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THE CONTROLLED CENTER DYNAMICS*

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Abstract. The center manifold theorem is a model reduction technique for determining the local asymptotic stability of an equilibrium of a dynamical system when its linear part is not hyperbolic. The overall system is asymptotically stable if and only if the center manifold dynamics is asymptotically stable. This allows for a substantial reduction in the dimension of the system whose asymptotic stability must be checked. Moreover, the center manifold and its dynamics need not be computed exactly; frequently, a low degree approximation is sufficient to determine its stability. The controlled center dynamics plays a similar role in determining local stabilizability of an equilibrium of a control system when its linear part is not stabilizable. It is a reduced order control system with a pseudoinput to be chosen in order to stabilize it. If this is successful, then the overall control system is locally stabilizable to the equilibrium. Again, usually low degree approximation suffices.

Key words. nonlinear systems, control bifurcations, center manifold theorem

AMS subject classifications. 93C10, 93C15, 37L10, 37N35

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1. Introduction.

Center manifold theory plays an important role in the study of the stability of dynamical systems when the equilibrium point is not hyperbolic. The center manifold is an invariant manifold of the differential equation which is tangent at the equilibrium point to the eigenspace of the neutrally stable eigenvalues. For instance, as the local dynamic behavior "transverse" to the center manifold is relatively simple since it is the one of the flows in the local stable (and unstable) manifolds, the center manifold method isolates the complicated asymptotic behavior by locating an invariant manifold tangent to the subspace spanned by the eigenspace of eigenvalues on the imaginary axis. In practice, one does not compute the center manifold and its dynamics exactly, since this requires the resolution of a quasi-linear PDE which is not easily solvable. In most cases of interest, an approximation of degree two or three of the solution is sufficient. Then we determine the reduced dynamics on the center manifold, study its stability, and conclude about the stability of the original system [24, 26, 21, 6, 15].

The combination of this theory with the normal form approach of Poincaré [25] was used extensively to study parameterized dynamical systems exhibiting bifurcations [27]. The center manifold theorem provides, in this case, a means of systematically reducing the dimension of the state spaces which need to be considered when analyzing bifurcations of a given type. In fact, after determining the center manifold, the analysis of these parameterized dynamical systems is based only on the restriction of the original system on the center manifold whose stability properties are the same as the ones of the full order system.

This approach was also adopted in control theory. The combination of the normal form approach for control systems [20] and center manifold theory enabled the analysis and stabilization of systems with one or two uncontrollable modes in continuous and

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discrete time [17, 18, 19, 14, 23, 9, 12, 13, 11, 10]. After using a linear feedback to asymptotically stabilize the linearly controllable part, it was possible to stabilize the whole system by focusing only on the restriction of the original control system on the center manifold, whose dimension equals the number of uncontrollable modes (i.e., one or two). This allows us to study the stabilizability and the synthesis of a controller for the full order system based on the linearly uncontrollable part.

In this paper, we generalize this approach to systems with any number of uncontrollable modes by introducing the *controlled center dynamics*. This controlled dynamics is a reduced order control system over which the control design for the full order system is performed and whose dimension is the number of uncontrollable modes. This allows us to reduce the complexity of the stabilization problem, as the dynamics of the linearly controllable part becomes stable by choosing a linear feedback that places its eigenvalues in the open left half-plane.

In practice, the controlled center dynamics will allow us to study the stabilizability and synthesizing stabilizing controllers for some classes of finite- or infinite-dimensional control systems based only on the study of a reduced order finite-dimensional control system given by the controlled center dynamics. Thus, this methodology can also be viewed as a reduction technique for some classes of controlled differential equations.

By deriving an explicit formula of the controlled center manifold and the controlled center dynamics, the link between feedbacks and the resulting center dynamics becomes clear. By changing the feedback, the stability properties of the controlled center dynamics will change, and thus the stability properties of the full order system will change too. Thus, choosing a feedback that stabilizes the controlled center dynamics allows us to stabilize the full order system.

The paper is organized as follows. In section 2, we define what is meant by the controlled center dynamics and show how a feedback will affect it. Then, in section 3, we apply this technique to stabilize systems with a transcontrollable bifurcation using a quadratic feedback and then using a piecewise linear feedback.

2. The controlled center dynamics.

Consider the nonlinear system

$$\dot{\zeta} = f(\zeta, v), \tag{2.1}$$

where the variable $\zeta \in \mathbb{R}^n$ is the state and $v \in \mathbb{R}$ is the input variable. The vector field $f(\zeta)$ is assumed to be C^k for some sufficiently large k.

Assume $f(0,0) = 0$, and suppose that the linearization of the system at the origin is (A, B),

$$A = \frac{\partial f}{\partial \zeta}(0,0), \qquad B = \frac{\partial f}{\partial v}(0,0),$$

with

$$\operatorname{rank}([B\ AB\ A^2B\ \cdots\ A^{n-1}B]) = n - r, \tag{2.2}$$

and $r > 0$. Moreover, assume that the system (2.1) has r uncontrollable modes on the imaginary axis. Let $\Sigma_{\mathcal{S}}$ denote the system (2.1) under the above assumptions.

The system $\Sigma_{\mathcal{S}}$ is not linearly controllable at the origin, and a change of some control properties may occur around this equilibrium point; this is called a control bifurcation if it is linearly controllable at other equilibria [23].

From linear control theory [16], we know that there exist a linear change of coordinates and a linear feedback transforming the system $\Sigma_{\mathcal{S}}$ to

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + \bar{f}_1(x_1, x_2, u), \\ \dot{x}_2 &= A_2 x_2 + B_2 u + \bar{f}_2(x_1, x_2, u), \end{aligned} \tag{2.3}$$

where $x_1 \in \mathbb{R}^r$, $x_2 \in \mathbb{R}^{n-r}$, $u \in \mathbb{R}$, $A_1 \in \mathbb{R}^{r\times r}$ is in the real Jordan form and its eigenvalues are on the imaginary axis, $A_2 \in \mathbb{R}^{(n-r)\times(n-r)}$, $B_2 \in \mathbb{R}^{(n-r)\times 1}$ are in the Brunovskỳ form, i.e.,

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and $\bar{f}_k(x_1, x_2, u) = O(x_1, x_2, u)^2$, for $k = 1, 2$.

Now consider the feedback given by

$$u(x_1, x_2) = \kappa(x_1) + K_2 x_2, \tag{2.4}$$

with κ a smooth function and $K_2 = \begin{bmatrix} k_{2,1} & \cdots & k_{2,n-r} \end{bmatrix}$.

