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A Critical Study of the Circuit Concept

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From Maxwell's equations, an expression for the complex power associated with a wire circuit is formulated and broken into a complex input power and a complex power into the external fields associated with the circuit, the latter including the radiated power. From these powers, the internal and external impedances of the circuit are obtained such that the current is not required to be everywhere in time phase within the circuit. This concept is extended to coupled circuits, bringing out some of the relations between some conventional methods for obtaining the driving point impedance of antenna arrays. The theory does not require the current distributions to be postulated, but in practical applications such a postulate becomes necessary unless the solution is obtained by a method such as the integral equation method. The resulting circuitry may readily be reduced to that for lumped elements. A more critical study of the impedance formulas is given in the appendix, based upon the reciprocity theorem which is derived therein.

THE CIRCUIT COMPLEX POWER

CONSIDER a circuit formed by a wire of radius a containing a slice between positions b and c (Fig. 1). Positions d and e may or may not coincide. The steady-state form of Maxwell's equations, along with Ohm's law, are postulated within and exterior to the wire, that is,

$$\begin{gathered}\nabla\cdot\bar{H}=0,\quad \nabla\cdot\bar{E}=\rho/\epsilon,\\ \nabla\times\bar{H}=\bar{\iota}+j\omega\epsilon\bar{E},\\ \nabla\times\bar{E}=-j\omega\mu\bar{H},\\ \bar{\iota}=\sigma\bar{E}.\end{gathered}\tag{1}$$

The solutions for $\bar{E}$ and $\bar{H}$ are given in terms of the retarded vector potential $\bar{A}$, with

$$\bar{A}=\frac{1}{4\pi}\int_V \bar{\iota}e(r_{21})dV,\; r_{21}=|\bar{r}_1-\bar{r}_2|,\; e(r_{21})=e^{-jkr_{21}}/r_{21},\tag{2}$$

being a solution of

$$[\nabla^2+k^2]\bar{A}=-\bar{\iota},\quad k=\omega(\mu\epsilon)^{\frac{1}{2}}=2\pi/\lambda.\tag{3}$$

Thus, since

$$\begin{gathered}\mu_0=4\pi(10^{-7})\text{ henries/meter},\\ \epsilon_0=1/36\pi(10^{-9})\text{ farads/meter},\\ \eta_0=(\mu_0/\epsilon_0)^{\frac{1}{2}}=120\pi\text{ ohms},\end{gathered}$$

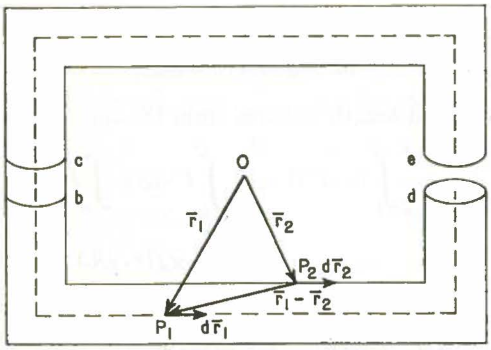

FIG. 1. Open or closed wire circuit.

for unity relative permeability and dielectric constants, defining the operator *deltil* by

$$\Diamond[\;]=[\nabla(\nabla\cdot\;)+k^2][\;],$$

$\bar{E}$ and $\bar{H}$ are given by

$$\begin{gathered}\bar{H}=\nabla\times\bar{A},\\ \bar{E}=\frac{1}{j\omega\epsilon_0}\nabla(\nabla\cdot\bar{A})-j\omega\mu_0\bar{A}=\frac{\eta_0}{jk}\Diamond\bar{A}.\end{gathered}\tag{4}$$

Now, denoting the complex conjugate by the superscript *,

$$\frac{1}{2}\int_V \bar{E}\cdot\bar{\iota}^*dV=\frac{1}{2\sigma}\int|\iota|^2dV=W_{Av},\tag{5}$$

in which V is the volume of the wire and W_{Av} is the time average power loss within the wire. Since

$$\bar{\iota}^*=\nabla\times\bar{H}^*+j\omega\epsilon\bar{E}^*$$

and

$$\nabla\cdot(\bar{E}\times\bar{H}^*)=\bar{H}^*\cdot\nabla\times\bar{E}-\bar{E}\cdot\nabla\times\bar{H}^*,$$

substitution into (5) yields

$$\begin{aligned}\frac{1}{2}\int_V \bar{E}\cdot\bar{\iota}^*dV&=\frac{1}{2}\int_V[j\omega\epsilon|\bar{E}|^2+\bar{E}\cdot\nabla\times\bar{H}^*]dV\\ &=\frac{1}{2}\int_V[j\omega\epsilon|\bar{E}|^2-j\omega\mu|\bar{H}|^2]dV-\frac{1}{2}\int_S\bar{E}\times\bar{H}^*\cdot d\bar{S},\end{aligned}$$

or

$$W_{Av}=j\omega[|U_E|-|U_H|]-\int_S\bar{P}\cdot d\bar{S},\tag{6}$$

where $|U_E|$ and $|U_H|$ are the peak energies stored in the electric and magnetic fields within the wire, respectively, and $\bar{P}$ is the complex Poynting vector. Hence,

$$W_{Av}=-Re\int_S\bar{P}\cdot d\bar{S}$$

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and

$$j\omega[|U_H|-|U_E|]=Im\int_S \bar{P}\cdot d\bar{S}. \quad (7)$$

Thus, by writing

$$\bar{P}=\bar{P}_0+\bar{P}_r,$$

the resultant field $\bar{E}$ may be broken into two components

$$\bar{E}=\bar{E}_0+\bar{E}_i,$$

such that

$$\frac{1}{2}\int \bar{E}_0\cdot\bar{\imath}^*dV=-\int_S \bar{P}_0\cdot d\bar{S}=\tilde{W}_0 \quad (8)$$

and

$$-\frac{1}{2}\int \bar{E}_i\cdot\bar{\imath}^*dV=j\omega[|U_H|-|U_E|]+\int_S \bar{P}_r\cdot d\bar{S} \quad (9)$$

$$=j\omega[|U_H|-|U_E|]+\tilde{W}_r$$

with $\tilde{W}_0$ being the complex power input to the wire and with $\tilde{W}_r$ being the complex power into the external fields associated with the circuit. The time average power radiated from the circuit is given by $Re[\tilde{W}_r]$. In other words, $\bar{E}_0$ becomes the applied field, or that

