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# On developing a higher-order family of double-Newton methods with a bivariate weighting function

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ABSTRACT

A high-order family of two-point methods costing two derivatives and two functions are developed by introducing a two-variable weighting function in the second step of the classical double-Newton method. Their theoretical and computational properties are fully investigated along with a main theorem describing the order of convergence and the asymptotic error constant as well as proper choices of special cases. A variety of concrete numerical examples and relevant results are extensively treated to verify the underlying theoretical development. In addition, this paper investigates the dynamics of rational iterative maps associated with the proposed method and an existing method based on illustrated description of basins of attraction for various polynomials.



## 1. Introduction

A large number of high-order multipoint methods for a given nonlinear equation $f(x) = 0$ have been developed since Traub [29] initiated the qualitative as well as the quantitative analyses of iterative methods in the 1960s. Petković et al. [25] recently collected and updated the state of the art of multipoint methods. Other works on multipoint methods can be found in [3–5,11,13,14,16,20,24,27]. The principal aim of this paper is to design a family of high-order methods costing only two derivatives and two functions. Described below in (1.1) is the well-known two-point fourth-order double-Newton method [15,29], which is a two-step Newton's method utilizing two derivatives and two functions:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}. \end{cases} \tag{1.1}$$

This method is only fourth-order. One can get fourth order methods requiring less information.

Among higher-order methods requiring only two derivatives and two functions, we find several three-point sixth-order methods in [5,23,31,20], being respectively shown in (1.2)–(1.5) below.

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - J_f(x_n)\cdot\frac{f(x_n)}{f'(x_n)}, \quad J_f(x_n) = \frac{3f'(y_n)+f'(x_n)}{6f'(y_n)-2f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{a(z_n-x_n)(z_n-y_n)+\frac{3}{2}J_f(x_n)f'(y_n)+(1-\frac{3}{2}J_f(x_n))f'(x_n)}, \quad a \in \mathbb{R}, \end{cases} \tag{1.2}$$

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$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \frac{2f(x_n)}{f'(x_n)+f'(y_n)}, \\ x_{n+1} = z_n - \frac{f'(x_n)+f'(y_n)}{3f'(y_n)-f'(x_n)} \cdot \frac{f(z_n)}{f'(x_n)}, \end{cases} \tag{1.3}$$

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \frac{9-5s}{10-6s}\frac{f(x_n)}{f'(y_n)}, \quad s = \frac{f'(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{a+bs}{c+ds+rs^2} \cdot \frac{f(z_n)}{f'(x_n)}, \end{cases} \tag{1.4}$$

where $a = (5c+3d+r)/2, b = (r-3c-d)/2, c+d+r \neq 0, a, b, c, d, r \in \mathbb{R}$.

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{1+\beta u}{1+(\beta-2)u}\frac{f(y_n)}{f'(x_n)} = y_n - G_f(u)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{1-u}{1-3u} \cdot \frac{f(z_n)}{f'(x_n)}. \end{cases} \tag{1.5}$$

where $u = \frac{f(y_n)}{f(x_n)}, G_f(u) = \frac{u(1+\beta u)}{1+(\beta-2)u}, \beta \in \mathbb{R}$.

**Definition 1.1** (*Error equation, asymptotic error constant, order of convergence*). Let $x_0, x_1, \ldots, x_n, \ldots$ be a sequence of numbers converging to $\alpha$. Let $e_n = x_n - \alpha$ for $n = 0, 1, 2, \ldots$. If constants $p \geqslant 1, c \neq 0$ exist in such a way that $e_{n+1} = ce_n^p + O(e_n^{p+1})$ called the *error equation*, then $p$ and $\eta = |c|$ are said to be the *order of convergence* and the *asymptotic error constant*, respectively. It is easy to find $c = \lim_{n\to\infty} \frac{e_{n+1}}{e_n^p}$. Some authors call $c$ the asymptotic error constant.

