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CONTROL SINGULARITIES OF CODIMENSIONS ONE AND TWO

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Abstract: We classify

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1. SINGULAR EQUILIBRIA OF DYNAMICAL SYSTEMS

We are interested in studying and classifying the simplest ways an equilibrium of a control system can be singular. Preliminary steps in this direction can be found in **?**. To understand what is meant by a singular equilibrium, let us first study those of a smooth dynamical system of the form

$$\dot{x} = f(x) \tag{1.1}$$

where $x \in I\!R^n$. We assume f is as smooth as is needed.

An equilibrium is a state x^e such that

$$f(x^e) = 0.$$

The linear approximating system to (1.1) around x^e is

$$\dot{z} = Fz \tag{1.2}$$

where

$$F = \frac{\partial f}{\partial x}(x^e).$$

This linear system has an equilibrium at $z^e = 0$.

The equilibrium x^e is said to be hyperbolic if all of the eigenvalues of F have nonzero real parts. The key result follows.

Definition 1.1. Two equilibria of two dynamical systems

$$\dot{x} = f(x) \tag{1.3}$$
$$0 = f(x^e) \tag{1.4}$$

$$\dot{z} = g(z) \tag{1.5}$$
$$0 = g(z^e) \tag{1.6}$$

are locally topologically conjugate if there is a local homeomorphism

$$z = \phi(x) \tag{1.7}$$

that carries trajectories to trajectories

$$z(t) = \phi(x(t)) \tag{1.8}$$

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for as long as they remain in the domain of the local homeomorphism.

Theorem 1.1. Grobman-Hartman Theorem
Suppose (1.1) has an hyperbolic equilibrium at x^e. Then it is locally topologically conjugate to its linear approximating system (1.2).

Therefore if x^e is a hyperbolic equilibrium then the local behavior of the dynamics (1.1) is similar to that of its linear approximating system (1.2) and the latter is well-understood. A singular equilibrium is one that is not hyperbolic. It is around such equilibria that the dynamics (1.1) can exhibit truly nonlinear behavior.

The simplest examples of singular equilibria have been studied and classified, see for example **?**, **?**. To explain what one means by simple, one must introduce some more concepts. Suppose we have an open set $\mathcal{X} \subset I\!R^n$, we would like to consider the set of all equilibria of all dynamical systems (1.1) that are defined on $\mathcal{X}$. This set is infinite dimensional and difficult to handle so instead we consider the k-jet bundle of all C^k maps $f : \mathcal{X} \to I\!R^n$. Given such a map f, at each $x \in \mathcal{X}$ we have the k-jet of f at x,

$$\left(x, f(x), \frac{\partial f}{\partial x}(x), \ldots, \frac{\partial^k f}{\partial x^k}(x)\right). \qquad (1.9)$$

The set of all k-jets of all such maps is a vector bundle over $\mathcal{X}$. Typically we assume that $k = 2$ or 3.

We are particularly interested in the equilibrium k-jets of the form

$$\left(x, 0, \frac{\partial f}{\partial x}(x), \ldots, \frac{\partial^k f}{\partial x^k}(x)\right). \qquad (1.10)$$

This is also a vector bundle over $\mathcal{X}$. Let $\mathcal{E}^k(\mathcal{X})$ denote the set of all equilibrium k-jets over $\mathcal{X}$. An equilibrium k-jet (1.10) is hyperbolic if

$$F = \frac{\partial f}{\partial x}(x)$$

has no zero or imaginary eigenvalues. Let $\mathcal{H}^k(\mathcal{X})$ denote the set of all hyperbolic equilibrium k-jets over $\mathcal{X}$. This is an open and dense subset of $\mathcal{E}^k(\mathcal{X})$. The compliment of $\mathcal{H}^k(\mathcal{X})$ in $\mathcal{E}^k(\mathcal{X})$ is the set $\mathcal{S}^k(\mathcal{X})$ of singular k-jets. This set is stratified, i.e., a union of disjoint submanifolds of varying codimensions.