Because (A_2, B_2) is controllable, the eigenvalues in the closed-loop system associated with the equation of x_2 can be placed at arbitrary given points in the complex plane by selecting values for K_2. If one of these controllable eigenvalues is placed in the right half-plane, the closed-loop system is unstable around the origin. Therefore, we assume that K_2 has the following property.

Property $\mathcal{P}$. The matrix $\bar{A}_2 = A_2 + B_2 K_2$ is Hurwitz.

Let us denote by $\mathcal{F}$ the feedback (2.4) with Property $\mathcal{P}$.

Now consider the closed-loop system (2.3)–(2.4) given by

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + \bar{f}_1(x_1, x_2, \kappa(x_1) + K_2 x_2), \\ \dot{x}_2 &= A_2 x_2 + B_2(\kappa(x_1) + K_2 x_2) + \bar{f}_2(x_1, x_2, \kappa(x_1) + K_2 x_2). \end{aligned} \tag{2.5}$$

This system possesses r eigenvalues on the imaginary axis and $n-r$ eigenvalues in the open left half-plane. Thus, a center manifold exists [6]. It is represented locally around the origin as

$$W^c = \{(x_1, x_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} | x_2 = \Pi(x_1), |x_1| < \delta, \Pi(0) = 0\} \tag{2.6}$$

for δ sufficiently small.

For any point (x_1, x_2) in W^c we have

$$x_2 = \Pi(x_1);$$

hence

$$\dot{x}_2 = \frac{\partial\,\Pi(x_1)}{\partial\,x_1}\dot{x}_1. \tag{2.7}$$

Since the points in W^c obey the dynamics generated by the closed-loop system (2.5), and since in W^c the feedback law (2.4) is

$$u(x_1, x_2)|_{x_2=\Pi(x_1)} = \kappa(x_1) + K_2\Pi(x_1),$$

then, substituting

$$\dot{x}_1 = A_1 x_1 + \bar{f}_1(x_1, \Pi(x_1), \kappa(x_1) + K_2\Pi(x_1)),$$
$$\dot{x}_2 = A_2\Pi(x_1) + B_2(\kappa(x_1) + K_2\Pi(x_1)) + \bar{f}_2(x_1, \Pi(x_1), \kappa(x_1) + K_2\Pi(x_1))$$

into (2.7) gives the PDE satisfied by Π and κ:

$$\begin{aligned} &\bar{A}_2\Pi(x_1) + B_2\kappa(x_1) + \bar{f}_2(x_1, \Pi(x_1), \kappa(x_1) + K_2\Pi(x_1)) \\ &= \frac{\partial \Pi}{\partial x_1}(x_1)\left(A_1 x_1 + \bar{f}_1(x_1, \Pi(x_1), \kappa(x_1) + K_2\Pi(x_1))\right). \end{aligned} \tag{2.8}$$

The center manifold theorem ensures that this equation has a local solution for any smooth $\kappa(x_1)$. The reduced dynamics of the closed-loop system (2.5) on the center manifold is given by

$$\dot{x}_1 = f_1(x_1; \kappa), \tag{2.9}$$

where

$$f_1(x_1; \kappa) = A_1 x_1 + \bar{f}_1(x_1, \Pi(x_1), \kappa(x_1) + K_2\Pi(x_1)).$$

According to the center manifold theorem, we know that if the dynamics (2.9) is locally asymptotically stable, then the closed-loop system (2.3)–(2.4) is locally asymptotically stable (see [6], for example).

The part of the feedback $\mathcal{F}$ given by $\kappa(x_1)$ determines the controlled center manifold $x_2 = \Pi(x_1)$ which in turn determines the dynamics (2.9). Hence the problem of stabilization of the system (2.3) reduces the problem of stabilizing the system (2.9) after solving the PDE (2.8), i.e., finding $\kappa(x_1)$ such that the origin of the dynamics (2.9) is asymptotically stable. Thus we can view $\kappa(x_1)$ as a pseudocontrol.

Since solving the PDE (2.8) is difficult, it is usually sufficient to approximate the center manifold. Using the Taylor expansion of Π and κ around $x_1 = 0$ permits one to have an approximation of the center manifold. Because κ starts with linear terms

$$\kappa(x_1) = K_1 x_1 + \kappa^{[2]}(x_1) + \cdots, \tag{2.10}$$

Π starts with linear terms

$$\Pi(x_1) = \Pi^{[1]} x_1 + \Pi^{[2]}(x_1) + \cdots. \tag{2.11}$$

The PDE implies that

$$\bar{A}_2\Pi^{[1]} + B_2 K_1 = \Pi^{[1]} A_1, \tag{2.12}$$

$$\begin{aligned} &\bar{A}_2\Pi^{[2]}(x_1) + B_2\kappa^{[2]}(x_1) + \bar{f}_2^{[2]}(x_1, \Pi^{[1]}x_1, K_1 x_1 + K_2\Pi^{[1]}x_1) \\ &= \frac{\partial \Pi^{[2]}}{\partial x_1}(x_1) A_1 x_1 + \Pi^{[1]}\bar{f}_1^{[2]}(x_1, \Pi^{[1]}x_1, K_1 x_1 + K_2\Pi^{[1]}x_1), \end{aligned} \tag{2.13}$$

and so on.

For any $\kappa^{[k]}(x_1)$, these linear equations are solvable for $\Pi^{[k]}(x_1)$ since the eigenvalues of $\bar{A}_2$ and A_1 do not coincide. In fact, K_2 in (2.4) is chosen such that $\Re(\sigma(\bar{A}_2)) < 0 = \Re(\sigma(A_1))$.