Three-point methods (1.2)–(1.5) possess rather complicated structures, as compared with two-point methods like (1.1). Among the existing methods requiring two derivatives and two functions, two-point methods of order higher than four are rarely found. As a result, this rareness gives us a strong motivation to design less complicated higher-order two-point methods using two derivatives and two functions. In this paper, our special attention is paid to the development of a general class of two-point higher-order extended double-Newton methods. To this end, by introducing a two-variable weighting function in the second step of (1.1), we propose a higher-order family of two-point methods in the following form:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - K_f(s,u) \cdot \frac{f(y_n)}{f'(y_n)}, \end{cases} \tag{1.6}$$

where the weighting function $K_f : \mathbb{C}^2 \to \mathbb{C}$ is holomorphic[26] in a neighborhood of $(1,0)$ with $s = \frac{f'(y_n)}{f'(x_n)} = 1 + O(e_n)$ and $u = \frac{f(y_n)}{f(x_n)} = O(e_n)$. In view of the fact that $s - 1 = O(e_n), u = O(e_n)$, Taylor series expansion of $K_f(s,u)$ about $(1,0)$ up to terms of several order in each variable will play an essential role in designing two-point sixth-order methods costing two derivatives and two functions.

Note that proposed scheme (1.6) requires four new function evaluations for $f(x_n), f(y_n), f'(x_n), f'(y_n)$. In Section 2, methodology and analysis is described for a new family of sixth-order methods with appropriate forms of $K_f$. Section 3 investigates some special cases of $K_f(s,u)$, Section 4 discusses the extraneous fixed points, while Section 5 presents numerical experiments and concluding remarks.

## 2. Method development and convergence analysis

This section deals with the main theorem and its proof describing the methodology and convergence behavior on iterative scheme (1.6).

**Theorem 2.1.** *Assume that $f : \mathbb{C} \to \mathbb{C}$ has a simple root $\alpha$ and is analytic [1] in a region containing $\alpha$. Let $\Delta = f'(\alpha)$ and $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$ for $j = 2, 3, \ldots$. Let $x_0$ be an initial guess chosen in a sufficiently small neighborhood of $\alpha$. Let $K_f : \mathbb{C}^2 \to \mathbb{C}$ be holomorphic in a neighborhood of $(1,0)$. Let $K_{ij} = \frac{1}{i!j!}\frac{\partial^{i+j}}{\partial s^i \partial u^j}K_f(s,u)|_{(s=1,u=0)}$ for $0 \leqslant i,j \leqslant 4$. If $K_{00} = 1, K_{01} = K_{10} = 0, K_{20} = \frac{3+K_{02}}{4}, K_{11} = 1 + K_{02}$, $K_{12} = \frac{1}{2}K_{03} + 2(K_{21} - 2K_{30} - 1)$ are satisfied, then iterative scheme (1.6) defines a family of two-point sixth-order methods satisfying the error equation below: for $n = 0, 1, 2, \ldots$,*

$$e_{n+1} = \left\{c_2^2 c_4 - \frac{(3+K_{02})}{4}c_2 c_3^2 + c_2^3 c_3\left(2K_{21} - 8K_{30} - \frac{1}{2}K_{03} - 9\right) + c_2^5 \phi\right\}e_n^6 + O(e_n^7), \tag{2.1}$$

*where $\phi = 8K_{31} + 2K_{13} - 4K_{22} - 16K_{40} - K_{04} + 14$.*

**Proof.** Taylor series expansion of $f(x_n)$ about $\alpha$ up to 6th-order terms with $f(\alpha) = 0$ leads us to:

$$f(x_n) = \Delta\{e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)\}. \tag{2.2}$$

It follows that

$$f'(x_n) = \Delta\{1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + O(e_n^7)\}. \tag{2.3}$$

