$

with initial condition

$$X(x,0) = X_0(x),$$

for some initial embedding X_0 of a given hypersurface M_0. Here $X_t : M^n \to \mathbb{R}^{n+1}$ is a family of smooth embeddings, possibly with boundary, H is the mean curvature of M_t at the point $X(x,t)$ and $\nu(x,t)$ is a smooth choice of unit normal vector. It is well known that the $H = \Delta_{M_t} X$ where Δ_{M_t} is the quasilinear Laplace-Beltrami operator on the manifold M_t which leads to the possibility of considering mean curvature flow as heat flow.

4 Structure of the thesis

The structure of this thesis is as follows:

In Chapter 1, the topic of the thesis, literature review and background are introduced.

In Chapter 2, we set out the majority of our notation. Furthermore, we briefly describe the geometric features of axially symmetric hypersurfaces that we will require in our analysis. In particular, we provide expressions for the Christoffel symbols and the gradient of the Weingarten map in this setting.

In Chapter 3, we detail the properties of nonlinear speed functions under consideration. Moreover, the system of evolution equations describing the position vector of the evolving hypersurface is equivalent to a scalar evolution equation for the corresponding height of the graph of the generating curve above the x_1 axis. We follow Huisken [39] to calculate all required evolution equations.

In Chapter 4, we discuss the flow problem and prove some preliminary results. We characterise the maximal time as the time of blow-up of the norm of the second fundamental form and we show that in this setting, as is the case with convex surfaces [3, 9], the singularity is Type I.

In Chapter 5, we obtain a lower bound on the rotational curvature that is inversely proportional to the height of the evolving graph; this implies a similar bound on the axial derivative of the graph function. We prove the crucial pinching estimate, listing several important corollaries including a bound on the ratio of the principal curvatures. We also apply curvature pinching to remove the convexity condition in Theorem 4.2 in the case of pure Neumann boundary conditions, and to show that the second axial derivative of the graph function is bounded inversely proportional to the square of the graph height. We turn our attention to contracting closed, convex, axially symmetric hypersurfaces, showing that such hypersurfaces contract under our class of flow to asymptotically spherical points in finite time. Moreover, if the contraction is self-similar, then the hypersurface must be a sphere. For $n \geq 3$ these are a new results: they generalise to higher dimensions the results in [9] and

[53] and can also be thought of as a relaxation of requirements on the speed or the initial data of other works. We consider a more general case where the speed F in (3.1) is replaced by F^k for constant $k > 0$.

In Chapter 6, we study the Sturmian theorem and apply it to show that the zeros of the second spatial derivative of the graph function are discrete and nonincreasing.

In Chapter 7, volume preserved curvature flow is introduced and the important evolution equations are computed. Bounds of some quantities, such as area and global term h, were obtained. Sturmian theorem is also applied in this case and we get boundedness of the gradient and discreteness of zeros of the second spatial derivative of the graph function.

Chapter 2

Notation

1 Introduction

We include in this Chapter some results and definitions. Then, we introduce geometry of axially symmetric hypersurfaces.

2 Parabolic equations

Parabolic equations are used for physical or mathematical problems. Heat equations are a well known example for parabolic equations. We consider a general parabolic equation of the form

$$u_t = F(x, t, u, Du, D^2u),$$

for some nonlinear function F defined on $\Gamma = \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$ where $\mathbb{R}^{n\times n}$ denotes the set of all real symmetric $n \times n$ matrices. We now consider a typical point (x, t, z, p, r) in Γ, then we can say the operator F is elliptic on some subset Γ_1 of Γ if the matrix $\dot{F}$ is positive definite. Particularly

$$\frac{\partial F}{\partial r_{ij}} > 0.$$

F is uniformly elliptic on Γ_1 if there are positive functions λ and Λ such that

$$\lambda|\eta|^2 \leq \frac{\partial F}{\partial r_{ij}}\eta_i\eta_j \leq \Lambda|\eta|^2,$$

for any $(x, t, z, p, r) \in \Gamma_1$, any positive definite vector η, and if the ratio Λ/λ is bounded on Γ_1.

3 Nonlinear parabolic comparison principle

In second order elliptic and parabolic partial differential equations, maximum principles are very useful tools to understand the behaviour of the solutions especially uniqueness, symmetry and boundedness. For much of this work we need only a well-known ordinary differential equation (ODE) comparison result for determining how spatial extrema behave over time (see, for example, [34] Section 3). We will need standard maximum principles for parabolic equations that appear for example in [29] We also need the following nonlinear parabolic comparison principle.

Theorem 2.1. *(Non-Linear Parabolic Comparison Principle)[58]*
Given a compact manifold M. Let $F : M \times [0, T) \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ denote a non-linear operator with $F = (x, t, z, p, r)$ satisfying

1. *F is differentiable with respect to z, p and r.*
2. *$\left(\dfrac{\partial F}{\partial r_{ij}}\right)$ is positive semidefinite.*

Let $f, g : M \times [0, T) \to \mathbb{R}$ be twice differentiable functions satisfying

$$\begin{aligned}\frac{\partial}{\partial t} f &\geqslant F(x, t, f, Df, D^2 f),\\ \frac{\partial}{\partial t} g &\leqslant F(x, t, g, Dg, D^2 g),\\ f &\geqslant g \ \text{ on } \partial M \times [0, T),\end{aligned}$$

if $f \geqslant g$ at $t = 0$ then $f \geqslant g$ on $M \times [0, T)$.

4 Function spaces

We will introduce some relevant function spaces and associated norms on M and on $M \times [0, T)$, as in [52]. Similar definitions for domains appear in [47]. For $k \in$

$\mathbb{N}$, $C^k(M)$ is the Banach space of real valued functions on M that are $k-$times continuously differentiable, equipped with the norm

$$\|u\|_{C^k(M)} = \sum_{|\beta|\le k} \sup_M |\overline{\nabla}^\beta u|.$$

Here β is a standard multi-index for partial derivatives and $\overline{\nabla}$ is the derivative on M. We further define, for $\alpha \in (0,1]$, $C^{k,\alpha}$ to be the space of functions $u \in C^k(M)$ such that the norm

$$\|u\|_{C^{k,\alpha}(M)} = \|u\|_{C^k(M)} + \sup_{|\beta|=k} \sup_{\substack{x,y\in M\\ x\neq y}} \frac{|\overline{\nabla}^\beta u(x) - \overline{\nabla}^\beta u(y)|}{|x-y|^\alpha},$$

is finite. Here $|x-y|$ is the distance between x and y in M. On the space-time domain $M\times I$ where $I=[a,b]\subset\mathbb{R}$, we denote by $C^k(M\times I)$ the space of real valued functions u which are $k-$times continuously differentiable with respect to x and $[\frac{k}{2}]-$times continuously differentiable with respect to t such that the norm

$$\|u\|_{C^k(M\times I)} = \sum_{|\beta|+2r\le k} \sup_{M\times I} |\overline{\nabla}^\beta D_t^r u|,$$

is finite, where D_t denotes the time derivative. Here $[\frac{k}{2}]$ is the largest integer not greater than $\frac{k}{2}$. We also denote by $C^{k,\alpha}(M\times I)$ the space of functions in $C^k(M\times I)$ such that the norm

$$\|u\|_{C^{k,\alpha}(M\times I)} = \|u\|_{C^k(M\times I)} + \sup_{|\beta|+2r=k} \sup_{\substack{(x,s),(y,t)\in M\times I\\ (x,s)\neq(y,t)}} \frac{|\overline{\nabla}^\beta D_t^r u(x,s) - \overline{\nabla}^\beta D_t^r u(y,t)|}{(|x-y|^2+|s-t|)^{\frac{\alpha}{2}}}$$

is finite. For most of this work we will have $M=[0,a]$ or $M=\mathbb{S}^n$.

5 Geometric background

In this part, we introduce the geometry that we need for hypersurfaces, axially symmetric hypersurfaces. The height of the graph we use will be introduced as in [26]. We will define geometric quantities like the metric, second fundamental form, the Weingarten map and state fundamental relations like the Codazzi equations.

5.1 Differential geometry

We will consider a family of smooth immersions $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ that defines the evolving $n-$dimensional hypersurface, $M_t = X(M^n, t)$. Suppose $i_1, ..., i_{n+1}$ is the standard basis in $\mathbb{R}^{n+1}$. We will define the induced metric and the second fundamental form on M_t by $g = (g_{ij})$ and $A = (h_{ij})$ respectively. The inner product in $\mathbb{R}^{n+1}$ will be used in form $\langle . , . \rangle$. We compute the metric components as

$$g_{ij}(x,t) = \langle \overline{\nabla}_i X(x,t), \overline{\nabla}_j X(x,t) \rangle, \qquad x \in M^n, \;\; t \in [0,T),$$

where $\overline{\nabla}$ is the covariant derivative on M^n. The second fundamental form is

$$h_{ij}(x,t) = \langle \overline{\nabla}_i \nu(x,t), \overline{\nabla}_j X(x,t) \rangle = -\langle \nu(x,t), \overline{\nabla}_i \overline{\nabla}_j X(x,t) \rangle,$$

where $\nu(x,t)$ is the outer unit normal to M_t. The induced connection on M_t is defined via the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\overline{\nabla}_i g_{jl} + \overline{\nabla}_j g_{il} - \overline{\nabla}_l g_{ij} \right),$$

and therefore the covariant derivative on M_t of a smooth tangent vector field $Y = Y_j \tau_j$ is given

$$\nabla_j Y = \sum_{k=1}^{n} \left(\overline{\nabla}_j Y_k + \Gamma^k_{ij} Y_i \right) \tau_k.$$

The divergence on $\mathbb{R}^{n+1}$ will be denoted by $\mathrm{div}_{\mathbb{R}}$ while the divergence on the manifold will be denoted by div.

5.2 Axially symmetric hypersurfaces

Consider the $n-$dimensional hypersurface M_0 in $\mathbb{R}^{n+1}$ obtained by rotating the graph of u_0 about the x_1 axis with Neumann boundary condition $u_0^{'}(0) = u_0^{'}(a) = 0$. The n-dimensional axially symmetric hypersurface M can be specified by a corresponding strict positive and suitably smooth function on the bounded interval $u : [0,a] \to \mathbb{R}$

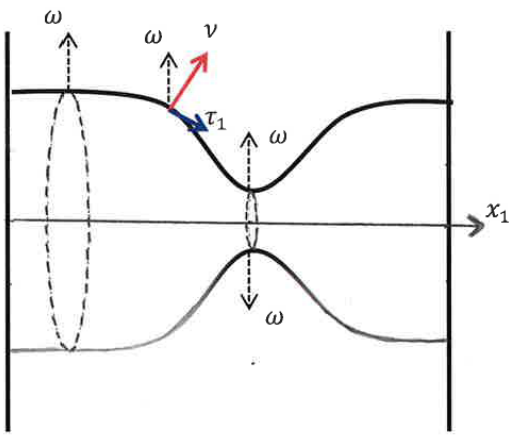

Figure 2.1: The hypersurface M_0

such that M is parametrised by $X : [0, a] \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}$, where

$$X(x, \omega) = (x, u(x)\,\omega). \tag{2.1}$$

We will assume that u is smooth enough on $[0, a]$ for all derivatives we use to make sense. Throughout the thesis, derivatives at the endpoints $x = 0$ and $x = a$ are interpreted naturally as one-sided.

In order to have evolution equations later on M_t, as in [39], we will consider $\tau_1, ..., \tau_n$ a local orthonormal frame on M_t which satisfies

$$\langle \tau_l, i_1 \rangle = 0 \quad \text{for} \quad l = 2, ..., n \quad \text{and} \quad \langle \tau_1, i_1 \rangle > 0. \tag{2.2}$$

Specifically, τ_1 is the tangent to the generating curve given by u in the direction of i_1 while $\tau_2, ..., \tau_n$ are tangents to the $n-1$ sphere.

We will define v as a gradient function which is a geometric quantity related to $\sqrt{1 + u_x^2}$ for more details see Section 2 in Chapter 4. More specific, we define $\omega = \frac{\hat{X}}{|\hat{X}|}$ as the unit outward normal to the cylinder over the $n-1$ dimensional sphere, see Figure 2.1, where

$$\hat{X} = X - \langle X, i_1 \rangle i_1. \tag{2.3}$$

Let

$$v = \langle \omega, \nu \rangle^{-1}, \tag{2.4}$$

and $y = \langle X, \omega \rangle$ where y is the height function

$$y = \sqrt{|X|^2 - \langle X, i_1 \rangle^2}. \tag{2.5}$$

We introduce

$$p = \langle \tau_1, i_1 \rangle y^{-1}, \qquad q = \langle \nu, i_1 \rangle y^{-1}, \tag{2.6}$$

where i_1 is the unit vector in the x_1 axis direction. As in [39] we can write

$$p^2 + q^2 = y^{-2}. \tag{2.7}$$

Particularly

$$q = -y' \, p, \tag{2.8}$$

and

$$\nabla_1 y = -qy, \tag{2.9}$$

in addition to

$$k = \langle \nabla_{\tau_1} \nu, \tau_1 \rangle = \frac{-y''}{(1 + {y'}^2)^{\frac{3}{2}}}, \quad p = \frac{1}{y\sqrt{1 + {y'}^2}}, \tag{2.10}$$

where p and k are the eigenvalues of the second fundamental form. There are $n-1$ eigenvalues equal to p.

y and u are two different interpretations of the same physical object. Clearly, $y(x,t)$ is the height function and $y : M^n \times [0,T) \to \mathbb{R}$ while $u(x,t)$ is the radius function such that $u : [0,a] \times [0,T) \to \mathbb{R}$ as in [43]. In a slight abuse of notation, we will often write u in place of y to emphasise the dependence of the graph function only on the axial direction.

The function $y = (|X|^2 - |\langle X, i_1 \rangle|^2)^{\frac{1}{2}}$ agrees with y_0 at time $t = 0$. M_0 remains axially symmetric and we will later have the evolution equations as a generalization of Huisken work [39].

Similar notations as in [37] are used. The metric and second fundamental form are given respectively as $g = g_{ij}$, $A = h_{ij}$ and $|A|$ the norm of the second fundamental form $|A|^2 = g^{ij}g^{lm}h_{il}h_{jm} = h^j_l h^l_j$ where g^{ij} is the (i,j)-entry of the inverse of the matrix (g_{ij}). The Weingarten map has entries $h^i_j = g^{ik}h_{kj}$ where we sum over repeated indices from 1 to n unless otherwise indicated. It has everywhere the useful diagonal structure. In detail the metric, second fundamental form and Weingarten map of M are given respectively by

$$g_{ij} = \begin{pmatrix} 1+u_x^2 & 0 \\ 0 & u^2\,\overline{\sigma}_{ij} \end{pmatrix}, \qquad h_{ij} = \begin{pmatrix} -\frac{u_{xx}}{\sqrt{1+u_x^2}} & 0 \\ 0 & \frac{u}{\sqrt{1+u_x^2}}\,\overline{\sigma}_{ij} \end{pmatrix},$$

$$\text{and} \;\; {h^i}_j = \begin{pmatrix} k & 0 \\ 0 & p\,I \end{pmatrix} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2\,I \end{pmatrix} = \begin{pmatrix} -\frac{u_{xx}}{[1+u_x^2]^{\frac{3}{2}}} & 0 \\ 0 & \frac{1}{u\sqrt{1+u_x^2}}\,I \end{pmatrix}$$
$$= \frac{1}{\sqrt{1+u_x^2}} \begin{pmatrix} -\left(\arctan\left(u_x\right)\right)_x & 0 \\ 0 & \frac{1}{u}I \end{pmatrix}, \tag{2.11}$$

where where $\overline{\sigma}_{ij}$ denotes the metric on $\mathbb{S}^{n-1}$ and I is the $(n-1)\times(n-1)$ identity matrix. Here and throughout the thesis we will write $u_x = \frac{\partial u}{\partial x}$ and $u_x^2 = \left(\frac{\partial u}{\partial x}\right)^2$. It is important to know that through this thesis we define, as in [39], $\kappa_1 = k$ and $\kappa_2 = \cdots = \kappa_n = p$ as axial and rotational curvatures respectively.

Because of the axial symmetry, many of the derivatives of the second fundamental form for axially symmetric surfaces are identically equal to zero. We compute them explicitly, via the Christoffel symbols.

Lemma 2.1. *In normal coordinates at any particular point, the only nonzero Christoffel symbols of the induced metric of axially symmetric hypersurfaces of the form* (2.1) *are*

$$\Gamma^1_{11} = \frac{u_x u_{xx}}{1+u_x^2},$$

and for any $k \geq 2$,

$$\Gamma^1_{kk} = -\frac{u\,u_x}{1+u_x^2}, \qquad \Gamma^k_{1k} = \Gamma^k_{k1} = \frac{u_x}{u}.$$

Proof: The required formulae follow by direct computation using

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\overline{\nabla}_i g_{jl} + \overline{\nabla}_j g_{il} - \overline{\nabla}_l g_{ij} \right),$$

and the facts that

$$\overline{\nabla}_1 g_{ij} = \frac{\partial}{\partial x_1} g_{ij} = \begin{cases} 2\,u_x\,u_{xx} & i=j=1, \\ 2\,u\,u_x\,\overline{\sigma}_{ij}, & i,j \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

and for $k \geq 2$, $\overline{\nabla}_k g_{ij} = 0$.

Therefore

$$\begin{aligned} \Gamma^1_{11} &= \frac{1}{2} g^{1l} [\overline{\nabla}_i g_{jl} + \overline{\nabla}_j g_{il} - \overline{\nabla}_l g_{ij}] \\ &= \frac{1}{2} g^{1l} [\overline{\nabla}_1 g_{1l} + \overline{\nabla}_1 g_{1l} - \overline{\nabla}_l g_{11}] \\ &= \frac{1}{2} g^{11} [\overline{\nabla}_1 g_{11} + \overline{\nabla}_1 g_{11} - \overline{\nabla}_1 g_{11}] \\ &= \frac{1}{2} \frac{1}{(1+u_x^2)} [2u_x u_{xx} \cancel{+2u_x u_{xx}} \cancel{-2u_x u_{xx}}] \\ &= \frac{u_x u_{xx}}{1+u_x^2} \quad . \end{aligned}$$

For any $k \geq 2$

$$\begin{aligned} \Gamma^1_{kk} &= \frac{1}{2} g^{1l} [\overline{\nabla}_k g_{kl} + \overline{\nabla}_k g_{kl} - \overline{\nabla}_l g_{kk}] \\ &= \frac{1}{2} g^{11} [0 + 0 - \overline{\nabla}_1 g_{kk}] \\ &= \frac{1}{2} \frac{1}{(1+u_x^2)} [-2uu_x] \\ &= -\frac{uu_x}{1+u_x^2}, \end{aligned}$$

and

$$
\begin{aligned}
\Gamma^k_{1k} &= \frac{1}{2} g^{kl} [\overline{\nabla}_1 g_{kl} + \overline{\nabla}_k g_{1l} - \overline{\nabla}_l g_{1k}] \\
&= \frac{1}{2} g^{kk} [\overline{\nabla}_1 g_{kk} + 0 + 0] \\
&= \frac{1}{2} \frac{1}{u^2} [2 u u_x] \\
&= \frac{u u_x}{u^2} = \frac{u_x}{u}.
\end{aligned}
$$

An example for other Christoffel symbols that is equal zero

$$
\begin{aligned}
\Gamma^2_{22} &= \frac{1}{2} g^{22} [\overline{\nabla}_2 g_{22} + \overline{\nabla}_2 g_{22} - \overline{\nabla}_2 g_{22}] \\
&= \frac{1}{2} g^{22} [0 + 0 + 0] \\
&= 0.
\end{aligned}
$$

□

Lemma 2.2. *The non-zero components of* $\nabla \mathcal{W}$ *for axially symmetric hypersurfaces of the form* (2.1) *are*

$$
\nabla_1 h^1{}_1 = \frac{-u_{xxx}}{(1+u_x^2)^{\frac{3}{2}}} + \frac{3\, u_x u_{xx}^2}{(1+u_x^2)^{\frac{5}{2}}},
$$

and

$$
\nabla_1 h^k{}_k = \frac{-u_x}{u^2 \sqrt{1+u_x^2}} - \frac{u_x u_{xx}}{u\,(1+u_x^2)^{\frac{3}{2}}} = \frac{u_x}{u}\,(\kappa_1 - \kappa_2)\,.
$$

Proof: We use the formula

$$
\nabla_i h^j{}_k = \overline{\nabla}_i h^j{}_k + \Gamma^j_{il} h^l{}_k - \Gamma^l_{ik} h^j{}_l \;,
$$

in addition to the expressions for h^j_k from (2.11) and Lemma 2.1. Therefore

$$
\begin{aligned}
\nabla_1 h_1^1 &= \overline{\nabla}_1 h_1^1 + \Gamma_{1l}^1 h_1^l - \Gamma_{11}^l h_l^1 \\
&= \overline{\nabla}_1 h_1^1 + \Gamma_{11}^1 h_1^1 - \Gamma_{11}^1 h_1^1 \\
&= \overline{\nabla}_1 h_1^1 \\
&= \frac{\partial}{\partial x}\left[-\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}\right] \\
&= \frac{-u_{xxx}(1+u_x^2)^{\frac{3}{2}} + 3u_{xx}(1+u_x^2)^{\frac{1}{2}} u_x u_{xx}}{(1+u_x^2)^3} \\
&= \frac{-u_{xxx}}{(1+u_x^2)^{\frac{3}{2}}} + \frac{3\,u_x u_{xx}^2}{(1+u_x^2)^{\frac{5}{2}}},
\end{aligned}
$$

and for $k \geqslant 2$

$$
\begin{aligned}
\nabla_1 h^k{}_k &= \overline{\nabla}_1 h^k{}_k + \Gamma_{1l}^k h^l{}_k - \Gamma_{1k}^l h^k{}_l \\
&= \overline{\nabla}_1 h^k{}_k + \Gamma_{1k}^k h^k{}_k - \Gamma_{1k}^k h^k{}_k \\
&= \overline{\nabla}_1 h^k{}_k \\
&= \frac{\partial}{\partial x}\left[\frac{1}{u\sqrt{1+u_x^2}}\right] \\
&= -\frac{u_x}{u^2\sqrt{1+u_x^2}} - \frac{u_x u_{xx}}{u(1+u_x^2)^{\frac{3}{2}}} \\
&= \frac{u_x}{u}\left[-\frac{1}{u\sqrt{1+u_x^2}} - \frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}\right] \\
&= \frac{u_x}{u}\left(\kappa_1 - \kappa_2\right).
\end{aligned}
$$

□

Chapter 3

Evolution equations and preliminary results

From the evolution of the position vector we can compute corresponding evolution equations for all geometric quantities associated with the evolving hypersurface. In this Chapter we introduce various useful evolution equations of the evolving hypersurface.

1 Fully nonlinear curvature flow

Let M_t be an evolving family of hypersurfaces in $\mathbb{R}^{n+1}$ with boundary, M_t is moved by a curvature flow if it is satisfies the nonlinear parabolic equation

$$\frac{\partial X}{\partial t}(x,t) = -F\left(\mathcal{W}(x.t)\right)\nu(x.t), \qquad x \in M,\ t \in \mathbb{R}, \tag{3.1}$$

where F is the fully nonlinear speed of M_t, $\mathcal{W}(x,t)$ denotes the matrix of the Weingarten map of the evolving hypersurface $M_t = X(\cdot,t)$ at the point $X(x,t)$.

In our study, we will consider the case in which the initial surface M_0 in $\mathbb{R}^n$ is axially symmetric generated by rotating graph u around x_1 axis and we will choose some structure conditions on F that give us similar properties as those for mean curvature flow. The goal is to study evolution equations of fully nonlinear curvature flow and when we move from the quasilinear equations of MCF to fully non linear evolution equations, we expect some of the same properties as well as some differences because of the additional nonlinearity. As with mean curvature flow, we expect the surface may develop singularities before shrinking to a point.

We will denote by $\left(\dot{F}^{kl}\right)$ in (3.1) the matrix of first partial derivatives of F with respect to the components of its argument:

$$\frac{\partial}{\partial s} F(A+sB)\Big|_{s=0} = \dot{F}^{kl}(A) B_{kl}.$$

Similarly for the second partial derivatives of F we write

$$\frac{\partial^2}{\partial s^2} F(A+sB)\Big|_{s=0} = \ddot{F}^{kl,rs}(A) B_{kl} B_{rs}.$$

We will also use the notation

$$\dot{f}^{i}(\kappa) = \frac{\partial f}{\partial \kappa_i}(\kappa) \text{ and } \ddot{f}^{ij}(\kappa) = \frac{\partial^2 f}{\partial \kappa_i \kappa_j}(\kappa),$$

because $F(\mathcal{W}) = f(\lambda(\mathcal{W}))$ which mean F is a symmetric function of the eigenvalue of $\mathcal{W}$. Unless otherwise indicated, throughout this work we will always evaluate partial derivatives of F at $\mathcal{W}$ and partial derivatives of f at $\kappa(\mathcal{W})$.

In a local orthonormal frame of eigenvectors of $\mathcal{W}$, we may write $\ddot{F}$ in terms of $\ddot{f}$ and $\dot{f}$ as follows, for any symmetric matrix B. The next Theorem is important because it is giving a relation between derivatives of eigenvalues and eigenvectors of symmetric matrices, and of functions of symmetric matrices defined in terms of their eigenvalues:

Theorem 3.1. *[8] Let f be a C^2 symmetric function defined on a symmetric region Ω in $\mathbb{R}^n$, Let $\overline{\Omega} = \{A \in Sym(n) : \lambda(A) \in \Omega\}$, and define $F : \overline{\Omega} \to \mathbb{R}$ by $F(A) = f(\lambda(A))$. Then at any diagonal $A \in \overline{\Omega}$ with distinct eigenvalues, the second derivative of F in direction $B \in Sym(n)$ is given by*

$$\ddot{F}^{pq,rs}(\mathcal{W}) B_{pq} B_{rs} = \ddot{f}^{pr} B_{pp} B_{rr} + 2 \sum_{p<r} \frac{\dot{f}^{p}(\kappa) - \dot{f}^{r}(\kappa)}{\kappa_p - \kappa_r} (B_{pr})^2. \tag{3.2}$$

This formula makes sense as a limit in the case of any repeated values of κ_i. For details of the proof, we refer the reader to [8].

We will denote by ∇ the covariant derivative on M_t and by $\overline{\nabla}$ the derivative on $[0,a]\times\mathbb{S}^{n-1}$. Since $M=[0,a]\times S^{n-1}$ we will write $\overline{\nabla}_1=\frac{\partial}{\partial x}$, while $\overline{\nabla}_j$, $j\geq 2$ denote the $\mathbb{S}^{n-1}$ derivatives.

The speed functions F that we consider have the following properties:

Conditions 1

i) $F(\mathcal{W})=f(\kappa(\mathcal{W}))$ where $\kappa\left(\mathcal{W}\right)$ gives the eigenvalues of $\mathcal{W}$ and f is a smooth, symmetric function of the eigenvalues κ of $\mathcal{W}$.

ii) f is defined on an open convex cone Γ containing the positive cone $\Gamma^+=\{\kappa=(\kappa_1,\ldots,\kappa_n):\kappa_i>0 \text{ for all } i\}$.

iii) f is strictly increasing $\frac{\partial f}{\partial\kappa_1}>0$ for each $i=1,\ldots,n$ at every point in Γ.

iv) f is positive and normalised, $f\left(1,\ldots,1\right)=1$.

v) f is homogeneous of degree 1: $f(K\kappa)=Kf(\kappa)$ for any $K>0$ and all $\kappa\in\Gamma$. For more details of homogeneity see Appendix Section 1.

vi) f is convex (sometimes we can remove this condition using Codazzi equations as in Andrews [5, 9], see Chapter 5).

Speeds which satisfy the above conditions are discussed in [53] and in [14]; in particular, having $\frac{\partial f}{\partial k_1}>0$ ensure equation (3.1) is parabolic modulo tangential diffeomorphism. Moreover, if we don't require (vi) then we can take linear combinations of examples requiring individual convexity or concavity. All these conditions, sometimes with some adjustments, have been used before in curvature contraction flows of convex hypersurfaces [3, 8, 9, 15, 16, 23, 36] and recently in flows of closed hypersurfaces not necessarily convex [12, 14, 13, 53]. Some example functions F are given in those papers; in particular many examples satisfying the above properties including positivity on a cone larger than the positive cone can be built from appropriate operations of the elementary symmetric functions of the principal curvatures.

Remarks:

i) p is positive and k can be either positive or negative such that F is positive.

ii) Conditions 1, iii) ensures existence at least for a short time of a solution to (3.1); with initial condition $u(\cdot, 0) = u_0$ and pure Neumann boundary conditions. We will name later in each case the exact conditions that is required and we refer the reader to a precise short time existence to the flow in this setting in Theorem 4.1.

For Theorem 4.2 concerning the behaviour of solutions at the maximal existence time, we require the following additional structure condition on F:

Condition 2

Suppose f satisfies $\lim_{z \to -\infty} f(z, 1, \ldots, 1) < 0$, where we allow the case that the limit is equal to $-\infty$.

Remark: For the above condition to be satisfied the cone Γ of definition of f must allow the above limit to be taken.

Some examples of f satisfying Condition 2 are

i) The mean curvature, $F = H$, so $f(z, 1, \ldots, 1) = z + (n-1)$.

ii) A fully nonlinear example, $F = \frac{1}{n+\eta\sqrt{n}} (H + \eta |A|)$ for any $\eta \in [0, 1)$. In this case

$$f = \frac{1}{n+\eta\sqrt{n}} \left\{ [\kappa_1 + (n-1)\kappa_2] + \eta\sqrt{\kappa_1^2 + (n-1)\kappa_2^2} \right\},$$

so $f(z, 1, \ldots, 1) = \frac{1}{n+\eta\sqrt{n}} \left\{ z + (n-1) + \eta\sqrt{z^2 + (n-1)} \right\}$.

iii) More generally, for a constant $\eta \in [0, 1)$ and $p \geq 1$ we have

$$f = \left(n + \eta\, n^{\frac{1}{p}} \right)^{-1} \left[\sum_{i=1}^{n} \kappa_i + \eta \left(\sum_{j=1}^{n} \kappa_j^p \right)^{\frac{1}{p}} \right].$$

Condition 2 is required only for our characterisation of the singular time, Theorem 4.2. Briefly, the purpose of this condition is as follows: in Lemma 4.2 we show that $F > 0$ is preserved under the evolution (3.1). The rotational curvatures $\kappa_j, j = 2, ..., n$ remain positive under the evolution. Using homogeneity, Condition 2 implies that $z = \frac{\kappa_1}{\kappa_2}$ does not become too negative, giving rise to a lower bound on κ_1 in terms of κ_2.

2 Evolution equations

We will compute all evolution equations that we will need later. The computation techniques are similar to [39] which depend on local coordinates not adapted frames. Evolution equations on the evolving surface will be used to analyse our flow behaviour, for general n. The following evolution equations are natural generalizations of Huisken work [39] for the mean curvature flow.

For consistency with previous work in this section we will denote coordinate derivatives by $\frac{\partial}{\partial x_i}$.

Lemma 3.1. *The evolution equation of the metric for the evolving hypersurface M_t satisfies*

$$\frac{\partial}{\partial t} g_{ij} = -2F h_{ij}.$$

Proof:

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= \frac{\partial}{\partial t} \left\langle \frac{\partial X}{\partial x_i}, \frac{\partial X}{\partial x_j} \right\rangle \\
&= 2 \left\langle \frac{\partial}{\partial t} \frac{\partial X}{\partial x_i}, \frac{\partial X}{\partial x_j} \right\rangle \\
&= 2 \left\langle \frac{\partial}{\partial x_i} (-F\nu), \frac{\partial X}{\partial x_j} \right\rangle \\
&= 2F \left\langle -\frac{\partial \nu}{\partial x_i}, \frac{\partial X}{\partial x_j} \right\rangle \\
&= -2F h_{ij}. \qquad (3.3)
\end{aligned}$$

□

Remark: Because the tangent vector $\frac{\partial X}{\partial x_i}$ satisfies $\left\langle \nu, \frac{\partial X}{\partial x_j} \right\rangle = 0$, therefore

$$\frac{\partial}{\partial x_i} \left\langle \nu, \frac{\partial X}{\partial x_j} \right\rangle = \left\langle \frac{\partial \nu}{\partial x_i}, \frac{\partial X}{\partial x_j} \right\rangle + \left\langle \nu, \frac{\partial^2 X}{\partial x_i \partial x_j} \right\rangle = 0,$$

and so

$$h_{ij} = \left\langle \frac{\partial \nu}{\partial x_i}, \frac{\partial x}{\partial x_j} \right\rangle = -\left\langle \nu, \frac{\partial^2 x}{\partial x_i \partial x_j} \right\rangle.$$

Corollary 3.1. $\frac{\partial}{\partial t} g^{ij} = 2Fh^{ij}$.

Proof: Using Lemma 3.1

$$\begin{aligned} \frac{\partial}{\partial t} g^{ij} &= -g^{im} \left(\frac{\partial}{\partial t} g_{mn} \right) g^{nj} \\ &= -g^{im} \left(-2Fh_{mn} \right) g^{nj} \\ &= 2Fh^{ij}. \end{aligned}$$

□

Lemma 3.2. *The evolution equation for the outer unit normal is given as follows*

$$\frac{\partial \nu}{\partial t} = \nabla F.$$

Proof: For any vector $Y = Y_j \tau_j$ we have

$$Y = \langle Y, \tau_i \rangle \tau_i,$$

then

$$\begin{aligned} \langle Y, \tau_k \rangle &= \langle Y, \tau_i \rangle \langle \tau_i, \tau_k \rangle \\ &= \langle Y, \tau_i \rangle g_{ik}, \end{aligned}$$

which will be used to compute the evolution equation of ν.

$$\begin{aligned} \frac{\partial \nu}{\partial t} &= \left\langle \frac{\partial \nu}{\partial t}, \frac{\partial X}{\partial x_i} \right\rangle \frac{\partial X}{\partial x_j} g^{ij} \\ &= -\left\langle \nu, \frac{\partial}{\partial t} \frac{\partial X}{\partial x_i} \right\rangle \frac{\partial X}{\partial x_j} g^{ij} \end{aligned}$$

$$\begin{aligned}\frac{\partial \nu}{\partial t} &= -\left\langle \nu, \frac{\partial}{\partial x_i}(-F\nu) \right\rangle \frac{\partial X}{\partial x_j} g^{ij} \\ &= \frac{\partial}{\partial x_i} F \frac{\partial X}{\partial x_j} g^{ij} \\ &= \nabla F.\end{aligned}$$

□

Corollary 3.2.

$$\frac{\partial}{\partial t} \langle \nu, i_1 \rangle = \mathcal{L} \langle \nu, i_1 \rangle + \dot{F}^{ij} h_j^k h_{ik} \langle \nu, i_1 \rangle ,$$

where the elliptic operator $\mathcal{L}$ *is given by* $\mathcal{L}x = \dot{F}^{kl} \nabla_k \nabla_l x$.

Proof: From Lemma 3.2 we have

$$\frac{\partial}{\partial t} \langle \nu, i_1 \rangle = \langle \nabla F, i_1 \rangle. \tag{3.4}$$

Because $\nabla_j \nu = h_j^k \tau_k$ we have

$$\nabla_i \nabla_j \nu = \nabla_i h_j^k \tau_k + h_j^k \nabla_i \tau_k = \nabla^k h_{ij} \tau_k - h_j^k h_{ik} \nu,$$

$$\dot{F}^{ij} \nabla_i \nabla_j \nu = \nabla F - \dot{F}^{ij} h_j^k h_{ik} \nu. \tag{3.5}$$

By substituting (3.5) into (3.4)

$$\begin{aligned}\frac{\partial}{\partial t} \langle \nu, i_1 \rangle &= \left\langle \left(\dot{F}^{ij} \nabla_i \nabla_j \nu + \dot{F}^{ij} h_j^k h_{ik} \nu \right), i_1 \right\rangle \\ &= \dot{F}^{ij} \nabla_i \nabla_j \langle \nu, i_1 \rangle + \dot{F}^{ij} h_j^k h_{ik} \langle \nu, i_1 \rangle . \end{aligned}\tag{3.6}$$

□

Lemma 3.3. *The second fundamental form of* M_t *evolves according to*

$$\frac{\partial}{\partial t} h_{ij} = \mathcal{L} h_{ij} + \dot{F}^{kl,rs} \nabla_i h_{rs} \nabla_j h_{kl} + \dot{F}^{kl} h_l^m h_{km} h_{ij} - 2F h_i^m h_{jm}.$$

Proof: We compute that

$$
\begin{aligned}
\frac{\partial}{\partial t} h_{ij} &= -\frac{\partial}{\partial t}\left\langle \frac{\partial^2 X}{\partial x_i \partial x_j}, \nu \right\rangle \\
&= -\left\langle \frac{\partial}{\partial t}\frac{\partial^2 X}{\partial x_i \partial x_j}, \nu \right\rangle - \left\langle \frac{\partial^2 X}{\partial x_i \partial x_j}, \frac{\partial}{\partial t}\nu \right\rangle \\
&= -\left\langle \frac{\partial^2}{\partial x_i \partial x_j}\frac{\partial X}{\partial t}, \nu \right\rangle - \left\langle \frac{\partial^2 X}{\partial x_i \partial x_j}, \frac{\partial}{\partial t}\nu \right\rangle \\
&= -\left\langle \frac{\partial^2}{\partial x_i \partial x_j}\left(-F\nu\right), \nu \right\rangle - \left\langle \frac{\partial^2 X}{\partial x_i \partial x_j}, \frac{\partial}{\partial t}\nu \right\rangle \\
&= \left\langle \left(\frac{\partial^2 F}{\partial x_i \partial x_j}\nu + F\frac{\partial^2 \nu}{\partial x_i \partial x_j} \right), \nu \right\rangle - \left\langle \frac{\partial^2 X}{\partial x_i \partial x_j}, \frac{\partial}{\partial t}\nu \right\rangle \\
&= \frac{\partial^2 F}{\partial x_i \partial x_j} + F\left\langle \frac{\partial^2 \nu}{\partial x_i \partial x_j}, \nu \right\rangle - \left\langle \frac{\partial^2 X}{\partial x_i \partial x_j}, \frac{\partial}{\partial t}\nu \right\rangle. \qquad (3.7)
\end{aligned}
$$

Using the Gauss-Weingarten relation (as for example in [37]) $\frac{\partial \nu}{\partial x_j} = h_{jl} g^{lk} \frac{\partial X}{\partial x_k} = h_j^m \frac{\partial X}{\partial x_m}$, $\frac{\partial^2 X}{\partial x_i \partial x_j} = \Gamma_{ij}^m \frac{\partial X}{\partial x_m} - h_{ij}\nu$ and Lemma 3.2 equation (3.7) becomes

$$
\begin{aligned}
\frac{\partial}{\partial t} h_{ij} &= \frac{\partial^2 F}{\partial x_i \partial x_j} + F\left\langle \frac{\partial}{\partial x_i}\frac{\partial \nu}{\partial x_j}, \nu \right\rangle - \left\langle \Gamma_{ij}^k \frac{\partial X}{\partial x_k} - h_{ij}\nu, \frac{\partial F}{\partial x_l}\frac{\partial X}{\partial x_m} g^{lm} \right\rangle \\
&= \frac{\partial^2 F}{\partial x_i \partial x_j} + F\left\langle \frac{\partial}{\partial x_i} h_{jl} g^{lk} \frac{\partial X}{\partial x_k}, \nu \right\rangle - \Gamma_{ij}^k \frac{\partial F}{\partial x_l}\left\langle \frac{\partial X}{\partial x_k}, \frac{\partial X}{\partial x_m} g^{lm} \right\rangle \\
&\quad + h_{ij}\frac{\partial F}{\partial x_l}\left\langle \nu, \frac{\partial X}{\partial x_m} \right\rangle g^{lm}. \qquad (3.8)
\end{aligned}
$$

Because $\left\langle \nu, \frac{\partial X}{\partial x_m} \right\rangle = 0$ we will have

$$
\begin{aligned}
\frac{\partial}{\partial t} h_{ij} &= \frac{\partial^2 F}{\partial x_i \partial x_j} + F h_{jl} g^{lk}\left\langle \frac{\partial}{\partial x_i}\frac{\partial X}{\partial x_k}, \nu \right\rangle - \Gamma_{ij}^k \frac{\partial F}{\partial x_k} \\
&= \frac{\partial^2 F}{\partial x_i \partial x_j} + F h_{jl} g^{lk}\left\langle \frac{\partial^2 X}{\partial x_i \partial x_k}, \nu \right\rangle - \Gamma_{ij}^k \frac{\partial F}{\partial x_k} \\
&= \frac{\partial^2 F}{\partial x_i \partial x_j} + F h_{jl} g^{lk}\left\langle \Gamma_{ik}^m \frac{\partial X}{\partial x_m} - h_{ik}\nu, \nu \right\rangle - \Gamma_{ij}^k \frac{\partial F}{\partial x_k} \\
&= \frac{\partial^2 F}{\partial x_i \partial x_j} + F h_{jl} g^{lk}\Gamma_{ik}^m \cancelto{0}{\left\langle \frac{\partial X}{\partial x_m}, \nu \right\rangle} - F h_{jl} g^{lk} h_{ik}\left\langle \nu, \nu \right\rangle - \Gamma_{ij}^k \frac{\partial F}{\partial x_k} \\
&\qquad \text{because } \nabla_i \nabla_j F = \frac{\partial^2 F}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial F}{\partial x_k} \\
&= \nabla_i \nabla_j F - F h_j^k h_{ik}. \qquad (3.9)
\end{aligned}
$$

Now, we need $\nabla_j F(\mathcal{W}) = \dot{F}^{kl}\nabla_j h_{kl}$ which leads to

$$\nabla_i\nabla_j F(\mathcal{W}) = \ddot{F}^{kl,rs}\nabla_i h_{rs}\nabla_j h_{kl} + \dot{F}^{kl}\nabla_i\nabla_j h_{kl}. \tag{3.10}$$

From interchanging covariant derivatives, see Appendix Section 3, we obtain

$$\begin{aligned}\mathcal{L}h_{kl} &= \dot{F}^{kl}\nabla_i\nabla_j h_{kl}\\ &= \dot{F}^{kl}\nabla_k\nabla_l h_{ij} + \dot{F}^{kl}h_{il}h_k^m h_{jm} - \dot{F}^{kl}h_i^m h_{kl}h_{jm}\\ &\quad + \dot{F}^{kl}h_{ij}h_l^m h_{km} - \dot{F}^{kl}h_i^m h_{kj}h_{lm}\\ &= \dot{F}^{kl}\nabla_k\nabla_l h_{ij} - Fh_i^m h_{jm} + \dot{F}^{kl}h_l^m h_{km}h_{ij}. \end{aligned}\tag{3.11}$$

Using (3.10) equation (3.9) becomes

$$\frac{\partial}{\partial t}h_{ij} = \ddot{F}^{kl,rs}\nabla_i h_{rs}\nabla_j h_{kl} + \dot{F}^{kl}\nabla_i\nabla_j h_{kl} - Fh_i^k h_{jk}, \tag{3.12}$$

and from (3.11) the evolution equation of h_{ij} is obtained as follows

$$\begin{aligned}\frac{\partial}{\partial t}h_{ij} &= \ddot{F}^{kl,rs}\nabla_i h_{rs}\nabla_j h_{kl} + \dot{F}^{kl}\nabla_k\nabla_l h_{ij} - Fh_i^m h_{jm} + \dot{F}^{kl}h_l^m h_{km}h_{ij} - Fh_i^k h_{kj}\\ &= \ddot{F}^{kl,rs}\nabla_i h_{rs}\nabla_j h_{kl} + \dot{F}^{kl}\nabla_k\nabla_l h_{ij} - 2Fh_i^m h_{jm} + \dot{F}^{kl}h_l^m h_{km}h_{ij}\\ &= \mathcal{L}h_{ij} + \ddot{F}^{kl,rs}\nabla_i h_{rs}\nabla_j h_{kl} - 2Fh_i^m h_{jm} + \dot{F}^{kl}h_l^m h_{km}h_{ij}. \end{aligned}\tag{3.13}$$

□

Corollary 3.3. *The Weingarten map evolves according to*

$$\frac{\partial}{\partial t}h_j^i = \mathcal{L}h_j^i + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_j h_{rs} + \dot{F}^{kl}h_l^m h_{km}h_j^i.$$

Proof:

$$\begin{aligned}\frac{\partial}{\partial t}h_j^i &= \frac{\partial}{\partial t}\left[g^{ik}h_{kj}\right]\\ &= \left[\frac{\partial}{\partial t}g^{ik}\right]h_{kj} + g^{ik}\left[\frac{\partial}{\partial t}h_{kj}\right] \quad \text{from Corollary 3.1 and Lemma 3.3,}\end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial t}h^i_j &= 2Fh^{ik}h_{kj} + g^{ik}\left[\mathcal{L}h_{kj} + \ddot{F}^{kl,rs}\nabla_i h_{kl}\nabla_j h_{rs} + \dot{F}^{kl}h^m_l h_{km}h_{kj} - 2Fh^l_k h_{lj}\right] \\
&= \mathcal{L}h^i_j + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_j h_{rs} + \dot{F}^{kl}h^m_l h_{km}h^i_j. \qquad (3.14)
\end{aligned}
$$

□

Lemma 3.4. *Under the flow* (3.1),

(i.) $\frac{\partial}{\partial t}H = \mathcal{L}H + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl}h^m_l h_{km}H.$

(ii.) $\frac{\partial}{\partial t}F = \mathcal{L}F + F\dot{F}^{kl}h^m_l h_{km}.$

Proof: Equations *(i)* and *(ii)* will be derived as in [3].