The degree k equations are

(2.14)
$$\bar{A}_2\Pi^{[k]}(x_1) + B_2\kappa^{[k]}(x_1) - \frac{\partial\Pi^{[k]}}{\partial x_1}(x_1)A_1x_1 = \sum_{j=1}^{k-1}\frac{\partial\Pi^{[k-j]}}{\partial x_1}(x_1)\tilde{f}_1^{[j+1]}(x_1) - \tilde{f}_2^{[k]}(x_1),$$

where

$$\tilde{f}_i(x_1) = \bar{f}_i(x_1, \Pi(x_1), \kappa(x_1) + K_2\Pi(x_1)).$$

Note that $\tilde{f}_i^{[j]}(x_1)$ depends only on $\Pi^{[1]}(x_1), \dots, \Pi^{[j-1]}(x_1)$ and $\kappa^{[1]}(x_1), \dots, \kappa^{[j-1]}(x_1)$. For $1 \le i \le r-1$, the ith row of these equations is

$$\Pi_{i+1}^{[k]}(x_1) = \frac{\partial\Pi_i^{[k]}}{\partial x_1}(x_1)A_1x_1 + \sum_{j=1}^{k-1}\frac{\partial\Pi_i^{[k-j]}}{\partial x_1}(x_1)\tilde{f}_1^{[j+1]}(x_1) - \tilde{f}_{2,i}^{[k]}(x_1). \tag{2.15}$$

The rth row is

$$\kappa^{[k]}(x_1) = \frac{\partial\Pi_r^{[k]}}{\partial x_1}(x_1)A_1x_1 + \sum_{j=1}^{k-1}\frac{\partial\Pi_r^{[k-j]}}{\partial x_1}(x_1)\tilde{f}_1^{[j+1]}(x_1) - \tilde{f}_{2,r}^{[k]}(x_1). \tag{2.16}$$

Note that $\Pi_1^{[k]}(x_1)$ determines $\Pi_2^{[k]}(x_1), \dots, \Pi_r^{[k]}(x_1), \kappa^{[k]}(x_1)$. Therefore we may change our point of view. Instead of viewing $\kappa^{[k]}(x_1)$ as determining $\Pi_1^{[k]}(x_1), \dots, \Pi_r^{[k]}(x_1)$, we can view $\Pi_1^{[k]}(x_1)$ as determining $\Pi_2^{[k]}(x_1), \dots, \Pi_r^{[k]}(x_1), \kappa^{[k]}(x_1)$. In other words, instead of viewing the feedback as determining the center manifold, we can view the first coordinate function of the center manifold as determining the other coordinate functions and the feedback.

Alternatively we can view Π_1 as a pseudocontrol and write the dynamics as

$$\dot{x}_1 = A_1x_1 + \bar{f}_1(x_1; \Pi_1). \tag{2.17}$$

We will call this dynamics the *controlled center dynamics*.

Now let us write explicitly the solution of (2.12) and (2.13) giving, respectively, the linear and the quadratic approximation of the center manifold of the closed-loop system (2.5).

2.1. Linear center manifold. In this section we solve (2.12), which gives the linear part of the center manifold, show how it is affected by the linear part of the feedback (2.4), and see how we can change the orientation of the center manifold through the linear part of the feedback (2.4).

Suppose the entries in K_2 are $K_{2,1}, K_{2,2}, \dots, K_{2,n-r}$. Then the characteristic polynomial, $P(\lambda)$, of the matrix $A_2 + B_2K_2$ is defined by

$$\begin{aligned} P(\lambda) &= \det\left(\lambda I_{(n-r)\times(n-r)} - A_2 - B_2K_2\right) \\ &= \lambda^{n-r} - K_{2,n-r}\lambda^{n-r-1} - \dots - K_{2,2}\lambda - K_{2,1}. \end{aligned} \tag{2.18}$$

The linear part of the feedback (2.4) is given by

$$u(x_1, x_2) = K_1x_1 + K_2x_2 + O(x_1, x_2)^2. \tag{2.19}$$

From (2.11), the linear part of the center manifold is given by

$$\Pi^{[1]}(x_1) = \Pi^{[1]} x_1,$$

and (2.12) is equivalent to the following system of equations:

$$\begin{aligned}
\Pi_2^{[1]} &= \Pi_1^{[1]} A_1, \\
\Pi_3^{[1]} &= \Pi_2^{[1]} A_1, \\
&\vdots \\
\Pi_{n-r}^{[1]} &= \Pi_{r-1}^{[1]} A_1, \\
0 &= \Pi_{n-r}^{[1]} A_1 - K_1 - K_{2,1}\Pi_1^{[1]} - \cdots - K_{2,n-r}\Pi_{n-r}^{[1]},
\end{aligned}$$

where $\Pi_i^{[1]}$ is the ith row vector in $\Pi^{[1]}$. Therefore,

$$\begin{aligned}
\Pi_2^{[1]} &= \Pi_1^{[1]} A_1, \\
\Pi_3^{[1]} &= \Pi_1^{[1]} A_1^2, \\
&\vdots \\
\Pi_{n-r}^{[1]} &= \Pi_1^{[1]} A_1^{n-r-1}, \\
0 &= -K_1 + \Pi_1^{[1]} A_1^{n-r} - K_{2,1}\Pi_1^{[1]} - K_{2,2}\Pi_1^{[1]} A_1 - \cdots - K_{2,n-r}\Pi_1^{[1]} A_1^{n-r-1} \\
&= -K_1 + \Pi_1^{[1]}(A_1^{n-r} - K_{2,1} I - K_{2,2} A_1 - \cdots - K_{2,n-r} A_1^{n-r-1}).
\end{aligned}$$

The last equation has the form of characteristic polynomial defined by (2.18).

To summarize, the linear part of the center manifold is defined by the following equations:

$$\begin{aligned}
\Pi_1^{[1]} &= K_1 P(A_1)^{-1}, \\
\Pi_i^{[1]} &= \Pi_1^{[1]} A_1^{i-1} \text{ for } i = 2, \ldots, n-r.
\end{aligned} \tag{2.20}$$

Note that $P(A_1)$ is always invertible for the following reason. The eigenvalues of $P(A_1)$ equal the values of $P(\lambda)$ evaluated at the eigenvalues of A_1. Since $\bar{A}_2 = A_2 + B_2 K_2$ is Hurwitz, the roots of the characteristic polynomial (2.18) are all in the open left half-plane. Since the eigenvalues of A_1 are all on the imaginary axis, which are different from the roots of $P(\lambda)$, we deduce that $P(A_1)$ has no zero eigenvalue. Thus, the matrix $P(A_1)$ is invertible.

THEOREM 2.1. *Given the feedback $\mathcal{F}$, the center manifold is given by*

$$x_2 = \Pi^{[1]} x_1 + O(x_1^2),$$

with the components of $\Pi^{[1]}$ *uniquely determined by* (2.20).

Now let us show that the orientation of the center manifold can be changed by changing K_1 in (2.10).

If we view the center manifold, represented by $x_2 = \Pi(x_1)$, as a submanifold in the space of $(x_1, x_2) \in \mathbb{R}^n$, we can say that the orientation of the center manifold at the origin is a basis of the orthogonal complement subspace of the tangent space of the center manifold. Indeed, the orientation of the center manifold at the origin is a set of vectors which are orthogonal to the manifold; they are linearly independent; and they generate a complement subspace of the manifold.