For simplicity, we will denote $e_n$ by $e$ from now on. With the aid of symbolic computation of Mathematica [32], we have:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2e^2 - 2(c_2^2 - c_3)e^3 + Y_4e^4 + Y_5e^5 + Y_6e_n^6 + O(e^7), \tag{2.4}$$

where $Y_4 = (4c_2^3 - 7c_2c_3 + 3c_4), Y_5 = -2(4c_2^4 - 10c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)$ and $Y_6 = (16c_2^5 - 52c_2^3c_3 + 33c_2c_3^2 + 28c_2^2c_4 - 17c_3c_4 - 13c_2c_5 + 5c_6)$. In view of the fact that $f'(y_n) = f'(x_n)|_{e_n \to (y_n - \alpha)}$, we get:

$$f'(y_n) = \Delta[1 + 2c_2^2e^2 - 4c_2(c_2^2 - c_3)e^3 + D_4e^4 + \Sigma_{i=4}^6 D_ie^i + O(e^7)], \tag{2.5}$$

where $D_4 = c_2(8c_2^3 - 11c_2c_3 + 6c_4), D_i = D_i(c_2, c_3, \ldots, c_6)$ for $5 \leqslant i \leqslant 6$. Hence we have:

$$s = \frac{f'(y_n)}{f'(x_n)} = 1 - 2c_2e + 3(2c_2^2 - c_3)e^2 - 4(4c_2^3 - 4c_2c_3 + c_4)e^3 + \Sigma_{i=4}^6 E_ie^i + O(e^7), \tag{2.6}$$

where $E_i = E_i(c_2, c_3, \ldots, c_6)$ for $4 \leqslant i \leqslant 6$. In view of the fact that $f(y_n) = f(x_n)|_{e_n \to (y_n - \alpha)}$, we get:

$$f(y_n) = \Delta[c_2e^2 - 2(c_2^2 - c_3)e^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e^4 + \Sigma_{i=5}^6 F_ie^i + O(e^7)], \tag{2.7}$$

where $F_i = F_i(c_2, c_3, \ldots, c_6)$ for $5 \leqslant i \leqslant 6$. Hence we have:

$$u = \frac{f(y_n)}{f(x_n)} = c_2e - (3c_2^2 - 2c_3)e^2 + (8c_2^3 - 10c_2c_3 + 3c_4)e^3 + \Sigma_{i=4}^6 G_ie^i + O(e^7), \tag{2.8}$$

where $G_i = G_i(c_2, c_3, \ldots, c_6)$ for $4 \leqslant i \leqslant 6$.

By direct substitution of $z_n, f(x_n), f(y_n), f'(x_n), f'(y_n)$ and $K_f(s,u)$ in (1.6), we find

$$x_{n+1} = y_n - K_f(s,u) \cdot \frac{f(y_n)}{f'(y_n)} = \alpha + (1 - K_{00})e^2 + \Sigma_{i=3}^6 \Gamma_ie^i + O(e^7), \tag{2.9}$$

where $\Gamma_i = \Gamma_i(c_2, c_3, \ldots, c_6, K_{j\ell}$, for $3 \leqslant i \leqslant 6, 0 \leqslant j, \ell \leqslant 4$.

Noting that $O(f(x_n)) = O(s-1) = O(u) = O(e)$ and $O(f(y_n)) = O(e^2)$, Taylor expansion of $K_f(s,u)$ about $(1,0)$ up to fourth-order terms in both variables yields after retaining up to fourth-order terms with $K_{14} = K_{23} = K_{24} = K_{32} = K_{33} = K_{34} = K_{41} = K_{42} = K_{43} = K_{44} = 0$:

$$\begin{aligned} K_f(s,u) = {} & K_{00} + K_{01}u + K_{02}u^2 + K_{03}u^3 + K_{04}u^4 + (s-1)(K_{10} + K_{11}u + K_{12}u^2 + K_{13}u^3) + (s-1)^2(K_{20} + K_{21}u + K_{22}u^2) \\ & + (s-1)^3(K_{30} + K_{31}u) + K_{40}(s-1)^4 + O(e^5). \end{aligned} \tag{2.10}$$