In particular $\mathcal{S}^k(\mathcal{X})$ contains exactly two submanifolds of codimension one in $\mathcal{E}^k(\mathcal{X})$. One is the zero singularities $\mathcal{Z}^k(\mathcal{X})$. These are equilibrium k-jets (1.10) where $\frac{\partial f}{\partial x}(x)$ has one zero eigenvalue, the other eigenvalues are not imaginary and a nondegeneracy condition is satisfied. After a suitable change of coordinates, a dynamical system realizing such a k-jet at an equilibrium can be brought to the form

$$\begin{aligned} \dot{x}_0 &= a_2 x_0^2 + O(x)^3 \\ \dot{x}_1 &= F_1 x_1 + O(x)^2 \end{aligned} \qquad (1.11)$$

where $x_0 \in I\!R,\ x_1 \in I\!R^{n-1}$ and F_1 is hyperbolic. The nondegeneracy condition is that $a_2 \neq 0$.

By a minimal unfolding of a singularity we mean a family of systems depending on as many parameters as the codimension of the singularity and whose k-jets touch all nearby strata of lower codimension in the set of equilibria. A minimal unfolding of the zero singularity (1.11) is

$$\begin{aligned} \dot{x}_0 &= \mu x_0 + a_2 x_0^2 + O(x)^3 \\ \dot{x}_1 &= F_1 x_1 + O(x)^2 \end{aligned} \qquad (1.12)$$

It depends on one parameter $\mu \in I\!R$ because $\mathcal{Z}^k(\mathcal{X})$ is of codimension one. At each value of the parameter there is one equilibrium $x = 0$ which is hyperbolic when $\mu \neq 0$.

The other set of singularities of codimension one is the imaginary singularities $\mathcal{I}^k(\mathcal{X})$. These are equilibrium k-jets (1.10) where $\frac{\partial f}{\partial x}(x)$ has one pair of nonzero, imaginary eigenvalues, the other eigenvalues have nonzero real part and a nondegeneracy condition is satisfied. After a suitable change of coordinates, a dynamical system realizing such a k-jet can be brought to the form

$$\begin{aligned} \dot{x}_0 &= \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} x_0 + \lambda_1 |x_0|^2 x_0 \\ &\quad + \lambda_2 |x_0|^2 \begin{bmatrix} -x_{0,2} \\ x_{0,1} \end{bmatrix} + O(x)^4 \\ \dot{x}_1 &= F_1 x_1 + O(x)^2 \end{aligned} \qquad (1.13)$$

where $x_0 \in I\!R^2,\ x_1 \in I\!R^{n-2}$ and F_1 is hyperbolic. The nondegeneracy conditions are that $\omega \neq 0$ and $\lambda_1 \neq 0$.

A minimal unfolding of this singularity is

$$\begin{aligned} \dot{x}_0 &= \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix} x_0 + \lambda_1 |x_0|^2 x_0 \\ &\quad + \lambda_2 |x_0|^2 \begin{bmatrix} -x_{0,2} \\ x_{0,1} \end{bmatrix} + O(x_0, x_1)^4 \\ \dot{x}_1 &= F_1 x_1 + O(x_0, x_1)^2. \end{aligned} \qquad (1.14)$$

It depends on one parameter $\mu \in I\!R$ because $\mathcal{I}^k(\mathcal{X})$ is of codimension one. When $\mu \neq 0$ the equilibrium is hyperbolic.

There are four submanifolds of singular equilibria of codimension two. The best known are the cusp singularities $\mathcal{C}^k(\mathcal{X})$. These are a degenerate zero singularities where $a_2 = 0$. After a suitable change of coordinates, a dynamical system realizing such a k-jet can be brought to the form

$$\begin{aligned} \dot{x}_0 &= a_3 x_0^3 + O(x)^4 \\ \dot{x}_1 &= F_1 x_1 + O(x)^2 \end{aligned} \tag{1.15}$$

where $x_0 \in I\!R,\ x_1 \in I\!R^{n-1}$ and F_1 is hyperbolic. The nondegeneracy condition is that $a_3 \neq 0$.