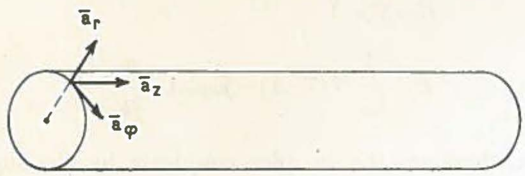

FIG. 2. Unit length volume of wire with vector directions.

component of $\bar{E}$ originating outside the wire which is required for maintaining

$$|U_H|-|U_E|=\text{constant},$$

as required for a steady-state condition, and $\bar{E}_i$ becomes the induced field, or the component of $\bar{E}$ which arises from the currents and charges within and on the wire itself. The induced field is given by

$$\bar{E}_i=\frac{\eta_0}{jk}\Diamond\bar{A}, \quad \bar{A}=\frac{1}{4\pi}\int_V \bar{\imath}e(r_{21})dV, \quad (10)$$

where V is the volume of the wire itself, whereas $\bar{E}_0$ may or may not be given by a similaı expression. For example, if the applied field arises from another circuit within the neighborhood, $\bar{E}_0$ will be given by a vector potential integrated over the volume of that circuit. But in this case, due to the interaction of the circuits, $\bar{E}_0$ also will be a function of the current in the given circuit. However, here it will be postulated that $\bar{E}_0$ is supplied by a slice generator inserted between terminals b and c such that if

$$\bar{\imath}=\bar{\imath}_0 f(P) \quad \text{for} \quad b\leq P\leq c,$$
$$\bar{\imath}=\bar{\imath}_0 \quad \text{for} \quad c\leq P\leq b,$$

then

$$\frac{1}{2}\int \bar{E}_0\cdot\bar{\imath}^*dV=-\frac{1}{2}\int_{V'} \bar{E}_0\cdot\bar{\imath}_0^*dV', \quad (11)$$

with V' being the volume of the slice and with P being a position within the circuit.

If the applied voltage V_0 is defined as

$$V_0=-\int_c^b \bar{E}_0\cdot d\bar{r}=\phi_b-\phi_c, \quad \phi=\frac{1}{j\omega\epsilon}\nabla\cdot\bar{A}, \quad (12)$$

then

$$\tilde{W}_0=\frac{1}{2}\int_V \bar{E}_0\cdot\bar{\imath}^*dV=-\frac{1}{2}\int_{V'} \bar{E}_0\cdot\bar{\imath}_0^*dV'$$

$$=-\frac{1}{2}\int_c^b \bar{E}_0\left(\int_S \bar{\imath}_0^*\cdot d\bar{S}\right)\cdot d\bar{r},$$

or the complex power input becomes

$$\tilde{W}_0=\tfrac{1}{2}V_0 I_0^*, \quad I_0=\int_S \bar{\imath}_0\cdot d\bar{S}, \quad (13)$$

it being tacitly assumed that $\bar{E}_0$ is cross-sectionally constant within the slice generator.

Thus, the input impedance becomes

$$Z_0=2\tilde{W}_0/|I_0|^2=V_0/I_0. \quad (14)$$

But from (8) and (10),

$$\tilde{W}_0=\frac{1}{2}\int_V \bar{E}\cdot\bar{\imath}^*dV-\frac{1}{2}\int_V \bar{E}_i\cdot\bar{\imath}^*dV$$

$$=\frac{1}{2}\int_V\left[\bar{E}\cdot\bar{\imath}^*+j\frac{\eta_0}{k}\Diamond\bar{A}\cdot\bar{\imath}^*\right]dV. \quad (15)$$

THE INTERNAL IMPEDANCE

Now the complex power input density $\tilde{W}_{01}$ per unit length at each position on the circuit where $\bar{\imath}\neq 0$, required to overcome the ohmic loss and to supply the reactive power due to the current through a cross section at that position, may be found by postulating the current and fields to extend one meter axially constant, with no retardation, within a right circular cylinder of radius a, and with a direction $\bar{a}_z$ equivalent to that of the wire at such a position, integrating over this volume and dividing by one meter of length. Since (Fig. 2)

$$dV=rd\phi dr(1)=dS',$$
$$d\bar{S}=\bar{a}_r dl(1)=\bar{a}_r a d\phi,$$

over the unit length volume, from (8) and (9),

$$\tilde{W}_{01}=\frac{1}{2}\int_V \bar{E}_0\cdot\bar{\imath}^*dV=-\int_S \bar{P}\cdot d\bar{S}+\int_S \bar{P}_r\cdot d\bar{S}$$

$$=\frac{1}{2}\int_S E\times\bar{H}^*\cdot d\bar{S}+\tilde{W}_{r1},$$

$$\tilde{W}_{01}-\tilde{W}_{r1}=-\frac{1}{2}\int_S E_{za}H_{\phi a}^*\bar{a}_z\times\bar{a}_\phi\cdot\bar{a}_r dS$$

$$=\tfrac{1}{2}E_{za}\int_0^{2\pi} H_{\phi a}^* a d\phi,$$

or since

$$\tilde{W}_{r1}=j\omega[|U_{E1}|-|U_{H1}|]-\frac{1}{2}\int_{S'}\bar{E}_i\cdot\bar{\iota}^*dS'\approx 0,$$

$$\tilde{W}_{01}=\tfrac{1}{2}E_{za}(2\pi aH_{\phi a}^*). \tag{16}$$

But since S' is the cross-sectional surface,

$$2\pi aH_{\phi a}^*=\oint\bar{H}_a^*\cdot d\bar{l}=\int_{S'}\nabla\times\bar{H}^*\cdot d\bar{S}'$$

$$=\int_{S'}\left[1-\frac{j\omega\epsilon}{\sigma}\right]\bar{\iota}^*\cdot d\bar{S}'\approx I_z^*, \tag{17}$$

and hence if the internal impedance per unit length of wire is defined as

$$Z_i=E_{za}/I_z, \tag{18}$$

substitution of (17) and (18) into (16) yields

$$\tilde{W}_{01}=\tfrac{1}{2}Z_i|I_z|^2. \tag{19}$$

To find I_z, first form the wave equations within the unit volume from (1), replacing $\bar{E}$ by $(1/\sigma)\bar{\iota}$, that is, write

$$\nabla\times\bar{\iota}=-j\omega\mu\sigma\bar{H}$$

$$\nabla\times\bar{H}=\left(1+\frac{j\omega\epsilon}{\sigma}\right)\bar{\iota}\approx\bar{\iota}. \tag{20}$$