By setting $K_{00} = 1$ from (2.9) along with $\Gamma_3 = 0$, we immediately solve for $K_{01}$ as

$$K_{10} = \frac{K_{01}}{2}. \tag{2.11}$$

By substituting $K_{00}, K_{10}$ into $\Gamma_4 = 0$, we find two independent relations below:

$$K_{01} = 0, \quad K_{02} - 2K_{11} + 4K_{20} - 1 = 0. \tag{2.12}$$

As a result, we find

$$K_{01} = 0, \quad K_{20} = \frac{1 - K_{02} + 2K_{11}}{4}. \tag{2.13}$$

By substituting $K_{00}, K_{10}, K_{01}, K_{20}$ into $\Gamma_5 = 0$, we find two independent relations below:

$$K_{11} - K_{02} - 1 = 0, \quad K_{12} - 2K_{21} + 4K_{30} - \frac{1}{2}K_{03} + 2 = 0. \tag{2.14}$$

Solving (2.14) for $K_{11}, K_{12}$ yields

$$K_{11} = 1 + K_{02}, \quad K_{12} = \frac{1}{2}K_{03} + 2(K_{21} - 2K_{30} - 1). \tag{2.15}$$

By substituting $K_{00}, K_{01}, K_{10}, K_{20}, K_{11}, K_{12}$ into $\Gamma_6$, we find:

$$\Gamma_6 = c_2^2c_4 - \frac{(3 + K_{02})}{4}c_2c_3^2 + c_2^3c_3\left(2K_{21} - 8K_{30} - \frac{1}{2}K_{03} - 9\right) + c_2^5\phi \tag{2.16}$$

with $\phi = 8K_{31} + 2K_{13} - 4K_{22} - 16K_{40} - K_{04} + 14$ describing (2.1). This completes the proof. □

## 3. Case studies

This section describes some interesting case studies based on different forms of weighting functions $K_f(s,u)$. Using relations (2.11), (2.13) and (2.15), the Taylor-polynomial form of $K_f(s,u)$ is easily given by

$$K_f(s,u) = 1+\frac{1}{4}(3+K_{02})S^2+K_{30}S^3+K_{40}S^4+\{(1+K_{02})S+K_{21}S^2+K_{31}S^3\}u$$
$$+\{K_{02}+\frac{1}{2}K_{03}+2(K_{21}-2K_{30}-1)S+K_{22}S^2\}u^2+\{K_{03}+K_{13}S\}u^3+K_{04}u^4, \tag{3.1}$$

where notations $S = s-1$ are introduced for simplicity. Special cases of $K_f(s,u)$ are considered here. In each case, relevant coefficients are determined based on relations (2.11)–(2.15), along with $\eta$ as the asymptotic error constant.

**Case 1:**

$$\begin{cases} K_f(s,u) = 1+a_1S^2+(a_2S+a_3S^2)u+(a_4+a_5S+a_6S^2)u^2, \\ a_1=\frac{1}{4}(a_4+3), a_2 = 1+a_4, a_5 = 2(a_3-1); a_3, a_4, a_6 = free, \\ \eta = (14-4a_6)c_2^5+(2a_3-9)c_2^3c_3-\frac{1}{4}(a_4+3)c_2c_3^2+c_2^2c_4. \end{cases} \tag{3.2}$$

In what follows, we consider three kinds of weighting functions as some interesting sub-cases with some choices of free parameters $a_3, a_4, a_6$.

**Sub-Case 1A:**

$$\begin{cases} K_f(s,u) = 1+\frac{3}{4}S^2+S(1+S)u, \\ a_3=1,\ a_4=0,\ a_6=0. \end{cases} \tag{3.3}$$

**Sub-Case 1B:**

$$\begin{cases} K_f(s,u) = 1+S(S-2)u-3u^2, \\ a_3=1,\ a_4=-3,\ a_6=0. \end{cases} \tag{3.4}$$

**Sub-Case 1C:**

$$\begin{cases} K_f(s,u) = 1+\frac{1}{2}S^2+S^2u-u^2, \\ a_3=1,\ a_4=-1,\ a_6=0. \end{cases} \tag{3.5}$$

**Case 2:**

$$\begin{cases} K_f(s,u) = 1+a_1S^2+a_2S^3+a_3Su+a_4u^2, \\ a_1=\frac{1}{4}(a_4+3),\ a_2=-\frac{1}{2},\ a_3=1+a_4,\ a_4=free, \\ \eta = 14c_2^5-(8a_2+9)c_2^3c_3-\frac{1}{4}(a_4+3)c_2c_3^2+c_2^2c_4. \end{cases} \tag{3.6}$$

In what follows, we consider two kinds of weighting functions as some interesting sub-cases with some choice of a free parameter $a_4$.