A minimal unfolding of the cusp singularity (1.15) is

$$\begin{aligned} \dot{x}_0 &= \mu_1 + \mu_2 x_0 + a_3 x_0^3 + O(x)^4 \\ \dot{x}_1 &= F_1 x_1 + O(x) \end{aligned} \tag{1.16}$$

It depends on two parameters $\mu_1,\ \mu_2$ because $\mathcal{C}^k(\mathcal{X})$ is of codimension two. It is called a cusp as the two dimensional parameter space is split by the cusp

$$4\mu_2^3 + 27 a_3 \mu_1^2 = 0. \tag{1.17}$$

On one side of this cusp the system (1.16) has one hyperbolic equilibrium and on the other side it has three hyperbolic equilibria. On the cusp itself except at $\mu_1 = \mu_2 = 0$, the system (1.16) has two equilibria, one hyperbolic and the other a zero singularity We refer the reader to **?** for more details.

After a suitable change of coordinates, a double zero singularity is of the form

$$\dot{x}_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_0 + \begin{bmatrix} 0 \\ b_1 x_{0,1} x_{0,2} + b_2 x_{0,2}^2 \end{bmatrix} + O(x)^3$$

$$\dot{x}_1 = F_1 x_1 + O(x)^2$$

where $x_0 \in I\!R^2,\ x_1 \in I\!R^{n-2}$ and F_1 is hyperbolic. The nondegeneracy condition is that $b_2 \neq 0$. The set of double zero singularities is denoted by $\mathcal{D}Z^k(\mathcal{X})$. A minimal unfolding of this singularity is

$$\dot{x}_0 = \begin{bmatrix} 0 & 1 \\ \mu_1 & \mu_2 \end{bmatrix} x_0 + \begin{bmatrix} 0 \\ b_1 x_{0,1} x_{0,2} + b_2 x_{0,2}^2 \end{bmatrix} + O(x)^3$$

$$\dot{x}_1 = F_1 x_1 + O(x)^2.$$

It depends on two parameters $\mu_1,\ \mu_2$ because $\mathcal{D}Z^k(\mathcal{X})$ is of codimension two. When the eigenvalues of

$$F_0(\mu) = \begin{bmatrix} 0 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}$$

are not on the imaginary axis, the equilibrium is hyperbolic. When the eigenvalues of $F_0(\mu)$ are on the imaginary axis and nonzero, the equilibrium is an imaginary singularity. When one eigenvalue of $F_0(\mu)$ is zero and the other is not zero then the equilibrium is a zero singularity. See **?** for more details.

The other two classes of singularities of codimension two are the zero-imaginary singularities $\mathcal{Z}I^k(X)$ and the double imaginary singularities $\mathcal{D}I^k(X)$. The linear part of the former have one zero eigenvalue, one pair of nonzero imaginary eigenvalues $\pm\omega i$ and the rest of the eigenvalues off the imaginary axis.

The linear part of the latter have two pairs of nonzero imaginary eigenvalues $\pm\omega_1 i,\ \pm\omega_2 i$ and the rest of the eigenvalues off the imaginary axis. The two pairs must be nonresonant, if $|\omega_1| \leq |\omega_2|$ then $|\omega_2| \neq k|\omega_1|$ for $k = 1, 2, 3$. See **?** and **?** for more details.

A parameterized dynamical system

$$\dot{x} = f(x, \pi),$$

where $x \in \mathcal{X} \subset I\!R^n, \pi \in \mathcal{P} \subset I\!R^p$, can have multiple equilibria x^e, π^e where

$$f(x^e, \pi^e) = 0.$$

Since this is n equations in $n + p$ unknowns we expect that around most (x^e, π^e), the set of equilibria is a p dimensional surface in $\mathcal{X} \times \mathcal{P}$. This is certainly true if the equilibrium (x^e, π^e) is hyperbolic for the implicit function theorem then asserts that locally the set of equilibria is a p surface parameterized by π. A classical bifurcation occurs when the set of equilibria intersects the set of singular equilibria. One usually requires the intersection to be nontangental.

Suppose $p = 1$. A fold (aka saddle-node) bifurcation occurs when the k-jets of a curve of equilibria intersect the set of zero singularities. A transcritical bifurcation occurs when the set of equilibria locally consists of two curves which cross at a zero singularity. A pitchfork bifurcation occurs when the set of equilibria locally consists of two curves which cross at a cusp singularity. A Hopf bifurcation occurs when the k-jets of a curve of equilibria intersect the set of imaginary singularities. A Bogdanov-Takens bifurcation occurs when the k-jets of a curve of equilibria intersect the set of double zero singularities. A fold-Hopf bifurcation occurs when the k-jets of a curve of equilibria intersect the set of zero-imaginary singularities. A double Hopf bifurcation occurs when the k-jets of a curve of equilibria intersect the set of double imaginary singularities.

