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Design of Some Active Compensators of Feedback Controls

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Table I

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
x(t)	1	0.91	0.82	0.75	0.685	0.62	0.563	0.512	0.465	0.42	0.37
β(t)	1	0.98	0.915	0.845	0.772	0.674	0.614	0.569	0.52	0.478	0.437

Table II

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
x(t)	1	0.99	0.96	0.91	0.86	0.8	0.73	0.67	0.61	0.55	0.5
α(t)	0	0.198	0.384	0.546	0.688	0.8	0.87	0.93	0.97	0.99	1
g(t)	2	1.98	1.85	1.65	1.48	1.28	1.06	0.9	0.75	0.6	0.5
β(t)	1	2.9	3.9	4.5	5	5.6	5.9	6	6	6.1	6.18

and

$$\phi^*(t)=x_0(t)-\int_o^T F[t,u,\phi^*(u)]du \qquad (74)$$

Subtracting equations 73 and 74 gives

$$[\phi^*(t)-\phi(t)]^2=\left[\int_o^T\{F[t,u,\phi(t)]-F[t,u,\phi^*(u)]\}du\right]^2 \qquad (75)$$

Therefore,

$$[\phi^*(t)-\phi(t)]^2\leqslant\int_o^T K^2(t,u)du \qquad (76)$$

$$\int_o^T[\phi^*(u)-\phi(u)]^2du$$

Let

$$\int_o^T[\phi^*(t)-\phi(t)]^2dt=S^2 \qquad (77)$$

Therefore,

$$[\phi^*(t)-\phi(t)]\leqslant\alpha^2(t)S^2 \qquad (78)$$

Substituting equation 78 in equation 76 and repeating n times, gives

$$\int_o^T[\phi^*(t)-\phi(t)]^2dt\leqslant S^2\frac{A^{2n}}{n!} \qquad (79)$$

Letting $n\rightarrow\infty$ gives

$$\int_o^T[\phi^*(i)-\phi(i)]^2dt=0 \qquad (80)$$

Hence the solution $\phi(t)$ or $x(t)$ is unique.

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Design of Some Active Compensators of Feedback Controls

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Summary: Certain networks capable of producing transfer functions with complex zeros (or complex poles) are presented. The equations for the s-plane loci of the complex singularities are developed and are equations of circles. Working equations are tabulated as an aid to design. Using these in conjunction with Carpenter's graphical construction, the network can be designed so that the roots of the compensated system lie at a specified location for a specified gain.

IN THE USUAL textbook on feedback control systems, and in most of the published literature, the problem of compensation network design is treated assuming that the network is passive, has real zeros and poles, and is to be cascaded in the main transmission channel. There are some noteworthy exceptions to the preceding statement; some authors treat compensation quite generally,[1,2] or treat specific networks that generate complex zeros and poles,[2] and a few papers[3,4,5] have presented some original concepts which are helpful in the solution of specific problems.

While passive compensators will always be useful, tacit restriction of compensators to passive devices should be discontinued, since the state of the art in the design of operational amplifiers makes active compensators quite practical. In like manner, restriction of the compensator device to have real zeros and real poles is no longer tenable; the problems of tolerances and adjustments which often made passive resonant circuits, bridged T, parallel T, and lattice networks impractical solutions are not critical considerations when an active network is used.

In this paper an attempt is made to point out various situations in which it is desirable (or even mandatory) to use a compensator with complex zeros. Active networks which are capable of generating complex zeros (or poles) are analyzed. It is shown that the loci of the singularities on the s-plane are circles, and the equations of these circles in terms of parameter values are tabulated. Procedures for the design of such compensators are developed and applied to specific cases.

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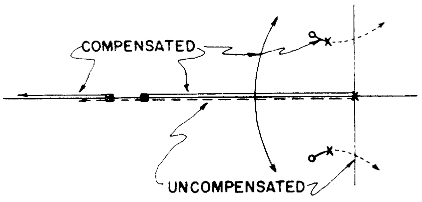

Fig. 1. Root locus system having a third-order actuator with complex poles

$G(s) = K/s(s^2 2w_n s + 1)$
X = poles, uncompensated
■ = poles of compensator
0 = zeros of compensator