**Sub-Case 2A:**

$$\begin{cases} K_f(s,u) = 1+\frac{1}{2}S^2-\frac{1}{2}S^3-u^2, \\ a_1=\frac{1}{2},\ a_2=-\frac{1}{2},\ a_3=0,\ a_4=-1. \end{cases} \tag{3.7}$$

**Sub-Case 2B:**

$$\begin{cases} K_f(s,u) = 1+\frac{3}{4}S^2-\frac{1}{2}S^3+Su, \\ a_1=\frac{3}{4},\ a_2=-\frac{1}{2},\ a_3=1,\ a_4=0. \end{cases} \tag{3.8}$$

**Case 3:**

$$\begin{cases} K_f(s,u) = \frac{r_0+r_1s+r_2s^2}{1+p_1s+p_2s^2}+au^2, \\ r_0=2+\frac{p_1}{2},\ r_1=-2, r_2=2+\frac{p_1}{2},\ p_2=1,\ a=-1,\ p_1(\neq -2)=free, \\ \eta = c_2^2c_4-5c_2^3c_3-\frac{1}{2}c_2c_3^2+2c_2^5\left(3+\frac{4}{p_1+2}\right). \end{cases} \tag{3.9}$$

In what follows, we consider four kinds of weighting functions as some interesting sub-cases with some choice of a free parameter $p_1$.

**Sub-Case 3A:**

$$\begin{cases} K_f(s,u) = \frac{2(1-s+s^2)}{(1+s^2)}-u^2, \\ r_0=2+\frac{p_1}{2},\ r_1=-2,\ r_2=2+\frac{p_1}{2},\ p_2=1,\ a=-1,\ p_1=0. \end{cases} \tag{3.10}$$

**Sub-Case 3B:**

$$\begin{cases} K_f(s,u) = -\frac{2s}{(1-4s+s^2)} - u^2, \\ r_0 = 2 + \frac{p_1}{2},\ r_1 = -2,\ r_2 = 2 + \frac{p_1}{2},\ p_2 = 1,\ a = -1,\ p_1 = -4. \end{cases} \tag{3.11}$$

**Sub-Case 3C:**

$$\begin{cases} K_f(s,u) = \frac{3s^2-2s+3}{(1+s)^2} - u^2, \\ r_0 = 2 + \frac{p_1}{2},\ r_1 = -2,\ r_2 = 2 + \frac{p_1}{2},\ p_2 = 1,\ a = -1,\ p_1 = 2. \end{cases} \tag{3.12}$$

**Sub-Case 3D:**

$$\begin{cases} K_f(s,u) = -\frac{(1+s)^2}{(1-6s+s^2)} - u^2, \\ r_0 = 2 + \frac{p_1}{2},\ r_1 = -2,\ r_2 = 2 + \frac{p_1}{2},\ p_2 = 1,\ a = -1,\ p_1 = -6. \end{cases} \tag{3.13}$$

**Case 4:**

$$\begin{cases} K_f(s,u) = \frac{a+b(s-1)+cu+m(s-1)^2+g(s-1)u+hu^2}{1+\mathrm{B}(s-1)+\mathrm{C}u+\mathrm{M}(s-1)^2+d(s-1)u+\mathrm{H}u^2}, \\ a = 1,\ b = \mathrm{B} = \frac{1}{2}(c+4),\ \mathrm{C} = c,\ h = \mathrm{H} - d + g - 1,\ \mathrm{M} = \frac{1}{4}(d + 4m - g - 2), \\ \text{where } m, g, c, d,\ \mathrm{H} \text{ are free}, \\ \eta = \frac{1}{4}c_2\{4c_2c_4 + c_3^2(d-g-2) - 4c_2^4(d - \mathrm{H} - 4m + g + 4) + 2c_2^2c_3(c-2)\}. \end{cases} \tag{3.14}$$

In the next section, we select free parameters $m, g, c, d, \mathrm{H}$ to position the extraneous fixed points on the imaginary axis based on the dynamics of basins of attraction associated with this $K_f$. The dynamical idea behind basins of attraction was initiated by Stewart [28] and followed by works of Chun et al.[6–8], Cordero et al. [10] and Neta et al.[21]. More recent results on basins of attraction can be found in [2,18,19].