Then, since $\nabla\cdot\bar{\iota}=0$ through this volume,

$$-\nabla\times\nabla\times\bar{\iota}=\nabla^2\bar{\iota}=j\omega\mu\sigma\bar{\iota},$$

or assuming the current is symmetrically distributed in azimuth,

$$\frac{d^2\iota_z}{dr^2}+\frac{1}{r}\frac{d\iota_z}{dr}-j\omega\mu\sigma\iota_z=0. \tag{21}$$

The solution of (21) in terms of the Bessel functions for an imaginary argument is

$$\iota_z=\iota_{za}\frac{I_0[j^{\frac{1}{2}}(\sqrt{2}/\delta)r]}{I_0[j^{\frac{1}{2}}(\sqrt{2}/\delta)a]},\quad \delta=\frac{1}{(\pi f\mu\sigma)^{\frac{1}{2}}}. \tag{22}$$

Integrating over a cross section for I_z,

$$I_z=-j\frac{2\pi a\delta\iota_{za}}{\sqrt{2}}\frac{\mathrm{Ber}'(\sqrt{2}a/\delta)+j\,\mathrm{Bei}'(\sqrt{2}a/\delta)}{\mathrm{Ber}(\sqrt{2}a/\delta)+j\,\mathrm{Bei}(\sqrt{2}a/\delta)}. \tag{23}$$

Hence,

$$Z_i=\frac{\iota_{za}}{\sigma I_z}=j\frac{\sqrt{2}R_s}{2\pi a}\frac{\mathrm{Ber}(\sqrt{2}a/\delta)+j\,\mathrm{Bei}(\sqrt{2}a/\delta)}{\mathrm{Ber}'(\sqrt{2}a/\delta)+j\,\mathrm{Bei}'(\sqrt{2}a/\delta)}, \tag{24}$$

with

$$R_s=\frac{1}{\sigma\delta}=(\pi f\mu/\sigma)^{\frac{1}{2}}.$$

At high frequencies,

$$Z_i=R_s/2\pi a(1+j), \tag{25}$$

and at very low frequencies,

$$Z_i=(1/\pi a^2\sigma)+j15k. \tag{26}$$

THE INPUT IMPEDANCE

Returning to (15), since the current has been postulated continuous through the slice generator of negligible thickness, by writing

$$\bar{I}\cdot d\bar{r}=I_0 f(P)dl,$$

substitution from (19) yields

$$\tilde{W}_0=\tfrac{1}{2}Z_i|I_0|^2\int_e^d|f(P)|^2dl+j\frac{60\pi}{k}\int_V\Diamond\bar{A}\cdot\bar{\iota}^*dV, \tag{27}$$

with $\bar{A}$ given by the integration of the retarded current over the volume of the wire with the exception of the cross section included in the element $\bar{\iota}^*dV$.

Now write,

$$\int_V\Diamond\bar{A}\cdot\bar{\iota}^*dV=\int_{V1}\Diamond\left[\int_{V2}\frac{\bar{\iota}_2e(r_{21})}{4\pi}dV_2\right]\cdot\bar{\iota}_1^*dV_1,$$

$$V_1=V_2=V, \tag{28}$$

in which the subscripts are introduced for distinguishing between the operations to be carried out at two different positions along the wire.

Furthermore, write

$$\begin{aligned}\bar{\iota}_2dV_2&=\iota_2dS_2d\bar{r}_2=(\bar{\iota}_2\cdot d\bar{S}_2)d\bar{r}_2,\\ \bar{\iota}_1dV_1&=\iota_1dS_1d\bar{r}_1=(\bar{\iota}_1\cdot d\bar{S}_1)d\bar{r}_1.\end{aligned} \tag{29}$$

But $d\bar{r}_1$ and $d\bar{r}_2$ also should actually be integrated cross sectionally along with $\bar{\iota}_1$ and $\bar{\iota}_2$. However, if $ka\ll 1$, they may be assumed essentially independent of their radial positions. Nevertheless, the question arises as to where their radial positions should be chosen. Choosing both positions along the same path is equivalent to postulating an infinitely thin wire and yields an infinite reactance. The conventional choice assumes one path along the axis and the other along the surface of the wire. If this is done, since $\bar{\iota}_2$ is a function of position two whereas the differentiations required by the operator $\Diamond_1$ are to be performed at position one,

$$\int_V\Diamond\bar{A}\cdot\bar{\iota}^*dV$$

$$=\frac{1}{4\pi}\oint_1\oint_2\left(\int_{S2}\bar{\iota}_2\cdot d\bar{S}_2\right)$$

$$\times\left(\int_{S1}\bar{\iota}_1^*\cdot d\bar{S}_1\right)\Diamond_1[e(r_{21})d\bar{r}_2]\cdot d\bar{r}_1$$

$$=\frac{|I_0|^2}{4\pi}\oint_1\oint_2 f(P_2)f(P_1)^*\Diamond_1[e(r_{21})d\bar{r}_2]\cdot d\bar{r}_1. \tag{30}$$

By selecting $d\bar{r}_1$ along the axis of the wire with $d\bar{r}_2$ along the inner periphery, substitution from (30) into

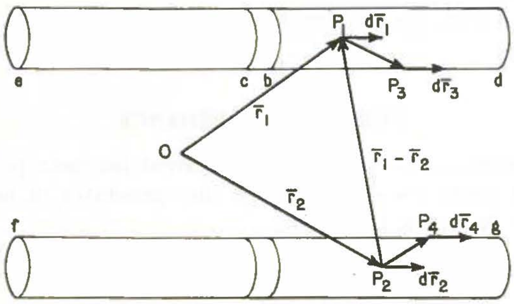

FIG. 3. Two indirectly coupled circuits.