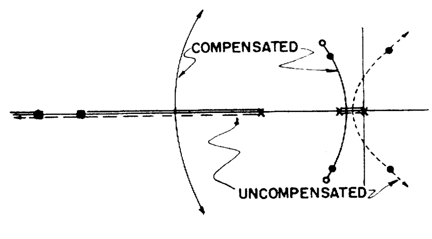

Fig. 2. Root locus for high-gain operation
● = root of closed-loop system

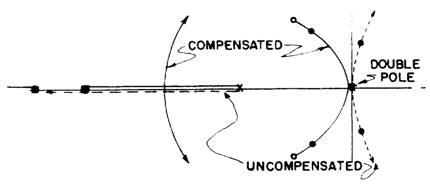

Fig. 3. Compensation of a type-2 system

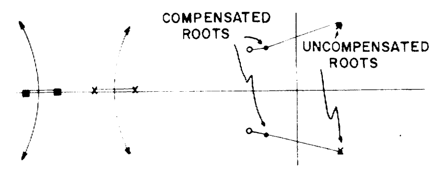

Fig. 4. Root locus for a problem in feedback compensation

Situations for Complex Zero Compensators

In the practical design of high-powered position-control loops (and speed-control loops) it is usually found that the loaded actuator unit is at least a third-order device with inherent complex poles. Typical examples are d-c shunt motors and hydraulic actuators. The root locus for such an actuator used in a positioning device is shown in Fig. 1. The loci from the complex poles enter the right-half plane at a low value of gain so accuracy and stability are incompatible. A cascaded compensator with complex zeros can effectively eliminate this problem, and the zeros need not be used to cancel the poles. Use of compensators with real zeros frequently will not even alleviate the problem and if it does the design normally requires several additional stages of amplification in either case.

A second practical situation arises when a high-order system (with all open-loop poles real) must be operated at high gain, yet specifications require that the dominant roots be complex, well damped, and of reasonably high frequency. An uncompensated root locus for this second case is shown in Fig. 2. If real zeros are used to reshape the locus it is often difficult to avoid a small real root which either dominates, or puts a long tail on the step response, or adversely affects the bandwidth. However complex zeros placed near the desired root location (as shown in Fig. 2) usually provide a satisfactory solution.

A third practical situation arises when the designer introduces an additional pure integration to produce a type-2 system. This technique has the advantage of insuring steady-state accuracy, but obviously makes the compensation problems more difficult. The root locus for this situation is shown in Fig. 3. The compensation problems are essentially the same as in Fig. 2 and require no additional examination.

A fourth situation can arise when feedback compensation is considered. A convenient block diagram[6] manipulation reduces the calculations to a problem in cascade compensation, the uncompensated linear system being represented by a single equivalent block. The poles of the transfer function of this block are the roots of the uncompensated system characteristic equation. Thus the problem might be described by a typical root locus as in Fig. 4. Usually the uncompensated system is unstable, thus yielding complex poles in the right-half plane. The compensator zeros must be chosen to force a root-locus segment from these poles into the left-half plane, and if possible through a desired root location. In general, the availability of complex zeros makes the compensation design much easier and the resulting performance more satisfactory.

Characteristics of Some Active Networks

Complex zeros and complex poles can be generated very simply using elementary lead and lag networks in conjunction with amplifiers. Two simple schemes are indicated in Fig. 5. The availability of complex singularities is readily seen from the algebra. For the parallel feed-forward case of Fig. 5(A)

$$\frac{V_o}{V_i} = \frac{A_1(s+Z_1)}{s+p_1} \mp \frac{A_2(s+Z_2)}{s+p_2} = \frac{A_1(s+Z_1)(s+p_2) \mp A_2(s+Z_2)(s+p_1)}{(s+p_1)(s+p_2)} \quad (1)$$

From equation 1 it is seen that the numerator is the difference between two quadratics, which is also a quadratic, and with proper choice of constants this quadratic can have complex roots. Thus, the transfer function of the active network can be made to have complex zeros.

For the feedback configuration of Fig. 5(B)

$$\frac{V_o}{V_i} = \frac{A_1(s+Z_1)/(s+p_1)}{1 \mp A_1 A_2 \dfrac{(s+Z_1)(s+Z_2)}{(s+p_1)(s+p_2)}}$$

$$\frac{V_o}{V_i} = \frac{A_1(s+Z_1)(s+p_2)}{(s+p_1)(s+p_2) \mp A_1 A_2 (s+Z_1)(s+Z_2)} \quad (2)$$

In equation 2 the denominator shows that complex poles can be made available by proper choice of parameter values.