To prove *(i)*

$$
\begin{aligned}
\frac{\partial}{\partial t}H &= g^j_i \frac{\partial}{\partial t}h^i_j \\
&= g^j_i\left[\mathcal{L}h^i_j + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_j h_{rs} + \dot{F}^{kl}h_{km}h^m_l h^i_j\right] \\
&= \mathcal{L}H + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl}h_{km}h^m_l H, \qquad (3.15)
\end{aligned}
$$

and to prove *(ii)*

$$
\begin{aligned}
\frac{\partial}{\partial t}F &= \dot{F}^j_i \frac{\partial}{\partial t}h^i_j \\
&= \dot{F}^j_i\left[\mathcal{L}h^i_j + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_j h_{rs} + \dot{F}^{kl}h_{km}h^m_l h^i_j\right] \\
&= \dot{F}^j_i\Big[\dot{F}^{kl}\nabla^i\nabla_j h^k_l + Fh^m_i h_{jm} - \dot{F}^{kl}h^m_l h_{km}h^i_j + \nabla^i\nabla_j F \\
&\quad -\dot{F}^{kl}\nabla^i\nabla_j h^k_l + \dot{F}^{kl}h_{km}h^m_l h^i_j\Big] \qquad \text{from (3.10) and (3.11)} \\
&= \dot{F}^j_i\left[Fh^m_i h_{jm} + \nabla^i\nabla_j F\right] \\
&= \dot{F}^j_i\left[\nabla^i\nabla_j F + Fh^m_i h_{jm}\right] \\
&= \mathcal{L}F + F\dot{F}^{ij}h^m_i h_{jm}. \qquad (3.16)
\end{aligned}
$$

□

Lemma 3.5. *As long as $y > 0$ we have*

(i.) $\frac{\partial}{\partial t}\langle X, i_1\rangle = \dot{F}^{kl}\nabla_k\nabla_l\langle X, i_1\rangle$.

(ii.) $\frac{\partial y}{\partial t} = \dot{F}^{kl}\nabla_k\nabla_l y - \frac{(n-1)\dot{F}^{22}}{y}$.

(iii.) $\frac{\partial q}{\partial t} = \dot{F}^{kl}\nabla_k\nabla_l q + \dot{F}^{kl}h_l^m h_{km} q - \Big\{\Big[2\dot{F}^{11}k - (n-1)\dot{F}^{22}p\Big]\, p$
$+ \Big[2\dot{F}^{11} - (n-1)\dot{F}^{22}\Big]\, q^2\Big\}\, q.$

(iv.) $\frac{\partial p}{\partial t} = \dot{F}^{kl}\nabla_k\nabla_l p + \dot{F}^{kl}h_l^m h_{km} p + 2\dot{F}^{11}q^2(k-p)$.

(v.) $\frac{\partial k}{\partial t} = \dot{F}^{kl}\nabla_k\nabla_l k + \dot{F}^{kl}h_l^m h_{km} k + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} - 2(n-1)\dot{F}^{11}q^2(k-p)$.

Proof: The first identity can be directly shown from (3.1). To prove *(ii)* we will need the following:

Using (2.5)

$$\begin{aligned}
\frac{\partial}{\partial t}y^2 &= \frac{\partial}{\partial t}\langle X, X\rangle - \frac{\partial}{\partial t}\langle X, i_1\rangle^2 \\
&= 2\left\langle \frac{\partial X}{\partial t}, X\right\rangle - 2\langle X, i_1\rangle\left\langle \frac{\partial X}{\partial t}, i_1\right\rangle \\
&= 2\left\langle \dot{F}^{kl}\nabla_k\nabla_l X, X\right\rangle - 2\langle X, i_1\rangle\left\langle \dot{F}^{kl}\nabla_k\nabla_l X, i_1\right\rangle \qquad (3.17)\\
&= 2\langle \mathcal{L}X, X\rangle - 2\langle X, i_1\rangle\langle \mathcal{L}X, i_1\rangle .
\end{aligned}$$

From (3.17) we can write

$$\begin{aligned}
\frac{\partial y}{\partial t} &= \frac{1}{2y}\frac{\partial y^2}{\partial t} \\
&= \frac{1}{2y}\left[2\langle \mathcal{L}X, X\rangle - 2\langle X, i_1\rangle\langle \mathcal{L}X, i_1\rangle\right] \\
&= \frac{1}{y}\left[\langle \mathcal{L}X, X\rangle - \langle X, i_1\rangle\langle \mathcal{L}X, i_1\rangle\right], \qquad (3.18)
\end{aligned}$$

and we calculate

$$\begin{aligned}
\mathcal{L}y^2 &= \dot{F}^{kl}\nabla_k\nabla_l y^2 \\
&= \dot{F}^{kl}\nabla_k\nabla_l\left[\langle X, X\rangle - \langle X, i_1\rangle^2\right] \\
&= \dot{F}^{kl}\nabla_k\left[2\langle \nabla_l X, X\rangle - 2\langle X, i_1\rangle\langle \nabla_l X, i_1\rangle\right],
\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}y^2 &= 2\dot{F}^{kl}\left[\langle \nabla_k\nabla_l X, X\rangle + \langle \nabla_l X, \nabla_k X\rangle - \langle \nabla_k X, i_1\rangle \langle \nabla_l X, i_1\rangle\right.\\
&\quad \left. - \langle X, i_1\rangle \langle \nabla_k \nabla_l X, i_1\rangle\right]\\
&= 2\langle \mathcal{L}X, X\rangle + 2\dot{F}^{kl}\langle \nabla_l X, \nabla_k X\rangle - 2\dot{F}^{kl}\langle \nabla_k X, i_1\rangle\langle \nabla_l X, i_1\rangle\\
&\quad - 2\dot{F}^{kl}\langle X, i_1\rangle\langle \nabla_k\nabla_l X, i_1\rangle\\
&= 2\langle \mathcal{L}X, X\rangle + 2\dot{F}^{kl}g_{kl} - 2\left[\dot{F}^{11}\langle \nabla_1 X, i_1\rangle^2 + (n-1)\dot{F}^{22}\langle \nabla_2 X, i_1\rangle^2\right]\\
&\quad - 2\langle X, i_1\rangle\langle \mathcal{L}X, i_1\rangle\\
&= 2\langle \mathcal{L}X, X\rangle + 2\ \mathrm{tr}_g\dot{F} - 2\left[\dot{F}^{11}\langle \tau_1, i_1\rangle^2\right] - 2\langle X, i_1\rangle\langle \mathcal{L}X, i_1\rangle\\
&\qquad \text{where } \mathrm{tr}_g\dot{F} = \dot{F}^{kl}g_{kl}\\
&= 2\langle \mathcal{L}X, X\rangle + \left[2\dot{F}^{11} + 2(n-1)\dot{F}^{22}\right] - 2\dot{F}^{11}p^2y^2 - 2\langle X, i_1\rangle\langle \mathcal{L}X, i_1\rangle\\
&= 2\langle \mathcal{L}X, X\rangle + \left[2\dot{F}^{11} + 2(n-1)\dot{F}^{22}\right]\\
&\quad - 2\dot{F}^{11}\left(1 - |\nabla y|^2\right) - 2\langle X, i_1\rangle\langle \mathcal{L}X, i_1\rangle. \qquad (3.19)
\end{aligned}
$$

Because

$$\nabla_1 y = -qy,$$

and

$$\nabla_i y = 0 \quad \text{for all } \ i = 2, \dots, n$$

we have

$$|\nabla y|^2 = |\nabla_1 y|^2 = q^2y^2. \qquad (3.20)$$

We know

$$
\begin{aligned}
\langle \nabla^i X, i_1\rangle^2 &= \langle \tau_i, i_1\rangle^2\\
&= y^2p^2\\
&= 1 - y^2q^2\\
&= 1 - |\nabla y|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
|\nabla X|^2 &= \langle \nabla^i X, \nabla_i X\rangle \\
&= \sum_i \langle \tau_i, \tau_i \rangle \\
&= u.
\end{aligned}
$$

Additionally, we need

$$
\begin{aligned}
\nabla_i y^2 &= \nabla_i(\langle X, X\rangle - \langle X, i_1\rangle^2) \\
&= 2\langle \nabla_i X, X\rangle - 2\langle X, i_1\rangle\langle \nabla_i X, i_1\rangle. \qquad (3.21)
\end{aligned}
$$

But $\nabla_l y^2 = 2y\nabla_l y$ and

$$
\begin{aligned}
\nabla_k \nabla_l y^2 &= \nabla_k(2y\nabla_l y) \\
&= 2\nabla_k y \nabla_l y + 2y\nabla_k\nabla_l y,
\end{aligned}
$$

so

$$
\begin{aligned}
\mathcal{L} y^2 &= \dot{F}^{kl}\nabla_k\nabla_l y^2 \\
&= 2\dot{F}^{kl}\nabla_k y\nabla_l y + 2y\dot{F}^{kl}\nabla_k\nabla_l y. \qquad (3.22)
\end{aligned}
$$

Therefore from (3.19) and (3.22)

$$
\begin{aligned}
\mathcal{L} y &= \dot{F}^{kl}\nabla_k\nabla_l y \\
&= \frac{1}{2y}\left[\dot{F}^{kl}\nabla_k\nabla_l y^2 - 2\dot{F}^{kl}\nabla_k y\nabla_l y\right] \\
&= \frac{1}{2y}\Big[2\langle \mathcal{L}X, X\rangle - 2\langle X, i_1\rangle\langle \mathcal{L}X, i_1\rangle + 2\dot{F}^{11} + 2(n-1)\dot{F}^{22} \\
&\quad -2\dot{F}^{11}(1 - |\nabla_1 y|^2) - 2\dot{F}^{11}|\nabla_1 y|^2\Big],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}y &= \frac{1}{y}\Big[\langle \mathcal{L}X, X\rangle - \langle X, i_1\rangle\langle \mathcal{L}X, i_1\rangle + \dot{F}^{11} + (n-1)\dot{F}^{22} - \dot{F}^{11}(1-|\nabla_1 y|^2) \\
&\quad -\dot{F}^{11}|\nabla_1 y|^2\Big] \\
&= \frac{1}{y}\Big[\langle \mathcal{L}X, X\rangle - \langle X, i_1\rangle\langle \mathcal{L}X, i_1\rangle + (n-1)\dot{F}^{22}\Big], \qquad (3.23)
\end{aligned}
$$

then from (3.18) and (3.23) we have

$$
\frac{\partial y}{\partial t} = \dot{F}^{kl}\nabla_k\nabla_l y - \frac{(n-1)\dot{F}^{22}}{y}. \qquad (3.24)
$$

We will use *(ii)* to prove *(iii)*; and because $\frac{\partial}{\partial t}qy = q\frac{\partial y}{\partial t} + y\frac{\partial q}{\partial t}$ we can write

$$
\frac{\partial q}{\partial t} = \frac{1}{y}\left[\frac{\partial}{\partial t}(qy) - q\frac{\partial y}{\partial t}\right]. \qquad (3.25)
$$

Because $q = \frac{\langle \nu, i_1\rangle}{y}$ with using (3.6) and (3.24) we compute the following

$$
\begin{aligned}
\frac{\partial q}{\partial t} &= \frac{1}{y^2}\left[y\frac{\partial}{\partial t}\langle \nu, i_1\rangle - \langle \nu, i_1\rangle\frac{\partial}{\partial t}y\right] \\
&= \frac{1}{y^2}\Big[y\dot{F}^{kl}\nabla_k\nabla_l\langle \nu, i_1\rangle + y\dot{F}^{kl}h_k^m h_{lm}\langle \nu, i_1\rangle - \langle \nu, i_1\rangle\dot{F}^{kl}\nabla_k\nabla_l y \\
&\quad + \frac{(n-1)}{y}\langle \nu, i_1\rangle\dot{F}^{22}\Big]. \qquad (3.26)
\end{aligned}
$$

We will use

$$
\begin{aligned}
\nabla_i q &= \nabla_i\left[\frac{\langle \nu, i_1\rangle}{y}\right] \\
&= \frac{1}{y^2}\left[y\nabla_i i\langle \nu, i_1\rangle - \langle \nu, i_1\rangle\nabla_i y\right] \\
&= \frac{1}{y^2}\left[h_1^i y^2 p - yq\delta_{i1}(-qy)\right] \\
&= h_1^i p + \delta_{i1}q^2, \qquad (3.27)
\end{aligned}
$$

so

$$
\nabla_1 q = h_1^1 p + \delta_{11}q^2 = kp + q^2,
$$

$$
\nabla_i q = 0, \text{ for all } \quad i = 2, \ldots, n
$$

and

$$
\begin{aligned}
|\nabla q|^2 &= |\nabla_1 q|^2 \\
&= (h_1^i p + \delta_{i1} q^2)(h_1^i p + \delta_{i1} q^2) \\
&= (h_1^i h_{i1} p^2 + 2h_1^i \delta_{i1} p q^2 + \delta_{i1} \delta_1^i q^4) \\
&= k^2 p^2 + 2kpq^2 + q^4 \\
&= (kp + q^2)^2. \qquad (3.28)
\end{aligned}
$$

We compute

$$
\begin{aligned}
\nabla_k \nabla_l q = \frac{1}{y^2} &\left[y \nabla_k \nabla_l \langle \nu, i_1 \rangle - \langle \nu, i_1 \rangle \nabla_k \nabla_l y \right] \\
&- \frac{2}{y^3} \left[y \nabla_k \langle \nu, i_1 \rangle - \langle \nu, i_1 \rangle \nabla_k y \right] \nabla_l y,
\end{aligned}
$$

so,

$$
\begin{aligned}
\mathcal{L} q &= \dot{F}^{kl} \nabla_k \nabla_l q \\
&= \frac{1}{y^2} \left[y \dot{F}^{kl} \nabla_k \nabla_l \langle \nu, i_1 \rangle - \langle \nu, i_1 \rangle \dot{F}^{kl} \nabla_k \nabla_l y \right] \\
&\quad - \frac{2}{y} \dot{F}^{kl} \nabla_k \left(\frac{\langle \nu, i_1 \rangle}{y} \right) \nabla_l y. \qquad (3.29)
\end{aligned}
$$

From (3.26) and (3.29)

$$
\begin{aligned}
\frac{\partial q}{\partial t} &= \frac{1}{y^2} \left[y \dot{F}^{kl} h_k^m h_{lm} \langle \nu, i_1 \rangle + \frac{(n-1)}{y} \langle \nu, i_1 \rangle \dot{F}^{22} \right] \\
&\quad + \frac{2}{y} \dot{F}^{kl} \nabla_k \frac{\langle \nu, i_1 \rangle}{y} \nabla_l y + \dot{F}^{kl} \nabla_k \nabla_l \frac{\langle \nu, i_1 \rangle}{y} \\
&= \dot{F}^{kl} \nabla_k \nabla_l q + \dot{F}^{kl} h_k^m h_{lm} \frac{\langle \nu, i_1 \rangle}{y} + \frac{(n-1)}{y^2} \dot{F}^{22} q + \frac{2}{y} \dot{F}^{11} \nabla_1 q \nabla_1 y \quad \text{from 3.6} \\
&= \dot{F}^{kl} \nabla_k \nabla_l q + \dot{F}^{kl} h_k^m h_{lm} q + \frac{(n-1)}{y^2} \dot{F}^{22} q + \frac{2}{y} \dot{F}^{11} (kp + q^2)(-qy) \\
&= \dot{F}^{kl} \nabla_k \nabla_l q + \dot{F}^{kl} h_k^m h_{lm} q + \frac{(n-1)}{y^2} \dot{F}^{22} q + \frac{2}{y} \dot{F}^{11} (kp)(-qy) + \frac{2}{y} \dot{F}^{11} (q^2)(-qy) \\
&= \dot{F}^{kl} \nabla_k \nabla_l q + \dot{F}^{kl} h_k^m h_{lm} q + \frac{(n-1)}{y^2} \dot{F}^{22} q - 2\dot{F}^{11} kpq - 2\dot{F}^{11} q^3. \qquad (3.30)
\end{aligned}
$$

Then

$$\begin{aligned}\frac{\partial q}{\partial t} &= \mathcal{L}q + \dot{F}^{kl}h_k^m h_{lm}q + q\frac{(n-1)}{y^2}\dot{F}^{22} - 2\dot{F}^{11}kpq - 2\dot{F}^{11}q^3\\ &= \mathcal{L}q + \dot{F}^{kl}h_k^m h_{lm}q + q\frac{(n-1)}{y^2}\dot{F}^{22} - 2\dot{F}^{11}kpq - \dot{F}^{11}q\left(y^{-2} - p^2\right) - \dot{F}^{11}q^3\\ &= \mathcal{L}q + \dot{F}^{kl}h_k^m h_{lm}q - 2\dot{F}^{11}kpq + q\frac{(n-1)}{y^2}\dot{F}^{22} - q\frac{\dot{F}^{11}}{y^2} + q\dot{F}^{11}p^2 - \dot{F}^{11}q^3\\ &= \mathcal{L}q + \dot{F}^{kl}h_k^m h_{lm}q - 2\dot{F}^{11}kpq - \dot{F}^{11}q^3 + q\dot{F}^{11}p^2 + \left[\frac{(n-1)\dot{F}^{22} - \dot{F}^{11}}{y^2}\right]q,\end{aligned} \tag{3.31}$$

or we can rewrite it as

$$\begin{aligned}\frac{\partial q}{\partial t} &= \mathcal{L}q + \dot{F}^{kl}h_k^m h_{lm}q - 2\dot{F}^{11}kpq - \dot{F}^{11}q^3 + q\dot{F}^{11}p^2 + \left[(n-1)\dot{F}^{22} - \dot{F}^{11}\right]\left(p^2+q^2\right)q\\ &\quad \text{because } y^{-2} = p^2 + q^2\\ &= \mathcal{L}q + \dot{F}^{kl}h_k^m h_{lm}q - 2\dot{F}^{11}kpq - \dot{F}^{11}q^3 + q\dot{F}^{11}p^2 + (n-1)\,\dot{F}^{22}p^2q - \dot{F}^{11}p^2q\\ &\quad + (n-1)\,\dot{F}^{22}q^3 - \dot{F}^{11}q^3\\ &= \mathcal{L}q + \dot{F}^{kl}h_k^m h_{lm}q + \Big[-2\dot{F}^{11}kp - \dot{F}^{11}q^2 + \dot{F}^{11}p^2 + (n-1)\dot{F}^{22}p^2 - \dot{F}^{11}p^2\\ &\quad +(n-1)\dot{F}^{22}q^2 - \dot{F}^{11}q^2\Big]q\\ &= \mathcal{L}q + \dot{F}^{kl}h_k^m h_{lm}q + \left[-2\dot{F}^{11}kp - \dot{F}^{11}q^2 + (n-1)\dot{F}^{22}p^2 + (n-1)\dot{F}^{22}q^2 - \dot{F}^{11}q^2\right]q\\ &= \mathcal{L}q + \dot{F}^{kl}h_k^m h_{lm}q + \left[-2\dot{F}^{11}kp - 2\dot{F}^{11}q^2 + (n-1)\dot{F}^{22}p^2 + (n-1)\dot{F}^{22}q^2\right]q\\ &= \mathcal{L}q + \dot{F}^{kl}h_k^m h_{lm}q + \left\{\left[-2\dot{F}^{11}k + (n-1)\dot{F}^{22}p\right]p + \left[-2\dot{F}^{11} + (n-1)\dot{F}^{22}\right]q^2\right\}q\\ &= \mathcal{L}q + \dot{F}^{kl}h_k^m h_{lm}q - \left\{\left[2\dot{F}^{11}k - (n-1)\dot{F}^{22}p\right]p + \left[2\dot{F}^{11} - (n-1)\dot{F}^{22}\right]q^2\right\}q.\end{aligned}$$

Therefore,

$$\frac{\partial q}{\partial t} = \mathcal{L}q + \dot{F}^{kl}h_k^m h_{lm}q - \left\{\left[2\dot{F}^{11}k - (n-1)\dot{F}^{22}p\right]p + \left[2\dot{F}^{11} - (n-1)\dot{F}^{22}\right]q^2\right\}q, \tag{3.32}$$

which gives the evolution equation for q.

We will compute now *(iv)* using (3.31)

$$\begin{aligned}\frac{\partial q^2}{\partial t} &= 2q\frac{\partial q}{\partial t} \\ &= 2q\mathcal{L}q + 2q\dot{F}^{kl}h_k^m h_{lm}q + 2\dot{F}^{11}\left(p^2 - q^2 - 2kp\right)q^2 + 2\left[\frac{(n-1)\dot{F}^{22} - \dot{F}^{11}}{y^2}\right]q^2.\end{aligned} \tag{3.33}$$

We need

$$\begin{aligned}\mathcal{L}q^2 &= \dot{F}^{kl}\nabla_k\nabla_l(q.q) \\ &= 2\dot{F}^{kl}\nabla_k(\nabla_l q.q) \\ &= 2\dot{F}^{kl}(\nabla_k\nabla_l q.q + \nabla_l q.\nabla_k q) \\ &= 2q\mathcal{L}q + 2\dot{F}^{kl}\nabla_k q\nabla_l q.\end{aligned} \tag{3.34}$$

From (3.34) and (3.33) we obtain

$$\begin{aligned}\frac{\partial q^2}{\partial t} = \mathcal{L}q^2 - 2\dot{F}^{kl}\nabla_k q\nabla_l q + 2q\dot{F}^{kl}h_k^m h_{lm}q + 2\dot{F}^{11}(p^2 - q^2 - 2kp)q^2 \\ + 2\left[\frac{(n-1)\dot{F}^{22} - \dot{F}^{11}}{y^2}\right]q^2.\end{aligned} \tag{3.35}$$

We compute also

$$\frac{\partial}{\partial t}y^{-2} = \frac{-2}{y^3}\frac{\partial}{\partial t}y = -\frac{2}{y^3}\left[\mathcal{L}y - \frac{(n-1)\dot{F}^{22}}{y}\right], \tag{3.36}$$

and because $\nabla_i y^{-2} = -\frac{2}{y^3}\nabla_i y$ we can have

$$\begin{aligned}\mathcal{L}y^{-2} &= \dot{F}^{kl}\nabla_k\nabla_l y^{-2} \\ &= \dot{F}^{kl}\nabla_k\left(\nabla_l y^{-2}\right) \\ &= \dot{F}^{kl}\nabla_k\left(-\frac{2}{y^3}\nabla_l y\right) \\ &= -2\dot{F}^{kl}\left[\nabla_k y^{-3}\nabla_l y + y^{-3}\nabla_k\nabla_l y\right]\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}y^{-2} &= -2\dot{F}^{kl}\left[-3y^{-4}\nabla_k y\nabla_l y + y^{-3}\nabla_k\nabla_l y\right] \\
&= \frac{6}{y^4}\dot{F}^{kl}\nabla_k y\nabla_l y - \frac{2}{y^3}\dot{F}^{kl}\nabla_k\nabla_l y \\
&= \frac{6}{y^4}\dot{F}^{kl}\nabla_k y\nabla_l y - \frac{2}{y^3}\mathcal{L}y. \qquad (3.37)
\end{aligned}
$$

From (3.36) and (3.37) we have

$$
\frac{\partial}{\partial t}y^{-2} = \mathcal{L}y^{-2} - \frac{6}{y^4}\dot{F}^{kl}\nabla_k y\nabla_l y + 2\frac{(n-1)\dot{F}^{22}}{y^4}. \qquad (3.38)
$$

Using (3.35) and (3.38)

$$
\begin{aligned}
\frac{\partial}{\partial t}p^2 &= \frac{\partial}{\partial t}\left(y^{-2} - q^2\right) \\
&= \mathcal{L}y^{-2} - \frac{6}{y^4}\dot{F}^{kl}\nabla_k y\nabla_l y + \frac{2(n-1)\dot{F}^{22}}{y^4} - \mathcal{L}q^2 + 2\dot{F}^{kl}\nabla_k q\nabla_l q - 2q\dot{F}^{kl}h_k^m h_{lm} q \\
&\quad - 2\dot{F}^{11}\left(p^2 - q^2 - 2kp\right)q^2 - 2\left[\frac{(n-1)\dot{F}^{22} - \dot{F}^{11}}{y^2}\right]q^2 \\
&= \mathcal{L}y^{-2} - \mathcal{L}q^2 - \frac{6}{y^4}\dot{F}^{kl}\nabla_k y\nabla_l y + 2\dot{F}^{kl}\nabla_k q\nabla_l q + \frac{2(n-1)\dot{F}^{22}}{y^4} - 2q\dot{F}^{kl}h_k^m h_{lm} q \\
&\quad - 2\dot{F}^{11}\left(p^2 - q^2 - 2kp\right)q^2 - 2\left[\frac{(n-1)\dot{F}^{22} - \dot{F}^{11}}{y^2}\right]q^2 \\
&= \mathcal{L}p^2 - \frac{6}{y^4}\dot{F}^{kl}\nabla_k y\nabla_l y + 2\dot{F}^{kl}\nabla_k q\nabla_l q + \frac{2(n-1)\dot{F}^{22}}{y^4} - 2q\dot{F}^{kl}h_k^m h_{lm} q \\
&\quad - 2\dot{F}^{11}\left(p^2 - q^2 - 2kp\right)q^2 - 2\left[\frac{(n-1)\dot{F}^{22} - \dot{F}^{11}}{y^2}\right]q^2. \qquad (3.39)
\end{aligned}
$$

From (3.39) we obtain

$$
\begin{aligned}
\frac{\partial p}{\partial t} &= \frac{1}{2p}\frac{\partial p^2}{\partial t} \\
&= \frac{1}{2p}\mathcal{L}p^2 - \frac{3}{py^4}\dot{F}^{kl}\nabla_k y\nabla_l y + \frac{1}{p}\dot{F}^{kl}\nabla_k q\nabla_l q + \frac{1}{p}\frac{(n-1)\dot{F}^{22}}{y^4} - \frac{q}{p}\dot{F}^{kl}h_k^m h_{ml} q \\
&\quad - \frac{\dot{F}^{11}}{p}\left(p^2 - q^2 - 2kp\right)q^2 - \frac{1}{p}\left[\frac{(n-1)\dot{F}^{22} - \dot{F}^{11}}{y^2}\right]q^2, \qquad (3.40)
\end{aligned}
$$

but

$$\begin{aligned}\mathcal{L}p^2 &= \dot{F}^{kl}\nabla_k\nabla_l\left(p^2\right)\\ &= \dot{F}^{kl}\nabla_k\left(2p\nabla_l p\right)\\ &= 2\dot{F}^{kl}\nabla_k p\nabla_l p + 2\dot{F}^{kl}p\nabla_k\nabla_l p\\ &= 2p\mathcal{L}p + 2\dot{F}^{kl}\nabla_k p\nabla_l p, \qquad (3.41)\end{aligned}$$

so

$$\mathcal{L}p = \frac{1}{2p}\mathcal{L}p^2 - \frac{1}{p}\dot{F}^{kl}\nabla_k p\nabla_l p. \qquad (3.42)$$

Furthermore, we need to compute the following

$$\begin{aligned}\nabla_i p^2 &= \nabla_i\left(y^{-2} - q^2\right)\\ &= -\frac{2}{y^3}\nabla_i y - 2q\nabla_i q\\ &= \frac{2}{y^3}\delta_{i1}qy - 2q\left[h_1^i p + q\delta_{1i}q\right] \quad \text{from 3.27}\\ &= \frac{2}{y^2}\delta_{i1}q - 2q\left[h_1^i p + q\delta_{1i}q\right]\\ &= 2\left(p^2+q^2\right)\delta_{i1}q - 2pqh_1^i - 2q^3\delta_{1i}\\ &= 2p^2q\delta_{i1} + 2q^3\delta_{i1} - 2pqh_1^i - 2q^3\delta_{1i}\\ &= 2p^2q\delta_{i1} - 2pqh_1^i. \qquad (3.43)\end{aligned}$$

From (3.43)

$$\begin{aligned}\nabla_i p &= \frac{1}{2p}\nabla_i p^2\\ &= pq\delta_{i1} - qh_{i1}, \qquad (3.44)\end{aligned}$$

so

$$\begin{aligned}&\nabla_1 p = pq - qh_1^1 = pq - qk = q\left(p-k\right),\\ &\nabla_i p = 0, \quad \text{for all} \quad i = 2,\ldots,n \qquad (3.45)\end{aligned}$$

and as a result,

$$
\begin{aligned}
|\nabla p|^2 &= |\nabla_1 p|^2 \\
&= \left(pq\delta_{i_1} - qh^i_{\ 1}\right)\left(pq\delta_{i_1} - qh^i_{\ 1}\right) \\
&= p^2q^2 - 2pq^2k + q^2k^2 \\
&= q^2\left(p^2 - 2pk + k^2\right) \\
&= q^2\left(p-k\right)^2 . \qquad (3.46)
\end{aligned}
$$

Therefore, the evolution equation of p can be computed as

$$
\begin{aligned}
\frac{\partial}{\partial t}p &= \mathcal{L}p + \frac{1}{p}\dot{F}^{kl}\nabla_k p\nabla_l p - \frac{3}{py^4}\dot{F}^{kl}\nabla_k y\nabla_l y - \frac{1}{p}\dot{F}^{kl}\nabla_k q\nabla_l q + \frac{1}{p}\frac{(n-1)\dot{F}^{22}}{y^4} \\
&\quad - \frac{q}{p}\dot{F}^{kl}h^m_k h_{lm} q - \frac{\dot{F}^{11}}{p}\left(p^2 - q^2 - 2kp\right)q^2 - \frac{1}{p}\left[\frac{(n-1)\dot{F}^{22} - \dot{F}^{11}}{y^2}\right]q^2 \\
&= \mathcal{L}p + \frac{1}{p}\left[\dot{F}^{11}|\nabla_1 p|^2 + (n-1)\dot{F}^{22}\cancelto{0}{|\nabla_2 p|^2}\right] - \frac{3}{py^4}\left[\dot{F}^{11}|\nabla_1 y|^2 + (n-1)\dot{F}^{22}\cancelto{0}{|\nabla_2 y|^2}\right] \\
&\quad + \frac{1}{p}\left[\dot{F}^{11}|\nabla_1 q|^2 + (n-1)\dot{F}^{22}\cancelto{0}{|\nabla_2 q|^2}\right] + \frac{1}{p}\frac{(n-1)\dot{F}^{22}}{y^4} - \frac{\dot{F}^{11}}{p}(p^2 - q^2 - 2kp)q^2 \\
&\quad - \frac{1}{p}\left[\frac{(n-1)\dot{F}^{22} - \dot{F}^{11}}{y^2}\right]q^2 - \frac{q}{p}\dot{F}^{kl}h^m_k h_{ml} q. \qquad (3.47)
\end{aligned}
$$

We need the following equations in order to simplify previous equation:

We know

$$
\dot{F}^{kl}\nabla_k q\nabla_l q = \dot{F}^{11}\left(\nabla_1 q\right)^2 = \dot{F}^{11}\left(pk + q^2\right)^2,
$$

and

$$
\begin{aligned}
\dot{F}^{kl}h^m_k h_{lm} &= \dot{F}^{11}h^m_1 h_{1m} + (n-1)\dot{F}^{22}h^m_2 h_{2m} \\
&= \dot{F}^{11}(h^1_1)^2 + (n-1)\dot{F}^{22}(h^2_2)^2 \\
&= \dot{F}^{11}k^2 + (n-1)\dot{F}^{22}p^2. \qquad (3.48)
\end{aligned}
$$

Now we simplify (3.47) to have the evolution equation of p

$$
\begin{aligned}
\frac{\partial p}{\partial t} &= \mathcal{L}p + \frac{1}{p}\dot{F}^{11}q^2\left(p-k\right)^2 - \frac{3}{py^2}\left(p^2+q^2\right)\dot{F}^{11}q^2y^2 + \frac{1}{p}\dot{F}^{11}\left(kp+q^2\right)^2 \\
&\quad + \frac{(n-1)}{p}\dot{F}^{22}\left(p^2+q^2\right)^2 - \frac{\dot{F}^{11}}{p}\left(p^2-q^2-2kp\right)q^2 - \frac{1}{p}\left[\frac{(n-1)\dot{F}^{22}-\dot{F}^{11}}{y^2}\right]q^2 \\
&\quad - \frac{1}{p}\dot{F}^{kl}h_k^m h_{lm}q^2 \\
&= \mathcal{L}p + \frac{1}{p}\dot{F}^{11}q^2\left(p-k\right)^2 - \frac{3}{p}\left(p^2+q^2\right)\dot{F}^{11}q^2 + \frac{1}{p}\dot{F}^{11}\left(kp+q^2\right)^2 \\
&\quad + \frac{(n-1)}{p}\dot{F}^{22}\left(p^2+q^2\right)^2 - \frac{\dot{F}^{11}}{p}\left(p^2-q^2-2kp\right)q^2 - \frac{1}{p}\left[\frac{(n-1)\dot{F}^{22}-\dot{F}^{11}}{y^2}\right]q^2 \\
&\quad - \frac{1}{p}\dot{F}^{kl}h_k^m h_{lm}\left(y^{-2}-p^2\right) \\
&= \mathcal{L}p + \frac{1}{p}\dot{F}^{11}q^2\left(p-k\right)^2 - \frac{3}{p}\left(p^2+q^2\right)\dot{F}^{11}q^2 + \frac{1}{p}\dot{F}^{11}\left(kp+q^2\right)^2 \\
&\quad + \frac{(n-1)}{p}\dot{F}^{22}\left(p^2+q^2\right)^2 - \frac{\dot{F}^{11}}{p}\left(p^2-q^2-2kp\right)q^2 - \frac{1}{p}\left[\frac{(n-1)\dot{F}^{22}-\dot{F}^{11}}{y^2}\right]q^2 \\
&\quad - \frac{1}{p}\dot{F}^{kl}h_k^m h_{lm}y^{-2} + \frac{1}{p}\dot{F}^{kl}h_k^m h_{lm}p^2 \\
&= \mathcal{L}p + \frac{1}{p}\dot{F}^{kl}h_k^m h_{lm}p^2 + \frac{1}{p}\dot{F}^{11}q^2\left(p-k\right)^2 - \frac{3}{p}(p^2+q^2)\dot{F}^{11}q^2 + \frac{1}{p}\dot{F}^{11}\left(kp+q^2\right)^2 \\
&\quad + \frac{(n-1)}{p}\dot{F}^{22}\left(p^2+q^2\right)^2 - \frac{\dot{F}^{11}}{p}\left(p^2-q^2-2kp\right)q^2 - \frac{1}{p}\left[\frac{(n-1)\dot{F}^{22}-\dot{F}^{11}}{y^2}\right]q^2 \\
&\quad - \frac{1}{p}\dot{F}^{kl}h_k^m h_{lm}y^{-2} \\
&= \mathcal{L}p + \frac{1}{p}\dot{F}^{kl}h_k^m h_{lm}p^2 + \frac{1}{p}\dot{F}^{11}q^2\left(p-k\right)^2 - \frac{3}{py^2}\left(p^2+q^2\right)\dot{F}^{11}q^2y^2 + \frac{1}{p}\dot{F}^{11}\left(kp+q^2\right)^2 \\
&\quad + \frac{(n-1)}{p}\dot{F}^{22}\left(p^2+q^2\right)^2 - \frac{\dot{F}^{11}}{p}\left(p^2-q^2-2kp\right)q^2 - \frac{1}{p}\left[\frac{(n-1)\dot{F}^{22}-\dot{F}^{11}}{y^2}\right]q^2 \\
&\quad - \frac{1}{p}\left[\dot{F}^{11}k^2 + (n-1)\dot{F}^{22}p^2\right]\left(p^2+q^2\right) \\
&= \mathcal{L}p + \dot{F}^{kl}h_k^m h_{lm}p + \dot{F}^{11}\left[\frac{q^2}{p}\left(p-k\right)^2 - 3\frac{q^2}{p}\left(p^2+q^2\right) + \frac{q^2}{p}\left(q^2+p^2\right) + \frac{1}{p}\left(kp+q^2\right)^2\right. \\
&\quad \left. -\frac{q^2}{p}\left(p^2-q^2-2kp\right) - \frac{k^2}{p}\left(p^2+q^2\right)\right] + (n-1)\dot{F}^{22}\left[\frac{1}{p}\left(p^2+q^2\right)^2 - \frac{q^2}{p}\left(p^2+q^2\right)\right. \\
&\quad \left. -\frac{p^2}{p}\left(p^2+q^2\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial p}{\partial t} &= \mathcal{L}p + \dot{F}^{kl}h_k^m h_{lm}p + \dot{F}^{11}\frac{1}{p}\left[q^2p^2 - 2pkq^2 + k^2q^2 - 3q^2p^2 - 3q^4 + k^2p^2 + 2kpq^2 + q^4\right.\\
&\quad \left. -q^2p^2 + q^4 + 2kpq^2 + q^4 + q^2p^2 - k^2p^2 - k^2q^2\right]\\
&\quad + (n-1)\dot{F}^{22}\left[\frac{p^4 + q^4 + 2q^2p^2 - q^2p^2 - q^4 - p^4 - q^2p^2}{p}\right]\\
&= \mathcal{L}p + \dot{F}^{kl}h_k^m h_{lm}p + \dot{F}^{11}\frac{1}{p}\left[-2q^2p^2 + 2kpq^2\right]\\
&= \mathcal{L}p + \dot{F}^{kl}h_k^m h_{lm}p + \dot{F}^{11}\left[2kq^2 - 2q^2p\right],
\end{aligned}
$$

which implies that

$$
\frac{\partial p}{\partial t} = \mathcal{L}p + \dot{F}^{kl}h_k^m h_{lm}p + 2\dot{F}^{11}q^2\left[k - p\right]. \tag{3.49}
$$

Considering $\frac{\partial p}{\partial t}$ and $\frac{\partial H}{\partial t}$ we will compute the evolution equation of k

$$
\begin{aligned}
\frac{\partial k}{\partial t} &= \frac{\partial}{\partial t}\left(H - (n-1)p\right)\\
&= \frac{\partial}{\partial t}H - (n-1)\frac{\partial}{\partial t}p\\
&= \mathcal{L}H + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl}h_{km}h_l^m H\\
&\quad - (n-1)\left[\mathcal{L}p + \dot{F}^{kl}h_{km}h_l^m p + 2\dot{F}^{11}q^2\left(k - p\right)\right]\\
&= \mathcal{L}[H - (n-1)p] + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl}h_{km}h_l^m[H - (n-1)p]\\
&\quad - 2(n-1)\dot{F}^{11}q^2(k-p),
\end{aligned}
$$

so

$$
\frac{\partial k}{\partial t} = \mathcal{L}k + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl}h_{km}h_l^m k - 2(n-1)\dot{F}^{11}q^2(k-p). \tag{3.50}
$$

□

Lemma 3.6. *Under the flow* (3.1),

$$
\frac{\partial}{\partial t}v = \mathcal{L}v - \dot{F}^{ij}h_j^k h_{ik}\; v + \frac{(n-1)}{y^2}v\dot{F}^{22} - 2v^{-1}\dot{F}^{ij}\nabla_i v\nabla_j v.
$$

Proof: Similar notation as in [43] will be used for this proof. Notation D will be used for the gradient in $\mathbb{R}^{n+1}$ and $D\omega$ is used to define the $(n+1)\times(n+1)$ matrix of first derivatives of ω.