THEOREM 2.2. *Given any* $(n-r)\times r$ *matrix of the form*

$$[\mathcal{M}_{(n-r)\times r}\ \ \mathcal{N}_{(n-r)\times(n-r)}],$$

its row vectors define the center manifold orientation at the origin for (2.3)–(2.19) *if and only if* $\mathcal{N}^{-1}$ *exists and* $\Pi^{[1]} = -\mathcal{N}^{-1}\mathcal{M}$ *satisfies* (2.20).

Proof. Suppose that $[\mathcal{M}_{(n-r)\times r}\ \ \mathcal{N}_{(n-r)\times(n-r)}]$ defines the orientation of the center manifold. Then it is orthogonal to the tangent space of the center manifold. It is known that the tangent space of the center manifold is given by its linear part

$$x_2 - \Pi^{[1]}x_1 = 0,$$

where $\Pi^{[1]}$ satisfies (2.20). In the (x_1, x_2)-space, a set of orthogonal vectors of the tangent space is the row vectors of $[-\Pi^{[1]}|\ I]$. Therefore, both $[-\Pi^{[1]}|\ I]$ and $[\mathcal{M}_{(n-r)\times r}\ \mathcal{N}_{(n-r)\times(n-r)}]$ generate the same space, which is orthogonal to the tangent space of the center manifold. Therefore, the row vectors of $[-\Pi^{[1]}\ I]$ are linear combinations of the row vectors in $[\mathcal{M}_{(n-r)\times r}\ \ \mathcal{N}_{(n-r)\times(n-r)}]$, i.e.,

$$[-\Pi^{[1]}|\ I] = \mathcal{N}^{-1}[\mathcal{M}_{(n-r)\times r}\ \ \mathcal{N}_{(n-r)\times(n-r)}].$$

So $\Pi^{[1]} = -\mathcal{N}^{-1}\mathcal{M}$, and it satisfies (2.20).

On the other hand, suppose $-\mathcal{N}^{-1}\mathcal{M}$ satisfies (2.20). By Theorem 2.1, the linear space

$$\mathcal{N}^{-1}\mathcal{M}\ x_1 + x_2 = 0$$

represents the linear part of the center manifold. It is the tangent space of the center manifold. Therefore, $[\mathcal{N}^{-1}\mathcal{M}\mid I]$, the row vectors in the coefficient matrix of this equation, form a basis of the orthogonal space. It is easy to check that the row vectors of $[\mathcal{M}\,\mathcal{N}]$ and $[\mathcal{N}^{-1}\mathcal{M}\mid I]$ generate the same vector space. Therefore, $[\mathcal{M}\ \ \mathcal{N}]$ defines the orientation of the center manifold. $\square$

Now consider the change of coordinates

$$\tilde{x}_{2,i} = x_{2,i} - \Pi_1^{[1]}A_1^{i-1}x_1, \qquad i = 1,\ldots,n-r; \tag{2.21}$$

then

$$\begin{aligned}\dot{\tilde{x}}_{2,i} &= \tilde{x}_{2,i+1}, \qquad i = 1,\ldots,n-r,\\ \dot{\tilde{x}}_{2,n-r} &= \sum_{i=1}^{n-r} k_{2,i}\tilde{x}_{2,i}.\end{aligned}$$

Hence, the coefficient K_1 has been removed from the x_2-part of the dynamics (2.3)–(2.19) by a change of coordinates. With $K_1 = 0$, we deduce from (2.20) that $\Pi^{[1]} = 0$. So the linear terms of the center manifold have been removed.

PROPOSITION 2.3. *Given any feedback* (2.19) *satisfying Property* $\mathcal{P}$, *and the change of coordinates* (2.21), *the center manifold is given by*

$$\tilde{x}_2 = O(x_1^2). \tag{2.22}$$

2.2. Quadratic approximation of the center manifold. In this section, we solve explicitly (2.13) giving the quadratic approximation of the center manifold and show how it is related to the quadratic part of the feedback (2.4).

Under a linear change of coordinates (2.21), the system is transformed into

$$\begin{aligned}\dot{x}_1 &= A_1 x_1 + \bar{f}_1^{[2]}(x_1, \tilde{x}_2 + \Pi^{[1]} x_1, \kappa^{[2]}(x_1)) + O(x_1, \tilde{x}_2)^3,\\ \dot{\tilde{x}}_2 &= A_2(\tilde{x}_2 + \Pi^{[1]} x_1) + B_2(K_1 x_1 + K_2 \tilde{x}_2 + K_2 \Pi^{[1]} x_1 + \kappa^{[2]}(x_1))\\ &\quad + \bar{f}_2^{[2]}(x_1, \tilde{x}_2 + \Pi^{[1]} x_1, u(x_1, \tilde{x}_2 + \Pi^{[1]} x_1)) - \Pi^{[1]} A_1 x_1\\ &\quad - \Pi^{[1]} \bar{f}_1^{[2]}(x_1, \tilde{x}_2 + \Pi^{[1]} x_1, u(x_1, \tilde{x}_2 + \Pi^{[1]} x_1)) + O(x_1, \tilde{x}_2)^3,\end{aligned}$$

in which u is the feedback defined by (2.4). Define a quadratic vector field $\tilde{f}_2^{[2]}(x_1, \tilde{x}_2)$ by

$$\begin{aligned}\tilde{f}_2^{[2]}(x_1, \tilde{x}_2) &= \bar{f}_2^{[2]}(x_1, \tilde{x}_2 + \Pi^{[1]} x_1, K_1 x_1 + K_2 \tilde{x}_2 + K_2 \Pi^{[1]} x_1)\\ &\quad - \Pi^{[1]} \bar{f}_1^{[2]}(x_1, \tilde{x}_2 + \Pi^{[1]} x_1, K_1 x_1 + K_2 \tilde{x}_2 + K_2 \Pi^{[1]} x_1).\end{aligned} \tag{2.23}$$

Then from (2.21) and (2.23), equation (2.3) is equivalent to

$$\begin{aligned}\dot{x}_1 &= A_1 x_1 + \bar{f}_1^{[2]}(x_1, \tilde{x}_2 + \Pi^{[1]} x_1, u(x_1, \tilde{x}_2 + \Pi^{[1]} x_1)) + O(x_1, \tilde{x}_2)^3,\\ \dot{\tilde{x}}_2 &= A_2 \tilde{x}_2 + B_2(K_2 \tilde{x}_2 + \kappa^{[2]}(x_1)) + \tilde{f}_2^{[2]}(x_1, \tilde{x}_2) + O(x_1, \tilde{x}_2)^3.\end{aligned} \tag{2.24}$$

In the $(x_1, \tilde{x}_2)$ coordinates, the center manifold has the form (2.22).