2. SINGULAR EQUILIBRIA OF CONTROL SYSTEMS

A control system is of the form

$$\dot{x} = f(x,u) \tag{2.1}$$

where $x \in \mathcal{X} \subset I\!R^n,\ u \in \mathcal{U} \subset I\!R^m$. We assume f is as smooth as is needed. An equilibrium is a pair (x^e, u^e) such that

$$f(x^e, u^e) = 0.$$

Given such a map f, at each pair $(x,u) \in \mathcal{X} \times \mathcal{U}$ we have the k-jet of f

$$\left((x,u), f(x,u), \frac{\partial f}{\partial(x,u)}(x,u), \dots, \frac{\partial^k f}{\partial(x,u)^k}(x,u)\right).$$

The set of all k-jets of all such maps is a vector bundle over $\mathcal{X} \times \mathcal{U}$.

We are particularly interested in the equilibrium k-jets of the form

$$\left((x,u), 0, \frac{\partial f}{\partial(x,u)}(x,u), \dots, \frac{\partial^k f}{\partial(x,u)^k}(x,u)\right).$$

This is also a vector bundle over $\mathcal{X} \times \mathcal{U}$. Let $\mathcal{E}^k(\mathcal{X} \times \mathcal{U})$ denote the set of all equilibrium k-jets over $\mathcal{X} \times \mathcal{U}$. It is convenient to denote

$$F = \frac{\partial f}{\partial x}(x,u), \qquad G = \frac{\partial f}{\partial u}(x,u).$$

An equilibrium k-jet is linearly controllable if

$$\operatorname{rank}\left[\, G\ FG\ F^2G\ \dots\ F^{n-1}G \,\right] = n.$$

Suppose $m = 1$. Let $\mathcal{LC}^k(\mathcal{X} \times \mathcal{U})$ denote the set of all linearly controllable equilibrium k-jets over $\mathcal{X} \times \mathcal{U}$. This is an open and dense subset of $\mathcal{E}^k(\mathcal{X} \times \mathcal{U})$. The compliment of $\mathcal{LC}^k(\mathcal{X} \times \mathcal{U})$ in $\mathcal{E}^k(\mathcal{X} \times \mathcal{U})$ is the set $\mathcal{SC}^k(\mathcal{X} \times \mathcal{U})$ of singular k-jets of control systems. This set is stratified, i.e., a union of disjoint submanifolds of varying codimensions. We would like to describe the strata of codimensions one and two.

There is one stratum of codimension one, the fold control singularities, $\mathcal{FC}^k(\mathcal{X} \times \mathcal{U})$. These are equilibrium k-jets where

$$\operatorname{rank}\left[\, G\ FG\ F^2G\ \dots\ F^{n-1}G \,\right] = n-1,$$

so there is one uncontrollable mode, and a nondegeneracy condition is satisfied. It is shown in **?** that after a suitable change of state coordinates and a state dependent change of input coordinates (aka invertible state feedback), a control system realizing such a k-jet at an equilibrium can be brought to the form

$$\begin{aligned} \dot{x}_0 &= \alpha x_0 + \gamma x_0 x_{1,1} + \sum_{j=1}^{n} \delta_j x_{1,j}^2 + O(x,u)^3 \\ \dot{x}_1 &= F_1 x_1 + G_1 u + O(x,u)^2 \end{aligned} \tag{2.2}$$

where $x_0 \in I\!R,\ x_1 \in I\!R^{n-1}$ and F_1, G_1 are in Brunovsky form,

$$F_1 = \begin{bmatrix} 0\ 1\ 0\ \dots\ 0 \\ 0\ 0\ 1\ \dots\ 0 \\ \ddots \\ 0\ 0\ 0\ \dots\ 1 \\ 0\ 0\ 0\ \dots\ 0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

For notational convenience we define $x_{1,n} = u$. The nondegeneracy conditions are that α and $\delta_1 \neq 0$. We could further divide $\mathcal{FC}^k(\mathcal{X} \times \mathcal{U})$ into those that are linearly stabilizable $\alpha < 0$ and those that are not $\alpha > 0$.