Firstly, we need some useful identities as following:

We define

$$\omega = (0, \omega_2, ..., \omega_n + 1) = \left(0, \frac{x_2}{|\hat{X}|}, ..., \frac{x_{n+1}}{|\hat{X}|}\right), \quad \text{for } |\hat{X}| \neq 0, \tag{3.51}$$

where $\hat{X}$ is defined as in equation (2.3), and

$$y = \langle X, \omega \rangle = \frac{1}{|\hat{X}|} \langle (x_1, ...x_{n+1}), (0, x_2, ..., x_{n+1}) \rangle = \frac{|\hat{X}|^2}{|\hat{X}|} = |\hat{X}| = \sqrt{x_2^2 + ... + x_{n+1}^2}, \tag{3.52}$$

then for $y \neq 0$

$$\frac{\partial y}{\partial x_i} = \begin{cases} 0, & \text{for } i = 1 \\ \frac{x_i}{y}, & \text{for } i \geq 2. \end{cases} \tag{3.53}$$

As a result,

$$Dy = \omega \quad \text{and} \quad \nabla y = \omega - \langle \omega, \nu \rangle. \tag{3.54}$$

Observe that, for $i, j \geqslant 2$

$$\frac{\partial}{\partial x_i}\left(\frac{x_j}{y}\right) = \frac{\delta_{ij}}{y} - \frac{x_j}{y^2}\frac{x_i}{y} = \frac{1}{y}(\delta_{ij} - \omega_i \omega_j).$$

Since

$$D\omega = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \frac{\partial \omega_2}{\partial x_1} & \frac{\partial \omega_2}{\partial x_2} & \dots & \frac{\partial \omega_2}{\partial x_{n+1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \omega_{n+1}}{\partial x_1} & \frac{\partial \omega_{n+1}}{\partial x_2} & \dots & \frac{\partial \omega_{n+1}}{\partial x_{n+1}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & \frac{\partial \omega_2}{\partial x_2} & \dots & \frac{\partial \omega_2}{\partial x_{n+1}} \\ & \vdots & \ddots & \vdots \\ 0 & \frac{\partial \omega_{n+1}}{\partial x_2} & \dots & \frac{\partial \omega_{n+1}}{\partial x_{n+1}} \end{bmatrix},$$

we will have

$$D\omega.\nu = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & \frac{\partial \omega_2}{\partial x_2} & \dots & \frac{\partial \omega_2}{\partial x_{n+1}} \\ & \vdots & \ddots & \vdots \\ 0 & \frac{\partial \omega_{n+1}}{\partial x_2} & \dots & \frac{\partial \omega_{n+1}}{\partial x_{n+1}} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{y}\left(\delta_{k2} - \omega_k \omega_2\right) \nu_k \\ \frac{1}{y}\left(\delta_{k3} - \omega_k \omega_3\right) \nu_k \\ \vdots \\ \frac{1}{y}\left(\delta_{kn+1} - \omega_k \omega_{n+1}\right) \nu_k \end{bmatrix},$$

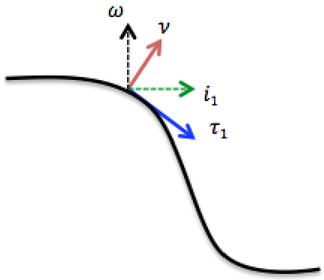

Figure 3.1: ω, ν, i_i and τ_1

with sum over repeated indices. Then, we show that

$$\langle \nu, D\omega.\nu \rangle = \sum_{i,k=2}^{n+1} \frac{1}{y} \left(\delta_{ki} - \omega_k \omega_i\right) \nu_k \nu_i = \frac{1}{y} \left(\sum_{i=2}^{n+1} \nu_i \nu_i - \langle \omega, \nu \rangle^2 \right).$$

Because

$$\nu = \langle \nu, i_1 \rangle i_1 + \langle \nu, \omega \rangle \omega = \langle \nu, i_1 \rangle i_1 + \cdots + \langle \nu, i_{n+1} \rangle i_{n+1},$$

we have

$$\langle \nu, \omega \rangle \omega = \sum_{k=2}^{n+1} \nu_k i_k,$$

which leads to

$$\langle \nu, \omega \rangle^2 = \langle \langle \nu, \omega \rangle \omega, \langle \nu, \omega \rangle \omega \rangle = \sum_{k=2}^{n+1} \nu_k \nu_k.$$

Therefore,

$$\langle \nu, D\omega.\nu \rangle = \frac{1}{y} \sum_{k=2}^{n+1} \nu_k \nu_k - \langle \nu, \omega \rangle^2 = 0. \tag{3.55}$$

Since $X = (x_1, x_2, \ldots, x_{n+1})$ we find

$$\langle X, D\omega.\nu \rangle = \sum_{i,k=2}^{n+1} \frac{1}{y} \left(\delta_{ik} - \omega_i \omega_k\right) \nu_k x_i = \frac{1}{y} \left(\sum_{i=2}^{n+1} \nu_i x_i - \langle \omega, X \rangle \langle \omega, \nu \rangle \right) \tag{3.56}$$

$$= \frac{1}{y} \left(\sum_{i=2}^{n+1} \nu_i x_i - \left\langle \frac{\hat{X} + 0 i_1}{|\hat{X}|}, \hat{X} + x_1 i_1 \right\rangle \left\langle \frac{\hat{X}}{|\hat{X}|}, \nu \right\rangle \right) \tag{3.57}$$

so

$$\langle X, D\omega.\nu\rangle = \frac{1}{y}\left(\sum_{i=2}^{n+1}\nu_i x_i - \frac{\hat{X}^2}{|\hat{X}|}\frac{1}{|\hat{X}|}\sum_{i=2}^{n+1}\nu_i x_i\right) = 0. \tag{3.58}$$

We know $y = \sqrt{x_2^2 + \cdots + x_{n+1}^2}$, so

$$\frac{\partial y}{\partial t} = \frac{1}{2y}\sum_{k=2}^{n+1} 2x_k \frac{\partial x_k}{\partial t} = -\frac{F}{y}\sum_{k=2}^{n+1} x_k \nu_k.$$

From (3.52), (3.54) and (3.58) we have

$$\frac{\partial y}{\partial t} = -\frac{F}{y}\langle\omega, X\rangle\langle\omega,\nu\rangle = -F\langle\omega,\nu\rangle.$$

Because the hypersurface is axially symmetric, $\langle\omega,\nu\rangle = \langle i_1, \tau_1\rangle = py$, see (2.6) and Figure 3.1, we find

$$\frac{\partial y}{\partial t} = -F\langle\omega,\nu\rangle = -Fpy. \tag{3.59}$$

Furthermore, as $\omega = \left(0, \frac{x_2}{y}, \ldots, \frac{x_{n+1}}{y}\right)$ we have

$$\begin{aligned}\frac{\partial\omega}{\partial t} &= \sum_{k=2}^{n+1}\left(-\frac{F}{y}\nu_k + \frac{x_k}{y^2}F\langle\omega,\nu\rangle\right) i_k \qquad \text{from (3.59)}\\ &= -\frac{F}{y}\sum_{k=2}^{n+1}(\nu_k - \omega_k\langle\omega,\nu\rangle)i_k. \end{aligned}\tag{3.60}$$

Consequently,

$$\begin{aligned}\left\langle\frac{\partial\omega}{\partial t}, \nu\right\rangle &= -\frac{F}{y}\left\langle\sum_{k=2}^{n+1}(\nu_k - \omega_k\langle\omega,\nu\rangle)\, i_k\ , \sum_{j=1}^{n+1}\nu_j i_j\right\rangle\\ &= -\frac{F}{y}\left(\sum_{k=2}^{n+1}\nu_k\nu_k - \langle\omega,\nu\rangle^2\right) = 0 \quad \text{from (3.55)}.\end{aligned}\tag{3.61}$$

From (2.4)

$$\begin{aligned}
\frac{\partial}{\partial t} v &= \frac{\partial}{\partial t} \langle \nu, \omega \rangle^{-1} \\
&= - \langle \nu, \omega \rangle^{-2} \left[\left\langle \frac{\partial}{\partial t} \nu, \omega \right\rangle + \left\langle \nu, \frac{\partial}{\partial t} \omega \right\rangle \right] \\
&= -v^2 \langle \nabla F, \omega \rangle, \text{ from Lemma 3.2.}
\end{aligned} \tag{3.62}$$

In order to have $\mathcal{L}v$ we need the following:

For any function ϕ

$$\begin{aligned}
\mathcal{L}\phi^{-1} &= \dot{F}^{ij} \nabla_i \nabla_j \phi^{-1} \\
&= -\dot{F}^{ij} \nabla_i \left[\frac{1}{\phi^2} \nabla_j \phi \right] \\
&= -\dot{F}^{ij} \left[-\frac{2}{\phi^3} \nabla_i \phi \nabla_j \phi + \frac{1}{\phi^2} \nabla_i \nabla_j \phi \right] \\
&= -\phi^{-2} \dot{F}^{ij} \nabla_i \nabla_j \phi + 2 \frac{1}{\phi^3} \dot{F}^{ij} \nabla_i \phi \nabla_j \phi \\
&= -\phi^{-2} \mathcal{L}\phi + 2\phi \dot{F}^{ij} \left(-\frac{\nabla_i \phi}{\phi^2} \right) \left(-\frac{\nabla_j \phi}{\phi^2} \right) \\
&= -\phi^{-2} \mathcal{L}\phi + 2\phi \dot{F}^{ij} \nabla_i \left(\frac{1}{\phi} \right) \nabla_j \left(\frac{1}{\phi} \right).
\end{aligned} \tag{3.63}$$

Replacing ϕ^{-1} by v we have

$$\begin{aligned}
\mathcal{L}v &= -v^2 \mathcal{L} v^{-1} + 2v^{-1} \dot{F}^{ij} \nabla_i v \nabla_j v \\
&= -v^2 \mathcal{L} \langle \nu, w \rangle + 2v^{-1} \dot{F}^{ij} \nabla_i v \nabla_j v.
\end{aligned} \tag{3.64}$$

Now we need the Weingarten relations

$$\overline{\nabla}_i \nu = \overline{\nabla}_{\tau_i} \nu = h_i^k \tau_k \quad , \quad \overline{\nabla}_i \tau_j = \overline{\nabla}_{\tau_i} \tau_j = -h_{ij} \nu + (\overline{\nabla}_i \tau_j)^\top, \tag{3.65}$$

and the fact that

$$\overline{\nabla}_\omega \tau_i = \overline{\nabla}_{\tau_i} \omega = [\omega, \tau_i] = 0, \tag{3.66}$$

where $\top$ indicates to the tangential component of a vector field and [,] is the Lie bracket. The calculation will be in normal coordinates where is $g_{ij} = \delta_{ij}$ and $(\overline{\nabla}_i \tau_j)^\top$

disappear at $X(x,t)$.

We start with

$$\begin{aligned}\overline{\nabla}_i\overline{\nabla}_j\nu &= \overline{\nabla}_i(h_j^k\tau_k) && (3.67)\\ &= (\overline{\nabla}_i h_j^k)\tau_k + h_j^k\overline{\nabla}_i\tau_k \\ &= \overline{\nabla}^k h_{ij}\tau_k - h_j^k h_{ik}\nu \quad \text{from Codazzi equation and (3.65)} \\ &= \nabla h_{ij} - h_j^k h_{ik}\nu, && (3.68)\end{aligned}$$

and we recall equation (3.5)

$$\mathcal{L}\nu = \dot{F}^{ij}\nabla_i\nabla_j\nu = \nabla F - \dot{F}^{ij}h_j^k h_{ik}\nu. \tag{3.69}$$

If we consider any unit normal vector $\eta = \eta_k i_k$ that satisfies

$$\begin{aligned}\overline{\nabla}_\eta\hat{X} = \eta_k\overline{\nabla}_{i_k}\hat{X} &= \sum_{k=1}^{n+1}\eta_k\sum_{j=2}^{n+1}\left(\frac{\partial x_j}{\partial x_k} + \overline{\Gamma}_{lk}^j x_l\right) i_j \\ &= \sum_{k=1}^{n+1}\eta_k\sum_{j=2}^{n+1}\delta_{kj}i_k = \sum_{j=2}^{n+1}\eta_k i_k = \eta - \langle\eta, i_1\rangle i_1. && (3.70)\end{aligned}$$

Note that $\overline{\Gamma}_{lk}^j$, the induced connection on $\mathbb{R}^{n+1}$, vanishes with orthonormal coordinates. Furthermore,

$$\begin{aligned}\eta\left(|\hat{X}|\right) = \eta_k i_k\left(|\hat{X}|\right) &= \eta_k\frac{\partial|\hat{X}|}{\partial x_k} \\ &= \eta_k\omega = \langle\eta, \omega\rangle,\end{aligned}$$

then we have

$$\begin{aligned}\overline{\nabla}_\eta\omega &= \overline{\nabla}_\eta\left(\frac{\hat{X}}{|\hat{X}|}\right) \\ &= \frac{1}{|\hat{X}|}\overline{\nabla}_\eta\hat{X} - \frac{1}{|\hat{X}|^2}\eta\left(|\hat{X}|\right)\hat{X} \\ &= \frac{1}{y}\left(\eta - \langle\eta, i_1\rangle i_1 - \langle\eta,\omega\rangle\omega\right).\end{aligned}$$

Suppose $\eta = \tau_1$ then we have

$$\overline{\nabla}_{\tau_1}\omega = \frac{1}{y}(\tau_1 - \langle\tau_1, i_1\rangle i_1 - \langle\tau_1, \omega\rangle\omega) = 0, \tag{3.71}$$

but if $\eta = \tau_1$ and $l \neq 1$ we have

$$\overline{\nabla}_{\tau_l}\omega = \frac{1}{y}(\tau_l - \langle\tau_l, i_1\rangle i_1 - \langle\tau_l, \omega\rangle\omega) = \frac{1}{y}\tau_l. \tag{3.72}$$

In normal coordinates, $\mathcal{L}\langle\nu, \omega\rangle$ can be calculated as

$$\begin{aligned}
\mathcal{L}\left\langle\nu, \omega\right\rangle &= \dot{F}^{ij}\overline{\nabla}_i\overline{\nabla}_j\langle\nu, \omega\rangle \\
&= \dot{F}^{ij}\overline{\nabla}_i\left[\langle\overline{\nabla}_j\nu, \omega\rangle + \langle\nu, \overline{\nabla}_j\omega\rangle\right] \\
&= \dot{F}^{ij}\left[\left\langle\overline{\nabla}_i\overline{\nabla}_j\nu, \omega\right\rangle + 2\left\langle\overline{\nabla}_i\nu, \overline{\nabla}_j\omega\right\rangle + \left\langle\nu, \overline{\nabla}_i\overline{\nabla}_j\omega\right\rangle\right], \\
&= \langle\mathcal{L}\nu, \omega\rangle + 2\dot{F}^{ij}\langle h_i^k\tau_k, \overline{\nabla}_j\omega\rangle + \dot{F}^{ij}\langle\nu, \overline{\nabla}_i\overline{\nabla}_\omega\tau_j\rangle \\
&\qquad \text{from (3.65) and .(3.66)} \\
&= \langle\nabla F - \dot{F}^{ij}h_j^k h_{ik}\nu, \omega\rangle + \dot{F}^{ij}h_i^k\langle\tau_k, \overline{\nabla}_j\omega\rangle + \dot{F}^{ij}h_k^j\langle\tau_j, \overline{\nabla}_k\omega\rangle \\
&\quad + \dot{F}^{ij}\langle\nu, \overline{\nabla}_\omega\overline{\nabla}_i\tau_j\rangle \\
&\qquad \text{from (3.69) and as } R^i_{jkl} = 0 \\
&= \langle\nabla F, \omega\rangle - \dot{F}^{ij}h_j^k h_{ik}\langle\nu, \omega\rangle + F^{ij}h_i^k\omega(g_{jk}) + \dot{F}^{ij}\langle\nu, \overline{\nabla}_\omega(-h_{ij}\nu + (\overline{\nabla}_i\tau_j)^\top)\rangle \\
&\qquad \text{from (3.65)} \\
&= \langle\nabla F, \omega\rangle - \dot{F}^{ij}h_j^k h_{ik}v^{-1} + \dot{F}^{ij}h_i^k\,\omega(g_{jk}) - \dot{F}^{ij}h_{ij}\left\langle\nu, \overline{\nabla}_\omega\nu\right\rangle \\
&\quad - \dot{F}^{ij}\left\langle\nu, \overline{\nabla}_\omega(h_{ij}).\nu\right\rangle + \dot{F}^{ij}\left\langle\nu, \overline{\nabla}_\omega((\overline{\nabla}_i\tau_j)^\top)\right\rangle \\
&= \langle\nabla F, \omega\rangle - \dot{F}^{ij}h_j^k h_{ik}v^{-1} + \dot{F}^{ij}h_i^k\,\omega(g_{jk}) \\
&\quad - \omega(h_{ij})\dot{F}^{ij}\left\langle\nu, \nu\right\rangle + \dot{F}^{ij}\left\langle\nu, \overline{\nabla}_\omega((\overline{\nabla}_i\tau_j)^\top)\right\rangle \\
&= \langle\nabla F, \omega\rangle - \dot{F}^{ij}h_j^k h_{ik}v^{-1} + \dot{F}^{ij}h_i^k\,\omega(g_{jk}) \\
&\quad - \omega(h_{ij})\dot{F}^{ij} + \dot{F}^{ij}\left\langle\nu, \overline{\nabla}_\omega((\overline{\nabla}_i\tau_j)^\top)\right\rangle.
\end{aligned} \tag{3.73}$$

Considering $\lambda_k\tau_k := (\overline{\nabla}_i\tau_i)^\top$, then we have

$$\left\langle\nu, \overline{\nabla}_\omega(\overline{\nabla}_i\tau_i)^\top\right\rangle = \langle\nu, \omega(\lambda_k)\tau_k\rangle + \langle\nu, \lambda_k\overline{\nabla}_\omega\tau_k\rangle = 0.$$

Therefore using equations (3.71), (3.72) and (3.66), equation (3.73)becomes

$$\mathcal{L}\langle \nu, \omega \rangle = \langle \nabla F, \omega \rangle - \dot{F}^{ij} h_j^k h_{ik} v^{-1} + \dot{F}^{ij} h_i^k \, \omega(g_{jk}) - \dot{F}^{ij} \omega(h_{ij}). \tag{3.74}$$

In order to have the last two terms in (3.74) we calculate in normal coordinates

$$\begin{aligned}
\omega(h_{ij}) &= -\omega\langle \nu, \overline{\nabla}_i \tau_j \rangle \\
&= \overline{\nabla}_\omega \langle \nu, \overline{\nabla}_i \tau_j \rangle \\
&= -\langle \nu, \overline{\nabla}_\omega \overline{\nabla}_i \tau_j \rangle - \langle \overline{\nabla}_\omega \nu, \overline{\nabla}_i \tau_j \rangle \\
&= -\langle \nu, \overline{\nabla}_i \overline{\nabla}_\omega \tau_j \rangle - \langle \overline{\nabla}_\omega \nu, -h_{ij}\nu + (\overline{\nabla}_i \tau_j)^\top \rangle \ \ \text{from (3.65)} \\
&= -\langle \nu, \overline{\nabla}_i \overline{\nabla}_\omega \tau_j \rangle + h_{ij} \langle \overline{\nabla}_\omega \nu, \nu \rangle \\
&= -\langle \nu, \overline{\nabla}_i \overline{\nabla}_j \omega \rangle \ \ \text{from (3.66)}.
\end{aligned}$$

As a result

$$\begin{aligned}
\dot{F}^{ij}\omega(h_{ij}) &= -\dot{F}^{ij} \langle \nu, \overline{\nabla}_i \overline{\nabla}_j \omega \rangle \ \ \text{where } \ i, j = 1, 2, \ldots, n \\
&= -\sum_{i=2}^{n} \dot{F}^{ii} \langle \nu, \overline{\nabla}_i \overline{\nabla}_i \omega \rangle \ \ i \neq 1 \ \text{ from (3.71) and (3.72)} \\
&= -\sum_{i=2}^{n} \dot{F}^{ii} \left\langle \nu, \overline{\nabla}_i \left(\frac{1}{y} \tau_i \right) \right\rangle \\
&= -\sum_{i=2}^{n} \frac{1}{y} \dot{F}^{ii} \langle \nu, \overline{\nabla}_i \tau_i \rangle \\
&= -\sum_{i=2}^{n} \frac{1}{y} \dot{F}^{ii} \left\langle \nu, -h_{ii}\nu + (\overline{\nabla}_i \tau_i)^\top \right\rangle \\
&= -\frac{1}{y}(n-1)\dot{F}^{22} \langle \nu, -h_{22}\nu \rangle \\
&= \frac{(n-1)}{y} h_{22} \dot{F}^{22} \\
&= \frac{(n-1)}{y} p \dot{F}^{22} \\
&= \frac{(n-1)}{y} \frac{1}{yv} \dot{F}^{22} \ \ \text{from (2.6)} \\
&= \frac{(n-1)}{y^2 v} \dot{F}^{22},
\end{aligned} \tag{3.75}$$

and

$$\begin{aligned}\dot{F}^{ij}\omega(g_{jk}) &= \dot{F}^{ij}\omega\langle\tau_j,\tau_k\rangle \\ &= \dot{F}^{ij}\langle\overline{\nabla}_\omega\tau_j,\tau_k\rangle - \dot{F}^{ij}\langle\tau_j,\overline{\nabla}_\omega\tau_k\rangle,\end{aligned}$$

and then

$$\begin{aligned}\dot{F}^{ij}h_i^k\omega(g_{jk}) &= \dot{F}^{ij}h_i^k\langle\overline{\nabla}_\omega\tau_j,\tau_k\rangle + \dot{F}^{ij}h_i^k\langle\tau_j,\overline{\nabla}_\omega\tau_k\rangle \\ &= \dot{F}^{ij}h_i^k\left\langle\frac{1}{y}\tau_j,\tau_k\right\rangle + \dot{F}^{ij}h_i^k\left\langle\tau_j,\frac{1}{y}\tau_k\right\rangle, \\ &\quad \text{from (3.71) and (3.72) where } j,k\neq 1 \\ &= 2\sum_{i=2}^{n}\dot{F}^{ij}h_i^k\left\langle\frac{1}{y}\tau_j,\tau_k\right\rangle \\ &= 2\sum_{i=2}^{n}\frac{\dot{F}^{ij}}{y}h_i^k g_{jk} \\ &= 2\sum_{i=2}^{n}\frac{\dot{F}^{ij}}{y}h_i^k\delta_{jk} \\ &= 2\frac{(n-1)}{y}p\dot{F}^{22} \\ &= 2\frac{(n-1)}{y}\frac{1}{yv}\dot{F}^{22} \quad \text{from (2.6)} \\ &= 2\frac{(n-1)}{y^2v}\dot{F}^{22}. \end{aligned} \tag{3.76}$$

From (3.76) and (3.75) the last two terms in (3.74) become

$$\dot{F}^{ij}h_i^k\omega(g_{jk}) - \dot{F}^{ij}\omega(h_{ij}) = \frac{(n-1)}{y^2v}\dot{F}^{22}. \tag{3.77}$$

Then we can write (3.74) as follows

$$\mathcal{L}\langle\nu,\omega\rangle = \mathcal{L}v^{-1} = \langle\nabla F,\omega\rangle - \dot{F}^{ij}h_j^k h_{ik}v^{-1} + \frac{(n-1)}{y^2v}\dot{F}^{22}. \tag{3.78}$$

Substituting this equation in (3.64) we obtain

$$\mathcal{L}v = -v^2\left[\langle \nabla F, \omega\rangle - \dot F^{ij} h_j^k h_{ik} v^{-1} + \frac{(n-1)}{y^2 v}\dot F^{22}\right] + 2v^{-1}\dot F^{ij}\nabla_i v \nabla_j v$$
$$= -v^2\langle \nabla F, \omega\rangle + \dot F^{ij} h_j^k h_{ik} v - \frac{(n-1)}{y^2} v\dot F^{22} + 2v^{-1}\dot F^{ij}\nabla_i v\nabla_j v. \qquad (3.79)$$

Comparing $-v^2\langle \nabla F, \omega\rangle$ from (3.79) and (3.62) we show the evolution equation of v

$$\frac{\partial}{\partial t} v = \mathcal{L}v - \dot F^{ij} h_j^k h_{ik} v + \frac{(n-1)}{y^2} v\dot F^{22} - 2v^{-1}\dot F^{ij}\nabla_i v\nabla_j v. \qquad (3.80)$$

□

Lemma 3.7. *We have the following evolution equation under the flow* (3.1)

$$\frac{\partial}{\partial t}\left(\frac{q}{F}\right) = \mathcal{L}\left(\frac{q}{F}\right) + \frac{2}{F}\dot F^{kl}\nabla_k F \nabla_l\left(\frac{q}{F}\right) - \left\{\left[2\dot F^{11}k - (n-1)\dot F^{22}p\right]p + \left[2\dot F^{11} - (n-1)\dot F^{22}\right]q^2\right\}\frac{q}{F}.$$

Proof: Starting with

$$\nabla_l\left(\frac{q}{F}\right) = \frac{1}{F^2}\left[F\nabla_l q - q\nabla_l F\right],$$

and

$$\nabla_k\nabla_l\left(\frac{q}{F}\right) = \frac{1}{F^2}\left[F\nabla_k\nabla_l q + \nabla_k F\nabla_l q - \nabla_k q\nabla_l F - q\nabla_k\nabla_l F\right] - \frac{2}{F^3}\left[F\nabla_l q - q\nabla_l F\right]\nabla_k F.$$

We have

$$\dot F^{kl}\nabla_k\nabla_l\left(\frac{q}{F}\right) = \frac{1}{F^2}\left[F\dot F^{kl}\nabla_k\nabla_l q - q\dot F^{kl}\nabla_k\nabla_l F\right] - \frac{2}{F}\dot F^{kl}\frac{1}{F^2}\left[F\nabla_l q - q\nabla_l F\right]\nabla_k F,$$

which can be written as

$$\mathcal{L}\left(\frac{q}{F}\right) = \frac{1}{F^2}\left[F\mathcal{L}q - q\mathcal{L}F\right] - \frac{2}{F}\dot F^{kl}\nabla_k\left(\frac{q}{F}\right)\nabla_l F. \qquad (3.81)$$

Then

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{q}{F}\right) &= \frac{F\frac{\partial}{\partial t}q - q\frac{\partial F}{\partial t}}{F^2} \\
&= \frac{F\left\{\mathcal{L}q + \dot{F}^{kl}h_k^m h_{ml}q - \left(\left[2\dot{F}^{11}k - (n-1)\dot{F}^{22}p\right]p + \left[2\dot{F}^{11} - (n-1)\dot{F}^{22}\right]q^2\right)q\right\}}{F^2} \\
&- \frac{q\left[\mathcal{L}F + F\dot{F}^{kl}h_{km}h_l^m\right]}{F^2} \\
&= \frac{F\mathcal{L}q - q\mathcal{L}F}{F^2} - \left(\left[2\dot{F}^{11}k - (n-1)\dot{F}^{22}p\right]p + \left[2\dot{F}^{11} - (n-1)\dot{F}^{22}\right]q^2\right)\frac{q}{F}.
\end{aligned}
\tag{3.82}
$$

From (3.81) we can write (3.82)

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{q}{F}\right) = \mathcal{L}\left(\frac{q}{F}\right) + \frac{2}{F}\dot{F}^{kl}\nabla_k F\nabla_l\frac{q}{F} - \Big\{\left[2\dot{F}^{11}k - (n-1)\dot{F}^{22}p\right]p \\
+ \left[2\dot{F}^{11} - (n-1)\dot{F}^{22}\right]q^2\Big\}\frac{q}{F}.
\end{aligned}
\tag{3.83}
$$

□

Lemma 3.8. *Under the flow* (3.1)

$$
\frac{\partial}{\partial t}\left(\frac{H}{F}\right) = \mathcal{L}\left(\frac{H}{F}\right) + \frac{2}{F}\dot{F}^{kl}\nabla_k F\nabla_l\frac{H}{F} + \frac{1}{F}\ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs}.
$$

Proof: We compute

$$
\begin{aligned}
\mathcal{L}\left(\frac{H}{F}\right) &= \dot{F}^{kl}\nabla_k\nabla_l\left(\frac{H}{F}\right) = \dot{F}^{kl}\nabla_k\frac{F\nabla_l H - H\nabla_l F}{F^2} \\
&= \dot{F}^{kl}\frac{F^2\left[\nabla_k F\nabla_l H + F\nabla_k\nabla_l H - \nabla_k H\nabla_l F - H\nabla_k\nabla_l F\right]}{F^4} \\
&- \quad \dot{F}^{kl}\frac{\left[F\nabla_l H - H\nabla_l F\right]2F\nabla_k F}{F^4} \\
&= \dot{F}^{kl}\left[\frac{F\nabla_k\nabla_l H - H\nabla_k\nabla_l F}{F^2} + \frac{-2F^2\nabla_k H\nabla_l F + 2FH\nabla_l F\nabla_k F}{F^4}\right],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}\left(\frac{H}{F}\right) &= \frac{F\mathcal{L}H - H\mathcal{L}F}{F^2} - 2\dot{F}^{kl}\frac{F^2\nabla_k H\nabla_l F - FH\nabla_l F\nabla_k F}{F^4} \\
&= \frac{F\mathcal{L}H - H\mathcal{L}F}{F^2} - 2\dot{F}^{kl}\nabla_k F\left[\frac{F\nabla_k H - H\nabla_l F}{F^3}\right] \\
&= \frac{F\mathcal{L}H - H\mathcal{L}F}{F^2} - \frac{2}{F}\dot{F}^{kl}\nabla_k F\left[\frac{F\nabla_l H - H\nabla_l F}{F^2}\right] \\
&= \frac{F\mathcal{L}H - H\mathcal{L}F}{F^2} - \frac{2}{F}\dot{F}^{kl}\nabla_k F\nabla_l\left(\frac{H}{F}\right),
\end{aligned}
\tag{3.84}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{H}{F}\right) &= \frac{F\mathcal{L}H - H\mathcal{L}F}{F^2} \\
&= \frac{F\left[\frac{\partial}{\partial t}H\right] - H\frac{\partial}{\partial t}F}{F^2},
\end{aligned}
$$

then

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{H}{F}\right) &= \frac{F\left[\mathcal{L}H + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl}h_{km}h^l_m H\right] - H\left[\mathcal{L}F + F\dot{F}^{kl}h_{km}h^m_l\right]}{F^2} \\
&= \frac{F\mathcal{L}H - H\mathcal{L}F}{F^2} + \frac{1}{F}\ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs}.
\end{aligned}
\tag{3.85}
$$

Using (3.84), equation (3.85) becomes

$$
\frac{\partial}{\partial t}\left(\frac{H}{F}\right) = \mathcal{L}\left(\frac{H}{F}\right) + \frac{2}{F}\dot{F}^{kl}\nabla_k F\nabla_l\left(\frac{H}{F}\right) + \frac{1}{F}\ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs}. \tag{3.86}
$$

□

Lemma 3.9. *We have the following evolution equation under the flow* (3.1)

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{k}{p}\right) = \mathcal{L}\left(\frac{k}{p}\right) + \frac{2}{p}\dot{F}^{kl}\nabla_k p\nabla_l\left(\frac{k}{p}\right) + \frac{1}{p}\ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} \\
- 2\dot{F}^{11}\frac{q^2}{p^2}[(n-1)p + k](k-p).
\end{aligned}
$$

Proof:

$$
\nabla_l\left(\frac{k}{p}\right) = \frac{1}{p^2}\left(p\nabla_l k - k\nabla_l p\right),
$$

$$\nabla_k \nabla_l \left(\frac{k}{p}\right) = \frac{1}{p^2}\left(p\nabla_k\nabla_l k - k\nabla_k\nabla_l p\right) - \frac{2}{p^3}\left(p\nabla_l k - k\nabla_l p\right)\nabla_k p,$$

$$\begin{aligned}\mathcal{L}\left(\frac{k}{p}\right) &= \dot{F}^{kl}\nabla_k p \nabla_l \left(\frac{k}{p}\right)\\ &= \frac{1}{p^2}\left(p\mathcal{L}k - k\mathcal{L}p\right) - \frac{2}{p}\dot{F}^{kl}\nabla_k p\nabla_l\left(\frac{k}{p}\right).\end{aligned} \tag{3.87}$$

Now we compute

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{k}{p}\right) &= \frac{p\frac{\partial}{\partial t}k - k\frac{\partial}{\partial t}p}{p^2}\\ &= \frac{p\left[\mathcal{L}k + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl}h_{km}h_l^m k - 2(n-1)\dot{F}^{11}q^2(k-p)\right]}{p^2}\\ &\quad - \frac{k\left[\mathcal{L}p + \dot{F}^{kl}h_{km}h_l^m p + 2\dot{F}^{11}q^2(k-p)\right]}{p^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{k}{p}\right) &= \frac{p\mathcal{L}k - k\mathcal{L}p}{p^2} + \frac{1}{p}\ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} - \frac{2(n-1)}{p}\dot{F}^{11}q^2(k-p)\\ &\quad - 2\frac{k}{p^2}\dot{F}^{11}q^2(k-p).\end{aligned} \tag{3.88}$$

From (3.87) into (3.88)

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{k}{p}\right) &= \mathcal{L}\left(\frac{k}{p}\right) + \frac{2}{p}\dot{F}^{kl}\nabla_k p\nabla_l\left(\frac{k}{p}\right) + \frac{1}{p}\ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs}\\ &\quad - 2\dot{F}^{11}q^2\left[\frac{(n-1)(k-p)}{p} + \frac{k(k-p)}{p^2}\right]\\ &= \mathcal{L}\left(\frac{k}{p}\right) + \frac{2}{p}\dot{F}^{kl}\nabla_k p\nabla_l \frac{k}{p} + \frac{1}{p}\ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs}\\ &\quad - 2\dot{F}^{11}q^2\left[\frac{(n-1)(k-p)p}{p^2} + \frac{k(k-p)}{p^2}\right]\\ &= \mathcal{L}\left(\frac{k}{p}\right) + \frac{2}{p}\dot{F}^{kl}\nabla_k p\nabla_l\left(\frac{k}{p}\right) + \frac{1}{p}\ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs}\\ &\quad - 2\dot{F}^{11}\frac{q^2}{p^2}[(n-1)p+k](k-p).\end{aligned} \tag{3.89}$$

□

Chapter 4

The Singularity

1 Introduction

Given an axially symmetric hypersurface of positive mean curvature, it is known that under mean curvature flow the surface evolves for a finite time until a singularity develops. In this Chapter we will consider an evolution by a fully nonlinear curvature flow of an axially symmetric hypersurface to see for which speeds we can show similar behaviour. This generalises work of Dziuk and Kawohl [26] for the mean curvature flow. In the case of shrinking convex hypersurface many examples of concave speed are given in [3], more in [8, 16]. Unfortunately our arguments of this chapter depend crucially on convexity of f. Convexity of the speed is essential for Lemma 4.3 and Theorem 4.2 in order to use this inequality $F \geqslant \frac{H}{n}$.

2 The evolving graph function

In a similar process as for the mean curvature flow, we add a tangential term to the normal evolution such that the flow problem with free boundary is well-posed". Short time existence of a solution then follows by standard theory (see, for example, [49]).

Our n-dimensional hypersurface M is axially symmetric about the x_1 axis, so there is a corresponding strictly positive and suitably smooth function $u : [0, a] \to \mathbb{R}$ such that M is parametrised by $X : [0, a] \times \mathbb{S}^{n-1} \times [0, T) \to \mathbb{R}^{n+1}$, which satisfies flow equation and maintain parametrisation

$$X(x_1,\omega) = (x_1, u(x_1)\omega),$$

where x_1 and ω independent of t.

For the evolution equations of u we need equation (3.51), (3.52) and (3.53) using the chain rule. Let $\overline{\omega} : [0,a] \times [0,T) \to \mathbb{R}^{n+1}$ be the unit outward normal of an n dimensional cylinder, intersecting the hypersurface at the point $u(x_1,t)$. The difference between $\overline{\omega}$ and ω that $\overline{\omega}$ is parametrized over the x_1 axis, while ω is parametrized over M^n.

$$X = x_1(t)i_1 + u(x_1(t),t)\overline{\omega}, \quad \overline{\omega} \in \mathbb{S}^{n-1},$$

$$\frac{\partial X}{\partial t} = \frac{\partial x_1}{\partial t} i_1 + \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t}\right)\overline{\omega},$$

$$-F\nu = \frac{\partial x_1}{\partial t} i_1 + \left(\frac{\partial u}{\partial t} + u_x\frac{\partial x_1}{\partial t}\right)\overline{\omega}.$$

Because $\nu = \frac{1}{\sqrt{1+u_x^2}}\left(-u_x i_1 + \overline{\omega}\right)$

$$-F\langle\nu,\nu\rangle = -\frac{\partial x_1}{\partial t}\frac{u_x}{\sqrt{1+u_x^2}} + \left(\frac{\partial u}{\partial t} + u_x\frac{\partial x_1}{\partial t}\right)\frac{1}{\sqrt{1+u_x^2}},$$

$$-F = \frac{\partial u}{\partial t}\frac{1}{\sqrt{1+u_x^2}},$$

$$\frac{\partial u}{\partial t} = -\sqrt{1+u_x^2}F, \tag{4.1}$$

which is the corresponding evolution equation for the graph height.

Equation (4.1) is a scalar evolution equation as opposed to (3.1) which is a system. In other words, if X is rotational symmetric with respect to x_1 axis, the equation can be written in the form of (4.1). Based on this situation, the evolution equation for u or y can be used because sometimes it is suitable to use one of them more than the other. Precisely, the changes of the radius when the surface is parametrized over the x_1 coordinate as an axially symmetric surface is described by the evolution equation for u while the changes of the height when the surface is parametrized over M^n is described by the evolution equation for y [43]. These two evolution equations are different because the evolution equation for u includes the term $\sqrt{1+u_x^2}$. At the same time, the evolution equation for y includes the term py which is related to $\frac{1}{\sqrt{1+u_x^2}}$

Using the degree one homogeneity of F and equation (2.11), equation (4.1) can be rewritten as

$$\frac{\partial u}{\partial t} = \dot{F}^{11}\frac{u_{xx}}{1+u_x^2} - \sum_{j=2}^{n}\dot{F}^{jj}\frac{1}{u} = \dot{F}^{11}\left(\arctan\left(u_x\right)\right)_x - \sum_{j=2}^{n}\dot{F}^{jj}\frac{1}{u}. \tag{4.2}$$

Since the matrix of the Weingarten map is everywhere diagonal, the matrix of $\dot{F}$ is everywhere diagonal and $\dot{F}^{kk} = \dot{f}^k$ for each k (see, for example, [8]), and the above evolution equation for u becomes

$$\frac{\partial u}{\partial t} = \dot{f}^1\left(\arctan\left(u_x\right)\right)_x - \sum_{j=2}^{n}\dot{f}^j\frac{1}{u}. \tag{4.3}$$

Our analysis will be performed as in [3] and [6]. We will need the following flow independent estimates.

Moreover, in view of symmetry, $\dot{f}^2\left(\kappa_1,\kappa_2,\ldots,\kappa_n\right) = \ldots = \dot{f}^n\left(\kappa_1,\kappa_2,\ldots,\kappa_n\right)$; throughout we will write $\dot{f}^2$.

Lemma 4.1. *Any function convex (concave) F satisfying Conditions 1 also satisfies*

(i.) $f \geq (\leq)\frac{1}{n}H$.

(ii.) $\sum_{k=1}^{n}\dot{f}_k = \operatorname{trace}\left(\dot{F}^{kl}\right) \leq (\geq) 1$.

Proof: Parts *(i)* and *(ii)* are proved in exactly the same way as in [68], Lemma 3.2 and Lemma 3.3. Here the proof works similarly, even when Γ is a larger convex cone than the positive cone.

To prove *(i)*, we know f homogeneous of degree 1 and $1 = f(1,\ldots,1) = \sum_i \frac{\partial f}{\partial k_i}(1,\ldots,1)$. Because of the symmetry of f we found $\frac{\partial f}{\partial k_i}(1,\ldots,1) = \frac{1}{n}$ for each i. Homogeneity of degree one means

$$f\left(K\kappa\right) = Kf\left(\kappa\right),$$

then

$$\frac{\partial f}{\partial \kappa_i}(K\kappa)\, K = K\frac{\partial f}{\partial \kappa_i}(\kappa)\,,$$

which means

$$\frac{\partial f}{\partial \kappa_i}(K\kappa) = \frac{\partial f}{\partial \kappa_i}(\kappa)\,.$$

Particularly $\frac{\partial f}{\partial k_i}(k,\ldots,k) = \frac{1}{n}$. Therefore,

$$\begin{aligned} f(\kappa_1,\ldots,\kappa_n) &= f(1+\kappa_1-1,\ldots,1+\kappa_n-1) \\ &= f(1,\ldots,1) + \sum_i \frac{\partial f}{\partial \kappa_i}(1,\ldots,1)(\kappa_i-1) + D^2 f|_{\tilde{y}}(\kappa-1,\kappa-1) \\ &\geqslant (\leqslant) 1 + \frac{1}{n}\sum_i (\kappa_i - 1) \\ &= \frac{1}{n}\sum_i \kappa_i \\ &= \frac{1}{n}H. \end{aligned}$$

To prove *(ii.)*, let $\kappa = (\kappa_1,\ldots,\kappa_n)$, $\kappa_1 \leq \cdots \leq \kappa_n$, for all $\mu > 0$ and $(\kappa_1+\mu,\ldots,\kappa_n+\mu) \in \Gamma$, for f convex

$$\begin{aligned} \mu + \kappa_{\max} &= (\mu + \kappa_{\max})\, f(1) \\ &= f(\mu+\kappa_{\max},\ldots,\mu+\kappa_{\max}) \\ &\geq f(\mu+\kappa_1,\ldots,\mu+\kappa_{\max}) \quad \text{since } \frac{\partial f}{\partial \kappa_i} > 0 \quad \text{for all } i \\ &= f(\kappa_1,\ldots,\kappa_n) + \mu\sum_i \frac{\partial f}{\partial \kappa_i}(\kappa) + \mu^2 \sum_{i,j} \frac{\partial^2 f}{\partial \kappa_i \partial \kappa_j}(\tilde{\kappa}) \\ &\geq f + \mu \sum_i \frac{\partial f}{\partial \kappa_i}(\kappa)\,, \quad \text{because of the convexity} \end{aligned}$$

then $\mu + \kappa_{\max} \geq f + \mu \sum_i \frac{\partial f}{\partial \kappa_i}(\kappa)$ and it can be written as

$$1 + \frac{\kappa_{\max}}{\mu} \geq \frac{f}{\mu} + \sum_i \frac{\partial f}{\partial \kappa_i}(\kappa),$$

holds for all $\mu > 0$ and letting $\mu \to \infty$ gives the result.