It satisfies the center manifold equation

$$A_2 \Pi^{[2]}(x_1) + B_2(K_2 \Pi^{[2]}(x_1) + \kappa^{[2]}(x_1)) + \tilde{f}_2^{[2]}(x_1, 0) = \frac{\partial \Pi^{[2]}(x_1)}{\partial x_1} A_1 x_1,$$

or, equivalently,

$$\begin{aligned}\Pi_{i+1}^{[2]}(x_1) &= \frac{\partial \Pi_i^{[2]}(x_1)}{\partial x_1} A_1 x_1 - \tilde{f}_{2,i}^{[2]}(x_1, 0) \text{ for } i = 2, 3, \ldots, n-r,\\ 0 &= \frac{\partial \Pi_{n-r}^{[2]}(x_1)}{\partial x_1} A_1 x_1 - \tilde{f}_{2,n-r}^{[2]}(x_1, 0) - K_2 \Pi^{[2]}(x_1) - \kappa^{[2]}(x_1).\end{aligned} \tag{2.25}$$

In the following, we adopt the matrix notation

$$\begin{aligned}\Pi_i^{[2]}(x_1) &= x_1^T Q_i x_1,\\ \tilde{f}_{2,i}^{[2]}(x_1, 0) &= x_1^T R_i x_1,\\ \kappa(x_1) &= x_1^T L x_1,\end{aligned} \tag{2.26}$$

where Q_i, R, and L are symmetric $r \times r$ matrices. Define a linear operator by

$$\mathcal{L}_{A_1}(Q) = A_1^T Q + Q A_1 \tag{2.27}$$

for all symmetric $r \times r$ matrices Q. Then (2.25) is equivalent to

$$\begin{aligned}Q_{i+1} &= \mathcal{L}_{A_1}(Q_i) - R_i \text{ for } i = 2, 3, \ldots, n-r,\\ 0 &= \mathcal{L}_{A_1}(Q_{n-r}) - R_{n-r} - K_{2,1} Q_1 - \cdots - K_{2,n-r} Q_{n-r} - L.\end{aligned}$$

Solving these equations, we have

$$
\begin{aligned}
Q_i &= \mathcal{L}_{A_1}^{i-1}(Q_1) - \sum_{j=0}^{i-2} \mathcal{L}_{A_1}^{j}(R_{i-j-1}), \\
P(\mathcal{L}_{A_1})Q_1 &= L + R_{(n-r)} + \sum_{i=2}^{n-r}\sum_{j=0}^{i-2} K_{2,i}\mathcal{L}_{A_1}^{j}(R_{i-j-1}).
\end{aligned}
\tag{2.28}
$$

To summarize, the equations (2.28) imply the following result on the quadratic approximation of the center manifold.

THEOREM 2.4. *If*

$$x_2 = \Pi^{[1]}(x_1) + \Pi^{[2]}(x_1) + O(x_1)^3$$

approximates the center manifold of (2.3)*, then* $\Pi^{[2]}(x_1)$ *is uniquely determined by the equations*

$$\Pi_i^{[2]}(x_1) = x_1^T Q_i x_1 \text{ for } i = 1, 2, \ldots, n-r,$$

where

$$
\begin{aligned}
Q_1 &= P(\mathcal{L}_{A_1})^{-1}\left(L + R_{n-r} + \sum_{i=2}^{n-r}\sum_{j=0}^{i-2} K_{2,i}\mathcal{L}_{A_1}^{j}(R_{i-j-1})\right), \\
Q_i &= \mathcal{L}_{A_1}^{i-1}(Q_1) - \sum_{j=0}^{i-2} \mathcal{L}_{A_1}^{j}(R_{i-j-1}),
\end{aligned}
$$

in which $\mathcal{L}_{A_1}$ *is the operator defined by* (2.27)*;* R_i *is from the quadratic dynamics and is defined by* (2.26) *and* (2.23)*;* L *is from the quadratic feedback and is defined by* (2.26)*; and* P *is the characteristic polynomial of* $A_2 + B_2K$ *given by* (2.18).

Similar to the derivation of the linear center manifold, the operator $P(\mathcal{L}_{A_1})$ is always invertible. The set of eigenvalues of the operator $\mathcal{L}_{A_1}$ is $\{\lambda_i + \lambda_j : \text{for } i, j = 1, \ldots, r\}$ with λ_ℓ, $\ell = 1, \ldots, r$, being the eigenvalues of A_1. Therefore, $\sigma(A_1) = 0$ implies that the eigenvalues of $\mathcal{L}_{A_1}$ are all on the imaginary axis. Since $\bar{A}_2$ is Hurwitz, the roots of $P(\lambda)$ are all in the left half-plane. They do not coincide with the eigenvalues of $\mathcal{L}_{A_1}$. Thus the eigenvalues of $P(\mathcal{L}_{A_1})$ given by $P(\lambda_i + \lambda_j)$, $i, j = 1, \ldots, r$, are nonzero. The linear operator, $P(\mathcal{L}_{A_1})$, from $\mathbb{R}^{r\times r}$ to $\mathbb{R}^{r\times r}$ must be invertible. The implicit differential equation (2.13) is thus solvable since $P(\mathcal{L}_{A_1})$ is invertible.

There are some special cases in which the center manifold is simpler. For instance, if (2.24) is in quadratic normal form (see [20]), then $\tilde{f}_2^{[2]}$ is independent of x_1. In this case, $\tilde{f}_2^{[2]}(x_1, 0) = 0$. Therefore, $R_i = 0$. Under the feedback

$$u = K_2 x_2 + x_1^T Q_{fb} x_1,$$

the quadratic approximation of the center manifold of (2.24) is

$$x_2 = \Pi^{[2]}(x_1),$$

where

$$
\begin{aligned}
\Pi_i^{[2]}(x_1) &= x_1^T Q_i x_1, \\
Q_1 &= P(\mathcal{L}_{A_1})^{-1}(Q_{fb}), \\
Q_i &= \mathcal{L}_{A_1}^{i-1}(Q_1).
\end{aligned}
$$

One special but useful case include systems with a zero uncontrollable mode (transcontrollable bifurcation). In the following section, the specific center manifolds of these systems are derived. The results in this section provide a tool to reduce a system to a low-dimensional center manifold. Feedback laws for the control of bifurcations can be derived based on the reduced system on the center manifold of the closed-loop system.

3. Stabilization of systems with transcontrollable bifurcation. In this section, we use the precedent results to stabilize systems with a transcontrollable bifurcation, i.e., those where $A_1 = 0 \in \mathbb{R}$.