The roots of the numerator of equation 1, and also of the denominator of equation 2, can be controlled with the gains A_1 and A_2. The effects of gain variations on the root locations can be shown on the root-locus plot by rewriting the functions as

$$\frac{A_1(s+Z_1)(s+p_2)}{A_2(s+Z_2)(s+p_1)} = \mp 1 \quad (3)$$

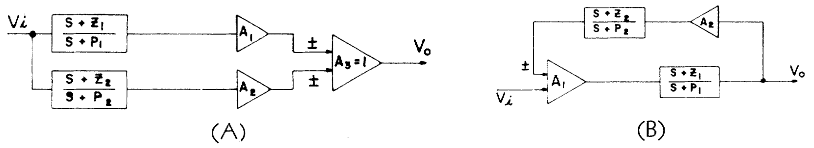

Fig. 5. Block diagrams of active compensators

A—Parallel feed-forward and sum
B—Positive or negative feedback loop

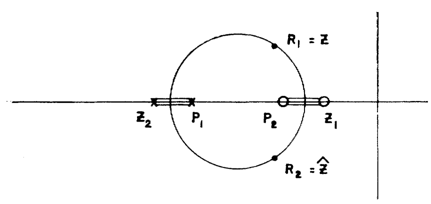

Fig. 6. Root locus for equation 3

Table I. Circle Equations for the Zeros of Active Networks

Compensator	Equation of the Root Locus of Complex Zeros
Lead-lead feed-forward difference compensator	$\left[x+\frac{p_{d2}Z_{d1}-p_{d1}Z_{d2}}{Z_{d2}-Z_{d1}+p_{d1}-p_{d2}}\right]^2+y^2=\frac{p_{d1}p_{d2}(Z_{d1}-Z_{d2})+Z_{d1}Z_{d2}(p_{d2}-p_{d1})}{Z_{d2}-Z_{d1}+p_{d1}-p_{d2}}+\left[\frac{p_{d2}Z_{d1}-p_{d1}Z_{d2}}{Z_{d2}-Z_{d1}+p_{d1}-p_{d2}}\right]^2$
Lag-lag feed-forward difference compensator	$\left[x+\frac{p_{g2}Z_{g1}-p_{g1}Z_{g2}}{Z_{g2}-Z_{g1}+p_{g1}-p_{g2}}\right]^2+y^2=\frac{p_{g1}p_{g2}(Z_{g1}-Z_{g2})+Z_{g1}Z_{g2}(p_{g2}-p_{g1})}{Z_{g2}-Z_{g1}+p_{g1}-p_{g2}}+\left[\frac{p_{g2}Z_{g1}-p_{g1}Z_{g2}}{Z_{g2}-Z_{g1}+p_{g1}-p_{g2}}\right]^2$
Lag-lead feed-forward summing compensator	$\left[x+\frac{p_dZ_g-p_gZ_d}{-Z_g+Z_d+p_g-p_d}\right]^2+y^2=\frac{p_dp_g(Z_g-Z_d)+Z_dZ_g(p_d-p_g)}{Z_d-Z_g+p_g-p_d}+\left[\frac{p_dZ_g-p_gZ_d}{Z_d-Z_d+p_g-p_d}\right]^2$

Note: The subscript d designates a lead network, g designates a lag network.

Table II. Circle Equations for the Poles of Active Networks

Compensator	Equation of the Root Locus of Complex Poles
Lead-lead negative-feedback compensator	$\left[x+\frac{p_{d1}p_{d2}-Z_{d1}Z_{d2}}{Z_{d1}+Z_{d2}-p_{d1}-p_{d2}}\right]^2+y^2=\frac{p_{d1}p_{d2}(Z_{d1}+Z_{d2})-Z_{d1}Z_{d2}(p_{d1}+p_{d2})}{Z_{d1}+Z_{d2}-p_{d1}-p_{d2}}+\left[\frac{p_{d1}p_{d2}-Z_{d1}Z_{d2}}{Z_{d1}+Z_{d2}-p_{d1}-p_{d2}}\right]^2$
Lag-lag negative-feedback compensator	$\left[x+\frac{p_{g1}p_{g2}-Z_{g1}Z_{g2}}{Z_{g1}+Z_{g2}-p_{g1}-p_{g2}}\right]^2+y^2=\frac{p_{g1}p_{g2}(Z_{g1}+Z_{g2})-Z_{g1}Z_{g2}(p_{g1}+p_{g2})}{Z_{g1}+Z_{g2}-p_{g1}-p_{g2}}+\left[\frac{p_{g1}p_{g2}-Z_{g1}Z_{g2}}{Z_{g1}+Z_{g2}-p_{g1}-p_{g2}}\right]^2$
Lag-lead positive-feedback compensator	$\left[x+\frac{p_dp_g-Z_dZ_g}{Z_d+Z_g-p_d-p_g}\right]^2+y^2=\frac{p_gp_d(Z_d+Z_g)-Z_dZ_g(p_d+p_g)}{Z_d+Z_g-p_d-p_g}+\left[\frac{p_dp_g-Z_dZ_g}{Z_d+Z_g-p_d-p_g}\right]^2$