Similarly for the concave case

$$\begin{aligned}
\mu + \kappa_{\min} &= (\mu + \kappa_{\min}) f(1) \\
&= f(\mu + \kappa_{\min}, \ldots, \mu + \kappa_{\min}) \\
&\leq f(\mu + \kappa_1, \ldots, \mu + \kappa_{\min}) \quad \text{since } \frac{\partial f}{\partial \kappa_i} > 0 \quad \text{for all } i \\
&= f(\kappa_1, \ldots, \kappa_n) + \mu \sum_i \frac{\partial f}{\partial \kappa_i}(\kappa) + \mu^2 \sum_{i,j} \frac{\partial^2 f}{\partial \kappa_i \partial \kappa_j}(\tilde{\kappa}) \\
&\leq f + \mu \sum_i \frac{\partial f}{\partial \kappa_i}(\kappa), \quad \text{because of the concavity}
\end{aligned}$$

then $\mu + \kappa_{\min} \leq f + \mu \sum_i \frac{\partial f}{\partial \kappa_i}(\kappa)$ and it can be written as

$$1 + \frac{\kappa_{\min}}{\mu} \leq \frac{f}{\mu} + \sum_i \frac{\partial f}{\partial \kappa_i}(\kappa),$$

holds for all $\mu > 0$ and letting $\mu \to \infty$ gives the result. □

3 Behaviour of the flow

In this section we are interested in solutions of (4.1) with the boundary conditions

$$u_x(0,t) = 0, \quad u_x(a,t) = g(t), \tag{4.4}$$

where g is a suitably smooth function. Although not necessary for the short time existence theorem, Theorem 4.1, we will, for the subsequent results, assume g is smooth, non-negative and non-increasing.

3.1 Short time existence

Our short time existence result for (4.1) is a special case of Theorem 8.5.4 from [49] whose proof uses semigroup theory. Similar results are presented for the case of mean curvature flow in [28] and [32].

Theorem 4.1. *Given an initial function* $u_0 \in C^2([0,a])\,(C^{2,\alpha}[0,a])$, *compatible with the boundary conditions* (4.4), *there exists a* $\delta > 0$ *such that there is a unique* *solution* $u \in C^2([0,a] \times [0,\delta))$ $(C^{2,\alpha}([0,a] \times [0,\delta))$ *to* (4.1), *with initial condition* $u(\cdot,0) = u_0$ *and satisfying the boundary conditions* (4.4).

Remarks:

i) Uniform parabolicity of f is not required for the above result; Condition 1, ii) suffices.

ii) Above we are using the standard notation for parabolic Hölder spaces, see section 4 Chapter 2.

iii) A similar short time existence result holds for Dirichlet or more general Robin boundary conditions, provided the initial data u_0 is compatible. Such a result is relevant for our later remarks, equation (4.20) concerning a mixed boundary value problem.

iv) We will not pursue the optimal smoothing affect of the nonlinear operator F here, except to note that the case of $u_0 \in C^2([0,a])$ above gives that the curvatures of the hypersurface M_t are continuous, so (4.3) is uniformly parabolic on a possibly shorter time interval $\left[0,\tilde{\delta}\right)$. The short time existence result in Chapter 14 of [47] then implies that $u \in C^{2,1}\left([0,a] \times \left(0,\tilde{\delta}\right)\right)$, moreover, classical Schauder estimates then provide higher short-time regularity provided F is sufficiently smooth. We will assume f is at least smooth enough for our maximum principle arguments to be valid. Importantly, we will use the C^2 version of Theorem 4.1 in characterising the maximal time T of existence (Theorem 4.2).

Note that, in the case of (4.1) under pure Neumann boundary conditions

$$u_x(t,0) = u_x(t,a) = 0, \tag{4.5}$$

or equivalently for a periodically deformed infinite cylinder, we know that u develops a singularity in finite time $T > 0$. Comparison with cylinders gives lower and upper bounds on the maximal existence time T.

We now turn our attention to initial hypersurfaces for which the generating function u_0 is non-decreasing. The next Lemma does not require f to be convex and it is generalised from Dziuk and Kawohl [26].

Lemma 4.2. *Consider* (4.1) *under the boundary conditions* (4.4)*, with F satisfying Conditions 1,* i) *to* iv)*. Let u_0 be at least $C^2([0,a])$.*

(i.) If

$$(u_0)_x \geq 0, \tag{4.6}$$

then $u_x(x,t) \geq 0$ for all $x \in [0,a]$, $t \in [0,T)$, that is, as long as a solution to (4.3) *exists.*

(ii.) Suppose that $f \geq 0$ everywhere on the initial hypersurface M_0, that is, u_0 satisfies

$$f\left(\frac{-(u_0)_{xx}}{\left(1+(u_0)_x^2\right)^{\frac{3}{2}}}, \frac{1}{u_0\sqrt{1+(u_0)_x^2}}, \ldots, \frac{1}{u_0\sqrt{1+(u_0)_x^2}}\right) \geq 0. \tag{4.7}$$

Then $u_t \leq 0$ for all $x \in [0,a]$, $t \in [0,T)$.

Proof: To prove *(i)* we differentiate (4.1) with respect to x

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial t} \\
&= \frac{\partial}{\partial x}\left[-\sqrt{1+u_x^2}F\right] \\
&= -\left(\frac{\partial}{\partial x}\sqrt{1+u_x^2}\right)F - \sqrt{1+u_x^2}\frac{\partial}{\partial x}F \\
&= -\left(1+u_x^2\right)^{\frac{-1}{2}} u_x u_{xx} F - \sqrt{1+u^2}\dot{F}_i^j \frac{\partial}{\partial x} h_j^i \\
&= -\left(1+u_x^2\right)^{\frac{-1}{2}} u_x u_{xx} F - \sqrt{1+u_x^2}\left[\dot{F}^{11}\frac{\partial}{\partial x}(k) + (n-1)\dot{F}^{22}\frac{\partial}{\partial x}(p)\right] \\
&\qquad \text{because } \frac{\partial}{\partial x} f(k,p) = \frac{\partial f}{\partial \kappa_1}\frac{\partial k}{\partial x} + (n-1)\frac{\partial f}{\partial \kappa_2}\frac{\partial p}{\partial x} \\
&= -\left(1+u_x^2\right)^{\frac{-1}{2}} u_x u_{xx} F - \sqrt{1+u_x^2}\left\{\dot{F}^{11}\frac{\partial}{\partial x}\left[\frac{-u_{xx}}{(1+u^2)^{\frac{3}{2}}}\right]\right. \\
&\qquad \left. +(n-1)\dot{F}^{22}\frac{\partial}{\partial x}\left[\frac{1}{u\sqrt{1+u_x^2}}\right]\right\}.
\end{aligned} \tag{4.8}$$

Computing $\frac{\partial k}{\partial x}$ and $\frac{\partial p}{\partial x}$ respectively we will have

$$
\begin{aligned}
\frac{\partial k}{\partial x} &= -\frac{\partial}{\partial x}\frac{u_{xx}}{\left(1+u_x^2\right)^{\frac{3}{2}}} \\
&= -\frac{u_{xxx}\left(1+u_x^2\right)^{\frac{3}{2}} - 3u_{xx}\left(1+u_x^2\right)^{\frac{1}{2}} u_x u_{xx}}{\left(1+u_x^2\right)^3} \\
&= \frac{-u_{xxx}\left(1+u_x^2\right)^{\frac{3}{2}} + 3u_{xx}\left(1+u_x^2\right)^{\frac{1}{2}} u_x u_{xx}}{\left(1+u_x^2\right)^3}. \qquad (4.9)
\end{aligned}
$$

Also

$$
\begin{aligned}
\frac{\partial p}{\partial x} &= \quad \frac{\partial}{\partial x}\frac{1}{u\sqrt{1+u_x^2}} \\
&= \frac{-u_x\left(1+u_x^2\right)^{\frac{1}{2}} - u\left(1+u_x\right)^{\frac{-1}{2}} u_x u_{xx}}{u^2\left(1+u_x^2\right)}, \qquad (4.10)
\end{aligned}
$$

from (4.9) and (4.10) into (4.8) with $v = u_x$ we find

$$
\begin{aligned}
v_t &= \frac{\partial}{\partial t} v \\
&= -\frac{v}{\sqrt{1+v^2}} v_x F \\
&\quad - \sqrt{1+v^2}\dot{F}^{11}\left[\frac{-v_{xx}\left(1+v^2\right)^{\frac{3}{2}} + 3(1+v^2)^{\frac{1}{2}} v v_x^2}{(1+v^2)^3}\right] \\
&\quad - (n-1)\sqrt{1+v^2}\dot{F}^{22}\left[\frac{-v(1+v^2)^{\frac{1}{2}} - u(1+v^2)^{\frac{-1}{2}} v v_x}{u^2(1+v^2)}\right] \\
&= -\frac{v}{\sqrt{1+v^2}} v_x \left[-\dot{F}^{11}\frac{v_x}{(1+v^2)^{\frac{3}{2}}} + (n-1)\dot{F}^{22}\frac{1}{u\sqrt{1+v^2}}\right] \\
&\quad + \frac{\dot{F}^{11}}{1+v^2} v_{xx} - 3\frac{\dot{F}^{11}}{(1+v^2)^2} v v_x^2 \\
&\quad + \frac{(n-1)\dot{F}^{22}}{u^2} v + (n-1)\frac{\dot{F}^{22}}{u(1+v^2)} v v_x \\
&= -\frac{2\dot{F}^{11}}{(1+v^2)^2} v v_x^2 + \frac{\dot{F}^{11}}{(1+v^2)} v_{xx} + (n-1)\frac{\dot{F}^{22}}{u^2} v.
\end{aligned}
$$

Therefore,

$$
v_t + \frac{2\dot{F}^{11}}{\left(1+v^2\right)^2} v v_x^2 - \frac{\dot{F}^{11}}{\left(1+v^2\right)} v_{xx} - (n-1)\frac{\dot{F}^{22}}{u^2} v = 0, \qquad (4.11)
$$

or it can be written as

$$\frac{\partial}{\partial t}v = \frac{\dot{f}^1}{1+v^2}v_{xx} - \frac{2\dot{f}^1}{(1+v^2)^2}vv_x^2 + \sum_{j=2}^{n}\frac{\dot{f}^j}{u^2}v. \tag{4.12}$$

From (4.4) we see that $v \geq 0$ for $x = 0$ and $x = a$ and (4.6) implies $v \geq 0$ for $t = 0$. If $v(x,0) = 0$ at any $x \in (0,a)$ then this is a local minimum and by (4.12), v does not decrease. Moreover, from (4.12), if v attains an interior zero minimum then v does not decrease. Hence $v \geq 0$ remains true under (4.1).

Similarly we prove *(ii)*; we instead differentiate (4.1) with respect to t

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\left[-\sqrt{1+u_x^2}F\right] \\
&= -\left(\frac{\partial}{\partial t}\sqrt{1+u_x^2}\right)F - \sqrt{1+u_x^2}(\frac{\partial}{\partial t}F) \\
&= -(1+u_x)^{\frac{-1}{2}}u_x u_{xt}F - \sqrt{1+u_x^2}\left[\dot{F}_i^j\left(\frac{\partial}{\partial t}h_j^i\right)\right] \\
&= -\left(1+u_x^2\right)^{\frac{-1}{2}}u_x u_{xt}F - \sqrt{1+u_x^2}\left[\dot{F}^{11}\frac{\partial}{\partial t}(k) + (n-1)\dot{F}^{22}\frac{\partial}{\partial t}p\right] \\
&= -(1+u_x^2)^{\frac{-1}{2}}u_x u_{xt}F - \sqrt{1+u_x^2}\left\{\dot{F}^{11}\frac{\partial}{\partial t}\left[\frac{-u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}\right]\right. \\
&\quad \left. +(n-1)\dot{F}^{22}\frac{\partial}{\partial t}\left[\frac{1}{u(1+u_x^2)^{\frac{1}{2}}}\right]\right\}.
\end{aligned} \tag{4.13}$$

Computing $\frac{\partial k}{\partial t}$ and $\frac{\partial p}{\partial t}$ respectively we will have

$$\begin{aligned}
\frac{\partial k}{\partial t} &= \frac{\partial}{\partial t}\left[-\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}\right] \\
&= -\frac{u_{xxt}(1+u_x^2)^{\frac{3}{2}} - 3u_{xx}(1+u_x^2)^{\frac{1}{2}}u_x u_{xt}}{(1+u_x^2)^3} \\
&= \frac{-u_{xxt}(1+u_x^2)^{\frac{3}{2}} + 3u_{xx}(1+u_x^2)^{\frac{1}{2}}u_x u_{xt}}{(1+u_x^2)^3},
\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}
\frac{\partial p}{\partial t} &= \frac{\partial}{\partial t}\left[\frac{1}{u\,(1+u_x^2)^{\frac{1}{2}}}\right] \\
&= \frac{-u_t\,(1+u_x^2)^{\frac{1}{2}} - u(1+u_x^2)^{\frac{-1}{2}}u_x u_{xt}}{u^2(1+u_x^2)}.
\end{aligned} \tag{4.15}$$

From (4.14) and (4.15) into (4.13) with $v = u_t$ we fined

$$
\begin{aligned}
v_t &= \frac{\partial}{\partial t} v \\
&= -\frac{1}{\sqrt{1+u_x^2}} u_x u_{xt} \left[\dot{F}^{11} \frac{-u_{xx}}{(1+u_x^2)^{\frac{3}{2}}} + (n-1) \dot{F}^{22} \frac{1}{u\sqrt{1+u_x^2}} \right] \\
&\quad - \sqrt{1+u_x^2} \dot{F}^{11} \left[\frac{-u_{xxt}(1+u_x^2)^{\frac{3}{2}} + 3u_{xx}(1+u_x^2)^{\frac{1}{2}} u_x u_{xt}}{(1+u_x^2)^3} \right] \\
&\quad - (n-1)\sqrt{1+u_x^2} \dot{F}^{22} \left[\frac{-u_t(1+u_x^2)^{\frac{1}{2}} - u(1+u_x^2)^{\frac{-1}{2}} u_x u_{xt}}{u^2(1+u_x^2)} \right] \\
&= \frac{\dot{F}^{11}}{(1+u_x^2)^2} u_x u_{xt} u_{xx} - \frac{(n-1)\dot{F}^{22}}{u\,(1+u_x)^2} u_x u_{xt} \\
&\quad + \frac{\dot{F}^{11}}{1+u_x^2} u_{xxt} - \frac{3\dot{F}^{11}}{(1+u_x)^2} u_{xx} u_x u_{xt} \\
&\quad + \frac{(n-1)\dot{F}^{22}}{u^2} u_t + \frac{(n-1)\dot{F}^{22}}{u(1+u_x^2)} u_x u_{xt} \\
&= \frac{-2\dot{F}^{11}}{(1+u_x^2)^2} u_x u_{xx} u_{xt} + \frac{\dot{F}^{11}}{1+u_x^2} u_{xxt} + \frac{(n-1)\dot{F}^{22}}{u^2} u_t \\
&= \frac{-2\dot{F}^{11}}{(1+u_x^2)^2} u_x u_{xx} v_x + \frac{\dot{F}^{11}}{1+u_x^2} v_{xx} + (n-1)\frac{\dot{F}^{22}}{u^2} v,
\end{aligned}
$$

satisfies the equation

$$
\frac{\partial}{\partial t} v = \frac{\dot{f}^1}{1+u_x^2} v_{xx} - \frac{2\dot{f}^1}{(1+u_x^2)^2} u_x u_{xx} v_x + \sum_{j=2}^{n} \dot{f}^j \frac{v}{u^2}. \tag{4.16}
$$

If $v = 0$ somewhere off the boundary at initial time $t = 0$ then this is a spatial maximum and v cannot go positive because $\frac{\dot{f}^1}{1+u_x^2} v_{xx} \leqslant 0$ which stops v from increasing. Suppose there is a first time when $v(x_0, t_0) = 0$ where $t_0 > 0$, the second and third term in (4.16) will disappear because of the local maximum. Applying the maximum principle to (4.16), this cannot occur at an interior point. At a boundary point, in view of (4.4), we have $v_x(0,t) = 0$ and $v_x(a,t) = g'(t) \leq 0$. By the Hopf Lemma (see, for example, [59]) a boundary maximum would have $v_x(0,t) < 0$ or $v_x(a,t) > 0$, so there can be no boundary maximum. We conclude that $v \leq 0$ is preserved. □

Remarks:

i) Since the cone of definition of f is larger than the positive cone, $f > 0$ does not immediately follow from Conditions 1 ii) and iii) via the Euler identity as in the case of convex hypersurfaces. Consequently, the above result Lemma 4.2, *(ii)* is useful because it implies that f does not become negative under the flow.

ii) In the case of pure Neumann conditions ($g \equiv 0$) we can construct an entire C^2 solution of (4.1) by reflecting $u : [0, a] \times [0, T)$ in the x_1 axis to create a spatially even function on $[-a, a] \times [0, T)$ and then extending this periodically in space to $\mathbb{R} \times [0, T)$. Such a construction is done in [18, 39]. We can then apply the maximum principle considering only interior extrema. In particular, the speed F evolves according to

$$\frac{\partial}{\partial t} F = \mathcal{L}F + \dot{F}^{kl} h_k^m h_{ml} F, \tag{4.17}$$

Applying the maximum principle to (4.17) we observe that F remains bounded below by its initial minimum, for more details of the proof see Corollary 5.1.

Further, under (3.1), as in [3] for example, the mean curvature evolves according to

$$\frac{\partial}{\partial t} H = \mathcal{L}H + \ddot{F}^{kl,rs} \nabla^i h_{kl} \nabla_i h_{rs} + \dot{F}^{kl} h_k^m h_{ml} H.$$

In the case that F is convex, that $H \geq 0$ remains true under (4.1) now follows directly by the maximum principle.

We may specify in a similar way as in [50] a condition on g which ensures that the solution u exists for a finite maximal time $T > 0$. Here we need F convex, so we can use Lemma 4.1, *(i)*. Of course, as commented earlier, in the special case that $g \equiv 0$ we know as in [26] that the maximal existence time is finite by comparing the solution of (4.1), with initial data u_0, with an enclosing cylinder; such a comparison does not require F convex nor F homogeneous.

Lemma 4.3. *Suppose in addition to* (4.6) *and* (4.7) *that*

$$\arctan g(0) < \frac{(n-1)\,a^2}{\int_0^a u_0(x)\,dx}. \tag{4.18}$$

Then the maximal existence time T of solution u to (4.1) *is finite.*

Proof: Otherwise a solution u exists and is positive for every finite time. We define the function $E : [0, \infty) \to \mathbb{R}_+$ by $E(t) = \int_0^a u(x,t)\,dx$. By differentiating $E(t)$ we obtain

$$\begin{aligned} E'(t) &= \int_0^a \frac{\partial u}{\partial t}(x,t)dx \\ &= \int_0^a \left(-\sqrt{1+u_x^2}F(\omega)\right) dx. \end{aligned} \tag{4.19}$$

Using Lemma 4.1,*(i)* in (4.19), we obtain

$$\begin{aligned} E'(t) &\le -\int_0^a \sqrt{1+u_x^2}\,\frac{H}{n}\,dx \\ &= -\frac{1}{n}\int_0^a \sqrt{1+u_x^2}\,H\,dx \\ &= \frac{1}{n}\int_0^a (\arctan u_x)_x\,dx - \frac{n-1}{n}\int_0^a \frac{1}{u}\,dx \\ &= \frac{1}{n}\arctan u_x\,|_0^a - \frac{n-1}{n}\int_0^a \frac{1}{u}dx \\ &= \frac{1}{n}\arctan u_x(a,t) - \frac{1}{n}\arctan u_x(0,t) - \frac{n-1}{n}\int_0^a \frac{1}{u}dx \\ &= \frac{1}{n}\arctan u_x(a,t) - \frac{n-1}{n}\int_0^a \frac{1}{u}dx \\ &= \frac{1}{n}\arctan g(t) - \frac{n-1}{n}\int_0^a \frac{1}{u}dx. \end{aligned}$$

By Hölder's inequality, $\int uv dx \le \left(\int u^2 dx\right)^{\frac{1}{2}} \left(\int v^2 dx\right)^{\frac{1}{2}}$

$$-\int_0^a \frac{1}{u}\,dx \int_0^a u\,dx \le -a^2,$$

$$-\int_0^a \frac{1}{u}\,dx \le -\frac{a^2}{\int_0^a u\,dx}.$$

Then

$$
\begin{aligned}
E'(t) &\leq \frac{1}{n}\arctan g(t) - \frac{n-1}{n}\frac{a^2}{\int_0^a u\,dx} \\
&\leq \frac{1}{n}\arctan g(0) - \frac{n-1}{n}\frac{a^2}{\int_0^a u_0\,dx} \\
&< 0.
\end{aligned}
$$

Integrating E'

$$
\begin{aligned}
\int_\delta^t E'(\tau)d\tau &\leq \int_\delta^t \left\{\frac{1}{n}\arctan g(0) - \frac{n-1}{n}\frac{a^2}{\int_0^a u_0}\right\} d\tau \\
\int_\delta^t E'(\tau)d\tau &\leq \left\{\frac{1}{n}\arctan g(0) - \frac{n-1}{n}\frac{a^2}{\int_0^a u_0}\right\}(t-\delta) \\
\lim_{\delta\to 0+}\int_\delta^t E'(\tau)d\tau &\leq \lim_{\delta\to 0+}\left\{\frac{1}{n}\arctan g(0) - \frac{n-1}{n}\frac{a^2}{\int_0^a u_0}\right\}(t-\delta) \\
\lim_{\delta\to 0+}[E(t)-E(\delta)] &\leq \left\{\frac{1}{n}\arctan g(0) - \frac{n-1}{n}\frac{a^2}{\int_0^a u_0}\right\}(t) \\
E(t)-E(0+) &\leq \left\{\frac{1}{n}\arctan g(0) - \frac{n-1}{n}\frac{a^2}{\int_0^a u_0}\right\}(t) \\
&\qquad\text{where } E(0+) = \lim_{\delta\searrow 0} E(\delta) \\
E(t) \leq E(0+) + \left\{\frac{1}{n}\arctan g(0+) - \frac{n-1}{n}\frac{a^2}{\int_0^a u_0}\right\}(t), &\quad \text{for all } t>0.
\end{aligned}
$$

We obtain

$$
E(t) \leq E(0) + \frac{1}{n}\left[\arctan g(0) - \frac{(n-1)\,a^2}{\int_0^a u_0\,dx}\right] t,
$$

which implies that E becomes negative in finite time, a contradiction. $\square$

Remarks:

i) If the boundary condition (4.4) is replaced by the mixed condition

$$
u_x(0,t) = 0,\ u(a,t) = h(t), \tag{4.20}
$$

for a positive function h which is bounded by $\frac{2(n-1)a}{\pi}$, then similar arguments show that again the flow speed remains non-positive and, using the energy $E(t) = \int_0^a u^2 dx$, the maximal existence time is finite.

ii) The second spatial derivative u_{xx} also satisfies a parabolic equation. The standard linear version of the Sturmian theorem gives that the number of zeros of u_{xx} does not increase during the evolution, see Lemma 6.1. Although we have a fully nonlinear case we can use the standard linear version of the Sturmian theorem because the coefficients are bounded. This tells us that the number of sign-changes of the axial curvature does not increase under the evolution, a property that could be of interest in applications. We refer the reader to [20, 31] for details of Sturmian theorem and its applications. We apply Sturmian theorem in Chapters 6 and 7 of this thesis.

4 Singularity

Now we characterise the maximal existence time T as the time of a curvature singularity, that is, when the norm $|A|$ of the second fundamental form becomes unbounded. More specifically, we show that if the axial curvature κ_1 does not blow up at $x = a$ as $t \to T$, then the rotational curvatures blow up at $x = 0$ and in view of the formula for κ_j, $j = 2, \ldots, n$, we must have $u\,(0, t) \to 0$ as $t \to T$. This result is also analogous to the corresponding result for the mean curvature flow in [50]. Critical to the argument in [50] was that the mean curvature remains positive under the evolution. We do not have this in general, but the structure condition on f, Condition 2, permits a similar deduction. In the case of pure Neumann conditions, we may instead see that the minimum of the mean curvature does not decrease under the evolution, as discussed in Remark ii) following Lemma 4.2.

Theorem 4.2. *Suppose that F satisfies Conditions 1 and 2. Suppose in addition to (4.6), (4.7) and (4.18) that $\lim_{t\to T} \kappa_1^2(a, t) < \infty$. Then $\lim_{t\to T} \kappa_j^2(0, t) = \infty$ for $j = 2, \ldots, n$.*

Proof: Suppose for the sake of establishing a contradiction that $\lim_{t\to T} u(0, t) = \delta > 0$. Then for $j = 2, \ldots, n$ we have that for all $(x, t) \in [0, a] \times [0, T)$,

$$\kappa_j^2(x, t) = \frac{1}{u^2(x, t)(1 + u_x^2(x, t))} \leq \frac{1}{u^2\,(x, t)} \leq \frac{1}{u^2\,(0, t)} \leq \frac{1}{\delta^2}. \tag{4.21}$$

It follows from Lemma 4.2, *(ii)* that under the evolution

$$f(\kappa_1, \kappa_2, \dots, \kappa_2) = \kappa_2 f\left(\frac{\kappa_1}{\kappa_2}, 1, \dots, 1\right) \geq 0.$$

Since the rotational curvatures $\kappa_2 > 0$, this means that for $z = \frac{\kappa_1}{\kappa_2}$,

$$f(z, 1, \dots, 1) \geq 0,$$

as long as the solution exists. Condition 2 on F implies therefore that

$$z = \frac{\kappa_1}{\kappa_2} \geq -c_0,$$

for some $c_0 > 0$, which, in terms of derivatives of u means

$$\frac{-u\, u_{xx}}{1 + u_x^2} \geq -c_0,$$

that is, in view of our assumption, we have on $[0, a] \times [0, T)$ that

$$\frac{u_{xx}}{1 + u_x^2} \leq \frac{c_0}{u} \leq \frac{c_0}{\delta}. \tag{4.22}$$

Multiplying equation (4.16) by $-e^{-\lambda t}$

$$\frac{\dot{f}^1}{1 + u_x^2} v_{xx}\left(-e^{-\lambda t}\right) - \frac{2\dot{f}^1 u_x u_{xx}}{\left(1 + u_x^2\right)^2} v_x\left(-e^{-\lambda t}\right) + \sum_{j=2}^{n} \frac{\dot{f}^j}{u^2} v\left(-e^{-\lambda t}\right) - v_t\left(-e^{-\lambda t}\right) = 0. \tag{4.23}$$

Set $w = -v\, e^{-\lambda t}$ we have, where $v = u_t$, then

$$w_t = -v_t e^{-\lambda t} + v\lambda e^{-\lambda t} \Rightarrow w_t = -v_t e^{-\lambda t} - \lambda w \Rightarrow v_t e^{-\lambda t} = -w_t - \lambda w,$$

$$w_x = -v_x e^{-\lambda t},$$

$$w_{xx} = -v_{xx} e^{-\lambda t},$$

so equation (4.23) can be written

$$\frac{\dot{f}^1}{1+u_x^2}w_{xx} - \frac{2\dot{f}^1 u_x u_{xx}}{(1+u_x^2)^2}w_x + \sum_{j=2}^{n}\frac{\dot{f}^j}{u^2}w - \lambda w - w_t = 0.$$

The function w satisfies the equation

$$\frac{\partial}{\partial t}w = \frac{\dot{f}^1}{1+u_x^2}w_{xx} - \frac{2\dot{f}^1}{(1+u_x^2)^2}u_x u_{xx} w_x + \left(\sum_{j=2}^{n}\dot{f}^j\frac{1}{u^2} - \lambda\right)w.$$

In view of Lemma 4.1, *(ii)* and our assumption, taking $\lambda > \frac{1}{\delta^2}$ ensures the coefficient of w is negative and so w cannot obtain an interior maximum. Further,

$$w\left(x,0\right) = \sqrt{1+u_x^2\left(x,0\right)}F \leq C\left(M_0\right). \tag{4.24}$$

Let us now show that w is bounded on the sides $x=0$ and $x=a$. We have

$$w(0,t) = \left(-\frac{\dot{f}^1 u_{xx}}{1+u_x^2} + \sum_{j=2}^{n}\dot{f}^j\frac{1}{u}\right)\bigg|_{(0,t)} e^{-\lambda t} \leq \frac{1}{u}\bigg|_{(0,t)} e^{-\lambda t},$$

and at $x=0$, $u_{xx} \geq 0$ in view of our assumption and Lemma 4.2, *(i)*, so using also Lemma 4.1, *(ii)* we have

$$w(0,t) \leq \frac{1}{\delta}.$$

Similarly, using our assumption and that $g\left(t\right)$ is nonincreasing, there is a non-positive constant $\alpha \leq u_{xx}\left(a,t\right)$ for all $t \in [0,T)$ and

$$w(a,t) \leq \left(-\dot{f}^1\alpha + \sum_{j=2}^{n}\dot{f}^j\frac{1}{u}\right)\bigg|_{(a,t)} e^{-\lambda t},$$

from which it follows using Lemma 4.1, *(ii)* that on $[0,T)$

$$w(a,t) \leq \left(\frac{1}{\delta} - \alpha\right)e^{-\lambda t} < \frac{1}{\delta} - \alpha.$$

Therefore, together with (4.24) we have an upper bound for w, that is

$$w = -u_t e^{-\lambda t} \leq \overline{C}\left(M_0, \delta, T\right),$$

and so on $[0,a] \times [0,T)$,

$$-u_t = \sqrt{1+u_x^2}F \leq \overline{C}e^{\lambda T}.$$

Using now Lemma 4.1, *(i)*, we obtain

$$\kappa_1 = \frac{-u_{xx}}{1+u_x^2} \leq n\overline{C}e^{\lambda T}.$$

Together with (4.22) we have on $[0,a] \times [0,T)$ that

$$\kappa_1^2 \leq \max\left(\frac{c_0^2}{\delta^2}, 4\overline{C}^2 e^{2\lambda T}\right). \tag{4.25}$$

Now the assumption together with Lemma 4.2, implies $u_T(x) = \lim_{t\to T} u(x,t) \geq 0$ exists, and (4.21) and (4.25) imply u_T generates a C^2 axially symmetric hypersurface which could be used as an initial hypersurface in the short time existence result, Theorem 4.1, contradicting the maximality of T. Thus our assumption is false and the theorem is proved. □

Remark: Since both Conditions 1 and 2 are required in Theorem 4.2 this imply that f is defined in the whole space.

5 Extension

In this section we are interested in generalising our earlier results to the case where the flow speed is homogeneous of degree $k > 0$, that is,

$$\frac{\partial X}{\partial t}(x,t) = -F^k(\mathcal{W}(x,t))\,\nu(x,t),$$

where F continues to satisfy Conditions 1. Similar flows of hypersurfaces have been considered before, particularly flows by powers of Gauss curvature and powers of the mean curvature and often for surfaces, usually in the context of convex initial data or translating solutions [1, 5, 6, 9, 15, 16, 23, 36, 41, 42, 61, 62, 63, 64].

The corresponding evolution equation of the graph function u is now

$$\frac{\partial u}{\partial t} = -\sqrt{1+u_x^2}\,F^k. \tag{4.26}$$

Under the flow (4.26), we have the following evolution equations.

Lemma 4.4.

(i.) $\frac{\partial}{\partial t}u_x = \frac{kF^{k-1}}{1+u_x^2}\dot{f}^1\,(u_x)_{xx}+(1-3k)\,\frac{F^{k-1}\dot{f}^1u_x}{(1+u_x^2)^2}\,((u_x)_x)^2+(n-1)\,(k-1)\,\frac{F^{k-1}\dot{f}^2u_x}{u(1+u_x^2)}\,(u_x)_x$
$+\,(n-1)\frac{kF^{k-1}\dot{f}^2}{u^2}u_x.$

(ii.) $\frac{\partial}{\partial t}u_t = \frac{kF^{k-1}}{1+u_x^2}\dot{f}^1\,(u_t)_{xx}+(1-3k)\,\frac{F^{k-1}\dot{f}^1u_xu_{xx}}{(1+u_x^2)^2}\,(u_t)_x+(n-1)\,(k-1)\,\frac{F^{k-1}\dot{f}^2u_x(u_t)_x}{u(1+u_x^2)}$
$+\,(n-1)\frac{kF^{k-1}\dot{f}^2}{u^2}u_t.$

(iii.) $\frac{\partial}{\partial t}F^k = \mathcal{L}F^k + k\dot{F}^{ij}h_i{}^m h_{mj}F^k.$

where we have used the notation $\mathcal{L} = kF^{k-1}\dot{F}^{ij}\nabla_i\nabla_j$.

Proof: To prove *(i)*

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial t}\\
&= \frac{\partial}{\partial x}\left[-\sqrt{1+u_x^2}F^k\right]\\
&= -\left(\frac{\partial}{\partial x}\sqrt{1+u_x^2}\right)F^k - \sqrt{1+u_x^2}\frac{\partial}{\partial x}F^k\\
&= -\left(1+u_x^2\right)^{\frac{-1}{2}}u_x(u_x)_xF^K - \sqrt{1+u^2}kF^{k-1}\dot{F}_i^j\frac{\partial}{\partial x}h_j^i\\
&= -(1+u_x^2)^{\frac{-1}{2}}u_x(u_x)_xF^k - kF^{k-1}\sqrt{1+u_x^2}\left[\dot{f}^1\frac{\partial}{\partial x}(k)+(n-1)\dot{f}^2\frac{\partial}{\partial x}(p)\right]\\
&= -(1+u_x^2)^{\frac{-1}{2}}u_x(u_x)_xF^k - kF^{k-1}\sqrt{1+u_x^2}\left\{\dot{f}^1\frac{\partial}{\partial x}\left[\frac{-(u_x)_x}{(1+u^2)^{\frac{3}{2}}}\right]\right.\\
&\quad\left.+(n-1)\dot{f}^2\frac{\partial}{\partial x}\left[\frac{1}{u\sqrt{1+u_x^2}}\right]\right\},
\end{aligned} \tag{4.27}$$

$$
\begin{aligned}
\frac{\partial}{\partial t}u_x =& -\frac{u_x}{\sqrt{1+u_{xx}^2}}(u_x)_x F^{k-1}F \\
& - kF^{k-1}\sqrt{1+u_x^2}\dot{f}^1 \left[\frac{-(u_x)_{xx}(1+u_x^2)^{\frac{3}{2}} + 3(1+u_x^2)^{\frac{1}{2}}u_x((u_x)_x)^2}{(1+_x^2)^3}\right] \\
& - (n-1)kF^{k-1}\sqrt{1+u_x^2}\dot{f}^2 \left[\frac{-u_x(1+u_x^2)^{\frac{1}{2}} - u(1+u_x^2)^{\frac{-1}{2}}u_x(u_x)_x}{u^2(1+u_x^2)}\right] \\
=& -\frac{u_x}{\sqrt{1+_x^2}}(u_x)_x F^{k-1}\left[-\dot{f}^1\frac{(u_x)_x}{(1+u_x^2)^{\frac{3}{2}}} + \dot{f}^2\frac{(n-1)}{u\sqrt{1+u_x^2}}\right] \\
& + kF^{k-1}\frac{\dot{f}^1}{1+u_x^2}(u_x)_{xx} - 3kF^{k-1}\frac{\dot{f}^1}{(1+u_x^2)^2}u_x((u_x)_x)^2 \\
& + (n-1)kF^{k-1}\frac{\dot{f}^2}{u^2}u_x + (n-1)kF^{k-1}\frac{\dot{f}^2}{u(1+u_x^2)}u_x(u_x)_x \\
=& \frac{kF^{k-1}}{1+u_x^2}\dot{f}^1\left(u_x\right)_{xx} + (1-3k)\frac{F^{k-1}\dot{f}^1 u_x}{\left(1+u_x^2\right)^2}\left(\left(u_x\right)_x\right)^2 \\
& + (n-1)(k-1)\frac{F^{k-1}\dot{f}^2 u_x}{u\left(1+u_x^2\right)} + (n-1)\frac{kF^{k-1}\dot{f}^2}{u^2}u_x.
\end{aligned} \tag{4.28}
$$

It is similar to prove *(ii)*

$$
\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial u}{\partial t} =& \frac{\partial}{\partial t}\left[-\sqrt{1+u_x^2}F^k\right] \\
=& -\left(\frac{\partial}{\partial t}\sqrt{1+u_x^2}\right)F^k - \sqrt{1+u_x^2}\left(\frac{\partial}{\partial t}F^k\right) \\
=& -(1+u_x)^{\frac{-1}{2}}u_x(u_x)_t F^k - \sqrt{1+u_x^2}kF^{k-1}\left[\dot{F}_i^j\left(\frac{\partial}{\partial t}h_j^i\right)\right] \\
=& -(1+u_x^2)^{\frac{-1}{2}}u_x(u_x)_t F^k - kF^{k-1}\sqrt{1+u_x^2}\left[\dot{f}^1\frac{\partial}{\partial t}(k) + (n-1)\dot{f}^2\frac{\partial}{\partial t}p\right] \\
=& -(1+u_x^2)^{\frac{-1}{2}}u_x(u_x)_t F^{k-1}F - kF^{k-1}\sqrt{1+u_x^2}\left\{\dot{f}^1\frac{\partial}{\partial t}\left[\frac{-(u_x)_x}{(1+u_x^2)^{\frac{3}{2}}}\right]\right. \\
& \left. +(n-1)\dot{f}^2\frac{\partial}{\partial t}\left[\frac{1}{u(1+u_x^2)^{\frac{1}{2}}}\right]\right\},
\end{aligned} \tag{4.29}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}u_t =& -\frac{1}{\sqrt{1+u_x^2}}u_x(u_x)_t F^{k-1}\left[\dot{f}^1\frac{-u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}+(n-1)\dot{f}^2\frac{1}{u\sqrt{1+u_x^2}}\right]\\
&-kF^{k-1}\sqrt{1+u_x^2}\dot{f}^1\left[\frac{-(u_x)_{xt}(1+u_x^2)^{\frac{3}{2}}+3u_{xx}(1+u_x^2)^{\frac{1}{2}}u_x(u_x)_t}{(1+u_x^2)^3}\right]\\
&-(n-1)kF^{k-1}\sqrt{1+u_x^2}\dot{f}^2\left[\frac{-u_t(1+u_x^2)^{\frac{1}{2}}-u(1+u_x^2)^{\frac{-1}{2}}u_x(u_x)_t}{u^2(1+u_x^2)}\right]\\
=& \frac{\dot{f}^1}{(1+u_x^2)^2}F^{k-1}u_x(u_x)_t u_{xx}-\frac{(n-1)\dot{f}^2}{u(1+u_x)^2}F^{k-1}u_x(u_x)_t\\
&+kF^{k-1}\frac{\dot{f}^1}{1+u_x^2}(u_x)_{xt}-kF^{k-1}\frac{3\dot{f}^1}{(1+u_x)^2}u_{xx}u_x(u_x)_t\\
&+kF^{k-1}\frac{(n-1)\dot{f}^2}{u^2}u_t+kF^{k-1}\frac{(n-1)\dot{f}^2}{u(1+u_x^2)}u_x(u_x)_t\\
=& \frac{kF^{k-1}}{1+u_x^2}\dot{f}^1\left(u_t\right)_{xx}+(1-3k)\frac{F^{k-1}\dot{f}^1u_xu_{xx}}{\left(1+u_x^2\right)^2}\left(u_t\right)_x\\
&+(n-1)(k-1)\frac{F^{k-1}\dot{f}^2u_x\left(u_t\right)_x}{u\left(1+u_x^2\right)}+(n-1)\frac{kF^{k-1}\dot{f}^2}{u^2}u_t. \qquad (4.30)
\end{aligned}
$$

To prove *(iii)*

$$
\begin{aligned}
\frac{\partial}{\partial t}F^k &= kF^{k-1}\frac{\partial}{\partial t}F\\
&= kF^{k-1}\dot{F}_i^j\frac{\partial}{\partial t}h_j^i\\
&= kF^{k-1}\dot{F}_i^j\left[\nabla^i\nabla_jF^k+F^kh^{im}h_{mj}\right]\\
&= kF^{k-1}\dot{F}_i^j\nabla^i\nabla_jF^k+kF^{k-1}\dot{F}_i^jF^kh^{im}h_{mj}\\
&= \mathcal{L}F^k+k\dot{F}^{ij}h_{mj}h_i^mF^k. \qquad (4.31)
\end{aligned}
$$

□

Using these equations and similar arguments as in the previous section we have the following consequences.

Corollary 4.1. *Suppose the initial hypersurface M_0 has $f > 0$ everywhere and consider the flow* (4.26)*.*

i) *With the boundary conditions* (4.4)*, $u_x \geq 0$ and $u_t \leq 0$ continue to hold under the flow.*

ii) *With the boundary conditions* (4.20), $u_t < 0$ *continues to hold under the flow.*

iii) *In the case of pure Neumann boundary conditions* (4.4) *with* $g \equiv 0$, *the minimum of* F *does not decrease under the flow.*

We have specified M_0 to have $f > 0$ strictly now, so our result Corollary 4.1, ii) is also a strict inequality. This is to ensure equation (4.26) is strictly parabolic for any $k > 0$, at least for a short time. Therefore a similar argument as before gives an equivalent local existence result to Theorem 4.1 in the case that the initial hypersurface satisfies $\min_{M_0} F > 0$.

In the case of pure Neumann boundary conditions we can compare the surface M_t evolving via (4.26) with an enclosing cylinder. The cylinder shrinks to a line segment in finite time, providing a sharp bound on the time T by which the solution hypersurface M_t must have ceased to exist.

Finally in the case that $k \leq 1$, pure Neumann boundary conditions and F satisfies Conditions 1 and 2 we classify the singularity at time T, using a similar argument as in the proof of Theorem 4.2.

Theorem 4.3. *Let* M_0 *be such that* $(u_0)_x \geq 0$ *and* $\min_{M_0} f > 0$. *Consider the flow* (4.26) *with* $k \leq 1$ *and pure Neumann boundary conditions. Suppose* $\lim_{t\to T} \kappa_1^2(a,t) < \infty$. *Then* $\lim_{t\to T} \kappa_j^2(0,t) = \infty$ *for* $j = 2, \ldots, n$.