A minimal unfolding is obtained by changing the x_0 dynamics to

$$\begin{aligned} \dot{x}_0 &= \alpha x_0 + \mu x_{1,1} + \gamma x_0 x_{1,1} + \sum_{j=1}^{n} \delta_j x_{1,j}^2 \\ &\quad + O(x,u)^3. \end{aligned} \tag{2.3}$$

When $\mu \neq 0$ the system is linearly controllable around the equilibrium $x = 0,\ u = 0$ but the integrated effect of the control on x_0 changes sign with μ.

Unlike a dynamical system, a control system typically has a continuum of equilibria because $f(x^e, u^e) = 0$ is n equations in $n + m$ unknowns. Since $m = 1$ the system (2.2) has a curve of equilibria conveniently parameterized by $x_{1,1}^e = \mu$,

$$\begin{aligned} x_0^e(\mu) &= -\frac{\delta_1}{\alpha}\mu^2 + O(\mu)^3 \\ x_{1,1}^e(\mu) &= \mu \\ x_{1,j}^e(\mu) &= O(\mu)^2 \\ u^e(\mu) &= O(\mu)^2. \end{aligned}$$

The linear approximating control system at the μ^{th} equilibrium is

$$\dot{z} = \begin{bmatrix} \alpha & 2\delta_1\mu\ 0 \dots 0 \\ 0 & F_1 \end{bmatrix} z + \begin{bmatrix} 0 \\ G_1 \end{bmatrix} v + O(\mu)^2$$

in displacement coordinates $z = x - x^e(\mu),\ v = u - u^e(\mu)$. This linear system is controllable except at $\mu = 0$ and similar to a minimal unfolding.

When the input u is scalar, $m = 1$ there are three strata of codimension two. The transcontrollable singularities, $\mathcal{TC}^k(\mathcal{X} \times \mathcal{U})$ are degenerate folds where the linearly uncontrollable eigenvalue is zero. After a suitable change of state coordinates and a state dependent change of input coordinates, a control system realizing such a k-jet at an equilibrium can be brought to the form

$$\begin{aligned}\dot{x}_0 &= \beta x_0^2 + \gamma x_0 x_{1,1} + \sum_{j=1}^{n} \delta_j x_{1,j}^2 + O(x,u)^3 \\ \dot{x}_1 &= F_1 x_1 + G_1 u + O(x,u)^2\end{aligned} \tag{2.4}$$

where $x_0 \in I\!R$, $x_1 \in I\!R^{n-1}$ and F_1, G_1 are in Brunovsky form. The nondegeneracy condition is that $\gamma^2 - 4\beta\delta_1 > 0$. This implies that $\dot{x}_0$ takes on both positive and negative values in any neighborhood of $x = 0$.

A minimal unfolding is obtained by changing the x_0 dynamics to

$$\begin{aligned}\dot{x}_0 &= \mu_1 x_0 + \mu_2 x_{1,1} + \beta x_0^2 + \gamma x_0 x_{1,1} + \sum_{j=1}^{n} \delta_j x_{1,j}^2 \\ &\quad + O(x,u)^3.\end{aligned} \tag{2.5}$$

When $\mu_2 \neq 0$ the system is linearly controllable around the equilibrium $x = 0,\ u = 0$. When $\mu_2 = 0$ but $\mu_1 \neq 0$ the equilibrium is a fold control singularity.

The next stratum of codimension two is the two real roots control singularities, $\mathcal{TRR}^k(\mathcal{X} \times \mathcal{U})$. After a suitable change of state coordinates and a state dependent change of input coordinates, a control system realizing such a k-jet at an equilibrium can be brought to the form

$$\begin{aligned}\dot{x}_0 &= F_0 x_0 + \Gamma x_0 x_{1,1} + \sum_{j=1}^{n} \Delta_j x_{1,j}^2 + O(x,u)^3 \\ \dot{x}_1 &= F_1 x_1 + G_1 u + O(x,u)^2\end{aligned} \tag{2.6}$$

where $x_0 \in I\!R^2$, $x_1 \in I\!R^{n-2}$, F_1, G_1 are in Brunovsky form,

$$F_0 = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}$$
$$\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}$$
$$\Delta_j = \begin{bmatrix} \delta_{1j} \\ \delta_{2j} \end{bmatrix}$$

The nondegeneracy conditions are that $\alpha_1 \neq 0,\ \alpha_2 \neq 0,\ \alpha_1 \neq \alpha_2,\ \alpha_1 \neq 2\alpha_2,\ \alpha_2 \neq 2\alpha_1$ and $F_0,\ \Delta_1$ is a controllable pair.