Table III. Complex Zero Relationships

Compensator	Working Equation	Numerator Root-Locus Equation	Real Poles	K_c
Lead-lead feed-forward difference compensator	$(Z_{d2}-Z_{d1}+p_{d1}-p_{d2})(x^2+y^2)+2x(p_{d2}Z_{d1}-p_{d1}Z_{d2})=p_{d1}p_{d2}(Z_{d1}-Z_{d2})+Z_{d1}Z_{d2}(p_{d2}-p_{d1})$	$\frac{A_1(s+Z_{d1})(s+p_{d2})}{A_2(s+Z_{d2})(s+p_{d1})}=1$	p_{d1}, p_{d2}	(A_1-A_2)
Lag-lag feed-forward difference compensator	$(Z_{g2}-Z_{g1}+p_{g1}-p_{g2})(x^2+y^2)+2x(p_{g2}Z_{g1}-p_{g1}Z_{g2})=p_{g1}p_{g2}(Z_{g1}-Z_{g2})+Z_{g1}Z_{g2}(P_{g2}-p_{g1})$	$\frac{A_1\alpha_1(s+Z_{g1})(s+p_{g2})}{A_2\alpha_2(s+Z_{g2})(s+p_{g1})}=1$	p_{g1}, p_{g2}	$(A_1\alpha_1-A_2\alpha_2)$
Lag-lead feed-forward summing compensator	$(Z_d-Z_g+p_g-p_d)(x^2+y^2)+2x(p_dZ_g-p_gZ_d)=Z_dZ_g(p_d-p_g)+p_dp_g(Z_g-Z_d)$	$\frac{A_1\alpha(s+Z_g)(s+p_d)}{A_2(s+Z_d)(s+p_g)}=-1$	p_d, p_g	$(A_1\alpha+A_2)$

Table IV. Complex Pole Relationships

Compensator	Working Equation	Denominator Root-Locus Equation	Real Zeros	K_c
Lead-lead negative-feedback compensator	$(Z_{d1}+Z_{d2}-p_{d1}-p_{d2})(x^2+y^2)+2x(p_{d1}p_{d2}-Z_{d1}Z_{d2})=p_{d1}p_{d2}(Z_{d1}+Z_{d2})-Z_{d1}Z_{d2}(p_{d1}+p_{d2})$	$\frac{A_1(s+Z_{d1})(s+Z_{d2})}{(s+p_{d1})(s+p_{d2})}=-1$	Z_{d1}, p_{d2}	$\frac{1}{1+A_1}$
Lag-lag negative-feedback compensator	$(Z_{g1}+Z_{g2}-p_{g1}-p_{g2})(x^2+y^2)+2x(p_{g1}p_{g2}-Z_{g1}Z_{g2})=p_{g1}p_{g2}(Z_{g1}+Z_{g2})-Z_{g1}Z_{g2}(p_{g1}+p_{g2})$	$\frac{A_1\alpha_1\alpha_2(s+Z_{g1})(s+Z_{g2})}{(s+p_{g1})(s+p_{g2})}=-1$	Z_{g1}, p_{g2}	$\frac{1}{1+A_1\alpha_1\alpha_2}$
Lag-lead positive-feedback compensator, lag in feedback path	$(Z_g+Z_d-p_g-p_d)(x^2+y^2)+2x(p_gp_d-Z_gZ_d)=p_gp_d(Z_d+Z_g)-Z_gZ_d(p_g+p_d)$	$\frac{A_1\alpha(s+Z_g)(s+Z_d)}{(s+p_g)(s+p_d)}=1$	Z_g, p_d	$\frac{1}{1-A_1\alpha}$
Lag-lead positive-feedback compensator, lead in feedback path	$(Z_g+Z_d-p_g-p_d)(x^2+y^2)+2x(p_gp_d-Z_gZ_d)=p_gp_d(Z_g+Z_d)-Z_gZ_d(p_d+p_g)$	$\frac{A_1\alpha(s+Z_g)(s+Z_d)}{(s+p_g)(s+p_d)}=1$	p_g, Z_d	$\frac{1}{1-A_1\alpha}$

$$\frac{A_1A_2(s+Z_1)(s+Z_2)}{(s+p_1)(s+p_2)}=\mp 1 \quad (4)$$

Equations 3 and 4 show that either zero or π loci may be used, depending on the circuit connections; also the relative magnitudes of the poles and zeros is important. (Note the Z_1 and p_2 are the *zeros* of equation 3, while Z_2 and p_1 are the *poles*.) Fig. 6 shows a root-locus plot for possible values in equation 3. The roots thus determined are *complex zeros* for equation 1. The poles of equation 1 would be p_1 and p_2. The locus of the complex zeros is precisely a circle. This is derived for a specific case in Appendix I. A tabular listing is also given for some circuit combinations which produce complex poles and zeros, together with the equations for the resulting circular loci. It is important to note that only certain filter combinations produce complex singularities; the majority of the active networks produce only real poles and zeros.