Proof: Suppose $\lim_{t\to T} u(0,t) = \delta > 0$. Then, as in the proof of Theorem 4.2,

$$\kappa_j^2(x,t) \leq \frac{1}{\delta^2},$$

for all $(x,t) \in [0,a] \times [0,T)$. It follows using Condition 2 in the same way as in the proof of Theorem 4.2 that on $[0,a] \times [0,T)$,

$$\frac{u_{xx}}{1+u_x^2} \leq \frac{c_0}{\delta},$$

for some finite $c_0 > 0$.

Multiplying equation *(ii)* Lemma 4.4 by $-e^{-\lambda t}$, For a constant λ to be chosen,

$$
\begin{aligned}
&\frac{kF^{k-1}}{1+u_x^2}\dot{f}^1 \left(u_t\right)_{xx}(-e^{-\lambda t}) + (1-3k)\frac{F^{k-1}\dot{f}^1 u_x u_{xx}}{\left(1+u_x^2\right)^2}\left(u_t\right)_x(-e^{-\lambda t}) \\
&\quad + (n-1)(k-1)\frac{F^{k-1}\dot{f}^2 u_x \left(u_t\right)_x}{u\left(1+u_x^2\right)}(-e^{-\lambda t}) + (n-1)\frac{kF^{k-1}\dot{f}^2}{u^2}u_t(-e^{-\lambda t}) \\
&\quad - u_{tt}(-e^{-\lambda t}) = 0,
\end{aligned}
$$

then

$$
\begin{aligned}
&\frac{kF^{k-1}}{1+u_x^2}\dot{f}^1 w_{xx} + (1-3k)\frac{F^{k-1}\dot{f}^1 u_x u_{xx}}{\left(1+u_x^2\right)^2}w_x \\
&\quad + (n-1)(k-1)\frac{F^{k-1}\dot{f}^2 u_x}{u\left(1+u_x^2\right)}w_x + (n-1)\frac{kF^{k-1}\dot{f}^2}{u^2}w - \lambda w - w_t = 0.
\end{aligned}
$$

Now the evolution equation for $w = -u_t e^{-\lambda t}$ can be written as before

$$
\begin{aligned}
\frac{\partial}{\partial t}w = &\frac{kF^{k-1}}{n\left(1+u_x^2\right)}\dot{f}^1 w_{xx} + (1-3k)\frac{F^{k-1}}{n(1+u_x^2)^2}u_x u_{xx}\dot{f}^1 w_x \\
&+ (k-1)\frac{n-1}{n}\frac{F^{k-1}}{u\left(1+u_x^2\right)}u_x \dot{f}^2 w_x + \left(\frac{(n-1)}{n}\frac{kF^{k-1}\dot{f}^2}{u^2} - \lambda\right)w. \quad (4.32)
\end{aligned}
$$

If we apply the maximum principle to (4.32), we need the right sign coefficient of the zero order term. Because of the convexity of F, we have trace$\dot{F} \leqslant 1$ and since $\dot{f}^1 > 0$ we further have $\dot{f}^2 \leqslant 1$, see Lemma 4.1. In addition, we need $\frac{n-1}{n}\frac{kF^{k-1}}{u^2}$ to be bounded. Since $k \leq 1$, F will have a negative power, then $F^{k-1} \leq \left(\min_{M_0} F\right)^{k-1}$ holds under the flow, by Corollary 4.1, iii). With the bound of u, see Theorem 4.1, we can again choose large λ such that the coefficient of w is negative, so w cannot obtain an interior maximum. The remainder of the proof is the same as for Theorem 4.2, where we now use our generalisation to the short time existence result of this section. □

Chapter 5

Curvature Pinching Estimate

1 Introduction

Using a maximum principle argument we can obtain "pinching ratio" bound. That is a bound on the supremum of the ratio of largest to smallest principal curvatures at each point over the surface.

In this Chapter we remove the convexity condition of the speed, but the speed is homogeneous of degree one in the principal curvatures and the boundary conditions are pure Neumann. Moreover, we classify the singularities of the flow of a larger class of axially symmetric hypersurfaces as Type I. Our approach to remove the convexity requirement on the speed is based upon earlier work of Andrews for evolving convex surfaces [5, 8]; in order to obtain a "curvature pinching estimate" the arguments may be adapted due to axial symmetry case. Constructions were also used in [52, 53] and very recently in [14] to control a pinching function under fully nonlinear curvature flow of nonconvex surfaces. The monotonicity of these curvature pinching functions is obtained via application of the maximum principle on the evolving surfaces; constructions use heavily the Codazzi equations and homogeneity of the speed and the pinching function which provide sufficient information in the case of surfaces. However, in this Chapter, we are able to establish curvature pinching for axially symmetric hypersurfaces, that is, for n-dimensional hypersurfaces with $\mathbb{S}^{n-1}$ symmetry, since enough of the gradient terms disappear from the evolution equation for the pinching function to permit a similar analysis to earlier work. Preservation of a pinching ratio under the flow implies uniform parabolicity of the flow equation and bounds above and below on all symmetric functions of the principal curvatures that are homogeneous of degree zero.

Let us briefly mention some related studies on classification of the singularity in curvature flow of hypersurfaces. In [39] singularity of the mean curvature flow of axially symmetric surfaces with Neumann boundary conditions were shown to be Type I. Later Huisken and Sinestrari obtained asymptotic convexity estimate [40] that states mean convex initial hypersurfaces without pinching estimate under mean curvature flow become weakly convex at a singularity. Additional descriptions of singularities in the positive mean curvature case were provided with the monotonicity formula of Huisken [39] and the Harnack inequality of Hamilton [35]. Unfortunately, monotonicity formulas are not available for other kinds of flows. Likewise, a Harnack inequality to classify Type II singularities is not generally available except for the sub-class of flows as in [4]. We refer the reader to [14, 13] for more information.

2 Elementary flow behaviour

As earlier the evolving graph function $u(x,t)$, describes an axially symmetric hypersurface flowing with speed in the normal direction as in (3.1). Because we have pure Neumann boundary conditions, we can reflect that the $x = 0$ plane to create an even graph function which we then extend to a periodic solution of (4.1) on $\mathbb{R} \times [0, T)$. Then in applying the maximum principle we need to only consider interior extrema. This idea was also used in [18, 39]. Here we use evolution equations mainly on the evolving hypersurface while in Chapter 4 we worked mainly on $[0, a] \times [0, T)$. Specifically, we have the following evolution equations.

Lemma 5.1. *Under the flow* (4.1),

(i.) $\frac{\partial}{\partial t} F = \mathcal{L} F + \dot{F}^{kl} h_k^m h_{ml} F.$

(ii.) $\frac{\partial}{\partial t} H = \mathcal{L} H + \ddot{F}^{kl,rs} \nabla^i h_{kl} \nabla_i h_{rs} + \dot{F}^{kl} h_k^m h_{ml} H.$

(iii.) $\frac{\partial}{\partial t} \kappa_2 = \mathcal{L} \kappa_2 + \dot{F}^{kl} h_{ml} h_k^m \kappa_2 + 2 \dot{F}^{11} q^2 \left(\kappa_1 - \kappa_2 \right).$

Proof: Equations *(i)* and *(ii)* are exactly as in Lemma 3.4. We note that we can actually compute the evolution of the individual principal curvatures because the Weingarten map is everywhere diagonal in our setting. Equations *(iii)* can be obtained as *(iv)* Lemma 3.5 as we have $(n-1)$ rotational curvature. $\square$

Applications of the maximum principle to Lemma 5.1 lead to the following.

Corollary 5.1. *Under the flow* (4.1),

i) *If* $F \geq 0$ *everywhere on* M_0*, then* $\min_{M_t} F \geq \min_{M_0} F$.

ii) *If* F *is convex then if* $H \geq 0$ *everywhere on* M_0*, then* $\min_{M_t} H \geq \min_{M_0} H$.

Proof: These are direct applications of the maximum principle in view of Conditions 1 ii) and iii). Because $\kappa = (\kappa_1, ..., \kappa_n) : \kappa_i > 0$ for all i and $\frac{\partial f}{\partial \kappa_1} > 0$ implies that

$$\dot{F}^{kl} h_{km} h_l^m = \dot{f}^1 \kappa_1^2 + (n-1)\, \dot{f}^2 \kappa_2^2 \geq 0. \tag{5.1}$$

For part ii), convexity of F ensures that the gradient term has the correct sign so this term $\ddot{F}^{kl,rs} \nabla^i h_{kl} \nabla_j h_{rs}$ is positive. The minimum of F is a Lipschitz continuous function, so is differentiable for almost every t. Therefore,

$$\frac{d}{dt} \min_{M_t} H \geq \dot{F}^{kl} h_l^m h_{km} \min_{M_0} H,$$

so if it is initially positive it remains positive. For details of this kind of ODE comparison we refer to [34]. $\square$

Remarks:

(i) As a consequence of our pinching estimate in Section 3 of this Chapter we will see that in fact $H > 0$ remains true under (4.1) even if F is not convex.

(ii) The Euler identity gives

$$f = \dot{f}^1 \kappa_1 + \dot{f}^2 \kappa_2 + \ldots + \dot{f}^n \kappa_n = \dot{f}^1 \kappa_1 + (n-1)\, \dot{f}^2 \kappa_2,$$

so in view of Conditions 1, iii), F remains positive and everywhere we must have at least one of the principal curvatures as positive and therefore $|A| > 0$

holds under the flow.

The evolution equation for κ_2 provides directly a uniform lower bound.

Proposition 5.1. *Under the flow* (4.1)*, the minimum of the rotational curvatures κ_2 does not decrease in time, that is*

$$\min_{M_t} \kappa_2 \geq \min_{M_0} \kappa_2 := c_0 > 0.$$

Proof: Since $q = -u_x\, \kappa_2$, we may rewrite Lemma 5.1, *(iii)* as

$$\frac{\partial}{\partial t}\kappa_2 = \mathcal{L}\kappa_2 + \dot{F}^{kl} h_{km} h^m_{\ l}\, \kappa_2 + 2\dot{F}^{11} u_x^2 \kappa_2^2 \left(\kappa_1 - \kappa_2\right).$$

Using Lemma 2.2, this can be rewritten as

$$\frac{\partial}{\partial t}\kappa_2 = \mathcal{L}\kappa_2 + \dot{F}^{kl} h_{km} h^m \kappa_2 + 2\dot{F}^{11} \frac{1}{u}\left(\frac{u_x}{1+u_x^2}\right) \nabla_1 \kappa_2.$$

As in (5.1), the zero order term is non-negative. Also, the coefficient of $\nabla_1\kappa_2$, while $u > 0$, is bounded because the function $p(t) = \frac{t}{1+t^2}$ is bounded as $-\frac{1}{2}(t^2+1) \leqslant t \leqslant \frac{1}{2}(t^2+1)$, under the flow (4.1) the minimum of κ_2 does not, by the maximum principle. □

Remark: In view of (2.11), precisely $\kappa_2 = \frac{1}{u\sqrt{1+u_x^2}}$ so proposition 5.1 implies

$$\min_{M_0} \kappa_2 = \min_{M_0} \frac{1}{u\sqrt{1+u_x^2}} = c_0 > 0,$$

then

$$\sqrt{1+u_x^2} \leq \frac{1}{c_0 u},$$

that is, while $u > 0$ the gradient u_x remains bounded. We will provide an analogous bound on u_{xx} in Section 3.

3 The pinching estimate

We first characterise the gradient terms at extrema of degree zero homogeneous functions of the curvatures evolving under (4.1). A similar result was established in [14] for surfaces that were not necessarily axially symmetric. Define $G : \Gamma \to \mathbb{R}$, $G(\mathcal{W}) = g(\kappa_1, \ldots, \kappa_n)$ as a curvature function that is smooth. We can compute the evolution equation as follows

$$\begin{aligned} \frac{\partial}{\partial t} G &= \dot{G}_i^j \frac{\partial}{\partial t} h_j^i \\ &= \dot{G}_i^j \left[\mathcal{L} h_j^i + \ddot{F}^{kl,rs} \nabla^i h_{kl} \nabla_j h_{rs} + \dot{F}^{kl} h_{km} h_l^m h_j^i \right] \\ &= \dot{G}^{ij} \left[\dot{F}^{kl} \nabla_k \nabla_l h_{ij} + \ddot{F}^{kl,rs} \nabla_i h_{kl} \nabla_j h_{rs} + \dot{F}^{kl} h_{km} h_l^m h_{ij} \right], \end{aligned} \tag{5.2}$$

but

$$\begin{aligned} \mathcal{L} G = \dot{F}^{kl} \nabla_k \nabla_l G &= \dot{F}^{kl} \nabla_k (\dot{G}_j^i \nabla_l h_j^i) \\ &= \dot{F}^{kl} \ddot{G}^{ij,rs} \nabla_l h_{rs} \nabla_k h_{ij} + \dot{F}^{kl} \dot{G}^{ij} \nabla_k \nabla_l h_{ij}, \end{aligned}$$

so we can have $\dot{F}^{kl} \dot{G}^{ij} \nabla_k \nabla_l h_{ij} = \mathcal{L} G - \dot{F}^{kl} \ddot{G}^{ij,rs} \nabla_l h_{rs} \nabla_k h_{ij}$ and then equation (5.2) becomes

$$\frac{\partial}{\partial t} G = \mathcal{L} G - \dot{F}^{kl} \ddot{G}^{ij,rs} \nabla_l h_{rs} \nabla_k h_{ij} + \dot{G}^{ij} \ddot{F}^{kl,rs} \nabla_i h_{kl} \nabla_j h_{rs} + \dot{F}^{kl} h_{km} h_l^m \dot{G}^{ij} h_{ij}. \tag{5.3}$$

If G is homogeneous of degree zero the last term in (5.3) disappears because of the Euler relation that gives $\dot{G}^{ij} h_{ij} = 0$, see Appendix Section 2, and then the evolution equation of G becomes

$$\frac{\partial}{\partial t} G = \mathcal{L} G - \dot{F}^{kl} \ddot{G}^{ij,rs} \nabla_l h_{rs} \nabla_k h_{ij} + \dot{G}^{ij} \ddot{F}^{kl,rs} \nabla_i h_{kl} \nabla_j h_{rs}. \tag{5.4}$$

For any smooth symmetric function $G(\mathcal{W}) = g(\kappa(\mathcal{W}))$ homogeneous of degree $\alpha \neq 0$ equation (5.3) becomes

$$\frac{\partial}{\partial t} G = \mathcal{L} G - \dot{F}^{kl} \ddot{G}^{ij,rs} \nabla_l h_{rs} \nabla_k h_{ij} + \dot{G}_i^j \ddot{F}^{kl,rs} \nabla_i h_{kl} \nabla_j h_{rs} + \alpha G \dot{F}^{kl} h_{km} h_l^m. \tag{5.5}$$

Lemma 5.2. *Let $G(\mathcal{W}) = g(\kappa(\mathcal{W}))$ be a smooth, symmetric, homogeneous of degree zero function in the principal curvatures of the axially symmetric hypersurface given by* (2.1). *At any stationary point of G for which $\dot{G}$ is nondegenerate,*

$$\left(\dot{G}^{ij}\ddot{F}^{kl,rs} - \dot{F}^{ij}\ddot{G}^{kl,rs}\right)\nabla_i h_{kl}\nabla_j h_{rs} = \frac{2\,f\,\dot{g}^1}{\kappa_2\left(\kappa_2-\kappa_1\right)}\left(\nabla_1 h_{22}\right)^2.$$

Proof: It follows by a short contradiction argument, as in [14], that wherever $\dot{G}$ is nondegenerate we have that $\kappa_1, \kappa_2 \neq 0$ and $\kappa_2 \neq \kappa_1$. Using orthonormal coordinates at a stationary point of G, from (3.2) we have the non-zero components of $\ddot{F}$ and similarly for $\ddot{G}$ as follows

$$\begin{aligned} \ddot{F}^{11,11} &= \ddot{f}^{11}; \qquad \ddot{F}^{11,22} = \ddot{F}^{22,11} = \ddot{f}^{12}, \\ \ddot{F}^{22,22} &= \ddot{f}^{22}; \qquad \ddot{F}^{12,12} = \ddot{F}^{21,21} = \frac{\dot{f}^2 - \dot{f}^1}{\kappa_2 - \kappa_1}. \end{aligned} \tag{5.6}$$

Using this with Lemma 2.2 we compute the following

$$\begin{aligned} R_1 &= \dot{F}^{kl}\ddot{G}^{pq,rs}\nabla_k h_{pq}\nabla_l h_{rs} \\ &= \dot{f}^1\ddot{g}^{11}(\nabla_1 h_{11})^2 \\ &\quad + \dot{f}^1\ddot{g}^{22}(\nabla_1 h_{22})^2 + ... + \dot{f}^1\ddot{g}^{nn}(\nabla_1 h_{nn})^2 \\ &\quad + (n-2)\dot{f}^1\ddot{g}^{22}\nabla_1 h_{22}\nabla_1 h_{22} + ... + (n-2)\dot{f}^1\ddot{g}^{2n}\nabla_1 h_{22}\nabla_1 h_{nn} \\ &\qquad\qquad \text{because } \ddot{g}^{22} = \ddot{g}^{23} = \cdots = \ddot{g}^{2n} \\ &\quad + 2\dot{f}^1\ddot{g}^{12}\nabla_1 h_{11}\nabla_1 h_{22} + ... + 2\dot{f}^1\ddot{g}^{1n}\nabla_1 h_{11}\nabla_1 h_{nn} \\ &\quad + 2\dot{f}^2\ddot{g}^{12,12}(\nabla_2 h_{12})^2 + ... + 2\dot{f}^2\ddot{g}^{1n,1n}(\nabla_2 h_{1n})^2 \\ &= \dot{f}^1\ddot{g}^{11}(\nabla_1 h_{11})^2 \\ &\quad + (n-1)\dot{f}^1\ddot{g}^{22}(\nabla_1 h_{22})^2 + (n-1)(n-2)\dot{f}^1\ddot{g}^{22}(\nabla_1 h_{22})^2 \\ &\qquad\qquad \text{because } \nabla_1 h_{22} = \nabla_1 h_{33} = \cdots = \nabla_1 h_{nn} \\ &\quad + 2(n-1)\dot{f}^1\ddot{g}^{12}\nabla_1 h_{11}\nabla_1 h_{22} \\ &\quad + 2(n-1)\dot{f}^2\ddot{g}^{12,12}(\nabla_2 h_{12})^2, \end{aligned}$$

so

$$
\begin{aligned}
R_1 = &\dot f^1 \ddot g^{11}(\nabla_1 h_{11})^2 \\
&+ (n-1)^2 \dot f^1 \ddot g^{22}(\nabla_1 h_{22})^2 \\
&+ 2(n-1)\dot f^1 \ddot g^{12}\nabla_1 h_{11}\nabla_1 h_{22} \\
&+ 2(n-1)\dot f^2 \ddot g^{12,12}(\nabla_2 h_{12})^2.
\end{aligned}
\tag{5.7}
$$

Because $\nabla_2 h_{12} = \nabla_1 h_{22}$ we rewrite R_1 as follows

$$
\begin{aligned}
R_1 = &\dot f^1 \ddot g^{11}(\nabla_1 h_{11})^2 + (n-1)^2 \dot f^1 \ddot g^{22}(\nabla_1 h_{22})^2 \\
&+ 2(n-1)\dot f^1 \ddot g^{12}\nabla_1 h_{11}\nabla_1 h_{22} + 2(n-1)\dot f^2 \ddot g^{12,12}(\nabla_1 h_{22})^2,
\end{aligned}
\tag{5.8}
$$

then

$$
\begin{aligned}
R_1 &= \dot f^1 \left[\ddot g^{11}(\nabla_1 h_{11})^2 + 2(n-1)\ddot g^{12}\nabla_1 h_{11}\nabla_1 h_{22} + (n-1)^2 \ddot g^{22}(\nabla_1 h_{22})^2\right] \\
&\quad + 2(n-1)\dot f^2 \ddot g^{12,12}(\nabla_1 h_{22})^2 \\
&= \dot f^1 \left[\ddot g^{11}(\nabla_1 h_{11})^2 + 2(n-1)\ddot g^{12}\nabla_1 h_{11}\nabla_1 h_{22} + (n-1)^2 \ddot g^{22}(\nabla_1 h_{22})^2\right] \\
&\quad + 2(n-1)\dot f^2 \left(\frac{\dot g^1 - \dot g^2}{\kappa_1 - \kappa_2}\right)(\nabla_1 h_{22})^2.
\end{aligned}
\tag{5.9}
$$

Similarly

$$
\begin{aligned}
R_2 &= \dot G^{kl}\ddot F^{pq,rs}\nabla_k h_{pq}\nabla_l h_{rs} \\
&= \dot g^1 \ddot f^{11}(\nabla_1 h_{11})^2 + (n-1)^2 \dot g^1 \ddot f^{22}(\nabla_1 h_{22})^2 \\
&\quad + 2(n-1)\dot g^1 \ddot f^{12}\nabla_1 h_{11}\nabla_1 h_{22} + 2(n-1)\dot g^2 \ddot f^{12,12}(\nabla_1 h_{22})^2 \\
&= \dot g^1 \left[\ddot f^{11}(\nabla_1 h_{11})^2 + 2(n-1)\ddot f^{12}\nabla_1 h_{11}\nabla_1 h_{22} + (n-1)^2 \ddot f^{22}(\nabla_1 h_{22})^2\right] \\
&\quad + 2(n-1)\dot g^2 \ddot f^{12,12}(\nabla_1 h_{22})^2 \\
&= \dot g^1 \left[\ddot f^{11}(\nabla_1 h_{11})^2 + 2(n-1)\ddot f^{12}\nabla_1 h_{11}\nabla_1 h_{22} + (n-1)^2 \ddot f^{22}(\nabla_1 h_{22})^2\right] \\
&\quad + 2(n-1)\dot g^2 \left(\frac{\dot f^1 - \dot f^2}{\kappa_1 - \kappa_2}\right)(\nabla_1 h_{22})^2,
\end{aligned}
\tag{5.10}
$$

so

$$
\begin{aligned}
R_2 - R_1 &= \left(\dot{G}^{ij}\ddot{F}^{kl,rs} - \dot{F}^{ij}\ddot{G}^{kl,rs}\right)\nabla_i h_{kl}\nabla_j h_{rs} \\
&= (\dot{g}^1\ddot{f}^{11} - \dot{f}^1\ddot{g}^{11})(\nabla_1 h_{11})^2 + 2(n-1)(\ddot{f}^{12}\dot{g}^1 - \dot{f}^1\ddot{g}^{12})\nabla_1 h_{11}\nabla_1 h_{22} \\
&\quad + (n-1)\left[(n-1)(\dot{g}^1\ddot{f}^{22} - \dot{f}^1\ddot{g}^{22}) + 2\frac{\cancel{\dot{g}^2\dot{f}^2} - \dot{g}^2\dot{f}^1 \cancel{- \dot{f}^2\dot{g}^2} + \dot{f}^2\dot{g}^1}{\kappa_2 - \kappa_1}\right](\nabla_1 h_{22})^2.
\end{aligned}
\tag{5.11}
$$

We know

$$
\nabla_k G = \dot{G}^{pq}\nabla_k h_{pq} = \dot{g}^1\nabla_k h_{11} + (n-1)\dot{g}^2\nabla_k h_{22},
$$

so

$$
\begin{aligned}
\nabla_1 G &= \dot{g}^1\nabla_1 h_{11} + (n-1)\dot{g}^2\nabla_1 h_{22}, \\
\nabla_2 G &= \dot{g}^1\nabla_2 h_{11} + (n-1)\dot{g}^2\nabla_2 h_{22},
\end{aligned}
$$

and since G is nondegenerate,

$$
\begin{aligned}
\nabla_1 h_{11} &= \frac{1}{\dot{g}^1}\left[\nabla_1 G - (n-1)\dot{g}^2\nabla_1 h_{22}\right], \\
\nabla_2 h_{22} &= \frac{1}{(n-1)\dot{g}^2}\left[\nabla_2 G - \dot{g}^1\nabla_2 h_{11}\right].
\end{aligned}
$$

Then (5.11) becomes

$$
\begin{aligned}
R_2 - R_1 = (\dot{g}^1\ddot{f}^{11} - \dot{f}^1\ddot{g}^{11})&\left[\frac{1}{\dot{g}^1}[\nabla_1 G - (n-1)\dot{g}^2\nabla_1 h_{22}\right]^2 \\
&+ 2(n-1)(\ddot{f}^{12}\dot{g}^1 - \dot{f}^1\ddot{g}^{12})\left[\frac{1}{\dot{g}^1}\left[\nabla_1 G - (n-1)\dot{g}^2\nabla_1 h_{22}\right]\right]\nabla_1 h_{22} \\
&+ 2(n-1)\left[\frac{-\dot{g}^2\dot{f}^1 + \dot{f}^2\dot{g}^1}{\kappa_2 - \kappa_1}\right](\nabla_1 h_{22})^2 \\
&+ (n-1)\left[(n-1)(\dot{g}^1\ddot{f}^{22} - \dot{f}^1\ddot{g}^{22})\right](\nabla_1 h_{22})^2,
\end{aligned}
$$

$$
\begin{aligned}
R_2 - R_1 = &\frac{1}{(\dot{g}^1)^2}\left[\dot{g}^1\ddot{f}^{11} - \dot{f}^1\ddot{g}^{11}\right](\nabla_1 G)^2 - \frac{2(n-1)}{(\dot{g}^1)^2}\nabla_1 G\dot{g}^2\nabla_1 h_{22}\left[\dot{g}^1\ddot{f}^{11} - \dot{f}^1\ddot{g}^1\right]\\
&+ \frac{1}{(\dot{g}^1)^2}(n-1)^2(\dot{g}^2)^2(\nabla_1 h_{22})^2\left[\dot{g}^1\ddot{f}^{11} - \dot{f}^1\ddot{g}^{11}\right]\\
&+ \frac{2(n-1)}{\dot{g}^1}\left[\ddot{f}^{12}\dot{g}^1 - \dot{f}^1\ddot{g}^{12}\right]\nabla_1 G\nabla_1 h_{22}\\
&- \frac{2(n-1)^2}{(\dot{g}^1)}\left[\ddot{f}^{12}\dot{g}^1 - \dot{f}^1\ddot{g}^{12}\right](\nabla_1 h_{22})^2(\dot{g}^2) + 2(n-1)\left[\frac{\dot{f}^2\dot{g}^1 - \dot{g}^2\dot{f}^1}{\kappa_2 - \kappa_1}\right](\nabla_1 h_{22})^2\\
&+ (n-1)\left[(n-1)(\dot{g}^1\ddot{f}^{22} - \dot{f}^1\ddot{g}^{22})\right](\nabla_1 h_{22})^2\\
= &\frac{1}{(\dot{g}^1)^2}\left[\dot{g}^1\ddot{f}^{11} - \dot{f}^1\ddot{g}^{11}\right](\nabla_1 G)^2\\
&+ \frac{2(n-1)}{\dot{g}^1}\left[(\ddot{f}^{12}\dot{g}^1 - \dot{f}^1\ddot{g}^{12}) - \frac{\dot{g}^2}{\dot{g}^1}(\dot{g}^1\ddot{f}^{11} - \dot{f}^1\ddot{g}^{11})\right]\nabla_1 G\nabla_1 h_{22}\\
&+ 2(n-1)\frac{\dot{f}^2\dot{g}^1 - \dot{g}^2\dot{f}^1}{\kappa_2 - \kappa_1}(\nabla_1 h_{22})^2\\
&+ (n-1)\left[\frac{(\dot{g}^2)^2}{(\dot{g}^1)^2}(n-1)(\dot{g}^1\ddot{f}^{11} - \dot{f}^1\ddot{g}^{11}) - 2\frac{\dot{g}^2}{\dot{g}^1}(n-1)(\ddot{f}^{12}\dot{g}^1 - \dot{f}^1\ddot{g}^{12})\right.\\
&\left. + (n-1)(\dot{g}^1\ddot{f}^{22} - \dot{f}^1\ddot{g}^{22})\right](\nabla_1 h_{22})^2\\
= &\frac{1}{(\dot{g}^1)^2}\left[\dot{g}^1\ddot{f}^{11} - \dot{f}^1\ddot{g}^{11}\right](\nabla_1 G)^2\\
&+ \frac{2(n-1)}{\dot{g}^1}\left[(\ddot{f}^{12}\dot{g}^1 - \dot{f}^1\ddot{g}^{12}) - \left(\frac{\dot{g}^2}{\dot{g}^1}\right)(\dot{g}^1\ddot{f}^{11} - \dot{f}^1\ddot{g}^{11})\right]\nabla_1 G\nabla_1 h_{22}\\
&+ 2(n-1)\frac{\dot{f}^2\dot{g}^1 - \dot{g}^2\dot{f}^1}{\kappa_2 - \kappa_1}(\nabla_1 h_{22})^2\\
&+ (n-1)^2\left[\left(\frac{\dot{g}^2}{\dot{g}^1}\right)^2(\dot{g}^1\ddot{f}^{11} - \dot{f}^1\ddot{g}^{11}) - 2\left(\frac{\dot{g}^2}{\dot{g}^1}\right)(\ddot{f}^{12}\dot{g}^1 - \dot{f}^1\ddot{g}^{12})\right](\nabla_1 h_{22})^2\\
&+ (n-1)^2\left[(\dot{g}^1\ddot{f}^{22} - \dot{f}^1\ddot{g}^{22})\right](\nabla_1 h_{22})^2. \qquad (5.12)
\end{aligned}
$$

At a spatial critical point of G the first two terms disappear.

For any smooth homogeneous symmetric function A of degree α the next identities are satisfied, see Section 2 Appendix for more details,

$$
\begin{aligned}
\dot{A}^1 y_1 + (n-1)\dot{A}^2 y_2 &= \alpha A,\\
\ddot{A}^{11} y_1 + (n-1)\ddot{A}^{12} y_2 &= (\alpha - 1)\dot{A}^1,\\
\ddot{A}^{12} y_1 + (n-1)\ddot{A}^{22} y_2 &= (\alpha - 1)\dot{A}^2,
\end{aligned}
$$

and then

$$\ddot{A}^{11}(y_1)^2 + 2(n-1)y_1y_2\ddot{A}^{12} + (n-1)^2\ddot{A}^{22}(y_2)^2 = (\alpha-1)(\dot{A}^1y_1 + (n-1)\dot{A}^2y_2)$$
$$= \alpha(\alpha-1)A.$$

Since g is homogeneous of degree zero, the Euler identity gives

$$\dot{g}^1\kappa_1 + (n-1)\,\dot{g}^2\kappa_2 = 0,$$

therefore, because G is nondegenerate we may write

$$\frac{\dot{g}^2}{\dot{g}^1} = -\frac{\kappa_1}{(n-1)\,\kappa_2},$$

and the coefficient of $(\nabla_1 h_{22})^2$ in (5.12) becomes

$$\begin{aligned}
R_2 - R_1 &= \left[\left(\frac{\kappa_1}{\kappa_2}\right)^2 \dot{g}^1\ddot{f}^{11} - 2\left(-\frac{\kappa_1}{\kappa_2}\right)(n-1)\dot{g}^1\ddot{f}^{12} + (n-1)^2\dot{g}^1\ddot{f}^{22}\right](\nabla_1 h_{22})^2 \\
&\quad + \left[\left(\frac{\kappa_1}{\kappa_2}\right)^2\left(-\dot{f}^1\ddot{g}^{11}\right) - 2\frac{-\kappa_1}{\kappa_2}(n-1)(-\dot{f}^1\ddot{g}^{12}) + (n-1)^2(-\dot{f}^1\ddot{g}^{22})\right](\nabla_1 h_{22})^2 \\
&\quad + 2(n-1)\frac{\dot{f}^2\dot{g}^1 - \dot{g}^2\dot{f}^1}{\kappa_2 - \kappa_1}(\nabla_1 h_{22})^2 \\
&= \frac{\dot{g}^1}{(\kappa_2)^2}\left[{\kappa_1}^2\ddot{f}^{11} + 2\kappa_1\kappa_2(n-1)\ddot{f}^{12} + (n-1)^2\ddot{f}^{22}\kappa_2^2\right](\nabla_1 h_{22})^2 \\
&\quad + \frac{\dot{f}^1}{\kappa_2^2}\left[-\kappa_1^2\ddot{g}^{11} - 2(n-1)\kappa_1\kappa_2\ddot{g}^{12} - (n-1)^2\ddot{g}^{22}\kappa_2^2\right](\nabla_1 h_{22})^2 \\
&\quad + 2(n-1)\frac{\dot{f}^2\dot{g}^1 - \dot{g}^2\dot{f}^1}{\kappa_2 - \kappa_1}(\nabla_1 h_{22})^2 \\
&= \frac{\dot{g}^1}{(\kappa_2)^2}\left[{\kappa_1}^2\ddot{f}^{11} + 2\kappa_1\kappa_2(n-1)\ddot{f}^{12} + (n-1)^2\ddot{f}^{22}\kappa_2^2\right](\nabla_1 h_{22})^2 \\
&\quad - \frac{\dot{f}^1}{\kappa_2^2}\left[\kappa_1^2\ddot{g}^{11} + 2(n-1)\kappa_1\kappa_2\ddot{g}^{12} + (n-1)^2\ddot{g}^{22}\kappa_2^2\right](\nabla_1 h_{22})^2 \\
&\quad + 2(n-1)\frac{\dot{f}^2\dot{g}^1 - \dot{g}^2\dot{f}^1}{\kappa_2 - \kappa_1}(\nabla_1 h_{22})^2, \qquad (5.13)
\end{aligned}$$

$$
\begin{aligned}
R_2 - R_1 = & \frac{\dot{g}^1}{(\kappa_2)^2}\left[{\kappa_1}^2\ddot{f}^{11} + 2\kappa_1\kappa_2(n-1)\ddot{f}^{12} + (n-1)^2\ddot{f}^{22}\kappa_2^2\right](\nabla_1 h_{22})^2 \\
& - \frac{\dot{f}^1}{\kappa_2^2}\left[\kappa_1^2\ddot{g}^{11} + 2(n-1)\kappa_1\kappa_2\ddot{g}^{12} + (n-1)^2\ddot{g}^{22}\kappa_2^2\right](\nabla_1 h_{22})^2 \\
& + 2(n-1)\frac{\dot{f}^2\dot{g}^1 - \dot{g}^2\dot{f}^1}{\kappa_2 - \kappa_1}(\nabla_1 h_{22})^2. \qquad (5.14)
\end{aligned}
$$

Since f is homogeneous of degree 1, the first line above is identically equal to zero, while since g is homogeneous of degree 0, the first square bracketed term on the second line above is also identically equal to zero. The remaining term in (5.14) is equal to

$$
\begin{aligned}
R_2 - R_1 &= 2(n-1)\frac{\dot{f}^2\dot{g}^1 - \dot{g}^2\dot{f}^1}{\kappa_2 - \kappa_1}(\nabla_1 h_{22})^2 \\
&= 2(n-1)\frac{\kappa_2\dot{f}^2\dot{g}^1 - \dot{g}^2\dot{f}^1\kappa_2}{\kappa_2(\kappa_2 - \kappa_1)}(\nabla_1 h_{22})^2 \\
&= 2(n-1)\frac{\kappa_2\dot{f}^2\dot{g}^1 - (-\frac{\kappa_1\dot{g}^1}{n-1})\dot{f}^1}{\kappa_2(\kappa_2 - \kappa_1)}(\nabla_1 h_{22})^2 \\
&= \frac{2(n-1)\dot{g}^1}{\kappa_2 - \kappa_1}\left[\dot{f}^2 + \frac{\kappa_1}{(n-1)\kappa_2}\dot{f}^1\right](\nabla_1 h_{22})^2 \\
&= \frac{2(n-1)\dot{g}^1}{\kappa_2(\kappa_2 - \kappa_1)}\left[\kappa_2\dot{f}^2 + \frac{\kappa_1}{(n-1)}\dot{f}^1\right](\nabla_1 h_{22})^2 \\
&= \frac{2\dot{g}^1}{\kappa_2(\kappa_2 - \kappa_1)}\left[(n-1)\kappa_2\dot{f}^2 + \kappa_1\dot{f}^1\right](\nabla_1 h_{22})^2 \\
&= \frac{2\dot{g}^1 F}{\kappa_2(\kappa_2 - \kappa_1)}(\nabla_1 h_{22})^2, \qquad (5.15)
\end{aligned}
$$

and we conclude that at an extremum of G,

$$
\left(\dot{G}^{ij}\ddot{F}^{kl,rs} - \dot{F}^{ij}\ddot{G}^{kl,rs}\right)\nabla_i h_{kl}\nabla_j h_{rs} = \frac{2f\dot{g}^1}{\kappa_2\left(\kappa_2 - \kappa_1\right)}\left(\nabla_1 h_{22}\right)^2.
$$

□

Theorem 5.1. *Under the flow* (3.1),

$$
H\left(x,t\right) \geq c_1\left|A\left(x,t\right)\right|,
$$

where $c_1 = \min\left(\min_{[0,a]}\frac{H}{|A|}\left(\cdot,0\right),1\right)$. *In particular, if* M_0 *has positive mean curvature, then* $H > 0$ *continues to hold under the flow.*

Proof: The function $G = \frac{H}{|A|}$ is homogeneous of degree zero, so it evolves under (3.1) according to

$$\frac{\partial}{\partial t} G = \mathcal{L}G + \left(\dot{G}^{ij}\ddot{F}^{kl,rs} - \dot{F}^{ij}\ddot{G}^{kl,rs}\right)\nabla_i h_{kl}\nabla_j h_{rs}, \tag{5.16}$$

see (5.4). We have $g(\kappa_1, \ldots, \kappa_n) = \frac{\kappa_1 + \ldots + \kappa_n}{\sqrt{\kappa_1^2 + \ldots + \kappa_n^2}}$ so

$$\begin{aligned}
\dot{g}^1(\kappa_1, \kappa_2, \ldots, \kappa_2) &= \frac{\partial g}{\partial \kappa_1} \\
&= \frac{\sqrt{\kappa_1^2 + \ldots + \kappa_n^2} - (\kappa_1 + \ldots + \kappa_n)\frac{1}{2}(\kappa_1^2 + \ldots + \kappa_n^2)^{\frac{-1}{2}} 2\kappa_1}{\kappa_1^2 + \ldots + \kappa_n^2} \\
&= \frac{\sqrt{\kappa_1^2 + (n-1)\kappa_2^2} - (\kappa_1 + (n-1)\kappa_2)[\kappa_1^2 + (n-1)\kappa_2^2]^{\frac{-1}{2}}\kappa_1}{\kappa_1^2 + (n-1)\kappa_2^2} \\
&= \frac{\kappa_1^2 + (n-1)\kappa_2^2 - \kappa_1^2 - (n-1)\kappa_1\kappa_2}{|A|^3} \\
&= \frac{(n-1)\,\kappa_2\,(\kappa_2 - \kappa_1)}{|A|^3}, \quad \text{and} \\
\dot{g}^2(\kappa_1, \kappa_2, \ldots, \kappa_2) &= \frac{\partial g}{\partial \kappa_2} \\
&= \frac{(n-1)\sqrt{\kappa_1^2 + \ldots + \kappa_n^2}}{\kappa_1^2 + \ldots + \kappa_n^2} \\
&\quad - \frac{(\kappa_1 + \ldots + \kappa_n)\frac{1}{2}(\kappa_1^2 + \ldots + \kappa_n^2)^{\frac{-1}{2}} 2(n-1)\kappa_2}{\kappa_1^2 + \ldots + \kappa_n^2} \\
&= \frac{(n-1)\sqrt{\kappa_1^2 + (n-1)\kappa_2^2}}{\kappa_1^2 + (n-1)\kappa_2^2} \\
&\quad - \frac{(\kappa_1 + (n-1)\kappa_2)[\kappa_1^2 + (n-1)\kappa_2^2]^{\frac{-1}{2}}(n-1)\kappa_2}{\kappa_1^2 + (n-1)\kappa_2^2} \\
&= \frac{(n-1)\kappa_1^2 + (n-1)^2\kappa_2^2 - (n-1)\kappa_1\kappa_2 - (n-1)^2\kappa_2^2}{|A|^3} \\
&= \frac{(n-1)\,\kappa_1\,(\kappa_1 - \kappa_2)}{|A|^3}.
\end{aligned} \tag{5.17}$$

Suppose now we are at a spatial minimum of G. At this point, $\dot{G}$ could be nondegenerate or degenerate. If $\dot{G}$ is nondegenerate, then using Lemma 5.2 we have

$$\left(\dot{G}^{ij}\ddot{F}^{kl,rs} - \dot{F}^{ij}\ddot{G}^{kl,rs}\right)\nabla_i h_{kl}\nabla_j h_{rs} = \frac{2\,(n-1)\,f}{|A|^3}\,(\nabla_1 h_{22})^2 > 0,$$

and the maximum principle applied to (5.16) gives that the minimum of G does not decrease. As used earlier, note that the minimum of G at M_T satisfies a corresponding ODE of (5.16) for almost every t by, for example, a result of Hamilton [34].

On the other hand, if $\dot{G}$ is degenerate, then from (5.17) either $\kappa_1 = 0$, or $\kappa_1 = \kappa_2$ (the case $\kappa_2 = 0$ does not occur in view of Proposition 5.1). It follows from the Euler identity and Corollary 5.1, *(i)* that wherever $\kappa_1 = \kappa_2$, the principal curvatures are positive.

- If $\kappa_1 = 0$ then
$$g(0, \kappa_2) = \frac{(n-1)\,\kappa_2}{\sqrt{(n-1)\,\kappa_2^2}} = \sqrt{n-1},$$
a positive lower bound on G.

- If $\kappa_1 = \kappa_2$ then
$$g(\kappa_2, \kappa_2) = \frac{n\,\kappa_2}{\sqrt{n\kappa_2^2}} = \sqrt{n},$$
so g achieves its absolute maximum, namely the equality case of the Cauchy-Schwarz inequality, at the supposed minimum. Thus g must be identically constant and M_t is umbilic, which is impossible.