(27) yields

$$\tilde W_{12}=\tfrac{1}{2}Z_i|I_0|^2\oint_e^d |f(P)|^2dl+j\frac{15}{k}|I_0|^2 \times\oint_e^d\oint_e^d f(P_2)f(P_1)^*\diamondsuit_1[e(r_{21})d\bar r_2]\cdot d\bar r_1, \quad (31)$$

from which it is possible to write an input impedance. However, if the path selections were reversed, the expression for the complex power would become

$$\tilde W_{21}=\tfrac{1}{2}Z_i|I_0|^2\oint_e^d |f(P)|^2dl+j\frac{15}{k}|I_0|^2 \times\oint_e^d\oint_e^d f(P_1)f(P_2)^*\diamondsuit_2[e(r_{12})d\bar r_1]\cdot d\bar r_2. \quad (32)$$

It may be demonstrated that,[1]

$$\diamondsuit_1[e(r_{21})d\bar r_2]\cdot d\bar r_1=\diamondsuit_2[e(r_{12})d\bar r_1]\cdot d\bar r_2, \quad (33)$$

which is the reciprocity theorem for the electromotive force and a current moment. Hence, it becomes apparent that unless the current distribution function becomes real, the two different selections seemingly yield different expressions for the complex power. To remove this seemingly paradoxical result, the final form for the complex power will be taken as the arithmetic mean of the powers for the two different choices. That is,

$$\tilde W=\tfrac{1}{2}(\tilde W_{12}+\tilde W_{21}), \quad (34)$$

or if f_m is the spatial root-mean-square current distribution and l is the length of the circuit, it follows from (14), (33), and (34), that since

$$\tfrac{1}{2}[f(P_1)f(P_2)^*+f(P_2)f(P_1)^*] =Re[f(P_1)f(P_2)^*]=Re[f(P_2)f(P_1)^*], \quad (35)$$

then

$$Z_0=lZ_if_m^2+j\frac{30}{k}\oint_e^d\oint_e^d Re[f(P_1)^*f(P_2)] \times\diamondsuit_1[e(r_{21})d\bar r_2]\cdot d\bar r_1. \quad (36)$$

COUPLED CIRCUITS

For two indirectly coupled circuits, such as for two open wire antennas (Fig. 3), the field applied to circuit one now consists of that applied from the slice generator plus that supplied from the currents and charges in circuit two. In other words, the resultant field $\bar E_1$ becomes

$$\bar E_1=\bar E_{01}+\bar E_{12}+\bar E_{i1}, \quad (37)$$

with

$$\bar E_{12}=\frac{\eta_0}{jk}\diamondsuit_1\bar A_{12},\quad \bar A_{12}=\int_{V_2}\frac{\bar\imath_2 e(r_{21})}{4\pi}dV_2. \quad (38)$$

Similarly,

$$\bar E_2=\bar E_{02}+\bar E_{21}+\bar E_{i2}, \quad (39)$$

with

$$\bar E_{21}=\frac{\eta_0}{jk}\diamondsuit_2\bar A_{21},\quad \bar A_{21}=\int_{V_1}\frac{\bar\imath_1 e(r_{12})}{4\pi}dV_1. \quad (40)$$

Letting $\tilde W_1$ and $\tilde W_2$ represent the complex powers of circuits one and two, respectively, and using the additional subscripts three and four, respectively, for indicating the order of path selections within the individual circuits,

$$\tilde W_{13}=\tfrac{1}{2}Z_{i1}l_1f_{1m}^2|I_{01}|^2 +j\frac{15}{k}|I_{01}|^2\oint_1\oint_3 f_1(P_1)^*f_1(P_3) \times\diamondsuit_1[e(r_{31})d\bar r_3]\cdot d\bar r_1+j\frac{15}{k}(I_{01}^*I_{02}) \times\oint_1\oint_2 f_1(P_1)^*f_2(P_2)\diamondsuit_1[e(r_{21})d\bar r_2]\cdot d\bar r_1, \quad (41)$$

$$\tilde W_{24}=\tfrac{1}{2}Z_{i2}l_2f_{2m}^2|I_{02}|^2 +j\frac{15}{k}|I_{02}|^2\oint_2\oint_4 f_2(P_2)^*f_2(P_4) \times\diamondsuit_2[e(r_{42})d\bar r_4]\cdot d\bar r_2+j\frac{15}{k}(I_{01}I_{02}^*) \times\oint_2\oint_1 f_2(P_2)^*f_1(P_1)\diamondsuit_2[e(r_{12})d\bar r_1]\cdot d\bar r_2. \quad (42)$$

Multiplying (41) by $2/I_{01}^*$ and (42) by $2/I_{02}^*$, a pair of mesh equations is obtained as follows:

$$I_{01}\left[Z_{i1}l_1f_{1m}^2+j\frac{30}{k}\oint_1\oint_3 f_1(P_1)^*f_1(P_3) \times\diamondsuit_1[e(r_{31})d\bar r_3]\cdot d\bar r_1\right]+I_{02}\left[j\frac{30}{k}\oint_1\oint_2 f_1(P_1)^*f_2(P_2) \times\diamondsuit_1[e(r_{21})d\bar r_2]\cdot d\bar r_1\right]=V_{01} \quad (43)$$

$$I_{01}\left[j\frac{30}{k}\oint_1\oint_2 f_1(P_1)f_2(P_2)^*\diamondsuit_2[e(r_{12})d\bar r_1]\cdot d\bar r_2\right] +I_{02}\left[Z_{i2}l_2f_{2m}^2+j\frac{30}{k}\oint_2\oint_4 f_2(P_2)^*f_2(P_4) \times\diamondsuit_2[e(r_{42})d\bar r_4]\cdot d\bar r_1\right]=V_{02} \quad (44)$$

[1] J. G. Chaney, "On the generalized circuit theory as applied to antennas and radiating lines," U. S. Naval Postgraduate School, Research Paper No. 1 (March, 1951).