From the tables of the circular loci it is seen that all of the equations for the zero loci are of the form:

$$\left[x+\frac{Z_1p_2-Z_2p_1}{Z_1-Z_2+p_2-p_1}\right]^2+y^2=\frac{Z_1Z_2(p_1-p_2)+p_1p_2(Z_2-Z_1)}{Z_1-Z_2+p_2-p_1}+\left[\frac{Z_1p_2-Z_2p_1}{Z_1-Z_2+p_2-p_1}\right]^2 \quad (5)$$

and the equations of the pole loci are of the form:

$$\left[x+\frac{p_1p_2-Z_1Z_2}{Z_1+Z_2-p_1-p_2}\right]^2+y^2=\frac{p_1p_2(Z_1+Z_2)-Z_1Z_2(p_1+p_2)}{Z_1+Z_2-p_1-p_2}+\left[\frac{p_1p_2-Z_1Z_2}{Z_1+Z_2-p_1-p_2}\right]^2 \quad (6)$$

Some further properties of these circular loci may be noted by inspection of equations 5 and 6. The denominators of the terms which define the centers of the circles may approach zero more or less independently of the sign or magnitude of the numerator of those terms. Therefore, the center of the circle may vary from negative to positive infinity. Similarly, denominators of the terms which define the squared radii of the circles may approach zero more or less independently of the sign or the magnitude of the numerator. Thus, it is possible to have radii which vary from zero to infinity. In the cases for which the squared radius is negative, of course, the circle is undefined. The denominators of the center and radii terms are the same, though, so as the center of the circle goes to infinity, the radius must also go to infinity. However, the center does not have to go to infinity at the same rate as the radius since the numerators of the terms are different. In fact, it is possible to have the center remain at the origin while the radius goes to infinity. The result of this is that, in general, it is possible to locate complex poles or zeros anywhere in the left- or right-half of the s-plane, providing the correct compensator is chosen. Some of the circles are restricted to the left-half s-plane by the nature of their root loci.

It is possible to use equation 5 or 6 in a trial and error fashion. By inspection select desired zero (pole) locations thus specifying x and y. Next choose arbitrarily (or from experience) an active circuit that seems suitable. Select two of the filter parameters (usually those which will be real poles or zeros in the composite transfer function). The two remaining parameters are determined by trial and error so that the equation is satisfied and the compensator is physically realizable. Such a technique seldom leads to optimum results, but it is usable. A significant simplification can be made as follows: For equation 5, let it be required that

$$Z_1p_2 = Z_2p_1 \triangleq K_1 \quad (7)$$

Then, equation 5 reduces to

$$x^2+y^2=K_1 \quad (8)$$

Application of equation 7 restricts the active compensator to two lag networks or two lead networks. In like manner, for equation 6 let it be required that

$$Z_1Z_2 = p_1p_2 \triangleq K_2 \quad (9)$$

Then equation 6 reduces to

$$x^2+y^2=K_2 \quad (10)$$

and the active compensator is restricted to use one lead and one lag network.

Designing the Compensator for a Prespecified Root Location

Relocation of the root loci of the characteristic equation so that they pass through a specified point is not in itself any great problem. Simultaneously meeting the requirements of dominance, steady-state accuracy, and transient response is, however, quite another matter. A considerable number of techniques exist for the simultaneous solution of the steady-state accuracy and root location problems.[7,8] These techniques apply only to compensation with negative real poles and zeros and do not easily extend to the use of complex poles and zeros. The technique presented here for complex zeros requires some intuition and trial and error. The techniques of solution, whether for complex poles or complex zeros, are practically identical, so only the complex zero case is given:

(a) Select the compensation path. Then determine the open-loop transfer function and manipulate the characteristic equation into the form

$$\frac{K_i \prod_{=1}^{n} (s+Z_i)}{s^m \prod_{j=1} (s+p_j)} G_c = -1$$

where G_c is the transfer function of the compensator.

(b) Plot the z_i, p_j, and s^m on the s-plane.

(c) Select the real poles of G_c so as to be most convenient: Convenience will be determined by the particular problem. Plot these poles on the s-plane. This will determine which of the complex zero-producing active networks is to be used.

(d) Select the desired locations, P and P', of two complex conjugate closed-loop roots. Plot P and P' on the s-plane. As mentioned previously, it is hoped that these roots will prove dominant, but there is no way of ensuring this.

(e) Calculate the phase angle at P or P' due to z_i, p_j, s^m, and the real poles of G_c. Then construct the locus of complex zeros, in the manner described in Appendix II, which will result in P and P' being on a 180-degree locus.