From [17, 18], we know that there exists a quadratic change of coordinates and feedback,

$$\begin{aligned} x &= z + \phi^{[2]}(z), \\ u &= v + \alpha^{[2]}(z, v), \end{aligned}$$

bringing the system (2.3) to a quadratic normal form

$$\begin{aligned} \dot{z}_1 &= \beta z_1^2 + \gamma z_1 z_{21} + \sum_{i=1}^{r+1} \delta_i z_{2i}^2 + O(z_1, z_2, v)^3, \\ \dot{z}_2 &= A_2 z_2 + B_2 v + O(z_1, z_2, v)^2, \end{aligned} \tag{3.1}$$

with $z_{2,r+1} = v$. Moreover, we know that this system has a transcontrollable bifurcation if $\gamma^2 - 4\beta\delta_1 > 0$ (see [17, 18]).

Now suppose that we use the linear feedback

$$v = K_1 z_1 + K_2 z_2,$$

and assume that the linear part of the center manifold is given by

$$z_2 = \Pi^{[1]} z_1.$$

Since $A_1 = 0$, we deduce from (2.20) that

$$\begin{aligned} \Pi_i^{[1]} &= 0, \qquad i = 2, \dots, r, \\ K_1 &= -K_{21}\Pi_1^{[1]}; \end{aligned} \tag{3.2}$$

so $\Pi_2^{[1]}, \dots, \Pi_r^{[1]}, K_1$ depend on $\Pi_1^{[1]}$.

Thus, the controlled center dynamics is

$$\dot{z}_1 = \left(\beta + \gamma \Pi_1^{[1]} + \delta_1 (\Pi_1^{[1]})^2\right) z_1^2 + O(z_1)^3.$$

Because $\gamma^2 - 4\beta\delta_1 > 0$, there are two choices of $\Pi_1^{[1]}$ such that $\beta + \gamma\Pi_1^{[1]} + \delta_1(\Pi_1^{[1]})^2 = 0$. After such a choice, the stability of the controlled center dynamics depends on cubic terms.

We use quadratic and cubic change of state coordinates and invertible quadratic and cubic feedback,

$$\begin{aligned} x &= z + \phi^{[2]}(z) + \phi^{[3]}(z), \\ u &= v + \alpha^{[2]}(z, v) + \alpha^{[3]}(z, v), \end{aligned}$$

to bring the system from linear normal form to quadratic and cubic normal form (see [23]):

$$
\begin{aligned}
\dot{z}_1 &= \beta z_1^2 + \gamma z_1 z_{21} + \sum_{i=1}^{r+1} \delta_i z_{2i}^2 + \bar{\beta} z_1^3 + \bar{\gamma} z_1^2 z_{21} + \sum_{i=1}^{r+1} \bar{\delta}_i z_1 z_{2i}^2 \\
&\quad + \sum_{i=1}^{r+1} \sum_{j=i}^{r+1} \bar{\epsilon}_{ij} z_{2i} z_{2j}^2 + O(z_1, z_2, v)^4, \\
\dot{z}_2 &= A_2 z_2 + B_2 v + O(z_1, z_2, v)^2.
\end{aligned} \tag{3.3}
$$

Let $\Sigma_{\mathcal{T}}$ denote this system. Because z_2 is linearly stabilizable, the quadratic and cubic terms will not affect the stability properties of the z_2-dynamics.

3.1. Stabilization using a quadratic feedback. Consider the quadratic feedback

$$
v = K_1 x_1 + K_2 x_2 + \kappa^{[2]}(z_1) \tag{3.4}
$$

to shape the linear and quadratic parts of the center manifold

$$
z_2 = \Pi^{[1]} z_1 + \Pi^{[2]}(z_1),
$$

which in turn shape the quadratic and cubic parts of the controlled center dynamics. The procedure to choose K_1 and K_2 in (3.4) is as follows. From Property $\mathcal{P}$, we know that K_2 is such that $\sigma(A+B_2K_2) < 0$. Moreover, we choose $\Pi_1^{[1]}$ so that the quadratic part of the controlled center dynamics is zero; then we deduce K_1 from (3.2).

We can choose $\Pi_1^{[2]}(z_1) = cz_1^2$ arbitrarily; then the controlled center dynamics is given by

$$
\dot{z}_1 = \left((\gamma + 2\delta_1 \Pi_1^{[1]})c + \bar{\beta} + \bar{\gamma}\Pi_1^{[1]} + \bar{\delta}_1 (\Pi_1^{[1]})^2 + \bar{\epsilon}_1 (\Pi_1^{[1]})^3 \right) z_1^3 + O(z_1)^4.
$$

There were two possible choices of $\Pi_1^{[1]}$ that canceled the quadratic part of controlled center dynamics. Since $\gamma^2 - 4\beta\delta_1 > 0$ there is at least one such $\Pi_1^{[1]}$ so that $\gamma + 2\delta_1\Pi_1^{[1]} \neq 0$.

Then we can choose c so that

$$
(\gamma + 2\delta_1 \Pi_1^{[1]})c + \bar{\beta} + \bar{\gamma}\Pi_1^{[1]} + \bar{\delta}_1 (\Pi_1^{[1]})^2 + \bar{\epsilon}_1 (\Pi_1^{[1]})^3 < 0
$$

and the controlled center dynamics is locally asymptotically stable, so the closed-loop system is locally asymptotically stable.

THEOREM 3.1. *Consider system* (3.3) *with* $\gamma^2 - 4\beta\delta_1 > 0$*; then the quadratic feedback* (3.4) *locally asymptotically stabilizes the system* $\Sigma_{\mathcal{T}}$.

3.2. Stabilization using a piecewise linear feedback. The quadratic controller (3.4) is not robust to small parameter variations because we must choose $\Pi_1^{[1]}$ such that

$$
\beta + \gamma\Pi_1^{[1]} + \delta_1 \left(\Pi_1^{[1]}\right)^2 = 0
$$

to cancel the quadratic part of the controlled center dynamics. A small variation of β, γ or δ_1 introduces quadratic terms in the controlled center dynamics, and hence instability.

Therefore we take an alternative approach. We use a piecewise linear feedback; i.e., κ is of class $\mathcal{C}^0$. So κ is not smooth as supposed previously, but we will see that our approach is still valid.