or

$$\begin{aligned} I_{01}Z_{13}+I_{02}Z_{12}' &= V_{01} \\ I_{01}Z_{21}'+I_{02}Z_{24} &= V_{02}. \end{aligned} \tag{45}$$

Again, inspection reveals that the self impedances are a function of the path selections and that the mutual impedances are asymmetric in the subscripts. But the reciprocity theorem[2] requires that

$$Z_{21}=Z_{12}.$$

In other words, the two current distribution functions are not mutually independent. Hence, the final forms of the equations again will be taken as the arithmetic means of the equations formed by the different orders of path selections, along with the interchanging of Z_{21}' and Z_{12}' in one set. Thus,

$$\begin{aligned} I_{01}\tfrac{1}{2}(Z_{13}+Z_{31})+I_{02}\tfrac{1}{2}(Z_{12}'+Z_{21}') &= V_{01}, \\ I_{01}\tfrac{1}{2}(Z_{21}'+Z_{12}')+I_{02}\tfrac{1}{2}(Z_{24}+Z_{42}) &= V_{02}, \end{aligned} \tag{46}$$

or

$$\begin{aligned} I_{01}Z_{11}+I_{02}Z_{12} &= V_{01}, \\ I_{01}Z_{21}+I_{02}Z_{22} &= V_{02}, \end{aligned} \tag{47}$$

with

$$Z_{11}=Z_{i1}l_1f_{1m}^2+j\frac{30}{k}\oint_1\oint_3 Re[f_1(P_1)^*f_1(P_3)] \times \Diamond_1[e(r_{31})d\bar{r}_3]\cdot d\bar{r}_1,$$

$$Z_{22}=Z_{i2}l_2f_{2m}^2+j\frac{30}{k}\oint_2\oint_4 Re[f_2(P_2)^*f_2(P_4)] \times \Diamond_2[e(r_{42})d\bar{r}_4]\cdot d\bar{r}_2,$$

$$Z_{12}=Z_{21}=j\frac{30}{k}\oint_1\oint_2 Re[f_1(P_1)^*f_2(P_2)] \times \Diamond_1[e(r_{21})d\bar{r}_2]\cdot d\bar{r}_1. \tag{48}$$

The circuit equations already formulated hold for any frequency provided the radius is small in comparison with the circuit length and provided $ka \ll 1$, and also, provided the circuit is within an isotropic medium and due cognizance is taken of the values of μ and ϵ, the latter having been taken above for free space. However, it is suggested that in cases where proximity effect should be considered, the actual path should be replaced by the centroidal current path. This would be equivalent to changing the radius of the wire, or in the mutual case, to slightly changing the spacing of the wires.

LUMPED ELEMENTS

If the current fails to vanish at the ends of a wire, Z_0 in (36) contains a transfer capacitative reactance, which may be illustrated by postulating

$$\begin{aligned} &f(P)=1, \quad e<P<d \\ &f(P)=0, \quad d<P<e \\ &|e^{-jkr_{12}}|\approx 1. \end{aligned} \tag{49}$$

[2] A. G. Clavier, Proc. Inst. Radio Engrs. 38, 1, 69 (1950).

Thus

$$j\frac{30}{k}\oint_e^d\oint_e^d \Diamond_1[e(r_{21})d\bar{r}_2]\cdot d\bar{r}_1 \approx -\frac{1}{j\omega\epsilon_0 4\pi}\oint_e^d\oint_e^d \nabla_1\left(\nabla_1\cdot\frac{d\bar{r}_2}{r_{21}}\right)\cdot d\bar{r}_1 + j\omega\mu_0\oint_e^d\oint_e^d\frac{d\bar{r}_2\cdot d\bar{r}_1}{4\pi r_{21}}. \tag{50}$$

Since

$$\nabla_1\cdot\frac{d\bar{r}_2}{r_{21}}=\left(\nabla_1\frac{1}{r_{21}}\right)\cdot d\bar{r}_2+\frac{1}{r_{21}}\nabla_1\cdot d\bar{r}_2=-\left(\nabla_2\frac{1}{r_{21}}\right)\cdot d\bar{r}_2,$$

and since Neumann's formula appears within the last term,

$$j\frac{30}{k}\oint_e^d\oint^d \Diamond_1\left(\frac{d\bar{r}_2}{r_{21}}\right)\cdot d\bar{r}_1$$

$$=\frac{1}{j\omega\epsilon_0 4\pi}\oint_e^d \nabla_1\left(\frac{1}{r_{d1}}-\frac{1}{r_{e1}}\right)\cdot d\bar{r}_1+j\omega L$$

$$=\frac{1}{j\omega\epsilon_0 4\pi}\left(\frac{1}{r_{dd}}-\frac{1}{r_{de}}-\frac{1}{r_{ed}}+\frac{1}{r_{ee}}\right)+j\omega L$$

$$=\frac{2}{j\omega}\left[\frac{1}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{r_{de}}\right)\right]+j\omega L \tag{51}$$

$$=\frac{2}{j\omega C}+j\omega L, \tag{52}$$

where

$$\frac{1}{C}=\frac{1}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{r_{de}}\right)$$

is the elastance of a spherical capacitor having an inner radius equivalent to the radius of the wire, and an outer radius equivalent to the distance between the ends of the wire. Two such capacitors appear in series since the integration passes twice around the circuit (Fig. 4).

If the circuit includes lumped elements, or concentrated R, L, and C, then R and L will appear merely by choosing the proper values of σ and μ over the resistor and inductor, respectively. However, due to the presence of the capacitor, the peak energy stored within the internal electric field of the circuit now exceeds that stored within the electric field contained within the wire itself. But the expressions for the complete power $\tilde{W}$ implies that all the energy of the electric field within the circuit itself must be within the wire. Hence, the reactive power of the capacitor must be subtracted from $\tilde{W}_0$. Thus, if V' now represents the volume within the capacitor, $\tilde{W}_0$ must contain the additional term $-j\omega(\epsilon/2)\int_{V'}|E|^2dV'$. For example, for a capacitor formed by two parallel plates of area S

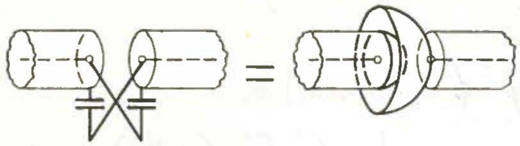

FIG. 4. Equivalent end capacitor.