(f) Choose two complex conjugate points, Q and Q', on this constructed locus. These will be the complex zeros of G_c.

(g) Determine the root-locus gain, K_{RL} which will cause roots to be at P and P' with the z_i, p_j, s^m, and the chosen real poles and complex zeros of G_c.

(h) Substitute K_{RL} and the real poles and complex zeros of G_c into the open-loop transfer function to determine if steady-state accuracy specifications are met. If not, choose two new locations of the complex zeros of G_c and repeat steps f and g.

(i) Determine x and y from the chosen locations of the complex zeros of G_c.

(j) Substitute x and y and the selected real poles of G_c into the appropriate equation of Table III to determine the algebraic relationship between the real zeros of the compensator passive networks.

(k) Construct a graph of this equation or evaluate it for several values of one variable and select any convenient, physically realizable values for the zeros of the passive networks.

(l) Substitute the real poles and zeros of the passive networks into the numerator root locus equation of Table III corresponding to the active network chosen. Plot this equation and the selected complex compensator zeros on another s-plane.

(m) Determine the value of A_1/A_2 which will cause the roots of the numerator of G_c to lie at the selected locations for the complex zeros.

(n) Using the results of steps g and m, A_1 and A_2 may be calculated and the compensator design is completed.

(o) The transient response must be computed and compliance with specifications determined.

Numerical Example

The example presented herein is straightforward in order to illustrate the use of active networks as compensators without becoming unnecessarily involved in other considerations. It is assumed that active compensation was necessary due to unspecified considerations. The subsections correspond to the steps in the technique of solution.

Given: The feedback control system of Fig. 7. $K_v=5.0$.

Requirements: Use cascade compensation. Desired root location to be such that $0.5 \leq \zeta \leq 0.7$ settling time$\cong$1.0 seconds. K_v not to be reduced.

Solution:

(a) The compensated open-loop transfer function of Fig. 7 is

$$F_o = \frac{K}{s(s^2+5s+100)} G_c$$

The characteristic equation of the uncompensated system is

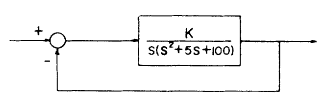

Fig. 7. Block diagram of a feedback control system

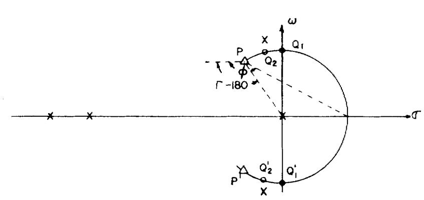

Fig. 8. s-plane constructions for designing the compensator

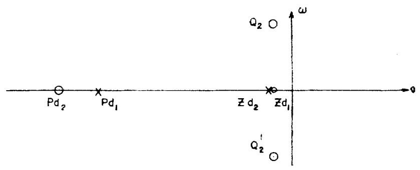

Fig. 9. s-plane pole-zero array

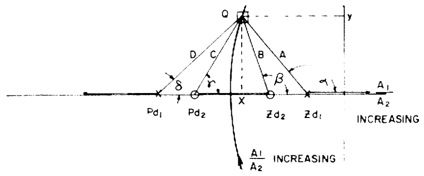

Fig. 10. Possible root locus for a lead-lead feed-forward compensator

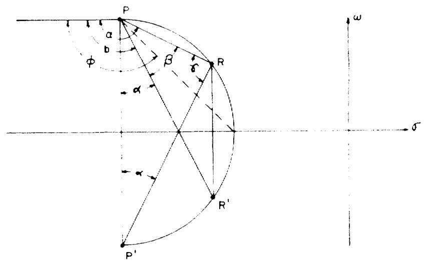

Fig. 11. Construction needed to obtain Carpenter's results

$$1+\frac{K}{s(s^2+5s+100)}=0$$

(b) The z_i, p_j, and s^m are plotted in Fig. 8.

(c) Select $p_{d1}=-25$, $p_{d2}=-30$, so as not to add any more poles in the region near the origin. p_{d1} and p_{d2} are also plotted in Fig. 8.

(d) Select the points $s=-5\pm j7$ to be P and P', the roots of the characteristic equation which will meet the requirements.

(e) From Fig. 8, the lead angle required at P is 307.3 degrees. The construction angles at P are

$\phi=153.65$ degrees

$\Gamma=307.3$ degrees

The arc of the circle on which complex compensating zeros can be located was constructed in Fig. 8 using these angles.

(f) In Fig. 8, choose $s=j8.8$ to be Q_1 and Q'_1 the locations of the complex compensator zeros.