It follows that G is bounded below by $c_1 := \min(\min_{M_0} G, 1)$. In the case that M_0 has positive mean curvature, $c_1 > 0$ and the second statement of the Lemma follows. $\square$

Remarks:

i) Theorem 5.1 and the Cauchy-Schwarz inequality imply
$$0 < c_0 \leq \frac{H}{|A|} \leq \frac{1}{\sqrt{n}},$$
under (3.1). If we restrict to $\{\kappa \in \Gamma : |A| = 1\}$, the curvatures remain within a compact subset. Since $\dot{f}$ is continuous and homogeneous of degree zero, this in turn implies (3.1) is uniformly parabolic; there are absolute constants $0 < \underline{C} < \overline{C} < 0$ such that for each i
$$\underline{C} \leq \dot{f}^i \leq \overline{C}, \tag{5.18}$$

is maintained under the flow.

ii) By the same argument as in i) above, any homogeneous of degree zero function of the principal curvatures is bounded above and below under the flow. In particular, considering the function $\frac{H}{F}$ we have

$$F \geq \underline{c}_0 H.$$

Similarly trace $\dot{F} = \sum_{i=1}^{n} \dot{f}^i$ is another homogeneous of degree zero function, so under the flow

$$\sum_{i=1}^{n} \dot{f}^i \leq \overline{c}_0.$$

(In the case that F is convex, it follows algebraically that one may take $\underline{c}_0 = \frac{1}{n}$ and $\overline{c}_0 = 1$ in the above two inequalities.) These inequalities were critical in the analysis in Theorem 4.2; in the case of pure Neumann boundary conditions they may be replaced by the above inequalities such that the results carry over for F homogeneous of degree 1 and not necessarily convex. Specifically we have the following partial singularity characterisation:

Theorem 5.2. *Let M_0 be an axially symmetric hypersurface given by* (2.1) *for some positive, nondecreasing function $u_0 \in C^2([0,a])$. Suppose F satisfies Conditions 1 and is everywhere nonnegative on M_0. There exists a unique solution $u \in C^2([0,a] \times [0,T))$, $T < \infty$ to* (4.1) *with pure Neumann boundary conditions. If, additionally,*

$$\lim_{z \to -\infty} f(z, 1, \ldots, 1) < 0,$$

where we allow the case that the limit is equal to $-\infty$, then if $\lim_{t \to T} \kappa_1^2(a,t) < \infty$ we have

$$\lim_{t \to T} \kappa_j^2(0,t) = \infty,$$

for $j = 2, \ldots, n$.

Together with some additional arguments, the pinching estimate of Theorem 5.1 may also be used to show the ratio $\frac{\kappa_1}{\kappa_2}$ remains bounded under (3.1).

Corollary 5.2. *Under the flow* (4.1), *the ratio* $\frac{\kappa_1^2}{\kappa_2^2}$ *remains bounded.*

Proof: We will prove this result by considering three cases separately. We know from Theorem 5.1 that $H > 0$ continues to hold under the flow, so

$$\kappa_1 > -(n-1)\,\kappa_2,$$

(in fact, a slightly stronger statement involving c_1 is possible from Theorem 5.1) and if $\kappa_1 < 0$, then we have

$$\kappa_1^2 \leq (n-1)^2\,\kappa_2^2.$$

If, instead, $0 \leq \kappa_1 \leq c_0$, then from Proposition 5.1 we have

$$0 < \kappa_1 \leq \kappa_2,$$

and so

$$\kappa_1^2 \leq \kappa_2^2 \leq (n-1)^2\,\kappa_2^2.$$

Finally, in the case $\kappa_1 \geq c_0$ we have

$$2\,(n-1)\,\kappa_1\kappa_2 \geq 2\kappa_1\kappa_2 \geq 2c_0^2,$$

and

$$H^2 - |A|^2 = 2\,(n-1)\,\kappa_1\kappa_2 + (n-1)\,(n-2)\,\kappa_2^2 \geq 2\,(n-1)\,\kappa_1\kappa_2,$$

so

$$\frac{H^2}{|A|^2} \geq 1 + \frac{2\,(n-1)\,\kappa_1\kappa_2}{|A|^2} \geq 1 + \varepsilon,$$

for some $\varepsilon > 0$, since the homogeneous of degree zero function $\frac{\kappa_1\kappa_2}{|A|^2}$ attains a positive minimum on the set $\{\kappa = (\kappa_1, \kappa_2) : |A| = 2c_0, \kappa_1, \kappa_2 \geq c_0\}$. Therefore

$$\frac{[\kappa_1 + (n-1)\,\kappa_2]^2}{\kappa_1^2 + (n-1)\,\kappa_2^2} \geq 1 + \varepsilon,$$

so

$$\kappa_1^2 + 2\,(n-1)\,\kappa_1\kappa_2 + (n-1)^2\,\kappa_2^2 \geq (1+\varepsilon)\left[\kappa_1^2 + (n-1)\,\kappa_2^2\right].$$

In other words,

$$\varepsilon\kappa_1^2 \leq \kappa_1^2 + 2\left(n-1\right)\kappa_1\kappa_2 + \left(n-1\right)^2\kappa_2^2 - \kappa_1^2 - (n-1)\kappa_2^2 - \varepsilon(n-1)\kappa_2^2,$$

$$\varepsilon\kappa_1^2 \leq \left(n-1\right)\left(n-2-\varepsilon\right)\kappa_2^2 + 2\left(n-1\right)\kappa_1\kappa_2.$$

Using $ab \leqslant \eta a^2 + \frac{1}{4\eta}b^2$ for any $\eta > 0$ we can consider $a = \kappa_1$ and $b = 2(n-1)\kappa_2$ in the last term of the previous equation and rewrite $2(n-1)\kappa_1\kappa_2 \leqslant \eta\kappa_1^2 + \frac{(n-1)^2}{\eta}\kappa_2^2$. Therefore,

$$\varepsilon\kappa_1^2 \leq \left(n-1\right)\left(n-2-\varepsilon\right)\kappa_2^2 + \eta\kappa_1^2 + \frac{(n-1)^2}{\eta}\kappa_2^2,$$

for any $\eta > 0$. Choosing $\eta = \frac{\varepsilon}{2}$ gives

$$(\varepsilon - \frac{\varepsilon}{2})\kappa_1^2 \leq \left(n-1\right)\left(n-2-\varepsilon\right)\kappa_2^2 + \frac{2(n-1)^2}{\varepsilon}\kappa_2^2,$$

$$\frac{\varepsilon}{2}\kappa_1^2 \leq \left(n-1\right)\left(n-2-\varepsilon+\frac{2(n-1)}{\varepsilon}\right)\kappa_2^2,$$

$$\kappa_1^2 \leq \frac{2}{\varepsilon}\left(n-1\right)\left(n-2-\varepsilon+\frac{2(n-1)}{\varepsilon}\right)\kappa_2^2.$$

We have shown that in all cases κ_1^2 is bounded by κ_2^2, by a constant depending only on n and M_0. This completes the proof. □

Corollary 5.3. *Under the flow* (4.1)*, there exists a constant C, depending only on n and M_0 such that*

$$\left(u_{xx}\right)^2 \leq \frac{C}{u^4}.$$

Proof: From (4.1) and Theorem 5.1 we have that under the flow, (from pinching estimate)

$$\frac{\kappa_1^2}{\kappa_2^2} = \frac{u_{xx}^2}{(1+u_x^2)^3}\frac{u^2(1+u_x^2)}{1} = \frac{u_{xx}^2 u^2}{(1+u_x^2)^2},$$

so

$$u_{xx}^2 = \frac{\kappa_1^2}{\kappa_2^2}\frac{(1+u_x^2)^2}{u^2},$$

because $\frac{\kappa_1^2}{\kappa_2^2} \geq C_1$. Then

$$u_{xx}^2 \leqslant C_1 \frac{(1+u_x^2)^2}{u^2},$$

is preserved. The result follows in view of the Remark after Proposition 5.1 Where $(1+u_x^2)^2 \leqslant \frac{1}{C_0 u}$ and then

$$u_{xx}^2 \leqslant \frac{C_1}{u^2} \frac{1}{C_0^2 u^2} \leqslant \frac{C_1}{C_0^2 u^4} \leqslant \frac{C}{u^4}.$$

□

4 The singularity

Given an initial hypersurface M_0 as in (2.1), with pure Neumann boundary conditions comparison with an enclosing cylinder also flowing under (3.1) shows that the maximal existence time T of a solution to (3.1) with initial hypersurface M_0 is finite. Moreover, as $t \to T$ we must have $u \to 0$, that is, the evolving hypersurface approaches the axis of rotation, because if not, then $u > 0$ at time T and then Proposition 5.1 and Corollary 5.3 imply u_x and u_{xx} are bounded, so M_T is a C^2 hypersurface which could be used in the short time existence result, contradicting the maximality of T. Therefore, there is some $x \in [0, a]$ such that $|A|^2 (x, t) \to \infty$ and $u(x, t) \to 0$ as $t \to T$.

Here we characterise the curvature singularity of an axially symmetric hypersurface with positive F evolving under (3.1) as Type I, analogous to the case of evolution of axially symmetric surfaces of positive mean curvature by the mean curvature flow [39].

Let $F_0 := \min_{M_0} F$. In view of uniform parabolicity, a short argument (Lemma 2.5 in [14]) shows that, under the flow (3.1),

$$F(\mathcal{W}(x,t)) \geq \frac{F_0}{\sqrt{1 - 2\underline{C} F_0^2 t}}.$$

In analogy with the case of mean curvature flow [39], we say a curvature singularity is Type I if there is a constant $C > 0$ such that

$$\lim_{t \to T} \max_{M_t} |A| \leq \frac{C}{\sqrt{T-t}}.$$

If the blow-up rate of $|A|$ is faster than the above right hand side the curvature singularity is said to be Type II.

Example: The blow-up rate of cylinders is Type I.

In the case of a cylinder, say $\kappa_1 = 0$, $\kappa_2 = \ldots = \kappa_n = \frac{1}{r}$, $u_x \equiv 0$ we know $X(x,t) = X(x, r\omega)$ where $\omega \in \mathbb{S}^{n-1}$ then $\frac{\partial X}{\partial t} = (0, \frac{\partial r}{\partial t}\omega)$. Normal vector is $(0,\omega)$ and the speed is $F(\mathcal{W}(x,t)) = f(0, \frac{1}{r}, \ldots, \frac{1}{r})$, so $(0, \frac{\partial r}{\partial t}\omega) = -f(0, \frac{1}{r}, \ldots, \frac{1}{r})(0,\omega)$ and (4.1) becomes

$$\frac{\partial r}{\partial t} = -f\left(0, \frac{1}{r}, \ldots, \frac{1}{r}\right) = -\frac{f_0}{r},$$

where $f_0 := f\,(0, 1, \ldots, 1)$. If the cylinder shrinks to a line at time T then

$$r\,(t) = \sqrt{2 f_0\,(T-t)},$$

and the curvature evolves according to

$$|A| = \frac{n-1}{\sqrt{2 f_0\,(T-t)}},$$

so the singularity is Type I.

Theorem 5.3. *Let M_0 be an axially symmetric hypersurface given by* (2.1) *for some positive function $u_0 \in C^2\,([0,a])$. Suppose F satisfies Conditions 1 and is everywhere strictly positive on M_0. If T is the maximal existence time then the norm of the second fundamental form satisfies*

$$\max_{M_t} |A|^2 \leq \frac{C}{T-t},$$

for all $t < T$.

Proof: We will use a modification of the argument as in Theorem 5.7 of [2] taking into account our more general flow speed F. In view of preserved curvature pinching, Theorem 5.1, there is a positive constant C such that

$$\frac{\dot{F}^{kp}\dot{F}^{l}_{p}h_{k}{}^{m}h_{ml}}{F^2}\leq C^2,$$

and this is because it is a positive homogeneous function of degree zero in a compact set, that obtains a positive minimum and maximum as in Remark ii) following Theorem 5.1 . This inequality states precisely that

$$\frac{\left(\dot{f}^1\right)^2 u_{xx}^2}{\left(1+u_x^2\right)^3}+\frac{(n-1)\left(\dot{f}^2\right)^2}{u^2\left(1+u_x^2\right)}\leq C^2\left[\frac{(n-1)\dot{f}^2}{u\sqrt{1+u_x^2}}-\frac{\dot{f}^1 u_{xx}}{\left(1+u_x^2\right)^{\frac{3}{2}}}\right]^2,$$

where the quantity inside the brackets on the right hand side, namely F, is strictly positive by Lemma 3.4, *(i)*. Neglecting the second term on the left hand side, it follows that

$$\frac{\dot{f}^1 u_{xx}}{1+u_x^2}\leq C\left[\frac{(n-1)\dot{f}^2}{u}-\frac{\dot{f}^1 u_{xx}}{1+u_x^2}\right],$$

and therefore

$$\frac{\dot{f}^1 u_{xx}}{1+u_x^2}\leq\frac{C}{C+1}\frac{(n-1)\dot{f}^2}{u}.$$

Using (4.2) we estimate

$$\begin{aligned}\frac{\partial u}{\partial t}&=\frac{\dot{f}^1}{1+u_x^2}u_{xx}-\frac{(n-1)\dot{f}^2}{u}\\&\leq\left[\frac{C}{C+1}-1\right]\frac{(n-1)\dot{f}^2}{u}=\frac{-1}{C+1}\frac{(n-1)\dot{f}^2}{u}\\&\leq\frac{-(n-1)\underline{C}}{C+1}\frac{1}{u}\qquad\text{from (5.18)}\\&=:-\frac{\delta}{u}.\end{aligned}$$

Now fix x and integrate:

$$\int_t^T u\frac{\partial}{\partial\tau}u(x,\tau)\,d\tau\leq-\int_t^T\delta\ d\tau$$

$$\int_t^T\frac{\partial}{\partial\tau}\frac{1}{2}u^2(x,\tau)\,d\tau\leq-\delta(T-t);$$

this implies

$$u^2\,(x,t) \geq u^2\,(x,T) + 2\delta\,(T-t)\,.$$

It follows that

$$\kappa_2^2\,(x,t) = \frac{1}{u^2(1+u_x^2)} \leq \frac{1}{u^2\,(x,t)} \leq \frac{1}{u^2\,(x,T)+2\delta\,(T-t)} \leq \frac{1}{2\delta\,(T-t)}\ ,$$

and in view of Corollary 5.2

$$\kappa_1^2\,(x,t) \leq C^2\kappa_2^2\,(x,t) \leq \frac{C^2}{2\delta\,(T-t)}.$$

The result follows. □

5 Closed, axially symmetric hypersurfaces

In this section we turn our attention to closed, convex, axially symmetric hypersurfaces without boundary evolving under (3.1). There has been much previous work on closed convex hypersurfaces contracting under flows such as (3.1), without the condition of axial symmetry.

In the case of speeds homogeneous of degree one in the principle curvatures, the famous result of Huisken for the mean curvature flow [37], proved the contraction of the convex hypersurfaces to a round point in finite time. A similar result for $n-$dimensional hypersurface was proved by Chow [24] but for a different speed function. For hypersurfaces in Euclidean space under fully nonlinear speeds [3], a general result is proved by Andrews, that any strictly convex compact initial hypersurface contracts to a spherical point in finite time under convex speed function of principal curvatures, also for a concave speed satisfying some other natural condition too. Later for the same author, in [8] the pinching estimate proved, for a wide class of flows of interest facilitating , convergence of convex hypersurfaces to spheres under various speed that are symmetric functions of curvature. In the special case of contracting surfaces, that is, $n=2$, without any second order conditions on the speed, nor any initial curvature, similar results were proven [9].

For other degrees of homogeneity, Schulze showed in [62] that for a closed convex hypersurface in $\mathbb{R}^{n+1}$ moved under mean curvature flow to a positive power k, it contracts to a point. Furthermore, he proved in [63] that if the initial ratio of the biggest and smallest principal curvatures is close enough to 1 everywhere then this is preserved under the flow. Moreover, by rescaling, it is found that the evolving surface contracts to a unit sphere. Additionally, a study was done by Schnurer [61] about the strictly convex surfaces that shrink to a spherical point, after rescaling, with normal velocity equal to $|A|^2$. Andrews and McCoy were able to show contraction to round points without a second order condition on the speed for hypersurfaces sufficiently close to spheres in [15], that is, for hypersurfaces already very strongly curvature pinched. In [16], a very general case was discussed where they proved that, in Euclidean space, weakly convex hypersurfaces shrinks to a spherical point or collapse to a line segment, containing cylindrical regions, under sufficient and different conditions of the speed function. Without any pinching estimate condition for $n = 2$, Andrews proved the convergence of a convex surface to a spherical point under Gauss curvature [5]. In [10], Andrews and Chen prove the contracting strictly convex surface to a spherical point under Gauss curvature speed function to the power $\frac{\alpha}{2}$. Another result by Andrews et. al. [11] was about Gauss curvature flow to any power $\alpha > \frac{1}{n+2}$ to prove the contraction of the convex hypersurface to fixed volume after the rescaling.

For compact convex hypersurfaces without boundary, short time existence of a solution of the flow equations follows that by standard modification writing a hypersurface as a graph function over $\mathbb{S}^n$, see [3] and [45]. When we fix a diffomorphism we remove degeneracy which means the evolution equations and the image hypersurface remains the same.

In this section, we extend Andrews' result [10] for surfaces to the case of axially symmetric hypersurfaces, concentrating on key steps in order to obtain a curvature pinching estimate, again without any second order condition on the speed.

We need the next Lemma for the relation between pinching ratio $r = \frac{\kappa_2}{\kappa_1}$ and function $G(\mathcal{W}) = \frac{n|A^0|^2}{H^2}$ where $|A^0|^2 = |A|^2 - \frac{1}{n}H^2$ is the trace-free norm of the second fundamental form, a natural pointwise measure for convex hypersurfaces of their closeness to a sphere.

Lemma 5.3. *If* $G(\mathcal{W}) = \frac{n|A^0|^2}{H^2}$ *where* $|A^0|^2 = |A|^2 - \frac{1}{n}H^2 = \frac{1}{n}\sum_{i\leqslant j}(\kappa_i - \kappa_j)^2$ *then we can write*

$$r = \frac{\kappa_1}{\kappa_2} = \frac{1}{n-1}\left[\frac{n}{1 \pm \sqrt{(n-1)G}} - 1\right].$$

Proof:

$$\begin{aligned} G &= \frac{n|A^0|^2}{H^2} \\ &= \frac{\sum_{i\leqslant j}(\kappa_i - \kappa_j)^2}{[\kappa_1 + (n-1)\kappa_2]^2} \quad i = 1 \quad j = 2, \ldots, n \\ &= \frac{(n-1)(\kappa_1 - \kappa_2)^2}{[\kappa_1 + (n-1)\kappa_2]^2} \\ &= \frac{(n-1)(r-1)^2}{[r + (n-1)]^2}, \end{aligned}$$

so

$$(n-1)G = \frac{(n-1)^2(r-1)^2}{(r+(n-1))^2},$$

$$\pm\sqrt{(n-1)G} = \frac{(n-1)(r-1)}{r+(n-1)},$$

$$\pm\sqrt{(n-1)G} = \frac{nr - n - r + 1}{r+(n-1)},$$

$$\pm\sqrt{(n-1)G} = \frac{nr}{r+(n-1)} - \frac{r+(n-1)}{r+(n-1)},$$

$$\pm\sqrt{(n-1)G} = \frac{n}{1+(n-1)r} - 1,$$

$$1 \pm \sqrt{(n-1)G} = \frac{n}{1+(n-1)r},$$

$\frac{n}{1+(n-1)r}$ cannot be equal to 0, therefore

$$1 + (n-1)r = \frac{n}{1 \pm \sqrt{(n-1)G}} \quad , \quad 1 \pm \sqrt{(n-1)G} \neq 0$$

$$(n-1)r = \frac{n}{1 \pm \sqrt{(n-1)G}} - 1,$$

$$r = \frac{1}{n-1}\left[\frac{n}{1 \pm \sqrt{(n-1)\,G}} - 1\right].$$

□

Theorem 5.4. *Let M_0 be a closed, smooth, strictly convex, axially symmetric n-dimensional hypersurface without boundary, $n \geq 2$ smoothly embedded in $\mathbb{R}^{n+1}$ by $X_0 : \mathbb{S}^n \to \mathbb{R}^{n+1}$. Let F satisfy Conditions 1. Then there exists a unique family of smooth, strictly convex, axially symmetric hypersurfaces $\{M_t = X_t(\mathbb{S}^n)\}_{0 \leq t < T}$ satisfying* (3.1)*, with initial condition $X(x,0) = X_0(x)$ for all $x \in \mathbb{S}^n$. The solution exists on a finite maximal time interval $[0,T)$ and the image converges uniformly to a point $p \in \mathbb{R}^{n+1}$ as $t \to T$. The rescaled maps $\frac{X_t - p}{\sqrt{2(T-t)}}$ converge smoothly and exponentially to an embedding $\tilde{X}_T$ whose image is equal to the unit sphere in $\mathbb{R}^{n+1}$ centred at the origin.*

Proof: The argument to obtain curvature pinching, in this setting positive bounds above and below on the ratio $\frac{\kappa_2}{\kappa_1}$ of the two potentially different principal curvatures, is very similar to that presented in [9], so we just point out the necessary adjustments. A suitable pinching function here is the natural generalisation of that in [9], namely

$$G(\mathcal{W}) = \frac{n\left|A^0\right|^2}{H^2}. \tag{5.19}$$

This function G corresponds to

$$g(\kappa(\mathcal{W})) = \frac{n\left(\kappa_1^2 + \ldots + \kappa_n^2\right) - \left(\kappa_1 + \ldots + \kappa_n\right)^2}{\left(\kappa_1 + \ldots + \kappa_n\right)^2}.$$

It can be written in terms of principle curvatures as

$$g(\kappa(\mathcal{W})) = \frac{(n-1)\left(\kappa_1^2 - \kappa_2^2\right)^2}{\left(\kappa_1 + (n-1)\kappa_2\right)^2}.$$

If we consider $g(\kappa) = \frac{n\left(\kappa_1^2+\ldots+\kappa_n^2\right)}{\left(\kappa_1+\ldots+\kappa_n\right)^2} - 1$ so

$$\begin{aligned}\frac{\partial g}{\partial \kappa_i} &= \frac{H^2 n 2\kappa_i - n|A|^2 2H}{H^4} \\ &= \frac{2n(H\kappa_i - |A|^2)}{H^3} \\ &= \frac{2n\{(\kappa_1 + (n-1)\kappa_2)\kappa_i - \kappa_1^2 - (n-1)\kappa_2^2\}}{H^3}. \end{aligned} \tag{5.20}$$

Denoting as earlier the curvature in the axially direction as κ_1, we have $\kappa_2 = \ldots = \kappa_n$ and, by slight abuse of notation, may rewrite

$$g(\kappa_1, \kappa_2) = \frac{n\kappa_1^2 + n(n-1)\kappa_2^2 - (\kappa_1 + (n-1)\kappa_2)^2}{(\kappa_1 + (n-1)\kappa_2)^2}.$$

(5.20) is used in order to compute the following

$$\begin{aligned}\dot{g}^1 &= \frac{2n\{(\kappa_1 + (n-1)\kappa_2)\kappa_1 - \kappa_1^2 - (n-1)\kappa_2^2\}}{H^3} \\ &= \frac{2n\{\kappa_1^2 + (n-1)\kappa_2\kappa_1 - \kappa_1^2 - (n-1)\kappa_2^2\}}{H^3} \\ &= \frac{2n(n-1)\kappa_2(\kappa_1 - \kappa_2)}{H^3} \quad \text{and} \\ \dot{g}^2 &= \frac{2n\{(\kappa_1 + (n-1)\kappa_2)\kappa_2 - \kappa_1^2 - (n-1)\kappa_2^2\}}{H^3} \\ &= \frac{2n\{\kappa_1\kappa_2 + (n-1)\kappa_2^2 - \kappa_1^2 - (n-1)\kappa_2^2\}}{H^3} \\ &= \frac{2n\kappa_1(\kappa_2 - \kappa_1)}{H^3},\end{aligned}$$

so

$$\dot{g}^1 = \frac{2n(n-1)\kappa_2(\kappa_1 - \kappa_2)}{H^3} \quad \text{and} \quad \dot{g}^2 = \frac{2n\kappa_1(\kappa_2 - \kappa_1)}{H^3}. \tag{5.21}$$

We can show that the maximum of G is not increasing in time: restricting ourselves initially to a short time interval on which M_t remains convex, at a maximum point (x_0, t_0) of G, $t_0 > 0$, we must have $G\big|_{(x_0,t_0)} > 0$ and therefore $\kappa_1 \neq \kappa_2$ at that point since otherwise, $G\big|_{(x,t_0)} \equiv 0$ so $\kappa_1 \equiv \kappa_2$ everywhere and because of the convexity M_{t_0} is a sphere. Therefore, $\dot{G}$ is nondegenerate at this maximum point and, in view of Lemma 5.2, we have that the gradient term in the evolution equation (5.16) for G, is equal to

$$\frac{2f\dot{g}^1}{\kappa_2(\kappa_2-\kappa_1)}(\nabla_1 h_{22})^2 = \frac{2f2n(n-1)\kappa_2(\kappa_1-\kappa_2)}{\kappa_2(\kappa_2-\kappa_1)H^3}(\nabla_1 h_{22})^2$$
$$= -\frac{4n(n-1)f}{H^3}(\nabla_1 h_{22})^2 < 0. \qquad (5.22)$$

It follows by the maximum principle that the maximum of G does not increase. At such a maximum point of G, there are two possibilities: $\kappa_1 > \kappa_2$ or $\kappa_1 > \kappa_2$. In the former case,

$$1 < r = \frac{\kappa_1}{\kappa_2} = \frac{1}{n-1}\left[\frac{n}{1-\sqrt{(n-1)G}} - 1\right],$$

so that G does not increase implies r does not increase. In the latter case

$$1 > r = \frac{\kappa_2}{\kappa_1} = \frac{1}{n-1}\left[\frac{n}{1+\sqrt{(n-1)G}} - 1\right],$$

so that G does not increase implies r does not decrease.

In other words, we have shown that the pinching ratio does not deviate further from 1 and so the ratio $\frac{\kappa_1}{\kappa_2}$ is bounded above and below by its initial extreme values. Pinching, together with the absolute lower bound on F (analogous to Corollary 5.1, (i)) gives that the evolving hypersurface remains convex, so the above argument applies up to time T.

Strong parabolic maximum principle will be used to show the strict improvement of the pinching ratio unless M_t is a sphere. Suppose G attained a new extremum at some (x_0, t_0), $t_0 > 0$. From strong maximum principle we have that G is identically constant. If this constant is 0 then the hypersurface is a sphere and it is done. On another hand if G is identically equal to a positive constant that $G = C$ then substituting in equation (5.4) we have $0 = 0 + \left(\dot{G}^{ij}\ddot{F}^{kl,rs} - \dot{F}^{ij}\ddot{G}^{kl,rs}\right)\nabla_i h_{kl}\nabla_j h_{rs}$ must be zero then

$$\left(\dot{G}^{ij}\ddot{F}^{kl,rs} - \dot{F}^{ij}\ddot{G}^{kl,rs}\right)\nabla_i h_{kl}\nabla_j h_{rs} \equiv 0,$$

and from equation (5.22)

$$\frac{4n\left(n-1\right)f}{H^{3}}\left(\nabla_{1}h_{22}\right)^{2}\equiv 0.$$

Since $\frac{4n(n-1)f}{H^{3}}$ is not zero we have

$$\nabla_{1}h_{22}\equiv 0.$$

We know $\nabla_{i}G=0$ so $\dot{G}^{kl}\nabla_{i}h_{kl}=0$ and therefore

$$0\equiv \dot{g}^{1}\nabla_{1}h_{11}+\left(n-1\right)\dot{g}^{2}\nabla_{1}h_{22}.$$

Since $\dot{g}^{1}\neq 0$ and $\dot{g}^{2}\neq 0$, see equation (5.21), we conclude

$$\nabla_{1}h_{11}\equiv 0.$$

In view of Lemma 2.2 and from a theorem of Lawson [46], it follows that M is a sphere.

In view of curvature pinching, the proof of Theorem 5.4 may be completed following the corresponding arguments in [15], since there no convexity condition on the speed was required. Convergence to a point follows by a contradiction argument as in [67]. Rescaling the hypersurfaces to fix the enclosed volume, for example, the rescaled hypesurfaces satisfy the same curvature pinching improvement since the quantity $\frac{\kappa_j}{\kappa_i}$ is homogeneous of degree zero. Therefore, rescaled evolution equation is uniformly parabolic. Then, there is an upper bound on rescaled F via a standard argument of Tso [67]. Alongside with pinching estimate, this gives a uniform upper bound on principle curvatures. The lower positive bound is obtained for rescaled F via Krylov-Safonov Harnack estimate [44] which can be considered also as a positive lower bound for all principle curvatures. Second spatial derivatives of local graph have uniform Hölder bounds following from Theorem 5 of [7]. We can then differentiate through the equation and get higher regularity by standard bootstrapping argument using Schauder estimate for uniformly parabolic PDE [Theorem 4.9 [47]].

In view of the monotone improvement of the pinching ratio, exponential convergence of the rescaled hypersurfaces to the sphere now follows using a linearisation about the sphere as in [52] for example. The metrics for all rescaled times τ are uniformly equivalent by a result of Hamilton [33] (see also [37]), as is the limiting metric $\tilde{g}_\infty$.

□

Remark: As in [9], there is a corresponding result if F is instead homogeneous of degree $\alpha > 1$, provided the initial hypersurface is sufficiently curvature pinched. Specifically, the proof proceeds as above, except that, in view of the homogeneity of F, we will recall (5.14) for the calculation in this case

$$\begin{aligned}
Q &= \left(\dot{G}^{ij}\ddot{F}^{kl,rs} - \dot{F}^{ij}\ddot{G}^{kl,rs}\right)\nabla_i h_{kl}\nabla_j h_{rs} \\
&= \frac{\dot{g}^1}{(k_2)^2}\left[{\kappa_1}^2\ddot{f}^{11} + 2\kappa_1\kappa_2(n-1)\ddot{f}^{12} + (n-1)^2\ddot{f}^{22}\kappa_2^2\right](\nabla_1 h_{22})^2 \\
&- \frac{\dot{f}^1}{\kappa_2^2}\left[\kappa_1^2\ddot{g}^{11} + 2(n-1)\kappa_1\kappa_2\ddot{g}^{12} + (n-1)^2\ddot{g}^{22}\kappa_2^2\right](\nabla_1 h_{22})^2 \\
&+ 2(n-1)\frac{\dot{f}^2\dot{g}^1 - \dot{g}^2\dot{f}^1}{\kappa_2 - \kappa_1}(\nabla_1 h_{22})^2.
\end{aligned} \tag{5.23}$$

Since f is homogeneous of degree α, the square brackets at the first line above is identically equal to $\alpha(\alpha-1)F$, see Appendix Section 2. Because g is homogeneous of degree 0, the square bracket on the second line above is identically equal to zero. Also using (5.15) for the last term in the previous equation considering F homogeneous of degree α (5.23) becomes

$$\begin{aligned}
Q &= \frac{\dot{g}^1}{(\kappa_2)^2}\left[\alpha(\alpha-1)F(\nabla_1 h_{22})^2\right] + \frac{2\dot{g}^1\alpha F}{\kappa_2(\kappa_2-\kappa_1)}(\nabla_1 h_{22})^2 \\
&= \frac{\alpha\dot{g}^1 F}{\kappa_2}\left[\frac{(\alpha-1)}{\kappa_2} - \frac{2}{(\kappa_1-\kappa_2)}\right](\nabla_1 h_{22})^2 \\
&= \frac{\alpha\dot{g}^1 F}{\kappa_2}\left[\frac{(\alpha-1)(\kappa_1-\kappa_2) - 2\kappa_2}{\kappa_2(\kappa_1-\kappa_2)}\right](\nabla_1 h_{22})^2,
\end{aligned} \tag{5.24}$$

from (5.21) to replace $\dot{g}^1$

$$
\begin{aligned}
Q &= \frac{2n(n-1)\kappa_2(\kappa_1-\kappa_2)}{H^3}\ \frac{\alpha F}{\kappa_2}\ \left[\frac{(\alpha-1)(\kappa_1-\kappa_2)-2\kappa_2}{\kappa_2(\kappa_1-\kappa_2)}\right](\nabla_1 h_{22})^2\\
&= \frac{2n\alpha(n-1)}{H^3}\ F\ \left[\frac{(\alpha-1)(\kappa_1-\kappa_2)-2\kappa_2}{\kappa_2}\right](\nabla_1 h_{22})^2\\
&= \frac{2n\alpha(n-1)}{H^3}\ F\ \left[\frac{(\alpha-1)\kappa_1-(\alpha-1)\kappa_2-2\kappa_2}{\kappa_2}\right](\nabla_1 h_{22})^2\\
&= \frac{2n\alpha(n-1)}{H^3}\ F\ \left[\frac{(\alpha-1)\kappa_1-(\alpha-1+2)\kappa_2}{\kappa_2}\right](\nabla_1 h_{22})^2\\
&= \frac{2n\alpha(n-1)}{H^3}\ F\ \left[(\alpha-1)\frac{\kappa_1}{\kappa_2}-(\alpha+1)\right](\nabla_1 h_{22})^2.
\end{aligned}
$$

The gradient term in the evolution equation for G now becomes

$$
\frac{2\,n\,\alpha\,(n-1)\,f}{H^3}\left[(\alpha-1)\,\frac{\kappa_1}{\kappa_2}-(1+\alpha)\right](\nabla_1 h_{22})^2\,.
$$

For this to be nonpositive requires the pinching ratio of the principal curvatures to be not greater than

$$
\frac{\alpha+1}{\alpha-1}=1+\frac{2}{\alpha-1}. \tag{5.25}
$$

□

6 Self-similar hypersurfaces

Some self-similar hypersurfaces flows are well known as a special solution for Mean curvature flow which means preserving the shape of the evolved hypersurface. Under flow self-similar hypersurface looks similar to the initial hypersurface at any time t. The pinching estimate can be used to show that convex, axially symmetric hypersurfaces contracting self-similarly under (3.1) are necessarily spheres. This complements other results on compact self-similar hypersurfaces contracting under curvature flows, such as those in [39, 53]. Specifically, in higher dimension under non-negative mean curvature flow if the compact hypersurface satisfied $H=\langle X,\nu\rangle$ then it is a sphere. Corresponding results were obtained for flows by powers of the Gauss curvature [6] and under fully nonlinear curvature flow [53]. Such hypersurfaces satisfy the corresponding elliptic equation

$$\langle X, \nu \rangle = F(\mathcal{W}), \tag{5.26}$$

and the characterisation as spheres may be deduced by considering the corresponding elliptic equation satisfied by the curvature pinching function. If F homogeneous of degree one we need the following equations in order to have the evolution equation of G for the next Lemma. The covariant derivative (5.26) gives

$$\begin{aligned}
\nabla_j F &= \dot{F}^{kl} \nabla_j h_{kl} \\
&= \langle X, \nabla_j \nu \rangle \\
&= \left\langle X, h_j^k e_k \right\rangle,
\end{aligned}$$

and so

$$\begin{aligned}
\nabla_i \nabla_j F = \ddot{F}^{kl,pq} \nabla_i h_{pq} \nabla_j h_{kl} + \dot{F}^{kl} \nabla_i \nabla_j h_{kl} &= \left\langle \nabla_i X, h_j^k e_k \right\rangle + \left\langle X, h_j^k \nabla_i e_k \right\rangle + \left\langle X, \nabla_i h_j^k e_k \right\rangle \\
&= h_j^k \langle \nabla_i X, e_k \rangle + h_j^k \langle X, \nabla_i e_k \rangle + \langle X, e_k \rangle \nabla^k h_{ij} \\
&= h_j^k \langle e_i, e_k \rangle + \left\langle X, h_j^k \nu \right\rangle + \langle X, e_k \rangle \nabla^k h_{ij} \\
&= h_j^k \delta_{ik} - F h_i^k h_{jk} + \langle X, e_k \rangle \nabla^k h_{ij} \\
&= h_{ij} - F h_i^k h_{jk} + \langle X, e_k \rangle \nabla^k h_{ij}, \quad (5.27)
\end{aligned}$$

then

$$\nabla_i \nabla_j F = \ddot{F}^{kl,pq} \nabla_i h_{pq} \nabla_j h_{kl} + \dot{F}^{kl} \nabla_i \nabla_j h_{kl} = h_{ij} - F h_i^k h_{jk} + \langle X, e_k \rangle \nabla^k h_{ij}. \tag{5.28}$$

Contracting (5.28) with $\dot{F}^{ij}$

$$\begin{aligned}
\mathcal{L} F = \dot{F}^{ij} \nabla_i \nabla_j F &= \dot{F}^{ij} [h_{ij} - F h_i^k h_{jk} + \langle X, e_k \rangle \nabla^k h_{ij}] \\
&= F - \dot{F}^{ij} F h_i^k h_{jk} + \langle X, e_k \rangle \nabla^k F \\
&= (1 - \dot{F}^{ij} h_i^k h_{jk}) F + \langle X, e_k \rangle \nabla^k F.
\end{aligned}$$

For general function G homogeneous of degree zero it is known that

$$\nabla_i \nabla_j G = \ddot{G}^{kl,pq} \nabla_i h_{kl} \nabla_j h_{pq} + \dot{G}^{kl} \nabla_i \nabla_j h_{pq}, \tag{5.29}$$

contracting (5.29) with $\dot{F}^{ij}$

$$\mathcal{L}G = \dot{F}^{ij} \nabla_i \nabla_j G = \dot{F}^{ij} \ddot{G}^{kl,pq} \nabla_i h_{kl} \nabla_j h_{pq} + \dot{F}^{ij} \dot{G}^{kl} \nabla_i \nabla_j h_{pq}, \tag{5.30}$$

and contracting (5.27) with $\dot{G}^{ij}$

$$\begin{aligned} \dot{G}^{ij} \ddot{F}^{kl,pq} \nabla_i h_{kl} \nabla_j h_{pq} + \dot{G}^{ij} \dot{F}^{kl} \nabla_i \nabla_j h_{kl} &= \dot{G}^{ij} h_{ij} - F \dot{G}^{ij} h_i^k h_{jk} + \langle X, e_k \rangle \dot{G}^{ij} \nabla^k h_{ij} \\ &= -F \dot{G}^{ij} h_i^k h_{jk} + \langle X, e_k \rangle \nabla^k G. \end{aligned} \tag{5.31}$$

Using interchanging covariant derivative, see Section 3 Appendix, and consider the homogeneity of F and G when we contract with $\dot{F}^{ij} \dot{G}^{kl}$ we have

$$\dot{F}^{ij} \dot{G}^{kl} \nabla_i \nabla_j h_{kl} = \dot{F}^{ij} \dot{G}^{kl} \nabla_k \nabla_l h_{ij} + F \dot{G}^{kl} h_{km} h_l^m. \tag{5.32}$$

From (5.32) into (5.30)

$$\mathcal{L}G = \dot{F}^{ij} \ddot{G}^{kl,pq} \nabla_i h_{kl} \nabla_j h_{pq} + \dot{F}^{ij} \dot{G}^{kl} \nabla_k \nabla_l h_{pq} + F \dot{G}^{kl} h_{km} h_l^m, \tag{5.33}$$

and from (5.31) and (5.33) we find that

$$\begin{aligned} \mathcal{L}G = {} & \dot{F}^{ij} \ddot{G}^{kl,pq} \nabla_i h_{kl} \nabla_j h_{pq} - \dot{G}^{ij} \ddot{F}^{kl,pq} \nabla_i h_{kl} \nabla_j h_{pq} - F \dot{G}^{ij} h_i^k h_{jk} + \langle X, e_k \rangle \nabla^k G \\ & + F \dot{G}^{kl} h_{km} h_m^l \\ = {} & (\dot{F}^{ij} \ddot{G}^{kl,pq} - \dot{G}^{ij} \ddot{F}^{kl,pq}) \nabla_i h_{kl} \nabla_j h_{pq} + \langle X, e_k \rangle \nabla^k G. \end{aligned} \tag{5.34}$$

Theorem 5.5. *If M is a closed, strictly convex, axially symmetric hypersurface satisfying (5.26), where $F(\mathcal{W}) = f(\kappa)$ is positive, symmetric and homogeneous of degree 1, then M is a unit sphere.*

Proof: In view of (5.26), the function G, as defined in (5.19), satisfies

$$\mathcal{L}G = \left(\dot{F}^{ij}\ddot{G}^{kl,pq} - \dot{G}^{ij}\ddot{F}^{kl,pq}\right)\nabla_i h_{kl}\nabla_j h_{pq} + \langle X, \nabla G\rangle . \tag{5.35}$$

Suppose that G obtains a local maximum on M. At such a local maximum point of G, it must be that $G > 0$ because otherwise $G \equiv 0$ and M is a sphere. Therefore, $\dot{G}$ is nondegenerate at this maximum point and in view of Lemma 5.2, we have

$$\begin{aligned}\left(\dot{F}^{ij}\ddot{G}^{kl,pq} - \dot{G}^{ij}\ddot{F}^{kl,pq}\right)\nabla_i h_{kl}\nabla_j h_{pq} &= \frac{2f\dot{g}^1}{\kappa_2\left(\kappa_2 - \kappa_1\right)}\left(\nabla_1 h_{22}\right)^2\\ &= \frac{4n\left(n-1\right)f}{H^3}\left(\nabla_1 h_{22}\right)^2 > 0,\end{aligned}$$

which is a contradiction to G having a local maximum. Therefore G must be identically constant, and if M is not a sphere $G > 0$ and everywhere on M we have $\kappa_1 \neq \kappa_2$. In this case, that the first term in (5.35) is identically equal to zero implies that $\nabla_1 h_{22} \equiv 0$, and since $\nabla G \equiv 0$ we have

$$0 \equiv \dot{g}^1\nabla_1 h_{11} + \left(n-1\right)\dot{g}^2\nabla_1 h_{22},$$

so $\nabla_1 h_{11} \equiv 0$ also because $\dot{g}^1 \neq 0$. In view of Lemma 2.2, it follows that M is a sphere. □

Remark: Again there is a corresponding result for speeds F homogeneous of degree $\alpha > 1$: if the closed, convex, axially symmetric hypersurface M satisfies (5.26) and has curvature pinching ratio not greater than (5.25), then it must be a sphere.