The control law has the form

$$v = K_1(z_1)z_1 + K_2 z_2 + O(z_1, z_2)^2, \tag{3.5}$$

with

$$K_1(z_1) = \begin{cases} \bar{k}_1, & z_1 \geq 0, \\ \tilde{k}_1, & z_1 < 0. \end{cases}$$

Under the feedback (3.5), the system (3.1) has $n-1$ eigenvalues with negative real parts ($\bar{A}_2$ is Hurwitz), and one zero-eigenvalue.

THEOREM 3.2. *Consider the closed-loop system* (3.1)–(3.5)*; then there exists a center manifold defined by* $z_2 = \Pi(z_1)$ *whose linear part is determined by the feedback* (3.5).

Proof. The linear part of the dynamics (3.1)–(3.5) is given by

$$\begin{aligned} \dot{z}_1 &= O(z_1, z_2)^2, \\ \dot{z}_2 &= B_2 K_1(z_1) z_1 + \bar{A}_2 z_2 + O(z_1, z_2)^2. \end{aligned} \tag{3.6}$$

Let $\Sigma_{\bar{k}_1}$ (resp., $\Sigma_{\tilde{k}_1}$) be the system (3.6) when $K_1(z_1) = \bar{k}_1$ (resp., $K_1(z_1) = \tilde{k}_1$) for all z_1. Since the system $\Sigma_{\bar{k}_1}$ (resp., $\Sigma_{\tilde{k}_1}$) is smooth and possesses one eigenvalue on the imaginary axis and $n-1$ eigenvalues in the open left half-plane, then, from the center manifold theorem, in a neighborhood of the origin, $\Sigma_{\bar{k}_1}$ (resp., $\Sigma_{\tilde{k}_1}$) has a center manifold $\overline{W}^c$ (resp., $\widetilde{W}^c$).

For $\Sigma_{\bar{k}_1}$, the center manifold is represented by $z_2 = \overline{\Pi}(z_1)$ for z_1 sufficiently small. Its equation is

$$\begin{aligned} \dot{z}_2 &= A_2 \overline{\Pi}(z_1) + B_2(\bar{k}_1 z_1 + K_2 \overline{\Pi}(z_1)) + O(z_1, z_2)^2 \\ &= \frac{\partial \overline{\Pi}(z_1)}{\partial z_1} \dot{z}_1 = O(z_1, z_2)^2. \end{aligned} \tag{3.7}$$

Since the linear part of the center manifold is of the form $z_2 = \overline{\Pi}^{[1]} z_1$ and its ith component is $z_{2,i} = \overline{\Pi}_i^{[1]} z_1$ for $i = 1, \ldots, n-1$, using (3.7) we obtain that $\overline{\Pi}_1^{[1]} = -\frac{\bar{k}_1}{k_{2,1}}$ and $\overline{\Pi}_i^{[1]} = 0$ for $2 \leq i \leq n-1$. Similarly for $\Sigma_{\tilde{k}_1}$, the center manifold is represented by $z_2 = \widetilde{\Pi}(z_1)$. Its linear part is given by $z_2 = \widetilde{\Pi}^{[1]} z_1$, whose components are defined by $\widetilde{\Pi}_1^{[1]} = -\frac{\tilde{k}_1}{k_{2,1}}$ and $\widetilde{\Pi}_i^{[1]} = 0$, for $2 \leq i \leq n-1$. Since $\bar{A}_2$ has no eigenvalues on the imaginary axis and $k_{2,1}$ is the product of all the eigenvalues of $\bar{A}_2$, then $k_{2,1} \neq 0$.

The center manifolds $\overline{W}^c$ and $\widetilde{W}^c$ intersect along the line $z_1 = 0$, since $\overline{\Pi}(z_1)|_{z_1=0} = 0$ and $\widetilde{\Pi}(z_1)|_{z_1=0} = 0$.

Hence, if we slice them along the line $z_1 = 0$ and then glue the part of $\overline{W}^c$ for which $z_1 > 0$ with the part of $\widetilde{W}^c$ for which $z_1 < 0$, along this line we deduce that, in an open neighborhood of the origin, the piecewise smooth system (3.6) has a piecewise smooth center manifold W_c. The linear part of the center manifold W_c is represented by $z_2 = \Pi^{[1]} z_1$. The ith component of z_2, $z_{2,i}$ is given by

$$z_{2,i} = \Pi_i^{[1]}(z_1) z_1, \tag{3.8}$$

with

$$\Pi_1^{[1]}(z_1) = -\frac{K_1(z_1)}{k_{2,1}} \text{ and } \Pi_i^{[1]}(z_1) = 0 \text{ for } i \geq 2. \qquad \square \tag{3.9}$$

Using (3.3) and (3.8), the controlled center dynamics is given by

$$\dot{z}_1 = \begin{cases} \Phi(\overline{\Pi}_1^{[1]})z_1^2 + O(z_1^3), & z_1 \geq 0, \\ \Phi(\widetilde{\Pi}_1^{[1]})z_1^2 + O(z_1^3), & z_1 < 0, \end{cases} \tag{3.10}$$

with Φ the function defined by $\Phi(X) = \beta + \gamma X + \delta_1 X^2$.

The following theorem shows that the origin of the system (3.10) can be made asymptotically stable.

THEOREM 3.3. *Consider system* (3.3) *with* $\gamma^2 - 4\beta\delta_1 > 0$*; then the piecewise linear feedback* (3.5) *locally asymptotically stabilizes the system* $\Sigma_{\mathcal{T}}$.

Proof. Since $\gamma^2 - 4\beta\delta_1 > 0$, and given any Φ_0 such that $0 < \Phi_0 < |\beta - \gamma^2/(4\delta)|$, there is a $\overline{\Pi}_1^{[1]}$ such that $\Phi(\overline{\Pi}_1^{[1]}) = -\Phi_0$ and a $\widetilde{\Pi}_1^{[1]}$ such that $\Phi(\widetilde{\Pi}_1^{[1]}) = \Phi_0$. The controlled center dynamics is then

$$\dot{z}_1 = -\Phi_0|z_1|z_1 + O(z_1)^3,$$

which is locally asymptotically stable.