(g) The root-locus gain at P using all poles and zeros is

$K_{RL}=3{,}250$

(h) Substituting this value of gain and the chosen compensator poles and zeros into the open-loop transfer function obtains $K_v=3.51$. This does not meet requirements. Therefore, choose two new complex compensator zeros a little closer to the desired root locations. Let these new locations be $s=-2.5\pm j8.4$, Q_2 and Q'_2. With these new zeros, the root-locus gain at P is

$K_{RL}=6{,}578$

Substituting this root-locus gain and the new complex zeros into the open-loop transfer function yields $K_v=6.72$. This meets the requirements.

(i) From the chosen locations of the complex zeros.

$x=-2.5$

$y=\pm 8.4$

(j) Substituting these values of x and y, p^{d}_1 and p_{d2} into the working equation of the lead-lead feed forward difference compensator of Table III yields

$$Z_{d1}=140.34\left[\frac{Z_{d2}+0.5466}{135.34-Z_{d2}}\right]$$

the algebraic relationship between z_{d1} and z_{d2}.

(k) Several values of z_{d2} were substituted into this equation and $z_{d1}=-2.5$, $z_{d2}=-3.0$ were selected as the zeros of the lead networks.

(l) Substituting the chosen z_{d1}, z_{d2}, p_{d1}, and p_{d2} into the numerator root-locus equation of the lead-lead feed-forward difference compensator of Table III yields

$$\frac{A_1(s+2.5)(s+30)}{A_2(s+3)(s+25)}=1$$

This relationship and the chosen complex compensator zeros are plotted in Fig. 9.

(m) From Fig. 9, $A_1/A_2=0.8368$ will cause the numerator zeros to be at the chosen complex conjugate locations.

(n) The values of A_1 and A_2 may now be computed from the equations

$A_1/A_2=0.8368$

$K(A_1-A_2)=K_{RL}=6578$

Since K is a variable-gain element, any values of K, A_1, and A_2 which satisfy these equations will yield satisfactory results.

(o) The characteristic equation of the compensated system may now be written as

$$\frac{6578(s+2.5+j8.4)(s+2.5-j8.4)}{s(s^2+10s+100)(s+25)(s+30)}=-1$$

The roots of this equation were obtained by calculation on a digital computer as

$s=-5.11+j7.26$

$s=-5.11-j7.26$

$s=-4.69+j11.69$

$s=-4.69-j11.69$

$s=-40.4$

The roots at $s-5.11+j7.26$ and $s=-5.11-j7.26$ corresponds to those selected early in the solution. There is some error due to the graphical techniques required in the solution, however the agreement is close enough for engineering work. The transient response may now be calculated and final agreement with requirements determined.

Conclusions

The active networks presented in this paper afford a simple method of generating complex poles and zeros. The locations of these complex poles and zeros may be accurately predicted and are easily varied by adjusting an amplifier gain or one or more of the passive elements of the networks. The varying of an amplifier gain is the more attractive of these two methods as it results in the complex poles or zeros moving along a circle which is precisely defined by the passive circuit elements.

Since the location of the complex poles and zeros is a function of amplifier gain, a requirement for a very stable amplifier is generated. This requirement may be relaxed somewhat if the compensated system is not too sensitive to the compensator pole and zero locations.

The lag-lead feed-forward summing compensator has characteristics which merit special attention. Since the gain K_c of this compensator is $A_1\alpha+A_2$, the possibility presents itself of using this compensator as the main transmission path amplifier. The complex zeros resulting from the network could be adjusted to best suit the needs of a specific system while the over-all gain remained independent.

Design of the active compensaton using the circular locus equations in conjunction with Carpenter's construction leads to accurate control of the complex roots of the compensated system.

Appendix I

Derivation of the Equations of the Root Loci of the Complex Poles and Zeros

For a lead-lead feed-forward difference compensator the equation of the complex zeros is derived as follows: In Fig. 10, let Q be a point with co-ordinates (x, y) which is on the root locus.

A zero-degree locus is required, therefore,

$\sphericalangle Q=2n\pi=-\alpha+\beta+\gamma-\zeta$

where $-\infty<n<\infty$ and is an integer.

$\sin(\sphericalangle Q)=0=\sin(-\alpha+\beta+\gamma-\zeta)$

Expanding by trigonometric identities:

$$\begin{aligned}0=&(\sin\beta\cos\alpha-\cos\beta\sin\alpha)\times\\&(\cos\gamma\cos\zeta+\sin\gamma\sin\zeta)+\\&(\cos\beta\cos\alpha+\sin\beta\sin\alpha)\times\\&(\sin\gamma\cos\zeta-\cos\gamma\sin\zeta)\end{aligned}$$