A minimal unfolding is obtained by changing the x_0 dynamics to

$$\begin{aligned}\dot{x}_0 = F_0 x_0 &+ \mu_1 x_{0,2} \mathbf{e}_{0,1} + \mu_2 x_{1,1} \mathbf{e}_{0,2} \\ &+ \Gamma x_0 x_{1,1} + \sum_{j=1}^{n} \Delta_j x_{1,j}^2 + O(x,u)^3\end{aligned}$$

When $\mu_1 \neq 0,\ \mu_2 \neq 0$ the equilibrium is linearly controllable. When $\mu_1 = 0$ but $\mu_2 \neq 0$ the equilibrium is a fold control singularity.

The third stratum, and last when $m = 1$, of codimension two is the two complex roots control singularities, $\mathcal{TCR}^k(\mathcal{X} \times \mathcal{U})$. They are similar to the two real roots control singularities except that

$$F_0 = \begin{bmatrix} \alpha & -\omega \\ \omega & \alpha \end{bmatrix}$$

where $\alpha \neq 0,\ \omega \neq 0$.

3. SINGULAR EQUILIBRIA OF MULTI-INPUT CONTROL SYSTEMS

In this section we consider the singular equilibria control systems with several inputs, $m > 1$. First we must define the controllability (aka Brunovsky) indices of a linear control system

$$\dot{x} = Fx + Gu.$$

The controllability matrix

$$\begin{bmatrix} G & FG & F^2G & \ldots & F^{n-1}G \end{bmatrix}$$

has n rows and nm columns. Starting from the left we delete a column if it is dependent on the columns to its left. After reordering we obtain a matrix of the form

$$\begin{bmatrix} G_1 & \ldots & F^{n_1-1}G_1 & \ldots & G_m & \ldots & F^{n_m-1}G_m \end{bmatrix}$$

where $n_1 \geq n_2 \geq \ldots \geq n_m \geq 0$. These integers are called the controllability indices. For generic $F,\ G$ they sum to n so that the system is linearly controllable. Also for a generic system, they differ by at most one. If k is the greatest integer not exceeding n/m then the indices are either $k+1$ or k. If either of these do not hold then the linear system is singular.

Henceforth for ease of exposition we shall assume that $m = 2$ and leave the general case to the reader. If n is even then the generic controllability indices are $n_1 = n_2 = n/2$. If n is odd then the generic controllability indices are $n_1 - 1 = n_2 = (n-1)/2$. A control shift singularity occurs when n_1 is larger and n_2 is smaller than the generic case.

The set of all control shift singularities $\mathcal{CS}^k(\mathcal{X} \times \mathcal{U})$ is a submanifold of $\mathcal{E}^k(\mathcal{X} \times \mathcal{U})$ of codimension one. After a suitable change of state coordinates and invertible state feedback, an equilibrium that realizes a control shift singularity can be brought to the form

$$\dot{x}_1 = F_1 x_1 + G_1 u_1 + O(x,u)^2 \tag{3.1}$$
$$\dot{x}_2 = F_2 x_2 + G_2 u_2 + O(x,u)^2 \tag{3.2}$$

where $x_1 \in I\!R^{n_1}$, $x_2 \in I\!R^{n_2}$, $F_1,\ G_1$ is in Brunovsky form with state dimension n_1 and

$F_2,\ G_2$ is in Brunovsky form with state dimension n_2. A minimal unfolding is

$$\dot{x}_1 = F_1x_1 + \mu x_{2,1}\mathbf{e_{1,1}} + G_1u_1 + O(x,u)^2$$
$$\dot{x}_2 = F_2x_2 + G_2u_2 + O(x,u)^2$$

where $\mathbf{e_{1,1}}$ is the unit column n_1 vector in the $x_{1,1}$ direction. If $\mu \neq 0$ then the controllability indices are $n_1 - 1,\ n_2 + 1$ and are generic.