and separated a distance s, inserted between terminals d and e,

$$-\frac{j\omega\epsilon}{2}\int_{V'}\bar{E}\cdot\bar{E}dV'=-\frac{j\omega\epsilon S|V_c|^2}{2s}=-\tfrac{1}{2}j\omega C|V_c|^2$$

$$=\frac{1}{2}\frac{(j\omega CV_c)(-j\omega CV_c^*)}{j\omega C}=\frac{1}{2}\frac{[I_0f(d)][I_0^*f(e)^*]}{j\omega C}$$

$$=\frac{|I_0|^2}{2j\omega C}|f_c|^2, \quad (53)$$

in which

$$C=S\epsilon/s, \quad f_c=f(d)=f(e).$$

Since the current has been postulated continuous through both the generator and the capacitor, if the axial dimensions of the resistor and the capacitor are excluded from l, the input impedance Z_0 becomes

$$Z_0=R|f_r|^2+\frac{|f_c|^2}{j\omega C}+lZ_if_m^2$$

$$+j\frac{30}{k}\oint_1\oint_2 Re[f(P_1)^*f(P_2)]\Diamond_1[e(r_{21})d\bar{r}_2]\cdot d\bar{r}_1, \quad (54)$$

with $j\omega L$ for the inductor given by the integration of the last term over that portion of the circuit forming the inductor. Thus, for a circuit having $f(P)=1$ and $|kr_{21}|\ll 1$, Z_0 reduces to

$$Z_0=R+j\left(\omega L-\frac{1}{\omega C}\right)+lZ_i, \quad (55)$$

with

$$L=\mu\oint_1\oint_2\frac{d\bar{r}_1\cdot d\bar{r}_2}{4\pi r_{12}}.$$

It becomes apparent that antenna circuits may be thought of in the same manner as low frequency lumped network circuits. Of course, in practice, the impedance formulas will require an *a priori* assumption of current distributions, which is sufficient for many engineering purposes, but not too accurate for certain broad band types of antennas. This method of finding the driving point impedance of an antenna array when postulating sinusoidal currents in open wire antennas is commonly referred to as the Carter circuit method.[3]

[3] P. S. Carter, Proc. Inst. Radio Engrs. 20, 1004 (1932).

OTHER CONVENTIONAL METHODS

It is quite interesting to note how some of the more conventional methods for finding antenna impedances and radiation resistances appear from the original formulation of the complex power expressions.

Suppose the antenna is well constructed so that the internal impedance is negligible with respect to the radiation or intrinsic impedance. Then, from (15),

$$\frac{1}{2}\int_V\bar{E}_0\cdot\bar{\imath}^*dV=-\frac{1}{2}\int_V\bar{E}_i\cdot\bar{\imath}^*dV, \quad (56)$$

or from (13) and (14), this is frequently written

$$Z_0=-\frac{1}{|I_0|^2}\int\bar{E}_i\cdot\bar{\imath}^*dV$$

$$=-\frac{1}{|I_0|^2}\oint\bar{E}_i\cdot\bar{I}^*dl. \quad (57)$$

Upon substituting from (10), this becomes the conventional method of taking the self impedance as the mutual impedance between the axis and an element on the surface, or if the paths are both chosen along the axis, it becomes the induced emf method for finding the radiation resistance R_r as

$$R_r=Re\left[\frac{1}{|I_0|^2}\oint\oint\frac{j\eta_0}{k}\Diamond\bar{A}\cdot\bar{I}^*dl^2\right]. \quad (58)$$

Also, since within the wire,

$$\nabla\times\bar{H}^*\approx\bar{\imath}^*, \quad (59)$$

$$R_r=-Re\left[\frac{1}{|I_0|^2}\int_V\bar{E}_i\cdot\nabla\times\bar{H}^*dV\right]$$

$$=\frac{1}{|I_0|^2}Re\left[\int_V\nabla\cdot(\bar{E}\times\bar{H}^*)dV\right]$$

$$=\frac{1}{|I_0|^2}Re\left[\int_S\bar{E}_i\times\bar{H}^*\cdot d\bar{S}\right], \quad (60)$$

which is the Poynting vector method for finding the radiation resistance.

Another interesting concept following from $Z_i\approx 0$ and the resulting equation,

$$\bar{E}=\bar{E}_0+\bar{E}_i=0,$$

is that the Poynting vector integration over the surface of the wire vanishes; that is, the power flow into the wire balances the power flow outward through the surface of the wire. But from (8), (9), and (56),

$$\int_{S'}\bar{P}_0\cdot d\bar{S}'=\int_S\bar{P}_i\cdot d\bar{S}, \quad (61)$$

with S' being the surface enclosing the gap and with S being the surface enclosing the wire. Hence, it is frequently concluded that the radiation from an antenna actually occurs at the gap.

Also, there is no conflict between the integral equation method for finding the driving point impedance of an antenna array and the circuit concept considered herein. For, consider the complex power density as given in (27) with the last term again taken as in (57). In other words, now let $\bar{W}_{01}$ be the total power density per unit length with $\int_{S'}\bar{E}_i\cdot\bar{\iota}^*dS'$ replaced by $\bar{E}_i\cdot\bar{I}^*$, that is,

$$\bar{W}_{01}=\tfrac{1}{2}Z_i|I|^2-\tfrac{1}{2}\bar{E}_i\cdot\bar{I}^*. \quad (62)$$

Integration along the wire with the current constant through the infinitely thin slice yields

$$\bar{W}_{01}=\frac{1}{2}\oint_e^b (Z_i\bar{I}-\bar{E}_i)\cdot\bar{I}^*dl,$$

or

$$-\frac{1}{2}\oint_c^b \bar{E}_0I_0^*\cdot d\bar{r}=\frac{1}{2}\oint_e^d (Z_i\bar{I}-\bar{E}_i)\cdot\bar{I}^*dl. \quad (63)$$