From Fig. 10,

$\sin\alpha = Y/A$ $\qquad$ $\sin\gamma = Y/C$

$\cos\alpha = \dfrac{X-Z_{d1}}{A}$ $\qquad$ $\cos\gamma = \dfrac{X-p_{d2}}{C}$

$\sin\beta = Y/B$ $\qquad$ $\sin\zeta = Y/D$

$\cos\beta = \dfrac{X-Z_{d2}}{B}$ $\qquad$ $\cos\zeta = \dfrac{X-p_{d1}}{D}$

Substituting, expanding, and collecting terms obtains the equation of a circle.

$$\left[X+\frac{p_{d2}Z_{d1}-p_{d1}Z_{d2}}{Z_{d2}-Z_{d1}+p_{d1}-p_{d2}}\right]^2+ y^2=\frac{p_{d1}p_{d2}(Z_{d1}-Z_{d2})+Z_{d1}Z_{d2}(p_{d2}-p_{d1})}{Z_{d2}-Z_{d1}+p_{d1}-p_{d2}}+\left[\frac{p_{d2}Z_{d1}-p_{d1}Z_{d2}}{Z_{d2}-Z_{d1}+p_{d1}-p_{d2}}\right]^2+\frac{ABCD}{y} \qquad (0)$$

This equation was derived with the $\sphericalangle Q=2n\pi$. Since identical results could have been obtained with $\sphericalangle Q=(2n-1)\pi$ it must be shown that for this case,

$\sphericalangle Q\pm(2n-1)\pi$

That this is the case may be seen from Fig. 10, since all of the $(2n-1)\pi$ loci are on the real axis.

Appendix II

Construction of the Locus of Complex Poles or Complex Zeros Producing a Constant Phase Angle at a Point in the s-Plane

It has been shown by Carpenter[5] that if the angle at a point in the s-plane due to an array of poles and zeros be known, it is possible to construct the locus of complex conjugate poles or complex conjugate zeros which will cause this point to be on a root locus. This locus of complex poles or zeros proves to be an arc of a circle. Carpenter's derivation is shown in Fig. 11.

Fig. 11 shows the necessary construction in proving the locus of complex conjugate

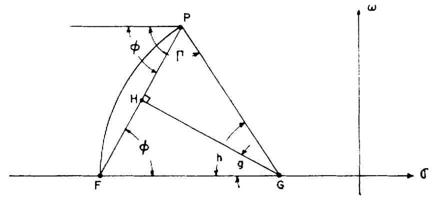

Fig. 12. Construction to locate the center of the circular arc

poles or zeros, R and R', producing a constant phase at a point P, in the s-plane, is an arc of a circle passing through that point. The center of the circle lies on the real axis.

From the figure:

$a = 90 \text{ degrees} + \alpha + \beta$

$b = 90 \text{ degrees} + \alpha$

$a+b=\beta+2\alpha+180=M=$ angle contributed by complex pair at point P. But $\beta+2\alpha+\gamma=180$ degrees. Subtracting $360-M=\gamma=$ a constant.

The vertex of γ, therefore, lies on an arc PP', of a circle whose center is on the real axis, by a fundamental theorem of plane geometry.

Since the arc always intersects the real axis, the arc may be quickly constructed by first locating that intersection. The intersection is found by constructing the angle

$\phi = M/2$

at the point P.

It can also be shown that the construction angle at the point P which locates the center of the circular arc on the real axis is

$\Gamma = M$

The necessary construction to show this is Fig. 12. F is the intersection of the circular arc with the real axis defined by the angle ϕ. HG is the perpendicular bisector of PF. Since the center of the circular arc is on the real axis, the intersection G must be the center of the circular arc, by plane geometry. From Fig. 12:

$\phi+g=90$ degrees

$2\phi+2g=180$ degrees

$2g=h$

Subtracting

$2\phi=180$ degrees

$h+\Gamma=180$ degrees

$\Gamma=180 \text{ degrees}-h$

Subtracting again

$2\phi=\Gamma=M$

thus proving the statement.

With the center of the circular arc and its point of intersection with the real axis known, from ϕ and K, the arc may be easily constructed.

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Multivariable Adaptive Control System

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Summary: A multivariable control system in which the parameters of the plant transfer function matrix are unknown functions of time is described in this paper. It is a sequel to a previous one[1] dealing with a single variable control system. A scheme is presented here in which the parameters of the plant transfer function are tracked and the controller transfer function matrix is adjusted in such a way as to keep the overall system optimized. The system may be realized in terms of ordinary analog computer elements. Information is given in a form which may be readily inserted into a controller for realization of a general type of adaptive control system.

THE WORD "ADAPTIVE" has been used with many shades of meaning. For the present purpose, the adaptive control system may be defined as one which may be described by sets of equations not necessarily known in detail, and one which is affected by the environment in which it resides. It is adaptive if it is capable of determining its performance equations and the effect

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