When $m = 2$ another stratum of control singularities is the set of double control fold singularities, $\mathcal{DCF}^k(\mathcal{X} \times \mathcal{U})$. This is of codimension two. These singularities have one mode that is not linearly controllable and the controllability indices are as close together as possible. After a change of coordinates and feedback a double control fold singularity takes the form

$$\dot{x}_0 = \alpha x_0 + O(x,u)^2$$
$$\dot{x}_1 = F_1x_1 + G_1u_1 + O(x,u)^2$$
$$\dot{x}_2 = F_2x_2 + G_2u_2 + O(x,u)^2$$

where $x_0 \in I\!R,\ x_1 \in I\!R^{n_1},\ x_2 \in I\!R^{n_2},\ F_1,\ G_1$ is in Brunovsky form with state dimension n_1 and $F_2,\ G_2$ is in Brunovsky form with state dimension n_2.

When n is odd then $n_1 = n_2 = (n-1)/2$ and a minimal unfolding is obtained by transforming the x_0 dynamics to

$$\dot{x}_0 = \alpha x_0 + \mu x_{1,1} + O(x,u)^2.$$

If $\mu \neq 0$ then the controllability indices are generic, $n_1 - 1 = n_2 = (n-1)/2$. Notice that the minimal unfolding depends on a single parameter even though the codimension is two.

When n is even then $n_1 - 1 = n_2 = (n-2)/2$ and a minimal unfolding is obtained by transforming the x_0 dynamics to

$$\dot{x}_0 = \alpha x_0 + \mu_1 x_{1,1} + \mu_2 x_{2,1} + O(x,u)^2.$$

If $\mu_2 \neq 0$ then the controllability indices are generic, $n_1 = n_2 = n/2$. If $\mu_2 = 0$ but $\mu_1 \neq 0$ then the controllability indices are $n_1 - 2 = n_2 = (n-2)/2$. Notice that the minimal unfolding depends on two parameters.

The set of all double control shift singularities $\mathcal{DCS}^k(\mathcal{X} \times \mathcal{U})$ is a submanifold of $\mathcal{E}^k(\mathcal{X} \times \mathcal{U})$ of codimension two when $m = 2$. For such singularities the first controllability index n_1 is two greater than the generic first index and the second index is two less than the generic second index. After a suitable change of state coordinates and invertible state feedback an equilibrium that realizes a double control shift singularity can be brought to the form (3.1). A minimal unfolding is

$$\dot{x}_1 = F_1x_1 + \mu_1 x_{2,1}\mathbf{e_{1,1}} + \mu_2 x_{2,2}\mathbf{e_{1,2}} + G_1u_1 + O(x,u)^2$$

$$\dot{x}_2 = F_2x_2 + G_2u_2 + O(x,u)^2$$

If $\mu_2 \neq 0$ then the controllability indices are $n_1 - 2,\ n_2 + 2$ and are generic. If $\mu_2 = 0$ but $\mu_1 \neq 0$ then the controllability indices are $n_1 - 1,\ n_2 + 1$ and the equilibrium is a control shift singularity.

Since the equilibrium condition $f(x^e, u^e) = 0$ is n equations in $n+m$ variables we expect that locally around most equilibria the set of equilibria is an m dimensional surface. By the implicit function theorem this is certainly true if the equilibrium is linearly controllable. Suppose for simplicity $m = 1$ so we expect a curve of equilibrium. When the k jet of an equilibrium intersects a control singularity, a control bifurcation occurs. A control system does not need a parameter to bifurcate because it has continua of equilibria. If the control system does have parameters

$$\dot{x} = f(x,u,\pi)$$

where $x \in \mathcal{X} \subset I\!R^n,\ u \in \mathcal{U} \subset I\!R^m,\ \pi \in \mathcal{P} \subset I\!R^p$. Then the equilibrium condition $f(x^e, u^e, \pi^e) = 0$ is n equations in $n + m + p$ variables so then typically the set of equilibria is locally an $m + p$ surface. When the k jets of these equilibria intersect a strata of control singularities a control bifurcation occurs, see **?** for more details.

4. CONCLUSION

We reviewed the classification of the singularities of dynamical systems of low codimension . Then we classified the singularities of control systems of low codimension.