Then, since the right member includes the slice, the expression $Z_i\bar{I}-\bar{E}_i$ may be assumed to vanish along the wire with a discontinuity in the scalar potential of $\phi_b-\phi_c$ existing at the gap. This, for a straight cylindrical antenna, yields the following differential equation in the vector potential $\bar{A}$,

$$(\partial^2A_z/\partial z^2)+k^2A_z=j\omega\epsilon_0Z_iI_z, \quad (64)$$

which is the equation usually obtained in the integral equation method by matching the external field to the internal field at the surface of the wire.[4]

By the integral equation method for finding the driving point impedance, an *a priori* current distribution is not postulated, and the mutual impedance of two antennas otherwise unaltered varies within the presence of other antennas. This variation in the mutual impedance also theoretically occurs in the Carter circuit method, for the current distribution functions are not assumed independent and the mutual impedances are subjected to the reciprocity theorem. However, in numerical applications using the circuit method, the current distribution is postulated *a priori*, which causes the mutual impedances to remain unaltered within the presence of other antennas. This approximation usually is sufficiently accurate for multiple arrays to more than offset the increased difficulty and approximations encountered in the integral equation method of determining the solution.

However, the theoretical circuit equations developed herein hold true even if the exact current distributions are first determined by some method such as the integral equation method, thus relating the integral equation concept to the more conventional circuit concept.

[4] R. King and C. W. Harrison, Jr., Proc. Inst. Radio Engrs. 31, 10, 548 (1943).

APPENDIX

Let $\bar{r}_{21}$ be the radius vector from a point P_2 to another distinct point P_1 within a three-dimensional space, and let $\psi(r_{21})$ be a function twice differentiable, then Chaney's identity may be stated,[1]

$$\bar{a}_1\cdot\nabla_1[\nabla_1\cdot\psi(r_{21})\bar{a}_2]=\bar{a}_2\cdot\nabla_2[\nabla_2\cdot\psi(r_{12})\bar{a}_1], \quad (65)$$

where $\bar{a}_1$ and $\bar{a}_2$ are any two unit vectors at points P_1 and P_2, respectively, where the subscripts on the vector operators indicate the points at which the differentiations occur, and where

$$\bar{r}_{12}=-\bar{r}_{21},\quad r_{21}=|\bar{r}_{21}|.$$

From (65), it immediately follows that

$$\frac{\eta_0}{j4\pi k}I_{02}f_2(P_2)\diamond_1[e(r_{21})d\bar{r}_2]\cdot d\bar{r}_1 = \frac{\eta_0}{j4\pi k}I_{02}f_2(P_2)\diamond_2[e(r_{12})d\bar{r}_1]\cdot d\bar{r}_2, \quad (66)$$

where $I_{02}f_2(P_2)$ is the current at P_2, where

$$e(r_{12})=e^{-jkr_{21}}/r_{21},$$

and where

$$\diamond=\nabla[\nabla\cdot+k^2],\quad k=2\pi/\lambda.$$

The left member of (66) gives the emf acting within a circuit element at P_1 due to a current moment at P_2, and the right member gives the emf acting within a circuit element at P_2 due to the removal of the current moment from P_2 to P_1. Hence, Eq. (66) is an analytic statement of the reciprocity theorem.

Also, from (66),

$$\oint_2 \frac{30}{jk}I_{02}f_2(P_2)\diamond_1[e(r_{21})d\bar{r}_2]\cdot d\bar{r}_1 = \oint_2 \frac{30}{jk}I_{02}f_2(P_2)\diamond_2[e(r_{12})d\bar{r}_1]\cdot d\bar{r}_2. \quad (67)$$

The left member of (67) gives the resultant emf at P_1 due to all the current in circuit *two* and the right member gives the total emf that would act around circuit *two* if each current moment of *two* were at P_1.

From (67),

$$j\frac{15I_{01}^*I_{02}}{k}\oint_1\oint_2 f_1(P_1)^*f_2(P_2)\diamond_1[e(r_{21})d\bar{r}_2]\cdot d\bar{r}_1 = j\frac{15I_{01}^*I_{02}}{k}\oint_1\oint_2 f_1(P_1)^*f_2(P_2)\diamond_2[e(r_{21})d\bar{r}_1]\cdot d\bar{r}_2. \quad (68)$$

Now, the left member of (68) gives the additional complex power which must be supplied by the generator of circuit *one* because of the presence of circuit *two*, provided the current in one is maintained the same as it was before circuit *two* was moved into place. However, this is physically impossible because the current distribution in *one* is actually altered by the presence of circuit *two*. The right member of (68) gives the complex power resulting from considering each current element of *two* removed to each point of *one* and finding the emf around *two* corresponding to each position of *one*, and then assuming this emf is acting at the corresponding point in *one*. In short, the right member transforms circuit *two* into circuit *one*.

A reversal of the order of the above integration yields,

$$j\frac{15I_{01}I_{02}^*}{k}\oint_1\oint_2 f_1(P_1)f_2(P_2)^*\diamond_2[e(r_{12})d\bar{r}_1]\cdot d\bar{r}_2 = j\frac{15I_{01}I_{02}^*}{k}\oint_1\oint_2 f_1(P_1)f_2(P_2)^*\diamond_1[e(r_{12})d\bar{r}_2]\cdot d\bar{r}_1. \quad (69)$$

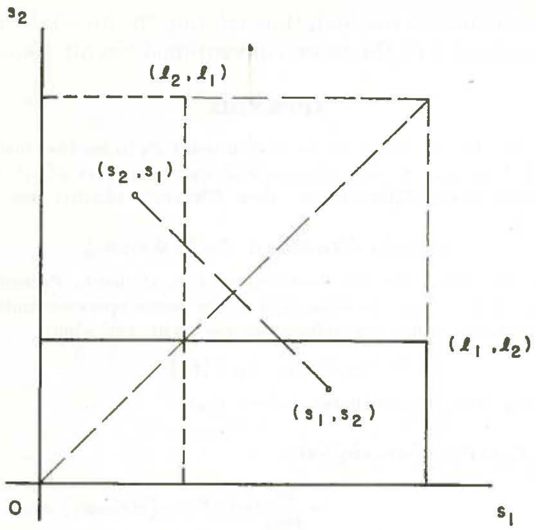

FIG. 5. Curvilinear rectangles.