Note that in this case (5.32) become

$$\dot{F}^{ij}\dot{G}^{kl}\nabla_i\nabla_j h_{kl} = \dot{F}^{ij}\dot{G}^{kl}\nabla_k\nabla_l h_{ij} + \alpha F\dot{G}^{kl}h_{km}h_l^m. \tag{5.36}$$

From (5.36) into (5.30)

$$\mathcal{L}G = \dot{F}^{ij}\ddot{G}^{kl,pq}\nabla_i h_{kl}\nabla_j h_{pq} + \dot{F}^{ij}\dot{G}^{kl}\nabla_k\nabla_l h_{pq} + \alpha F\dot{G}^{kl}h_{km}h_l^m, \tag{5.37}$$

and from (5.31) and (5.37) we find that

$$
\begin{aligned}
\mathcal{L}G &= \dot{F}^{ij}\ddot{G}^{kl,pq}\nabla_i h_{kl}\nabla_j h_{pq} - \dot{G}^{ij}F^{kl,pq}\nabla_i h_{kl}\nabla_j h_{pq} - F\dot{G}^{ij}h_i^k h_{jk} + \langle X, e_k\rangle \nabla^k G \\
&\quad + \alpha F\dot{G}^{kl}h_{km}h_m^l \\
&= (\dot{F}^{ij}\ddot{G}^{kl,pq} - \dot{G}^{ij}\ddot{F}^{kl,pq})\nabla_i h_{kl}\nabla_j h_{pq} + (\alpha - 1)F\dot{G}^{kl}h_l^m h_{km} + \langle X, e_k\rangle \nabla^k G.
\end{aligned}
$$

□

Chapter 6

Sturmian Theorem

1 Introduction

The set of solutions of parabolic partial differential equations can be studied by different methods. One of them was developed in 1936 when Sturm studied the evolution of zeros and zero sets $\{x : f(x,t) = 0\}$ for solutions $f(x,t)$ of partial differential equations of parabolic type

$$f_t = f_{xx} + q(x), f \quad for \quad x \in [0, 2\pi], \quad t > 0, \tag{6.1}$$

with the Dirichlet boundary condition $f = 0$ at $x = 0$ and $x = 2$ and smooth initial data at $t = 0$. A detailed description of the zero set of a solution of (6.1) was given by Sturmian theorem. He showed that the number of zeros (considering multiplicity) was nonincreasing with time. Later in 1980s, the Sturmian theorem attracted more attention in both linear and nonlinear equations. The number of sign changes can be considered rather than the number of zero sets for the function and they will be the same if all zeros are simple. Sometimes it is better to use the number of sign changes because it shows that is not increasing even if it is not guaranteed that the number is finite.

We need a priori estimates to use the Sturmian theorem because they ensure the coefficients of our equation have the right behaviour to allow us to apply the (linear) Sturmian theorem. Sturmian theorem tells us the number of zeros of a solution to an equation does not increase, and they are distinct, but still doesn't actually say anything about existence. The point is, if there is a solution, and if it satisfies the conditions for Sturmian, then the Sturmian theorem can be applied.

2 Linear case

Assume $f : \mathbb{R}^2 \to \mathbb{R}$ to be a solution of

$$f_t = a(x,t)f_{xx} + b(x,t)f_x + c(x,t)f, \tag{6.2}$$

on $Q = \{(x,t) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1, 0 \leqslant t \leqslant \tilde{T}\}$ with Dirichlet boundary conditions $f(0,t) = 0 = f(1,t)$. The number of zeros of $f(\cdot,t)$ is defined as the supremum of all k such that there exist $0 < x_1 < x_2 < \cdot\cdot\cdot\cdot < x_k < 1$ with

$$f(x_i,t)f(x_{i+1},t) < 0, i = 1, 2, \cdot\cdot\cdot, k-1.$$

For $t \in (0,\tilde{T})$ let

$$Z_t(f) = \{x \in \mathbb{R} : f(x,t) = 0\},$$

be the zero set of f. The following Theorem was proved by Angenent in [17]

Theorem 6.1. ***(Angenent)*** *Assume the coefficients of* (6.2) *to satisfy*

$$a > 0, a, a^{-1}, a_t, a_z, a_{zz} \in L^\infty,$$

$$b, b_t, b_z \in L^\infty,$$

$$c \in L^\infty \textit{and } |f(x,t)| \leqslant A\exp(Bx^2).$$

Then for each $t \in (0,\tilde{T})$ the zero set Z_t of f is a discrete subset of $\mathbb{R}$. Moreover if at (x_0,t_0) both $f$and f_x vanish, then there is a neighbourhood $N = [x_0-\epsilon, x_0+\epsilon] \times [t_0-\delta, t_0+\delta]$ of (x_0,t_0) such that

1. *$f \neq 0$ on the side of N, i.e. $f(x_0 \pm \epsilon, t) \neq 0$ for $|t-t_0| \leq \delta$.*

2. *$f(\cdot, t+\delta)$ has at most one zero in the interval $[x_0-\epsilon, x_0+\epsilon]$.*

3. $f(\cdot, t-\delta)$ *has at least two zeros in the interval* $[x_0 - \epsilon, x_0 + \epsilon]$.

The last two parts of the above Theorem are equivalent to saying that if (x_0, t_0) is a multiple zero of f, then for all $0 < t_1 < t_0 < t_2 < \tilde{T}$, the strict inequality $Z_{t_2} \subset Z_{t_1}$ holds, so the size of Z_t is strictly decreasing at $t = t_0$. This version of the Sturmian theorem was used in domains. The condition $|f(x,t)| \leq A\exp(Bx^2)$ was used to restrict the analysis to a class of functions which have a fixed growth at infinity. For bounded domains Angenent proved the following version of the Sturmian theorem.

Theorem 6.2. ***(Angenent)*** *Let* $f : [0,1] \times [0, \tilde{T}] \to \mathbb{R}$ *be a bounded solution to* (6.2) *which satisfies either Dirichlet, Neumann or periodic boundary conditions. Assume the coefficients of* (6.2) *to satisfy*

$$a > 0, a, a^{-1}, a_t, a_z, a_{zz} \in L^\infty,$$

$$b, b_t, b_z \in L^\infty,$$

$$c \in L^\infty,$$

and in addition, in the case of Neumann boundary conditions, assume that $a \equiv 1$ *and* $b \equiv 0$*, Let* z_t *denote the number of zeros of* $f(\cdot, t)$ *in* $[0,1]$*. Then*

1. for $t > 0$*,* z_t *is finite.*

2. if (x_0, t_0)*is a multiple zero of* f*, then for all* $t_1 < t_0 < t_2$ *we have* $Z_{t_1} > Z_{t_2}$*.*

It is possible for an equation of type (6.2) to be reduced to an equation of type (6.1), so that $a \equiv 1$ *and* $b \equiv 0$*. This reduction proceeds in two steps. First we introduce a new coordinate*

$$y = \int_0^x (s,t)^{\frac{1}{2}} ds.$$

In the y, t *coordinates* f *satisfies*

$$f_t = f_{yy} + \tilde{b}(y,t) f_z + \tilde{c}(y,t) f,$$

where $\tilde{b}$ *and* $\tilde{c}$ *satisfy the same conditions as* b *and* c *in the previous Theorem. Next* *substitute*

$$g(y,t) = \exp[\frac{1}{2}\int_0^y \tilde{b}(s,t)ds]f(y,t).$$

Then g satisfies $g_t = g_{yy} + \tilde{a}(y,t)g$ for suitable $\tilde{a}$.

Angenent's and others results showed that the number of points in zero set of a solution for a parabolic equation is non increasing [2]. Applying such a result is one of the goals of this thesis. In view of a priori estimates we are able to use the linear Sturmian theorem for example as in Tai-Chia et all. [48]. For the sake of completeness we include it here.

Theorem 6.3. *Let $u : [-L\pi, L\pi] \times [0,\infty)$ be a non-trivial classical solution of*

$$u_t = a(x,t)u_{xx} + b(x,t)u_z + c(x,t)u, \tag{6.3}$$

with the periodic boundary condition. Assume that a, b, c satisfy the condition

$$a, a^{-1}, a_t, a_z, a_{xx}, b, b_t, b_x, c \in L^\infty_{loc}([-L\pi, L\pi] \times [0,\infty)). \tag{6.4}$$

Let $z(t)$ denote the number of zeros of $u(.,t)$ in $(-L\pi, L\pi]$, i.e., $z(t)$ is the number of points $x \in (-L\pi, L\pi]$ such that $u(x,t) = 0$. Then

1. *For all $t > 0, z(t)$ is finite.*

2. *$z(t)$ is non-increasing in $t \in [0,\infty)$.*

3. *If (x_0, t_0), $t_0 > 0$, is a multiple zero of u, then for all $t_1 < t_0 < t_2$, we have $z(t_1) > z(t_2)$.*

3 Applying Sturmian theorem for fully nonlinear curvature flow

As an application of the Sturmian theorem we can apply Angenents theorem to obtain the following statement about that the zeros of u_{xx} that are discrete and nonincreasing.

We recall the evolution equation for u

$$\frac{\partial u}{\partial t} = \frac{\dot{f}^1}{1+u_x^2} u_{xx} - \frac{(n-1)\,\dot{f}^2}{u}. \tag{6.5}$$

Lemma 6.1. *Assume M_t to be a smooth surface solving* (6.5)*. Assume in addition that $u(x_1,t) \geq \epsilon, \epsilon > 0$, for $0 \leq x_1 \leq a, t \in (0,T^{'}), T^{'} < T$. Then the set $Z_t(u_{xx}) = \{x_1 \in \mathbb{R} : u_{xx}(x_1,t) = \frac{\partial^2 u}{\partial x \partial x} = 0\}$ is a discrete set in $[a,b]$, for all $t \in (0,T^{'})$. Moreover the number of zeros of u_{xx} is a nonincreasing function of time.*

Proof: Differentiating (6.5) with respect to x we find that u_x satisfies

$$\begin{aligned}\frac{\partial u_x}{\partial t} &= \frac{\partial}{\partial t}\frac{\partial u}{\partial x}\\ &= \frac{\dot{F}^{11}}{(1+u_x^2)} u_{xxx} + \frac{(n-1)\dot{F}^{22}}{u^2} u_x - \frac{2\dot{F}^{11}}{(1+u_x^2)^2} u_x u_x^2. \end{aligned} \tag{6.6}$$

By differentiating with respect to x we find that $\eta = u_{xx}$ satisfies

$$\begin{aligned}\frac{\partial \eta}{\partial t} &= \frac{\partial u_{xx}}{\partial t} = \frac{\partial}{\partial x}\frac{\partial u_x}{\partial t}\\ &= \frac{\dot{F}^{11}}{(1+u_x^2)} u_{xxxx} - 2\frac{\dot{F}^{11}}{(1+u_x^2)^2} u_x u_{xx} u_{xxx} + \frac{(n-1)\dot{F}^{22}}{u^2} u_{xx} - 2\frac{(n-1)\dot{F}^{22}}{u^3} u_x^2\\ &\quad - 2\frac{\dot{F}^{11}}{(1+u_x^2)^2} u_{xx}^3 - 4\frac{\dot{F}^{11}}{(1+u_x^2)^2} u_x u_{xx} u_{xxx} + 8\frac{\dot{F}^{11}}{(1+u_x^2)^3} u_x^2 u_{xx}^3\\ &= \frac{\dot{F}^{11}}{(1+u_x^2)} \eta_{xx} - 6\frac{\dot{F}^{11}}{(1+u_x^2)^2} u_x \eta \eta_x + \frac{(n-1)\dot{F}^{22}}{u^2} \eta - 2\frac{(n-1)\dot{F}^{22}}{u^3} u_x^2\\ &\quad - 2\frac{\dot{F}^{11}}{(1+u_x^2)^2} \eta^3 + 8\frac{\dot{F}^{11}}{(1+u_x^2)^3} u_x^2 \eta^3\\ &= \frac{\dot{F}^{11}}{(1+u_x^2)} \eta_{xx} - \left(6\frac{\dot{F}^{11}}{(1+u_x^2)^2} u_x \eta\right)\eta_x + \Bigg[(n-1)\frac{\dot{F}^{22}}{u^2} - 2\frac{\dot{F}^{11}}{(1+u_x^2)^2}\eta^2\\ &\quad +8\frac{\dot{F}^{11}}{(1+u_x^2)^3} u_x^2 \eta^2\Bigg]\eta - 2\frac{(n-1)\dot{F}^{22}}{u^3} u_x^2. \end{aligned} \tag{6.7}$$

As this is a nonlinear equation, we resort to the intersection comparison method discussed in [2]. Suppose we have two axially symmetric surfaces evolving by fully nonlinear curvature, then we can define the difference $w(x_1,t) = u_{xx}^*(x_1,t) - u_{xx}^{**}(x_1,t)$, where u^*, u^{**} are the respective radius functions of the first and the second surfaces. The difference $w(x_1,t)$ satisfies (6.3) where the coefficients conditions as in

(6.4). Now, we have strict parabolicity condition (5.18). While $u \geq \epsilon > 0$ we have bounds for first and second derivatives of u from earlier estimates, so we have a classical solution and curvatures are bounded. The $\dot{f}$ are functions of curvatures on compact set and therefore they are bounded. As a result, boundedness of the coefficients functions required by Sturmian theorem is satisfied. Thus we can apply Theorem 6.3 and conclude that the intersections are discrete and are nonincreasing in time. In particular if the function w has a multiple zero at (x_0, t_0) then for all $0 < t_1 < t_0 < t_2 < T$, the strict inequality $Z_{t_1}(w) > Z_{t_2}(w)$ holds, so $Z_t(w)$ is strictly decreasing at $t = t_0$. Finally as $u^{**} = c$ (i.e. $\eta^{**} = 0 = \eta^{**}_x = \eta^{**}_{xx}$) is a solution of (6.7) where c is a constant (in this case $u^*_x = 0$ and the last term of (6.7) vanishes), we can conclude that the zeros of $u_{xx} := u^*_{xx}$ are discrete and nonincreasing in time. □

Without loss of generality, assume that there is ι local minmima of u_x for the hypersurface and $\iota+1$ local maxima and both of them depending on the zeros of the u_{xx}. We set $\{\imath_j(t)\}_1 \leq j \leq \iota$ as a minmima of u_x and $\{\chi_j(t)\}_1 \leq j \leq \iota$ as a maxima of u_x, by the implicit function theorem each $\imath_j$ and χ_j is a continuous function of time, so

$$0 < \imath_1(t) < \imath_2(t) < \cdots < \imath_\iota(t) < a,$$

$$0 < \chi_1(t) < \chi_2(t) < \cdots < \chi_{\iota+1}(t) < a.$$

Lemma 6.2. *(Convergence of zeros of u_{xx}). The limits*

$$\lim_{t\to T} \imath_j(t) = \imath_j(T) \qquad \text{and} \qquad \lim_{t\to T} \chi_j(t) = \chi_j(T),$$

exist.

Proof: This proof is similar to (Lemma 5.1 [2]) and it will be included here as follows:

Assume that $\imath_j(t)$ does not converge as $t \to T$. Then $\liminf_{t\to T} \imath_j(t) < \limsup_{t\to T} \imath_j(t)$, and we can choose an $x_0 \in (\liminf_{t\to T} \imath_j(t), \limsup_{t\to T} \imath_j(t))$. Since $\imath_j(t)$ is continuous, there is an infinite sequence of times $t_k \to T$ at which $\imath_j(t_k) = x_0$, and at which therefore $u_{xx}(x_0, t_k) = 0$ holds.

Consider the family of curves u_x on $[0, a]$ and $\tilde{u}_x$ obtained by reflecting u_x in the hyperplane $x_1 = x_0$, so reflected curve is another solution i.e. no corner. Here $\tilde{u}_x(x_1, t) = u_x(2x_0 - x_1, t)$, and $\tilde{u}_x$ is defined on $[2x_0 - b, 2x_0]$. The curve $\tilde{u}_x$ corresponds to $\frac{\partial}{\partial x_1} u(2x_0 - x_1, t)$ because it satisfies the same equation (6.6) which allow us to look later at the difference between them. From Lemma 6.1 number of the zeros of u_x where $x_1 \in [0, a]$ and $\tilde{u}_x$ where $x_1 \in [2x_0 - a, 2x_0]$ is finite and non-increasing. Therefore the number of zeros of $\omega = u_x - \tilde{u}_x$ where $x_1 \in [\max(0, 2x_0 - a), \min(a, 2x_0)]$ is finite and non-increasing as well. Specifically, choosing a suitable small neighbourhood gives one zero of $\omega = u_x - \tilde{u}_x$ at the reflection point, and no other zeros in the neighbourhood. Moreover, the number of zeros of ω drops from 1 to 0, from Theorem 6.1, because u_x and $\tilde{u}_x$ intersect non-transversally. However, at time t_{k+1} we have $u_x(x_0, t_{k+1}) = 0$ and the reflected one also has $\tilde{u}_x(x_0, t_{k+1}) = 0$ i.e. a zero of ω which is a contradiction. Therefore, we must have $\liminf_{t_* \to T} \imath_j(t) = \limsup_{t_* \to T} \imath_j(t)$ As a result, $\lim_{t_* \to T} \imath_j(t)$ exists. We must therefore conclude that the $\imath_j(t)$ converges after all. The same argument also shows that $\chi_j(T)'s$ converge. □

Chapter 7

Volume Preserving Curvature Flows

1 Introduction

This Chapter will be about volume preserving curvature flows of hypersurfaces with Neumann boundary conditions on parallel planes. The main results of this Chapter are boundedness of the global term h and of the first derivative of the graph function. Discreteness of zeros of the second derivative of the graph function is obtained also. Sometimes we will need the speed function F to be concave and we will indicate where this is necessary. The n-dimensional axially symmetric hypersurface M can be specified by a corresponding strict positive and suitably smooth function on the bounded interval $u : [0, a] \to \mathbb{R}$ such that M is parametrised by $X : [0, a] \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}$. We define M_t as an evolving family hypersurfaces in $\mathbb{R}^{n+1}$ with boundary. We consider the family of maps $X_t = X(., t)$ evolving according to

$$\frac{\partial X}{\partial t}(x, t) = \{h(t) - F(\mathcal{W}(x, t))\}\nu(x, t), \qquad x \in M_t, t \in \mathbb{R}, \tag{7.1}$$

$$X(., 0) = X_0,$$

where F is fully nonlinear satisfying Conditions 1 *i)* to *vi)* and

$$h(t) = \frac{\int_{M_t} F(\mathcal{W}) d\mu_t}{\int_{M_t} d\mu_t},$$

$d\mu_t$ is the surface area element on M_t. Equation (7.1) ensures that the flow preserves the volume enclosed by M_0.

Remarks:

i) It is clear from evolution equation (7.1) the symmetry of the speed function F and the normal.

ii) Under the flow, the enclosed volume V, is preserved.

Let $E_t \subset G$ is $(n-1)$-dimensional set where G is the domain. As in ([18], Section 1), extending (7.1) to a vector field using the first variation formula and the divergence theorem

$$\begin{aligned} \frac{\partial}{\partial t} V_t &= \int_{E_t} \operatorname{div} \frac{\partial X}{\partial t} dx \\ &= \int_{\partial E_t} \frac{\partial X}{\partial t} . \nu \; d\mu_t \\ &= \int_{E_t} (h - F) \; d\mu_t \\ &= 0. \end{aligned}$$

From standard parabolic theory, the flow exists for a short time $0 < t < t_1$. In addition, we write $[0, T)$ as the maximal time interval when the flow exists. A fixed point argument can be used to handle the global term $h(t)$ to obtain short time existence, see for example [52].

2 Evolution equations

Lemma 7.1. *The evolution equation of the metric under the flow* (7.1)

$$\frac{\partial}{\partial t} g_{ij} = 2\,(h - F)\, h_{ij}.$$

Proof:

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= 2\left\langle \frac{\partial}{\partial t}\frac{\partial X}{\partial x_i}, \frac{\partial X}{\partial x_j}\right\rangle \\
&= 2\left\langle \frac{\partial}{\partial x_i}\left(h\nu - F\nu\right), \frac{\partial X}{\partial x_j}\right\rangle \\
&= 2\left[h\left\langle \frac{\partial \nu}{\partial x_i}, \frac{\partial X}{\partial x_j}\right\rangle + F\left\langle -\frac{\partial}{\partial x_i}\nu, \frac{\partial X}{\partial x_j}\right\rangle\right] \\
&= 2hh_{ij} - 2Fh_{ij} \\
&= 2(h-F)h_{ij}. \qquad (7.2)
\end{aligned}$$

□

Lemma 7.2. *The evolution equation for the outer unit normal under the flow* (7.1) *is given as follows*

$$\frac{\partial \nu}{\partial t} = \nabla F.$$

Proof:

$$\begin{aligned}
\frac{\partial \nu}{\partial t} &= \left\langle \frac{\partial \nu}{\partial t}, \frac{\partial X}{\partial x_i}\right\rangle \frac{\partial X}{\partial x_j} g^{ij} \\
&= -\left\langle \nu, \frac{\partial}{\partial t}\frac{\partial X}{\partial x_i}\right\rangle \frac{\partial X}{\partial x_j} g^{ij} \\
&= -\left\langle \nu, \frac{\partial}{\partial x_i}\left(h\nu - F\nu\right)\right\rangle \frac{\partial X}{\partial x_j} g^{ij} \\
&= \frac{\partial}{\partial x_i} F \frac{\partial X}{\partial x_j} g^{ij} \\
&= \nabla F. \qquad (7.3)
\end{aligned}$$

□

Lemma 7.3. *under the flow* (7.1)

(i.) $\frac{\partial}{\partial t} h_{ij} = \mathcal{L} h_{ij} + \ddot{F}^{kl,rs}\nabla_i h_{rs}\nabla_j h_{kl} + \dot{F}^{kl} h_k^m h_{ml} h_{ij} + (h-2F)\, h_i^k h_{kj}.$

(ii.) $\frac{\partial}{\partial t} h_j^i = \mathcal{L} h_j^i + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_j h_{rs} + \dot{F}^{kl} h_{km} h_l^m h_j^i - h h_m^i h_j^m.$

(iii.) $\frac{\partial}{\partial t} H = \mathcal{L} H + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl} h_{km} h_l^m H - h|A|^2.$

(iv.) $\frac{\partial}{\partial t} F = \mathcal{L} F - (h-F)\, \dot{F}^{kl} h_{km} h_l^m.$

Proof: Firstly, we need (3.10), (8.8) and (3.11). Since $\frac{\partial}{\partial t}h_{ij} = \nabla_i\nabla_j F + (h-F)\, h_i^k h_{kj}$ we obtain the evolution equation of h_{ij} as following

$$
\begin{aligned}
\frac{\partial}{\partial t}h_{ij} &= \ddot{F}^{kl,rs}\nabla_i h_{rs}\nabla_j h_{kl} + \dot{F}^{kl}\nabla_i\nabla_j h_{kl} + (h-F)\, h_i^k h_{kj} \text{ from (3.10)} \\
&= \ddot{F}^{kl,rs}\nabla_i h_{rs}\nabla_j h_{kl} + \dot{F}^{kl}\nabla_k\nabla_l h_{ij} - F h_i^m h_{mj} + \dot{F}^{kl} h_k^m h_{ml} h_{ij} \\
&\quad + (h-F) h_i^k h_{kj} \text{ from (3.11)} \\
&= \ddot{F}^{kl,rs}\nabla_i h_{rs}\nabla_j h_{kl} + \mathcal{L}h_{ij} + \dot{F}^{kl} h_k^m h_{ml} h_{ij} + (h-2F) h_i^k h_{kj} \\
&= \mathcal{L}h_{ij} + \ddot{F}^{kl,rs}\nabla_i h_{rs}\nabla_j h_{kl} + \dot{F}^{kl} h_k^m h_{ml} h_{ij} + (h-2F) h_i^k h_{kj}. \qquad (7.4)
\end{aligned}
$$

To prove *(ii)*

$$
\begin{aligned}
\frac{\partial h_j^i}{\partial t} &= \frac{\partial}{\partial t}\left[g^{ik}h_{kj}\right] \\
&= -2(h-F)h^{ik}h_{kj} + g^{ik}\Big[\mathcal{L}h_{kj} + \ddot{F}^{kl,rs}\nabla_i h_{rs}\nabla_j h_{kl} + \dot{F}^{kl} h_k^m h_{ml} h_{ij} \\
&\quad + (h-2F) h_k^l h_{lj}\Big] \\
&= \mathcal{L}h_j^i + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_j h_{rs} + \dot{F}^{kl} h_{km} h_l^m h_j^i - h h_m^i h_j^m. \qquad (7.5)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{\partial}{\partial t}H &= g_i^j \frac{\partial}{\partial t}h_j^i \\
&= g_i^j\left[\mathcal{L}h_j^i + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_j h_{rs} + \dot{F}^{kl} h_{km} h_l^m h_j^i - h h_m^i h_j^m\right] \\
&= \mathcal{L}H + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl} h_{km} h_l^m H - h g_i^j h_m^i h_j^m \\
&= \mathcal{L}H + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl} h_{km} h_l^m H - h|A|^2, \qquad (7.6)
\end{aligned}
$$

which gives *(iii)*.

To prove *(iv)*

$$
\begin{aligned}
\frac{\partial}{\partial t}F &= \dot{F}_i^j \frac{\partial}{\partial t}h_j^i \\
&= \dot{F}_i^j\left[\mathcal{L}h_j^i + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_j h_{rs} + \dot{F}^{kl} h_{km} h_l^m h_j^i - h h_m^i h_j^m\right],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}F = \dot{F}_i^j \Big[& \dot{F}^{kl}\nabla^k\nabla_l h_j^i + F h_i^m h_{mj} - \dot{F}^{kl} h_k^m h_{ml} h_j^i \qquad \text{from (3.11)} \\
& + \nabla^i \nabla_j F - \dot{F}^{kl}\nabla^i\nabla_j h_{kl} + \dot{F}^{kl} h_{km} h_l^m h_j^i - h h_m^i h_j^m \Big] \quad \text{from (3.10)} \\
= \dot{F}_i^j & \left[F h_i^m h_{mj} + \nabla^i \nabla_j F - h h_m^i h_j^m \right] \\
= \dot{F}_i^j & \left[\nabla^i \nabla_j F - (h - F) h^{ik} h_{kj} \right] \\
= \mathcal{L}F & - (h - F)\dot{F}^{kl} h_{km} h_l^m. \qquad (7.7)
\end{aligned}
$$

□

Lemma 7.4. *As long as* $y > 0$ *we have*

(i.) $\frac{\partial}{\partial t}\langle X, i_1\rangle = \mathcal{L}\langle X, i_1\rangle + hqy.$

(ii.) $\frac{\partial y}{\partial t} = \mathcal{L}y - \frac{(n-1)}{y}\dot{F}^{22} + hpy.$

(iii.) $\frac{\partial q}{\partial t} = \mathcal{L}q + \dot{F}^{ij}h_i^k h_{kj} q - \Big\{ \Big[-(n-1)\dot{F}^{22} + 2\dot{F}^{11} \Big] q^2 + \Big[-(n-1)\dot{F}^{22} p + h + k2\dot{F}^{11} \Big] p \Big\} q.$

(iv.) $\frac{\partial p}{\partial t} = \mathcal{L}p + \dot{F}^{ij}h_i^k h_{kj} p + 2\dot{F}^{11} q^2 (k - p) - hp^2.$

Proof: To prove *(i)*

$$
\begin{aligned}
\frac{\partial}{\partial t}\langle X, i\rangle &= \left\langle \frac{\partial}{\partial t} X, i_1 \right\rangle \\
&= \langle h(t)\nu - F\nu, i\rangle \\
&= \langle h(t)\nu, i\rangle + \langle -F\nu, i\rangle \\
&= h(t)\langle \nu, i\rangle + \langle \dot{F}^{ij}\nabla_i\nabla_j X, i_1\rangle \\
&= h(t)qy + \dot{F}^{ij}\nabla_i\nabla_j\langle X, i_1\rangle \\
&= hqy + \mathcal{L}\langle X, i_1\rangle, \qquad (7.8)
\end{aligned}
$$

and to prove *(ii)*

$$
\begin{aligned}
\frac{\partial}{\partial t}y^2 &= \frac{\partial}{\partial t}\langle X, X\rangle - \frac{\partial}{\partial t}\langle X, i_1\rangle^2 \\
&= 2\left\langle \frac{\partial X}{\partial t}, X \right\rangle - 2\langle X, i_1\rangle \left\langle \frac{\partial X}{\partial t}, i_1 \right\rangle.
\end{aligned}
$$

Because $\frac{\partial y}{\partial t} = \frac{1}{2y}\frac{\partial y^2}{\partial t}$ we can compute the following

$$\begin{aligned}
\frac{\partial y}{\partial t} &= \frac{1}{2y}\left[2\left\langle \frac{\partial X}{\partial t}, X\right\rangle - 2\left\langle X, i_1\right\rangle \left\langle \frac{\partial X}{\partial t}, i_1\right\rangle\right] \\
&= \frac{1}{y}\left[\langle \frac{\partial X}{\partial t}, X\rangle - \langle X, i_1\rangle\langle \frac{\partial X}{\partial t}, i_1\rangle\right] \\
&= \frac{1}{y}\left[\langle h\nu - F\nu, X\rangle - \langle X, i_1\rangle\langle h\nu - F\nu, i_1\rangle\right] \\
&= \frac{1}{y}\left\{\langle h\nu, X\rangle + \langle -F\nu X\rangle - \langle X, i_1\rangle\left[hqy + \dot{F}^{ij}\nabla_i\nabla_j\langle X, i_1\rangle\right]\right\} \\
&= \frac{1}{y}\left[\langle h\nu, X\rangle + \langle -F\nu, X\rangle - hqy\langle X, i_1\rangle - \langle X, i_1\rangle\dot{F}^{ij}\nabla_i\nabla_j\langle X, i_1\rangle\right] \\
&= \frac{1}{y}\langle h\nu, X\rangle + \frac{1}{y}\dot{F}^{ij}\langle\nabla_i\nabla_j X, X\rangle - hq\langle X, i_1\rangle - \frac{1}{y}\langle X, i_1\rangle\dot{F}^{ij}\nabla_i\nabla_j\langle X, i_1\rangle \\
&= \frac{1}{y}\langle h\nu, X\rangle + \frac{1}{y}\langle\mathcal{L}X, X\rangle - hq\langle X, i_1\rangle - \frac{1}{y}\langle X, i_1\rangle\langle\mathcal{L}X, i_1\rangle. \qquad (7.9)
\end{aligned}$$

From (3.23) and

$$\frac{\partial y}{\partial t} = \frac{1}{y}\langle h\nu, X\rangle - hq\langle X, i_1\rangle + \mathcal{L}y - \frac{(n-1)}{y}\dot{F}^{22},$$

then

$$\begin{aligned}
\frac{\partial y}{\partial t} &= \mathcal{L}y - \frac{(n-1)}{y}\dot{F}^{22} + \frac{1}{y}\langle h\nu, X\rangle - h\frac{\langle\nu, i_1\rangle}{y}\langle X, i_1\rangle \\
&= \mathcal{L}y - \frac{(n-1)}{y}\dot{F}^{22} + \frac{1}{y}h\left[\langle\nu, X\rangle - \langle\nu, i_1\rangle\langle X, i_1\rangle\right] \\
&= \mathcal{L}y - \frac{(n-1)}{y}\dot{F}^{22} + \frac{1}{y}h\left[\langle X, \nu - \langle\nu, i_1\rangle i_1\rangle\right].
\end{aligned}$$

Using

$$\nu = \langle\nu, i_1\rangle i_1 + \langle\nu, \omega\rangle\omega,$$

we have

$$\begin{aligned}
\frac{\partial y}{\partial t} &= \mathcal{L}y - \frac{(n-1)}{y}\dot{F}^{22} + \frac{1}{y}h\{\langle X, \langle\nu, \omega\rangle\omega\rangle\} \\
&= \mathcal{L}y - \frac{(n-1)}{y}\dot{F}^{22} + \frac{1}{y}h\{\underbrace{\langle X, \omega\rangle}_{y}\underbrace{\langle\nu, \omega\rangle}_{py}\},
\end{aligned}$$

which show the evolution equation of y

$$\frac{\partial y}{\partial t} = \mathcal{L}y - \frac{(n-1)}{y}\dot{F}^{22} + hpy. \tag{7.10}$$

To prove *(iii)*, we recall (3.6)

$$\frac{\partial}{\partial t}\langle \nu, i_1\rangle = \dot{F}^{ij}\nabla_i\nabla_j\langle \nu, i_1\rangle + \dot{F}^{ij}h_i^k h_{kj}\langle \nu, i_1\rangle, \tag{7.11}$$

and because $q = \frac{\langle \nu, i_1\rangle}{y}$ we compute the following

$$\begin{aligned}\frac{\partial q}{\partial t} &= \frac{1}{y^2}\left[y\frac{\partial}{\partial t}\langle \nu, i_1\rangle - \langle \nu, i_1\rangle\frac{\partial}{\partial t}y\right]\\ &= \frac{1}{y^2}\Big[y\dot{F}^{ij}\nabla_i\nabla_j\langle \nu, i_1\rangle + y\dot{F}^{ij}h_i^k h_{kj}\langle \nu, i_1\rangle - \langle \nu, i_1\rangle\dot{F}^{ij}\nabla_i\nabla_j y + \frac{(n-1)}{y}\langle \nu, i_1\rangle\dot{F}^{22}\\ &\quad -\langle \nu, i_1\rangle hpy\Big]. \end{aligned}\tag{7.12}$$

From (3.29) and (7.12)

$$\begin{aligned}\frac{\partial q}{\partial t} &= \frac{1}{y^2}\left[y\dot{F}^{ij}h_i^k h_{kj}\langle \nu, i_1\rangle + \frac{(n-1)}{y}\langle \nu, i_1\rangle\dot{F}^{22} - \langle \nu, i_1\rangle hpy\right]\\ &\quad + \frac{2}{y}\dot{F}^{ij}\nabla_i\frac{\langle \nu, i_1\rangle}{y}\nabla_j y + \mathcal{L}q\\ &= \mathcal{L}q + \dot{F}^{ij}h_i^k h_{kj}q + \frac{(n-1)}{y^2}\dot{F}^{22}q - hpq + \frac{2}{y}\dot{F}^{11}\nabla_1 q\nabla_1 y \quad \text{from 7.11}\\ &= \mathcal{L}q + \dot{F}^{ij}h_i^k h_{kj}q + (n-1)\dot{F}^{22}(q^2+p^2)q - hpq + \frac{2}{y}\dot{F}^{11}\nabla_1 q(-qy)\\ &= \mathcal{L}q + \dot{F}^{ij}h_i^k h_{kj}q + (n-1)\dot{F}^{22}(q^2+p^2)q - hpq - 2\dot{F}^{11}(kp+q^2)q\\ &= \mathcal{L}q + \dot{F}^{ij}h_i^k h_{kj}q - \left[-(n-1)\dot{F}^{22}(q^2+p^2) + hp + 2\dot{F}^{11}(kp+q^2)\right]q\\ &= \mathcal{L}q + \dot{F}^{ij}h_i^k h_{kj}q - \Big\{\left[-(n-1)\dot{F}^{22} + 2\dot{F}^{11}\right]q^2\\ &\quad + \left[-(n-1)\dot{F}^{22}p + h + k2\dot{F}^{11}\right]p\Big\}\,q. \end{aligned}\tag{7.13}$$

To prove *(iv)*

$$\begin{aligned}\frac{\partial}{\partial t}y^{-2} &= -\frac{2}{y^3}\left[\dot{F}^{ij}\nabla_i\nabla_j y - \frac{(n-1)}{y}\dot{F}^{22} + hpy\right]\\ &= -\frac{2}{y^3}\mathcal{L}y + \frac{2(n-1)}{y^4}\dot{F}^{22} - \frac{2}{y^2}hp. \end{aligned}\tag{7.14}$$

Using (3.37) we have

$$\frac{\partial}{\partial t} y^{-2} = \mathcal{L} y^{-2} - \frac{6}{y^4} \dot{F}^{ij} \nabla_i y \nabla_j y + \frac{2(n-1)\dot{F}^{22}}{y^4} - \frac{2}{y^2} hp.$$

Also from (7.13)

$$\frac{\partial}{\partial t} q^2 = 2q \left[\mathcal{L} q + \dot{F}^{ij} h_i^k h_{kj} q + (n-1)\dot{F}^{22}(q^2+p^2)q - hpq - 2\dot{F}^{11}(kp+q^2)q \right],$$

using (3.34) we can write

$$\begin{aligned}\frac{\partial}{\partial t} q^2 = \mathcal{L} q^2 &- 2\dot{F}^{ij} \nabla_i q \nabla_j q + 2q\dot{F}^{ij} h_i^k h_{kj} q - 2(n-1)\dot{F}^{22} \left(q^2+p^2\right) q^2 \\ &- 2hpq^2 - 4\dot{F}^{11} \left(kp+q^2\right) q^2, \end{aligned} \tag{7.15}$$

and then

$$\begin{aligned}
\frac{\partial}{\partial t} p^2 &= \frac{\partial}{\partial t} y^{-2} - \frac{\partial}{\partial t} q^2 \\
&= \mathcal{L} y^{-2} - \frac{6}{y^4} \dot{F}^{ij} \nabla_i y \nabla_j y + \frac{2(n-1)\dot{F}^{22}}{y^4} - \frac{2}{y^{-2}} hp - \mathcal{L} q^2 + 2\dot{F}^{ij} \nabla_i q \nabla_j q \\
&\quad - 2\dot{F}^{ij} h_i^k h_{kj} q^2 - 2(n-1)\dot{F}^{22} \left(q^2+p^2\right) q^2 + 2hpq^2 + 4\dot{F}^{11} \left(kp+q^2\right) q^2 \\
&= \mathcal{L} p^2 - \frac{6}{y^4} \dot{F}^{11} \nabla_1 y \nabla_1 y + 2(n-1)\dot{F}^{22} \left(q^2+p^2\right)^2 - 2hp \left(q^2+p^2\right) + 2\dot{F}^{11} |\nabla_1 q|^2 \\
&\quad - 2\dot{F}^{ij} h_i^k h_{kj} q^2 - 2(n-1)\dot{F}^{22} q^4 - 2(n-1)\dot{F}^{22} p^2 q^2 + 2hpq^2 + 4\dot{F}^{11} kpq^2 \\
&\quad + 4\dot{F}^{11} q^4 \\
&= \mathcal{L} p^2 - \frac{6\dot{F}^{11}}{y^4} \left(-qy\right)\left(-qy\right) + 2(n-1)\dot{F}^{22} q^4 + 4(n-1)\dot{F}^{22} p^2 q^2 + 2(n-1)\dot{F}^{22} p^4 \\
&\quad - 2hpq^2 - 2hp^3 + 2\dot{F}^{11} k^2 p^2 + 4\dot{F}^{11} q^4 + 4\dot{F}^{11} kpq^4 - 2\dot{F}^{ij} h_i^k h_{kj} q^2 \\
&\quad - 2(n-1)\dot{F}^{22} q^4 - 2(n-1)\dot{F}^{22} p^2 q^2 + 2hpq^2 + 4\dot{F}^{11} kpq^2 + 4\dot{F}^{11} q^4 \\
&= \mathcal{L} p^2 - 2\dot{F}^{11} k^2 q^2 - 2(n-1)\dot{F}^{22} p^2 q^2 - 6\dot{F}^{11} q^4 - 6\dot{F}^{11} q^2 p^2 + 2(n-1)\dot{F}^{22} q^4 \\
&\quad + 4(n-1)\dot{F}^{22} p^2 q^2 + 2(n-1)\dot{F}^{22} p^4 - 2hpq^2 - 2hp^3 + 2\dot{F}^{11} k^2 p^2 + 2\dot{F}^{11} q^4 \\
&\quad + 4\dot{F}^{11} kpq^2 - 2(n-1)\dot{F}^{22} q^4 - 2(n-1)\dot{F}^{22} p^2 q^2 + 2hpq^2 + 4\dot{F}^{11} kpq^2 + 4\dot{F}^{11} q^4 \\
&= \mathcal{L} p^2 - 2\dot{F}^{11} k^2 q^2 - 6\dot{F}^{11} q^2 p^2 + 2(n-1)\dot{F}^{22} p^4 - 2hp^3 + 2\dot{F}^{11} k^2 p^2 + 8\dot{F}^{11} kpq^2.
\end{aligned}$$

But $\dot{F}^{ij}h^k_j h_{kj}p^2 = \dot{F}^{11}k^2p^2 + (n-1)\dot{F}^{22}p^4$, so

$$\begin{aligned}\frac{\partial p^2}{\partial t} &= \mathcal{L}p^2 - 2\dot{F}^{11}k^2q^2 - 6\dot{F}^{11}q^2p^2 + 2\dot{F}^{ij}h^k_j h_{kj}p^2 - 2\dot{F}^{11}k^2p^2 - 2hp^3 + 2\dot{F}^{11}k^2p^2 \\ &\quad + 8\dot{F}^{11}kpq^2 \\ &= \mathcal{L}p^2 + 2\dot{F}^{ij}h^k_j h_{kj}p^2 - 2\dot{F}^{11}k^2q^2 - 6\dot{F}^{11}q^2p^2 - 2hp^3 + 8\dot{F}^{11}kpq^2. \end{aligned} \tag{7.16}$$

From (3.42) and (3.45) we have

$$\mathcal{L}p^2 = 2p\mathcal{L}p + 2\dot{F}^{11}q^2\left(p^2 - 2pk + k^2\right), \tag{7.17}$$

then equation (7.16) becomes

$$\begin{aligned}\frac{\partial p^2}{\partial t} &= 2p\mathcal{L}p + 2\dot{F}^{11}q^2p^2 - 4\dot{F}^{11}q^2pk + 2\dot{F}^{11}q^2k^2 \\ &\quad + 2\dot{F}^{ij}h^k_j h_{kj}p^2 - 2\dot{F}^{11}k^2q^2 - 6\dot{F}^{11}q^2p^2 - 2hp^3 + 8\ddot{F}^{11}k^2q^2 \\ &= 2p\mathcal{L}p + 2\dot{F}^{ij}h^k_j h_{kj}p^2 + 4\dot{F}^{11}kpq^2 - 4\dot{F}^{11}q^2p^2 - 2hp^3 \\ &= 2p\mathcal{L}p + 2\dot{F}^{ij}h^k_j h_{kj}p^2 + 4\dot{F}^{11}\left(k-p\right)pq^2 - 2hp^3. \end{aligned}$$

Using $\frac{\partial p^2}{\partial t} = 2p\frac{\partial p}{\partial t}$, it is concluded that

$$\frac{\partial p}{\partial t} = \mathcal{L}p + \dot{F}^{ij}h^k_j h_{kj}p + 2\dot{F}^{11}q^2\left(k-p\right) - hp^2. \tag{7.18}$$

□

Lemma 7.5. *Under the flow* (7.1) *we have the following evolution equation*

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{q}{F}\right) = \mathcal{L}\left(\frac{q}{F}\right) + \frac{2}{F}\dot{F}^{ij}\nabla_i\left(\frac{q}{F}\right)\nabla_j F - \frac{1}{F}\Big\{\Big[-(n-1)\dot{F}^{22} + 2\dot{F}^{11}\Big]q^2 \\ + \Big[-(n-1)\dot{F}^{22}p + h + 2k\dot{F}^{11}\Big]p\Big\}q + \frac{q}{F^2}h\dot{F}^{ij}h^k_j h_{kj}.\end{aligned}$$