To show the local asymptotic stability of the closed-loop system, we make the change of coordinates

$$z_{2\text{new}} = z_{2\text{old}} - \Pi(z_1).$$

In these new coordinates the system becomes

$$\begin{aligned} \dot{z}_1 &= -\Phi_0|z_1|z_1 + \bar{g}_1(z_1, z_2), \\ \dot{z}_2 &= \bar{A}_2 z_2 + \bar{g}_2(z_1, z_2), \end{aligned}$$

where $\bar{g}_i(z_1, 0) = 0$, $\frac{\partial \bar{g}_i}{\partial z_2}(0,0) = 0$. So given any $\epsilon > 0$, there exists a $\delta > 0$ such that if $|z| < \delta$, then

$$|\bar{g}_i(z_1, 0)| < \epsilon |z_2|.$$

Since $\bar{A}_2$ is Hurwitz, there exists a unique P such that

$$P\bar{A}_2 + \bar{A}_2^T P = -I.$$

Since P is positive definite, then there exists $0 < m \leq M$ such that[1]

$$m|z_2|^2 \leq z_2^T P z_2 \leq M|z_2|^2.$$

Let V be the composite Lyapunov function [22]

$$V(z_1, z_2) = \frac{1}{2}z_1^2 + \sqrt{z_2^T P z_2}\,;$$

[1]We can choose $m = \lambda_{\min}(P)$ and $M = \lambda_{\max}(P)$, the smallest and the largest eigenvalue of P, respectively.

then

$$\frac{d}{dt}V(z_1,z_2) \le -\Phi_0|z_1|z_1^2 + |z_1||\bar{g}_1(z_1,z_2)| + \frac{1}{2\sqrt{m}}\left(-|z_2| + 2M|\bar{g}_2(z_1,z_2)|\right)$$
$$\le -\Phi_0|z_1|z_1^2 - \frac{|z_2|}{4\sqrt{m}} - \left(\frac{1}{4\sqrt{m}} - \epsilon\left(\delta + \frac{M}{\sqrt{m}}\right)\right)|z_2|.$$

By choosing ϵ such that $\frac{1}{4\sqrt{m}} - \epsilon(\delta + \frac{M}{\sqrt{m}}) > 0$, then $\frac{d}{dt}V(z_1,z_2) < 0$. So the origin of the closed-loop system (3.3)–(3.5) is asymptotically stable. □

With this approach, we generalize the results in [4], where the authors used a piecewise linear optimal controller to stabilize a special class of systems of the form (3.1).

REFERENCES

[1] E. H. Abed and J.-H. Fu, *Local feedback stabilization and bifurcation control, part* I. *Hopf bifurcation*, Systems Control Lett., 7 (1986), pp. 11–17.

[2] E. H. Abed and J.-H. Fu, *Local feedback stabilization and bifurcation control, part* II. *Stationary bifurcation*, Systems Control Lett., 8 (1987), pp. 467–473.

[3] D. Aeyels, *Stabilization of a class of nonlinear systems by a smooth feedback control*, Systems Control Lett., 5 (1985), pp. 289–294.

[4] E. G. Al'brekht and S. B. Mironova, *Optimal stabilization in the critical case of a single zero root*, J. Appl. Math. Mech., 61 (1997), pp. 709–715.

[5] S. Behtash and S. Sastry, *Stabilization of nonlinear systems with uncontrollable linearization*, IEEE Trans. Automat. Control, 33 (1988), pp. 585–590.

[6] J. Carr, *Application of Centre Manifold Theory*, Springer, New York, Berlin, 1981.

[7] F. Colonius and W. Kliemann, *Controllability and stabilization of one-dimensional systems near bifurcation points*, Systems Control Lett., 24 (1995), pp. 87–95.

[8] O. Fitch, *The Control of Bifurcation with Engineering Applications*, Ph.D. thesis, Naval Postgraduate School, Monterey, CA, 1997.

[9] B. Hamzi, *Analyse et commande des systèmes non linéaires non commandables en première approximation dans le cadre de la théorie des bifurcations*, Ph.D. thesis, University of Paris XI-Orsay, France, 2001.

[10] B. Hamzi, J.-P. Barbot, S. Monaco, and D. Normand-Cyrot, *Nonlinear discrete-time control of systems with a Naimark-Sacker bifurcation*, Systems Control Lett., 44 (2001), pp. 245–258.

[11] B. Hamzi and W. Kang, *Resonant terms and bifurcations of nonlinear control systems with one uncontrollable mode*, Systems Control Lett., 49 (2003), pp. 267–278.

[12] B. Hamzi and A. J. Krener, *Practical stabilization of systems with a fold control bifurcation*, in New Trends in Nonlinear Dynamics and Control and Their Applications, W. Kang, C. Borges, and M. Xiao, eds., Springer, Berlin, 2003, pp. 37–48.

[13] B. Hamzi, W. Kang, and A. J. Krener, *Control of center manifolds*, in Proceedings of the 42nd IEEE Conference on Decision and Control, Vol. 3, Maui, HI, 2003, pp. 2065–2070.

[14] B. Hamzi, W. Kang, and J.-P. Barbot, *Analysis and control of Hopf bifurcations*, SIAM J. Control Optim., 42 (2004), pp. 2200–2220.

[15] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. 840, Springer, New York, Berlin, 1981.

[16] T. Kailath, *Linear Systems*, Prentice–Hall, Englewood Cliffs, NJ, 1979.

[17] W. Kang, *Bifurcation and normal form of nonlinear control systems, Part* I, SIAM J. Control Optim., 36 (1998), pp. 193–212.

[18] W. Kang, *Bifurcation and normal form of nonlinear control systems, Part* II, SIAM J. Control Optim., 36 (1998), pp. 213–232.

[19] W. Kang, *Bifurcation control via state feedback for systems with a single uncontrollable mode*, SIAM J. Control Optim., 38 (2000), pp. 1428–1452.

[20] W. Kang and A. J. Krener, *Extended quadratic controller normal form and dynamic state feedback linearization of nonlinear systems*, SIAM J. Control Optim., 30 (1992), pp. 1319–1337.

[21] A. Kelley, *The stable, center-stable, center, center-unstable, unstable manifold*, J. Differential Equations, 3 (1967), pp. 546–570.

[22] H. K. Khalil, *Nonlinear Systems*, 2nd ed., Prentice–Hall, Upper Saddle River, NJ, 1996.

[23] A. J. Krener, W. Kang, and D. E. Chang, *Control bifurcations*, in Proceedings of the 39th IEEE Conference on Decision and Control, Vol. 1, Sydney, Australia, 2000, pp. 1602–1607.

[24] V. A. Pliss, *The reduction principle in the theory of stability of motion*, Soviet Math., 5 (1964), pp. 247–250.

[25] H. Poincaré, *Sur les propriétés des fonctions définies par les équations aux différences partielles*, 1879, in Oeuvres, Gauthier-Villars, Paris, 1929, pp. XCIX–CX.

[26] A. N. Sositaisvili, *Bifurcations of topological type of a vector field near a singular point*, Trud. Sem. Petrovsk., 1 (1975), pp. 279–309.

[27] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Texts Appl. Math. 2, Springer, New York, 1990.