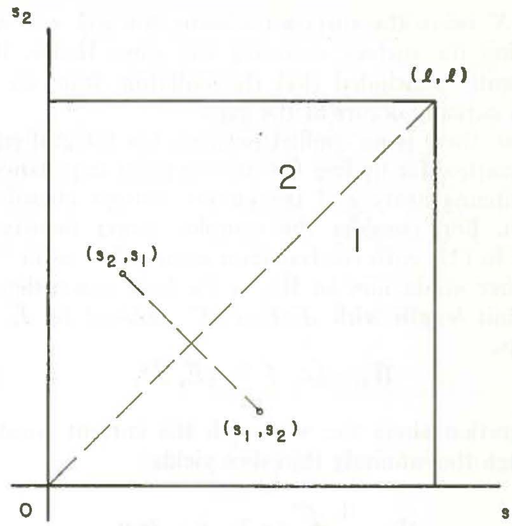

FIG. 6. Curvilinear square.

Thus, from (68) and (69), each generator sees its own circuit plus the other circuit transformed into its own circuit. Indeed, it might be stated that it is this transformation which causes the current distributions to alter in such a manner as to bring about an equality of the two mutual impedances.

Now consider the applied voltage to exist only in circuit *two* at position P_{02}. The voltage at a point P_{01} is given by the negative left member of (67). If the circuit element at P_{01} is replaced by the applied voltage V_2, as far as the circuits are concerned, this is equivalent to interchanging the emf in the elements of P_{01} and P_{02}. Hence, from (66), the current at P_{02} becomes the same as the current that formerly existed at P_{01}. Thus, one obtains the system of equations,

$$\begin{aligned} &I_{01}Z_{11}+I_{02}Z_{12} &&=0\\ &I_{01}Z_{21}+I_{02}Z_{22} &&-V_2=0\\ &I_{01}Z_{12} &&+I_{01}'Z_{11}-V_2=0\\ &I_{01}Z_{22} &&+I_{01}'Z_{21}=0 \end{aligned} \tag{70}$$

for which a necessary and sufficient condition for their having a solution is that[2] $Z_{12}=Z_{21}$.

The equality of Z_{12} and Z_{21} requires that

$$\oint_1\oint_2 f_1(P_1)^*f_2(P_2)\diamond_1[e(r_{21})d\bar{r}_2]\cdot d\bar{r}_1 = \oint_1\oint_2 f_1(P_1)f_2(P_2)^*\diamond_2[e(r_{12})d\bar{r}_1]\cdot d\bar{r}_2. \tag{71}$$

In terms of arc lengths, by letting

$$d\bar{r}_1=d\bar{s}_1,\quad d\bar{r}_2=d\bar{s}_2,\quad \bar{a}_1\cdot\bar{a}_2=\cos[\theta(s_1,s_2)],$$
$$e(r_{12})=g(s_1,s_2)=g(s_2,s_1)=e(r_{21}),$$

Eq. (71) may be written as

$$\int_0^{l_1}\int_0^{l_2} f_1(s_1)^*f_2(s_2)\left[\frac{-\partial^2}{\partial s_1\partial s_2}+k^2\cos\theta\right]g(s_2,s_1)ds_1ds_2,$$
$$=\int_0^{l_2}\int_0^{l_1} f_1(s_1)f_2(s_2)^*\left[\frac{-\partial^2}{\partial s_2\partial s_1}+k^2\cos\theta\right]g(s_1,s_2)ds_1ds_2, \tag{72}$$

which may be interpreted as integrations over curvilinear rectangles (Fig. 5). Since

$$f_1(s_1)f_2(s_2)^*=[f_1(s_1)^*f_2(s_2)]^*$$

and

$$\left[\frac{-\partial^2}{\partial s_1\partial s_2}+k^2\cos\theta\right]g(s_2,s_1)=\left[\frac{-\partial^2}{\partial s_2\partial s_1}+k^2\cos\theta\right]g(s_2,s_1),$$

it follows that the current distribution functions must be such that

$$\int_0^{l_1}\int_0^{l_2} f_1(s_1)^*f_2(s_2)\left[\frac{-\partial^2}{\partial s_1\partial s_2}+k^2\cos\theta\right]g(s_2,s_1)ds_2ds_1$$
$$=\int_0^{l_1}\int_0^{l_2} Re[f_1(s_1)^*f_2(s_2)]\left[\frac{-\partial^2}{\partial s_1\partial s_2}+k^2\cos\theta\right]g(s_1,s_2)ds_1ds_2. \tag{73}$$

The retention of only the real part of the product current functions in (36) may similarly be justified. Here, the functional forms are the same. Letting

$$f(s_1)^*f(s_2)=u(s_1,s_2)+jv(s_1,s_2),$$

the integral of (31) may be written

$$A=\oint_e^d\oint_e^d f(P_1)^*f(P_2)\diamond_1[e(r_{21})d\bar{r}_2]\cdot d\bar{r}_1$$
$$=\int_0^l\int_0^l f(s_1)^*f(s_2)\left[\frac{-\partial^2}{\partial s_1\partial s_2}+k^2\cos\theta\right]g(s_1,s_2)ds_2ds_1$$

or

$$A=\left(\int_{S_1}+\int_{S_2}\right)[u(s_1,s_2)+jv(s_1,s_2)]$$
$$\times\left[\frac{-\partial^2}{\partial s_1\partial s_2}+k^2\cos\theta\right]g(s_1,s_2)ds_2ds_1, \tag{74}$$

where S_1 is the surface below the diagonal of the curvilinear square and S_2 is the surface above the diagonal (Fig. 6).

Integrating simultaneously over positions symmetric with respect to the diagonal and considering

$$f(s_2)^*f(s_1)=u(s_1,s_2)-jv(s_1,s_2),$$

$$A=2\int_{S_1}u(s_1,s_2)\left[\frac{-\partial^2}{\partial s_1\partial s_2}+k^2\cos\theta\right]g(s_1,s_2)ds_2ds_1$$
$$=\oint_e^d\oint_e^d Re[f(P_1)^*f(P_2)]\diamond_1[e(r_{21})d\bar{r}_2]\cdot d\bar{r}_1. \tag{75}$$