Proof: Using (3.81) the evolution equation of $\frac{\partial}{\partial t}\left(\frac{q}{F}\right)$ can be calculated as follows

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{q}{F}\right) &= \frac{1}{F^2}\left[F\frac{\partial q}{\partial t} - q\frac{\partial F}{\partial t}\right] \\
&= \frac{F}{F^2}\left\{\mathcal{L}q + \dot{F}^{ij}h_i^k h_{kj} q - \left(\left[-(n-1)\dot{F}^{22} + 2\dot{F}^{11}\right] q^2 + [-(n-1)\right.\right. \\
&\quad \left.\left.\dot{F}^{22}p + h + k2\dot{F}^{11}\right] p\right) q\Big\} - \frac{q}{F^2}\left[\mathcal{L}F - (h-F)\dot{F}^{ij}h_j^k h_{kj}\right] \\
&= \frac{1}{F^2}\left[F\mathcal{L}q - q\mathcal{L}F\right] + \frac{F}{F^2}\left\{\dot{F}^{ij}h_i^k h_{kj} q - \left(\left[-(n-1)\dot{F}^{22} + 2\dot{F}^{11}\right] q^2\right.\right. \\
&\quad \left.\left.+\left[-(n-1)\dot{F}^{22}p + h + k2\dot{F}^{11}\right] p\right) q\right\} - \frac{q}{F^2}\left[-(h-F)\dot{F}^{ij}h_j^k h_{kj}\right] \\
&= \mathcal{L}(\frac{q}{F}) + \frac{2}{F}\dot{F}^{ij}\nabla_i\left(\frac{q}{F}\right)\nabla_j F - \frac{1}{F}\left\{\left[-(n-1)\dot{F}^{22} + 2\dot{F}^{11}\right] q^2 + [-(n-1)\right. \\
&\quad \left.\dot{F}^{22}p + h + 2k\dot{F}^{11}\right] p\Big\} q + \frac{q}{F^2}h\dot{F}^{ij}h_j^k h_{kj}. \qquad (7.19)
\end{aligned}
$$

□

Lemma 7.6. *Under the flow* (7.1)

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{H}{F}\right) = \mathcal{L}\left(\frac{H}{F}\right) + \frac{2}{F}\dot{F}^{kl}\nabla_k F\nabla_l\left(\frac{H}{F}\right) + \frac{1}{F}\ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} \\
- \frac{h}{F^2}\left(F|A|^2 - H\dot{F}^{kl}h_{km}h_l^m\right).
\end{aligned}
$$

Proof: Using (3.84) we compute

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{H}{F}\right) &= \dot{F}^{kl}\nabla_k\nabla_l\left(\frac{H}{F}\right) = \frac{F\mathcal{L}H - H\mathcal{L}F}{F^2} - \frac{2}{F}\dot{F}^{kl}\nabla_k F\nabla_l\left(\frac{H}{F}\right) \\
&= \frac{F\left[\mathcal{L}H + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl}h_{km}h_l^m H - h|A|^2\right]}{F^2} \\
&\quad - \frac{H\left[\mathcal{L}F - (h-F)\dot{F}^{kl}h_{km}h_l^m\right]}{F^2} \\
&= \mathcal{L}\left(\frac{H}{F}\right) + \frac{2}{F}\dot{F}^{kl}\nabla_k F\nabla_l\left(\frac{H}{F}\right) + \frac{1}{F}\ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} - \frac{h}{F^2}F|A|^2 \\
&\quad + \frac{H}{F^2}h\dot{F}^{kl}h_{km}h_l^m \\
&= \mathcal{L}\left(\frac{H}{F}\right) + \frac{2}{F}\dot{F}^{kl}\nabla_k F\nabla_l\left(\frac{H}{F}\right) + \frac{1}{F}\ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} \\
&\quad - \frac{h}{F^2}\left(F|A|^2 - H\dot{F}^{kl}h_{km}h_l^m\right). \qquad (7.20)
\end{aligned}
$$

□

Lemma 7.7. *Under the flow* (7.1)

$$\frac{\partial k}{\partial t} = \mathcal{L}k + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl}h_{km}h_l^m k + h\left[(n-1)p^2 - |A|^2\right]$$
$$- 2(n-1)\dot{F}^{11}q^2\left(k-p\right).$$

Proof: By the use of (7.6) and (7.18) we compute

$$\begin{aligned}\frac{\partial k}{\partial t} &= \frac{\partial}{\partial t}\left[H-(n-1)p\right]\\ &= \mathcal{L}H + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl}h_{km}h_l^m H - h|A|^2 - (n-1)\left[\mathcal{L}p\right.\\ &\quad \left.+\dot{F}^{kl}h_{km}h_l^m p + 2\dot{F}^{11}q^2(k-p) - hp^2\right]\\ &= \mathcal{L}k + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl}h_{km}h_l^m k - h|A|^2 - 2(n-1)\dot{F}^{11}q^2(k-p)\\ &\quad + (n-1)hp^2\\ &= \mathcal{L}k + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl}h_{km}h_l^m k + h\left[(n-1)p^2 - |A|^2\right]\\ &\quad - 2(n-1)\dot{F}^{11}q^2(k-p).\end{aligned} \tag{7.21}$$

□

Lemma 7.8. *We have the following evolution equation under the flow* (7.1)

$$\frac{\partial}{\partial t}\left(\frac{k}{p}\right) = \mathcal{L}\left(\frac{k}{p}\right) + \frac{2}{p}\dot{F}^{kl}\nabla_k\left(\frac{k}{p}\right)\nabla_l p + \frac{1}{p}\ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + h\left[(n-1)p - \frac{1}{p}|A|^2 + k\right]$$
$$- 2\frac{q^2}{p^2}\dot{F}^{11}\left[(k-p)(k+(n-1)p)\right].$$

Proof: Using (3.87) we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{k}{p}\right) &= \frac{1}{p^2}\left[p\frac{\partial}{\partial t}k - k\frac{\partial}{\partial t}p\right]\\ &= \frac{1}{p^2}\left\{p\left[\mathcal{L}k + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl}h_{km}h_l^m k + h\left[(n-1)p^2 - |A|^2\right]\right.\right.\\ &\quad \left.\left.-2(n-1)\dot{F}^{11}q^2(k-p)\right]\right\}\\ &\quad - \frac{1}{p^2}\left\{k\left[\mathcal{L}p + \dot{F}^{kl}h_k^m h_{ml}p + 2\dot{F}^{11}q^2(k-p) - hp^2\right]\right\},\end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{k}{p}\right) &= \mathcal{L}\left(\frac{k}{p}\right) + \frac{2}{p}\dot{F}^{kl}\nabla_k\left(\frac{k}{p}\right)\nabla_l p \\
&\quad + \frac{1}{p}\left\{h\left[(n-1)p^2 - |A|^2\right] - 2(n-1)\dot{F}^{11}q^2(k-p)\right\} \\
&\quad + \frac{1}{p}\ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} - 2\frac{k}{p^2}\dot{F}^{11}q^2(k-p) + hk \\
&= \mathcal{L}\left(\frac{k}{p}\right) + \frac{2}{p}\dot{F}^{kl}\nabla_k\left(\frac{k}{p}\right)\nabla_l p + \frac{1}{p}\ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + h\left[(n-1)p - \frac{1}{p}|A|^2 + k\right] \\
&\quad - 2\frac{k}{p^2}\dot{F}^{11}q^2(k-p) - \frac{1}{p}2(n-1)\dot{F}^{11}q^2\,(k-p) \\
&= \mathcal{L}\left(\frac{k}{p}\right) + \frac{2}{p}\dot{F}^{kl}\nabla_k\left(\frac{k}{p}\right)\nabla_l p + \frac{1}{p}\ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + h\left[(n-1)p - \frac{1}{p}|A|^2 + k\right] \\
&\quad - 2\frac{q^2}{p^2}\dot{F}^{11}\left[(k-p)\,(k+(n-1)p)\right]. \qquad (7.22)
\end{aligned}
$$

□

3 Evolving graph function

We use the chain rule to obtain the evolution equation for u. Let $\overline{\omega} : [0, a] \times [0, T) \to \mathbb{R}^{n+1}$ be the unit outward normal of an n dimensional cylinder, which intersects the hypersurface at the point $u(x_1, t)$. Here $\overline{\omega}$ is parametrized over the x_1 axis, whereas ω is parametrized over M^n. As in Chapter 4, we obtain the corresponding evolution equation for the graph height but here using speed function $(h - F)$

$$
\frac{\partial u}{\partial t} = \sqrt{1 + \left(\frac{\partial u}{\partial x_1}\right)^2}\,(h - F(\mathcal{W}))\,, \qquad (7.23)
$$

which means

$$
\frac{\partial u}{\partial t} = \sqrt{1 + \left(\frac{\partial u}{\partial x_1}\right)^2}\,h + \dot{F}^{11}\frac{u_{xx}}{1+u_x^2} - \sum_{j=2}^{n}\dot{F}^{jj}\frac{1}{u}. \qquad (7.24)
$$

4 The lower bound of the surface area

From (2.11) we can compute the following

$$
\det g_{ij} = u^{2(n-1)}\left(1 + u_x^2\right)w_n^2,
$$

and therefore

$$\mu = \sqrt{\det g_{ij}} = u^{n-1}\sqrt{1+u_x^2}\ w_n.$$

Then

$$|M_t| = w_n \int_0^a u^{n-1}\sqrt{1+u_x^2}\ dx,$$

where w_n is area of $n-1$ dimensional sphere. Assuming u is bounded away from 0, $u \geqslant c_5 > 0$

$$\begin{aligned} |\mu_t| &\geqslant w_n \int_0^a u^{n-1} dz \\ &\geqslant w_n c_5^{n-1} a \\ &> 0. \end{aligned} \tag{7.25}$$

As a result, we obtain a lower bound of the surface area which will be used later for the bound of h.

5 Estimate on h

Because h is a global term, it is very important to bound it. In the following, the bound of h is obtained in similar way as in [18].

Proposition 7.1. *Assume M_t to be a smooth, axially symmetric hypersurface solving (7.1), with concave speed function f and a radius function $u(x_1, t) \geqslant c_5 > 0$. Then there is a constant c_2 such that the mean value of the fully nonlinear curvature satisfies*

$$0 \leqslant h(t) \leqslant c_2,$$

with c_2 constant depends on the initial hypersurface.

Proof: It is important here to have F concave which means $F \leqslant \frac{1}{n}H$, see Lemma 4.1. Therefore

$$\int F d\mu_t \leqslant \int \frac{1}{n} H d\mu_t.$$

Now, as in [18] using the parametrisation of M_t by its radius function $u \geqslant c_5 > 0$

we know

$$H = -\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}} + \frac{n-1}{u(1+u_x^2)^{\frac{1}{2}}}.$$

Clearly

$$\begin{aligned} h(t) =& \frac{\int_{M_t} F(\mathcal{W}) d\mu_t}{\int_{M_t} d\mu_t} \\ \leqslant& \frac{\int_{M_t} H(\mathcal{W}) d\mu_t}{\int_{M_t} d\mu_t} \\ =& \frac{1}{|M_t|} \int_{M_t} (k + (n-1)p)\, d\mu_t \\ =& \frac{1}{|M_t|} \int_{M_t} \left(-\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}} + \frac{(n-1)}{u(1+u_x^2)^{\frac{1}{2}}} \right) d\mu_t \\ =& \frac{\int_0^a \left(-\frac{u_{xx}}{(1+u_x^2)} u^{n-1} + (n-1) u^{n-2} \right) dx}{\int_0^a u^{n-1} (1+u_x^2)^{\frac{1}{2}} dx}. \end{aligned} \tag{7.26}$$

For the second term of (7.26) because $p = \frac{1}{u(1+u_x^2)^{\frac{1}{2}}} \leqslant \frac{1}{u} \leqslant \frac{1}{c_5}$ we have

$$\begin{aligned} 0 \leqslant \frac{1}{|M_t|} \int_{M_t} (n-1)p \; d\mu_t \leqslant& \frac{1}{\int_{M_t} d\mu_t} \int_{M_t} (n-1) \frac{1}{u} \; d\mu_t \\ \leqslant& \frac{n-1}{u} \frac{\int_{M_t} d\mu_t}{\int_{M_t} d\mu_t} \leqslant \frac{n-1}{c_5} \leqslant c(n, c_5). \end{aligned} \tag{7.27}$$

For the first term we know $k = -\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}} = -\frac{d}{dx}(\arctan u_x)$ and then

$$\begin{aligned} \frac{1}{|M_t|} \int_{M_t} k \; d\mu_t =& \frac{1}{|M_t|} \int_{M_t} -\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}} \; d\mu_t \\ =& \frac{n w_n}{|M_t|} \int_0^a -\frac{u_{xx}}{(1+u_x^2)} u^{n-1} \; dx \\ =& \frac{n w_n}{|M_t|} \int_0^a -\frac{d}{dx}(\arctan u_x) u^{n-1} \; dx. \end{aligned} \tag{7.28}$$

Using $\int_0^a u dv = [uv]_0^a - \int_0^a v du$ equation (7.28) become

$$\frac{1}{|M_t|} \int_{M_t} k \; d\mu_t = \frac{n(n-1) w_n}{|M_t|} \int_0^a (\arctan u_x) u_x u^{n-2} \; dx, \tag{7.29}$$

because $(\arctan u_x)|_{x=0} = 0$ and also $(\arctan u_x)|_{x=a} = 0$, Neumann boundary conditions. In addition, $0 \leqslant (\arctan u_x)u_x \leqslant \frac{\pi}{2}|u_x| \leqslant \frac{\pi}{2}\sqrt{1+u_x^2}$ so

$$\begin{aligned} \frac{1}{|M_t|}\int_{M_t} k \; d\mu_t &\leqslant \frac{n(n-1)w_n}{|M_t|}\frac{\pi}{2}\int_0^a \sqrt{1+u_x^2}u^{n-2} \; dx \\ &\leqslant \frac{(n-1)}{|M_t|}\frac{\pi}{2}\int_M \frac{1}{u} \; d\mu_t \leqslant c'(n, c_5). \end{aligned} \tag{7.30}$$

Now combining the two terms it is concluded that

$$0 \leqslant h(t) = \frac{\int_{M_t} F(\mathcal{W})d\mu_t}{\int_{M_t} d\mu_t} \leqslant \frac{\int_{M_t} H(\mathcal{W})d\mu_t}{\int_{M_t} d\mu_t} \leqslant c_2.$$

□

6 A gradient estimate

As in [27] we consider now $v = \langle \nu, \omega \rangle^{-1}$, see (3.51) where ω is the unit outward normal to the cylinder over the $n-1$ dimensional sphere, to be used in order to have some bounds for some quantities such as yv and v as in [18] and [43] respectively.

Lemma 7.9. *There exists a constant c_3 depending only on the initial hypersurface, where $yv < c_3$.*

Proof: We recall evolution equation of v from Lemma 3.6

$$\frac{\partial}{\partial t}v = \mathcal{L}v - \dot{F}^{ij}h_j^k h_{ki} \; v + \frac{(n-1)}{y^2}v\dot{F}^{22} - 2v^{-1}\dot{F}^{ij}\nabla_i v \nabla_j v. \tag{7.31}$$

Using c_2 as in Section 5 we compute

$$
\begin{aligned}
\frac{\partial}{\partial t}(yv - c_2 t) &= y\frac{\partial v}{\partial t} + v\frac{\partial y}{\partial t} - c_2 \\
&= y\left[\mathcal{L}v - \dot{F}^{ij}h_j^k h_{ki}\ v + \frac{(n-1)}{y^2}v\dot{F}^{22} - 2v^{-1}\dot{F}^{ij}\nabla_i v\nabla_j v\right] \\
&\quad + v\left[\dot{F}^{ij}\nabla_i\nabla_j y - \frac{(n-1)}{y}\dot{F}^{22} + hpy\right] - c_2 \text{ from Lemma 7.4} \\
&= y\mathcal{L}v - y\dot{F}^{ij}h_j^k h_{ki}\ v + y\frac{(n-1)}{y^2}v\dot{F}^{22} - 2\frac{y}{v}\dot{F}^{ij}\nabla_i v\nabla_j v \\
&\quad + v\mathcal{L}y - \frac{v(n-1)}{y}\dot{F}^{22} + hpyv - c_2 \\
&= y\mathcal{L}v + v\mathcal{L}y - y\dot{F}^{ij}h_j^k h_{ki}\ v - 2\frac{y}{v}\dot{F}^{ij}\nabla_i v\nabla_j v + hpyv - c_2 \\
&= y\mathcal{L}v + v\mathcal{L}y - y\dot{F}^{ij}h_j^k h_{ki}\ v - 2\frac{y}{v}\dot{F}^{ij}\nabla_i v\nabla_j v + hpy\langle \nu, w\rangle^{-1} - c_2.
\end{aligned}
\tag{7.32}
$$

We need

$$
\begin{aligned}
\mathcal{L}(yv) = \dot{F}^{ij}\nabla_i\nabla_j(yv) &= \dot{F}^{ij}\nabla_i\left[v\nabla_j y + y\nabla_j v\right] \\
&= \dot{F}^{ij}\left[\nabla_i v\nabla_j y + v\nabla_i\nabla_j y + \nabla_i y\nabla_j v + y\nabla_i\nabla_j v\right] \\
&= \dot{F}^{ij}\left[2\langle\nabla_i v, \nabla_j y\rangle + v\nabla_i\nabla_j y + y\nabla_i\nabla_j v\right] \\
&= 2\dot{F}^{ij}\langle\nabla_i v, \nabla_j y\rangle + v\mathcal{L}y + y\mathcal{L}v.
\end{aligned}
\tag{7.33}
$$

Also as $\nabla(yv) = (\nabla(y)v + y\nabla(v))$ then

$$
\begin{aligned}
\langle\nabla v, \nabla(yv)\rangle &= \langle\nabla v, \nabla(y)v\rangle + \langle\nabla v, y\nabla v\rangle \\
&= v\langle\nabla v, \nabla(y)\rangle + y\langle\nabla v, \nabla v\rangle \\
&= v\langle\nabla v, \nabla(y)\rangle + y\nabla_i v\nabla_j v.
\end{aligned}
\tag{7.34}
$$

As a result of (7.34), we can write

$$
2\langle\nabla v, \nabla(y)\rangle = \frac{2}{v}\langle\nabla v, \nabla(yv)\rangle - \frac{2y}{v}\nabla_i v\nabla_j v.
\tag{7.35}
$$

By substituting (7.35) into (7.33)

$$\begin{aligned}\mathcal{L}(yv) &= \dot{F}^{ij}\left[\frac{2}{v}\langle\nabla v, \nabla(yv)\rangle - \frac{2y}{v}\nabla_i v\nabla_j v\right] + v\mathcal{L}y + \mathcal{L}v \\ &= v\mathcal{L}y + \mathcal{L}v + \dot{F}^{ij}\frac{2}{v}\langle\nabla v, \nabla(yv)\rangle - \dot{F}^{ij}\frac{2y}{v}\nabla_i v\nabla_j v. \qquad (7.36)\end{aligned}$$

Therefore from (7.36) and (7.32)

$$\begin{aligned}\frac{\partial}{\partial t}(yv - c_2 t) =& \mathcal{L}(yv) - \dot{F}^{ij}\frac{2}{v}\langle\nabla v, \nabla(yv)\rangle + \dot{F}^{ij}\frac{2y}{v}\nabla_i v\nabla_j v - y\dot{F}^{ij}h_j^k h_{ki}\ v \\ & - 2\frac{y}{v}\dot{F}^{ij}\nabla_i v\nabla_j v + hpy\langle\nu, w\rangle^{-1} - c_2 \\ =& \mathcal{L}(yv) - \dot{F}^{ij}\frac{2}{v}\langle\nabla v, \nabla(yv)\rangle - y\dot{F}^{ij}h_j^k h_{ki}\ v + hpy\langle\nu, w\rangle^{-1} - c_2 \\ & \text{because} \quad py = \langle\nu, w\rangle \\ =& \mathcal{L}(yv) - \dot{F}^{ij}\frac{2}{v}\langle\nabla v, \nabla(yv)\rangle - y\dot{F}^{ij}h_j^k h_{ki}\ v + h - c_2 \\ =& \mathcal{L}(yv - c_2 t) - \dot{F}^{ij}\frac{2}{v}\langle\nabla v, \nabla(yv - c_2 t)\rangle - y\dot{F}^{ij}h_j^k h_{ki}\ v + h - c_2.\end{aligned}$$

At a maximum we have $\mathcal{L}(yv - c_2 t) \leqslant 0$ and $\nabla(yv - c_2 t) = 0$. Since the zero order term is non positive then

$$\frac{\partial}{\partial t}(yv - c_2 t) \leqslant \mathcal{L}(yv - c_2 t) - \dot{F}^{ij}\frac{2}{v}\langle\nabla v, \nabla(yv - c_2 t)\rangle.$$

By the result of Hamilton [34] for almost every t, we have

$$\frac{d}{dt}\max_{M_t}(yv - c_2 t) \leqslant 0.$$

Therefore,

$$\max_{M_t}(yv - c_2 t) \leqslant \max_{M_0} yv =: c_4,$$

then

$$yv - c_2 t \leqslant c_4,$$

and

$$yv \leqslant c_2 t + c_4 \leqslant c_2 T + c_4 \leqslant c_3.$$

□

For the following Proposition we need to assume $0 < c_5 \leqslant y \leqslant R$.

Proposition 7.2. *If we assume $v \leqslant v_0$ on the initial surface M_0, then*

$$\max_{t>0} v \leqslant c_6(n, c_5, R, v_0).$$

Proof: Using the Neumann boundary conditions at $x = 0$ and $x = a$, we will consider equivalently the evolution of periodic surfaces $\tilde{M}_t$ defined along the whole x_1 axis. We assume that the product $u^2 v$ attains a maximum, denoted by K, on $\tilde{M}_{t_1}$ for $t_1 > 0$. We need to prove at this maximum point that $(\frac{\partial}{\partial t} - \mathcal{L})(y^2 v) \leqslant 0$ if K is large enough.

From Lemma 3.6 and Lemma 7.4 we have

$$\frac{\partial}{\partial t} v = \mathcal{L} v - \dot{F}^{ij} h_j^k h_{ki}\ v + \frac{(n-1)}{y^2} v \dot{F}^{22} - 2v^{-1} \dot{F}^{ij} \nabla_i v \nabla_j v, \tag{7.37}$$

and

$$\frac{\partial y}{\partial t} = \dot{F}^{ij} \nabla_i \nabla_j y - \frac{(n-1)}{y} \dot{F}^{22} + hpy.$$

We need that $\frac{\partial y^2}{\partial t} = 2y \frac{\partial y}{\partial t}$ to compute the next equation

$$\begin{aligned}
\frac{\partial}{\partial t}\left(y^2 v\right) &= y^2 \frac{\partial v}{\partial t} + v \frac{\partial y^2}{\partial t} \\
&= y^2 \frac{\partial v}{\partial t} + v 2y \frac{\partial y}{\partial t}, \\
&= y^2 \left[\mathcal{L} v - \dot{F}^{ij} h_j^k h_{ki}\ v + \frac{(n-1)}{y^2} v \dot{F}^{22} - 2v^{-1} \dot{F}^{ij} \nabla_i v \nabla_j v\right] \\
&\quad + v \left[2y \dot{F}^{ij} \nabla_i \nabla_j y - 2y \frac{(n-1)}{y} \dot{F}^{22} + 2yhpy\right] \\
&= y^2 \left[\mathcal{L} v - \dot{F}^{ij} h_j^k h_{ki}\ v + \frac{(n-1)}{y^2} v \dot{F}^{22} - 2v^{-1} \dot{F}^{ij} \nabla_i v \nabla_j v\right] \\
&\quad + v \left[2y \dot{F}^{ij} \nabla_i \nabla_j y - 2y \frac{(n-1)}{y} \dot{F}^{22} + 2yhv^{-1}\right], \quad \text{because } py = \langle v, w \rangle = v^{-1}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(y^2 v\right) &= y^2\mathcal{L}v - y^2\dot{F}^{ij}h_j^k h_{ki}\ v + (n-1)v\dot{F}^{22} - 2\frac{y^2}{v}\dot{F}^{ij}\nabla_i v\nabla_j v \\
&\quad + v2y\dot{F}^{ij}\nabla_i\nabla_j y - 2v(n-1)\dot{F}^{22} + 2hy \\
&= y^2\mathcal{L}v - y^2\dot{F}^{ij}h_j^k h_{ki}\ v + (n-1)v\dot{F}^{22} - 2\frac{y^2}{v}\dot{F}^{ij}\nabla_i v\nabla_j v \\
&\quad + v\left[\dot{F}^{ij}\nabla_i\nabla_j y^2 - 2\dot{F}^{ij}\nabla_i y\nabla_j y\right] - 2v(n-1)\dot{F}^{22} + 2hy \\
&\quad \text{because } \dot{F}^{ij}\nabla_i\nabla_j y^2 = \dot{F}^{ij}\nabla_i[2y\nabla_j y] = 2\dot{F}^{ij}\nabla_i y\nabla_j y + 2y\dot{F}^{ij}\nabla_i\nabla_j y \\
&= y^2\mathcal{L}v + v\mathcal{L}y^2 - y^2\dot{F}^{ij}h_j^k h_{ki}\ v - 2\frac{y^2}{v}\dot{F}^{ij}\nabla_i v\nabla_j v \\
&\quad - 2v\dot{F}^{ij}\nabla_i y\nabla_j y - v(n-1)\dot{F}^{22} + 2hy. \qquad (7.38)
\end{aligned}
$$

We also need

$$
\begin{aligned}
\mathcal{L}\left(y^2 v\right) &= \dot{F}^{ij}\nabla_i\nabla_j\left(y^2 v\right) = \dot{F}^{ij}\nabla_i[v2y\nabla_j y + y^2\nabla_j v] \\
&= \dot{F}^{ij}\left[2\nabla_i y(\nabla_j y)v + v2y\nabla_i\nabla_j y + 2y\nabla_i v\nabla_j y + 2y\nabla_i y\nabla_j v + y^2\nabla_i\nabla_j v\right] \\
&= \dot{F}^{ij}\left[2\nabla_i y(\nabla_j y) + 2y\nabla_i\nabla_j y\right]v + \dot{F}^{ij}\nabla_i v\nabla_j y^2 + \dot{F}^{ij}\nabla_i y^2\nabla_j v \\
&\quad + y^2\dot{F}^{ij}\nabla_i\nabla_j v \\
&= (\dot{F}^{ij}\nabla_i\nabla_j y^2)v + \dot{F}^{ij}\nabla_i v\nabla_j y^2 + \dot{F}^{ij}\nabla_i y^2\nabla_j v + y^2\mathcal{L}v \\
&= v\mathcal{L}y^2 + 2\dot{F}^{ij}\langle\nabla_i v, \nabla_j y^2\rangle + y^2\mathcal{L}v. \qquad (7.39)
\end{aligned}
$$

Also as $\nabla\left(y^2 v\right) = (\nabla(y^2)v + y^2\nabla(v))$ we have

$$
\begin{aligned}
\langle\nabla v, \nabla\left(y^2 v\right)\rangle &= \langle\nabla v, \nabla(y^2)v\rangle + \langle\nabla v, y^2\nabla v\rangle \\
&= v\langle\nabla v, \nabla(y^2)\rangle + y^2\langle\nabla v, \nabla v\rangle \\
&= v\langle\nabla v, \nabla(y^2)\rangle + y^2\nabla_i v\nabla_j v. \qquad (7.40)
\end{aligned}
$$

As a result of (7.40), we can write

$$
2\langle\nabla v, \nabla(y^2)\rangle = \frac{2}{v}\langle\nabla v, \nabla(y^2 v)\rangle - \frac{2y^2}{v}\nabla_i v\nabla_j v. \qquad (7.41)
$$

By substituting (7.41) into (7.39)

$$\begin{aligned}\mathcal{L}\left(y^2v\right) &= v\mathcal{L}y^2 + \dot{F}^{ij}\left[\frac{2}{v}\langle\nabla v, \nabla(y^2v)\rangle - \frac{2y^2}{v}\nabla_i v\nabla_j v\right] + y^2\mathcal{L}v \\ &= v\mathcal{L}y^2 + y^2\mathcal{L}v + \dot{F}^{ij}\frac{2}{v}\langle\nabla v, \nabla(y^2v)\rangle - \dot{F}^{ij}\frac{2y^2}{v}\nabla_i v\nabla_j v. \qquad (7.42)\end{aligned}$$

Therefore from (7.42) and we have (7.38)

$$\begin{aligned}\frac{\partial}{\partial t}(y^2v) &= \mathcal{L}\left(y^2v\right) - \dot{F}^{ij}\frac{2}{v}\langle\nabla v, \nabla(y^2v)\rangle + 2\dot{F}^{ij}\frac{y^2}{v}\nabla_i v\nabla_j v - y^2\dot{F}^{ij}h_j^k h_{kj}\ v \\ &\quad - 2\frac{y^2}{v}\dot{F}^{ij}\nabla_i v\nabla_j v - 2v\dot{F}^{ij}\nabla_i y\nabla_j y - v(n-1)\dot{F}^{22} + 2hy \\ &= \mathcal{L}(y^2v) - \dot{F}^{ij}\frac{2}{v}\langle\nabla v, \nabla(y^2v)\rangle - y^2\dot{F}^{ij}h_j^k h_{kj}\ v \\ &\quad - 2v\dot{F}^{ij}\nabla_i y\nabla_j y - v(n-1)\dot{F}^{22} + 2hy. \qquad (7.43)\end{aligned}$$

Similarly as in [43], we throw away $y^2\dot{F}^{ij}h_j^k h_{kj}\ v$. We have now

$$\frac{\partial}{\partial t}\left(y^2v\right) \leqslant -v(n-1)\dot{F}^{22} + 2hy. \qquad (7.44)$$

We need to show that $\dot{f}^2$ is bounded below at the maximum.

From first order term condition at maximum $\nabla_i\left(y^2v\right) = 0$, where is $v = \sqrt{1+u_x^2}$ we have

$$2uu_xv + u^2\frac{1}{2}\frac{2u_xu_{xx}}{\sqrt{1+u_x^2}} = 0,$$

$$2\sqrt{1+u_x^2} + u\frac{u_{xx}}{\sqrt{1+u_x^2}} = 0,$$

$$\frac{2}{u\sqrt{1+u_x^2}} + \frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}} = 0,$$

which can be written in terms of curvatures as

$$2\kappa_2 - \kappa_1 = 0,$$

so $2\kappa_2 = \kappa_1$. Since $\dot{f}^2$ is homogeneous of degree zero we have

$$\dot{f}^2\left(\kappa_1, \kappa_2, \ldots, \kappa_2\right) = \dot{f}^2\left(2\kappa_2, \kappa_2, \ldots, \kappa_2\right) = \dot{f}^2\left(2, 1, \ldots, 1\right).$$

Because $\dot{f}^2$ is a strictly positive function, see Conditions 1 iii), in particular $\dot{f}^2(2, 1, \ldots, 1) = C > 0$ where C is an absolute constant.

Now, at a maximum K of the product $y^2 v$; in particular, $v \geqslant \frac{K}{R^2}$ at this point, since $c_5 \leqslant y \leqslant R$. We also have $0 \leqslant h \leqslant c_2(n, c_5)$ (see Proposition 7.1).
We deduce that $\frac{\partial}{\partial t}(y^2 v) \leqslant 0$ at the maximum point provided

$$K > \frac{2R^3 c_2(n, c_5)}{(n-1)C}.$$

Therefore

$$\max_{t>0}\left(y^2 v\right) \leqslant \frac{2R^3 c_2(n, c_5)}{(n-1)C},$$

and

$$\max_{t>0} v \leqslant \max\left(\frac{2R^3 c_2(n, c_5)}{(n-1)C c_5^2}, \frac{R^2 v_0}{c_5^2}\right).$$

□

7 Application of the Sturmian theorem

Sturmian theorem will be used to show that the zeros of u_{xx} are discrete and non-increasing.

Lemma 7.10. *Assume M_t to be a smooth surface solving* (7.24). *Assume in addition that $u(x_1, t) \geq \epsilon, \epsilon > 0$, for $a \leq x_1 \leq b, t \in (0, T'), T' < T$. Then the set $Z_t(u_{xx}) = \{x_1 \in \mathbb{R} : u_{xx}(x_1, t) = \frac{\partial^2 u}{\partial x^2} = 0\}$ is a discrete set in $[a, b]$, for all $t \in (0, T')$. Moreover the number of zeros of u_{xx} is a nonincreasing function of time.*

Proof: Differentiating (6.5) with respect to x we find that u_x satisfies

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial t} \\
&= \frac{\partial}{\partial x}\left[\sqrt{1 + \left(\frac{\partial u}{\partial x_1}\right)^2} h + \dot{F}^{11}\frac{u_{xx}}{1 + u_x^2} - \sum_{j=2}^{n} \dot{F}^{jj}\frac{1}{u}\right] \\
&= -\frac{1}{2}(1 + u_x^2)^{\frac{-1}{2}} 2u_x u_{xx} h + \frac{\dot{F}^{11}}{(1 + u_x^2)} u_{xxx} + \frac{(n-1)\dot{F}^{22}}{u^2} u_x - \frac{2\dot{F}^{11}}{(1 + u_x^2)^2} u_x u_x^2 \\
&= -(1 + u_x^2)^{\frac{-1}{2}} u_x u_{xx} h + \frac{\dot{F}^{11}}{(1 + u_x^2)} u_{xxx} + \frac{(n-1)\dot{F}^{22}}{u^2} u_x - \frac{2\dot{F}^{11}}{(1 + u_x^2)^2} u_x u_x^2.
\end{aligned}$$

By differentiating with respect to x we find that

$$
\begin{aligned}
\frac{\partial u_{xx}}{\partial t} &= \frac{\partial}{\partial x}\frac{\partial u_x}{\partial t} \\
&= \left[-\frac{1}{2}(1+u_x^2)^{\frac{-3}{2}}2u_xu_{xx}u_xu_{xx} + (1+u_x^2)^{\frac{-1}{2}}u_{xx}u_{xx} + (1+u_x^2)^{\frac{-1}{2}}u_xu_{xxx}\right]h \\
&\quad + \frac{\dot{F}^{11}}{(1+u_x^2)}u_{xxxx} - 2\frac{\dot{F}^{11}}{(1+u_x^2)^2}u_xu_{xx}u_{xxx} + \frac{(n-1)\dot{F}^{22}}{u^2}u_{xx} - 2\frac{(n-1)\dot{F}^{22}}{u^3}u_x^2 \\
&\quad - 2\frac{\dot{F}^{11}}{(1+u_x^2)^2}u_{xx}^3 - 4\frac{\dot{F}^{11}}{(1+u_x^2)^2}u_xu_{xx}u_{xxx} + 8\frac{\dot{F}^{11}}{(1+u_x^2)^3}u_x^2u_{xx}^3 \\
&= \left[-\frac{1}{(1+u_x^2)^{\frac{3}{2}}}u_x^2u_{xx}^2 + \frac{1}{\sqrt{(1+u_x^2)}}u_{xx}^2 + \frac{1}{\sqrt{(1+u_x^2)}}u_xu_{xxx}\right]h \\
&\quad + \frac{\dot{F}^{11}}{(1+u_x^2)}u_{xxxx} - 2\frac{\dot{F}^{11}}{(1+u_x^2)^2}u_xu_{xx}u_{xxx} + \frac{(n-1)\dot{F}^{22}}{u^2}u_{xx} - 2\frac{(n-1)\dot{F}^{22}}{u^3}u_x^2 \\
&\quad - 2\frac{\dot{F}^{11}}{(1+u_x^2)^2}u_{xx}^3 - 4\frac{\dot{F}^{11}}{(1+u_x^2)^2}u_xu_{xx}u_{xxx} + 8\frac{\dot{F}^{11}}{(1+u_x^2)^3}u_x^2u_{xx}^3.
\end{aligned}
$$

Let $\eta = u_{xx}$ then

$$
\begin{aligned}
\frac{\partial \eta}{\partial t} &= \left[-\frac{1}{(1+u_x^2)^{\frac{3}{2}}}u_x^2\eta^2 + \frac{1}{\sqrt{(1+u_x^2)}}\eta^2 + \frac{1}{\sqrt{(1+u_x^2)}}u_x\eta_x\right]h \\
&\quad + \frac{\dot{F}^{11}}{(1+u_x^2)}\eta_{xx} - 6\frac{\dot{F}^{11}}{(1+u_x^2)^2}u_x\eta\eta_x + \frac{(n-1)\dot{F}^{22}}{u^2}\eta - 2\frac{(n-1)\dot{F}^{22}}{u^3}u_x^2 \\
&\quad - 2\frac{\dot{F}^{11}}{(1+u_x^2)^2}\eta^3 + 8\frac{\dot{F}^{11}}{(1+u_x^2)^3}u_x^2\eta^3,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \eta}{\partial t} &= \frac{\dot{F}^{11}}{(1+u_x^2)}\eta_{xx} - \left[6\frac{\dot{F}^{11}}{(1+u_x^2)^2}u_x\eta + \frac{h}{\sqrt{(1+u_x^2)}}u_x\right]\eta_x \\
&\quad + \left[-\frac{h}{(1+u_x^2)^{\frac{3}{2}}}u_x^2\eta + \frac{h}{\sqrt{(1+u_x^2)}}\eta + \frac{(n-1)\dot{F}^{22}}{u^2} - 2\frac{\dot{F}^{11}}{(1+u_x^2)^2}\eta^2\right. \\
&\quad \left. +8\frac{\dot{F}^{11}}{(1+u_x^2)^3}u_x^2\eta^2\right]\eta - 2\frac{(n-1)\dot{F}^{22}}{u^3}u_x^2.
\end{aligned} \tag{7.45}
$$

Because h is bounded as in Section 5 we may apply Sturmian theorem similarly as an unconstrained case, see Lemma 6.1. □

Lemma 7.11. *(Convergence of zeros of u_{xx}). The limits*

$$\lim_{t \to T} \imath_j(t) = \imath_j(T) \qquad \text{and} \qquad \lim_{t \to T} \chi_j(t) = \chi_j(T),$$

exist.

Proof: because we have bound on h the proof is a repetition of Lemma 6.2

Chapter 8

Appendix

1 Homogeneity

Function f is homogeneous of a positive degree α if

$$f(k\kappa) = k^\alpha f(\kappa),$$

for any $k > 0$. Euler's theorem for a smooth homogeneous function of degree α is very useful and states that

$$\sum_i \frac{\partial f(\kappa)}{\partial \kappa_i}\kappa_i = \alpha f(\kappa_i),$$

and this is because

$$\alpha f(\kappa) = \frac{d}{ds}|_{s=1} s^\alpha f(\kappa) = \frac{d}{ds}|_{s=1} f(s\kappa) = \sum_{i=1}^{n} \frac{\partial f(\kappa)}{\partial \kappa_i}\kappa_i.$$

2 F homogeneous of degree $\alpha > 1$

For speed function $f(\kappa_1, \ldots, \kappa_n)$ homogeneous of degree α which means $f(k\kappa) = k^\alpha f(\kappa)$ differentiating f respecting to κ_i

$$\frac{\partial f}{\partial \kappa_i} = \sum_i \frac{\partial f}{\partial y_i}(k\kappa)\kappa_i = \alpha k^{\alpha-1} f(\kappa). \tag{8.1}$$

Euler's theorem for (8.1) at $k = 1$ imply

$$\frac{\partial f}{\partial \kappa_i} = \sum_i \frac{\partial f}{\partial y_i}(\kappa)\kappa_i = \alpha f(\kappa). \tag{8.2}$$

Differentiating (8.1) again

$$\frac{\partial^2 f}{\partial \kappa_i^2} = \sum_{i,j} \frac{\partial^2 f}{\partial y_i \partial y_j}(k\kappa)\kappa_i\kappa_j = \alpha(\alpha-1)k^{\alpha-2}f(\kappa), \tag{8.3}$$

evaluate (8.3) at $k = 1$

$$\frac{\partial^2 f}{\partial \kappa_i^2} = \sum_{i,j} \frac{\partial^2 f}{\partial y_i \partial y_j}(\kappa)\kappa_i\kappa_j = \alpha(\alpha-1)f(\kappa). \tag{8.4}$$

In our case where $\kappa = (\kappa_1, \kappa_2 \ldots, \kappa_n) = (\kappa_1, (n-1)\kappa_2)$ (8.4) gives Euler's theorem for the second derivative of the homogeneous function f of degree α as follows

$$\begin{aligned}\frac{\partial^2 f}{\partial \kappa_i^2} &= \dot{f}^{11}\kappa_1^2 + 2(n-1)\dot{f}^{12}\kappa_1\kappa_2 + \sum_{i,j=2} \dot{f}^{22}\kappa_2^2 \\ &= \dot{f}^{11}\kappa_1^2 + 2(n-1)\dot{f}^{12}\kappa_1\kappa_2 + (n-1)^2\dot{f}^{22}\kappa_2^2 \\ &= \alpha(\alpha-1)f(\kappa). \end{aligned} \tag{8.5}$$

3 Interchange of two covariant derivatives

The Riemann curvature tensor can be written in terms of Gauss equation

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk},$$

for standard interchange of second covariant derivatives of two tensor we obtain

$$\nabla_k \nabla_i h_{lj} - \nabla_i \nabla_k h_{lj} = R_{kilm}g^{mn}h_{nj} + R_{kijm}g^{mn}h_{nl}, \tag{8.6}$$

and the Codazzi equations are

$$\nabla_i h_{kl} = \nabla_k h_{il} = \nabla_l h_{ik},$$

using Codazzi equations we have

$$\nabla_k \nabla_l h_{ij} = \nabla_k \nabla_i h_{lj}, \tag{8.7}$$

and then

$$\begin{aligned}
\nabla_k \nabla_l h_{ij} &= \nabla_k \nabla_i h_{lj} \\
&= \nabla_i \nabla_k h_{lj} + R_{kilm} g^{mn} h_{nj} + R_{kijm} g^{mn} h_{nl} \quad \text{from (8.6)} \\
&= \nabla_i \nabla_k h_{lj} + h_{kl} h_{im} g^{mn} h_{nj} - h_{km} h_{il} g^{mn} h_{nj} + h_{kj} h_{im} g^{mn} h_{nl} \\
&\quad - h_{km} h_{ij} g^{mn} h_{nl} . \quad \text{from Gauss equation}
\end{aligned} \tag{8.8}$$

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