## 4. Extraneous fixed points

Multipoint iterative methods [17] solving a nonlinear equation of the form $f(x) = 0$ can be written as

$$x_{n+1} = R_f(x_n), \tag{4.1}$$

where $R_f$ is the iteration function whose fixed points are zeros of $f(x)$ under consideration. The iteration function $R_f$, however, might possess other fixed points that are not zeros of $f$. Such fixed points different from zeros of $f$ are called the *extraneous fixed points* [12,30] of the iteration function $R_f$. Extraneous fixed points may form attractive cycles and periodic orbits to display chaotic dynamics of the basin of attraction under investigation. This motivates our technical selection of appropriate parameters among free parameters $m, g, c, d, H$ of $K_f$ in Case 4 of the preceding section via intensive analysis of the extraneous fixed points under some constraints.

The proposed method (1.6) can be put in the form:

$$x_{n+1} = R_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} H_f(x_n), \tag{4.2}$$

where $H_f(x_n) = 1 + \frac{u}{s} K_f(s,u)$ can be regarded as a weighting function of the classical Newton's method. It is obvious that $\alpha$ is a fixed point of $R_f$. The points $\xi \neq \alpha$ for which $H_f(\xi) = 0$ are extraneous fixed points of $R_f$. If we look at $H_f$ in Case 1, it contains one free parameter $a_6$. Thus all of its listed three subcases with the same value of $a_6 = 0$ are the same dynamically. If we look at $H_f$ in Case 2, it contains no free parameter. Thus all of its three subcases are the same dynamically. If we look at $H_f$ in Case 3, it contains one free parameter $p_1$. Thus all four subcases are dynamically different from each other. We pay a special attention to Case 4 with a bivariate second-order rational weighting function $K_f$. Indeed, this case allows us to get $H_f$ in the form of a bivariate second-order rational function with some free parameters.

We are ready to impose some constraints on the extraneous fixed points to be determined from the zeros of $H_f$ in order to select free parameters $m, g, c, d, \mathrm{H}$ for a bivariate second-order rational weighting function $K_f$ under Case 4 in Section 3. Chun et al. [9] imposed constraints on the extraneous fixed points all of which should lie on the imaginary axis. It is also worth to observe the dynamic behavior near the extraneous fixed points on the imaginary axis which is the boundary of two basins of two roots for the typical quadratic polynomial $f(z) = z^2 - 1$ as well as to observe the dynamics of zeros of $f$ in the basins of attraction under investigation. Closely following the approach of Chun et al. [9], we would like to position all the extraneous fixed points on the imaginary axis. To this end, we first find the explicit form of $K_f$ in (3.14):

$$K_f(s,u) = \frac{1 + (2 + \frac{c}{2})(s-1) + cu + m(s-1)^2 + g(s-1)u + (g - d + \mathrm{H} - 1)u^2}{1 + (2 + \frac{c}{2})(s-1) + cu + \frac{1}{4}(4m - g + d - 2)(s-1)^2 + d(s-1)u + \mathrm{H}u^2}. \tag{4.3}$$

We then construct $H_f(x_n) = 1 + \frac{u}{s}K_f(s,u)$ in (4.2) below:

$$H_f(x_n) = 1 + \frac{u}{s}\cdot\frac{1 + (2+\frac{c}{2})(s-1) + cu + m(s-1)^2 + g(s-1)u + (g-d+\mathtt{H}-1)u^2}{1 + (2+\frac{c}{2})(s-1) + cu + \frac{1}{4}(4m-g+d-2)(s-1)^2 + d(s-1)u + \mathtt{H}u^2}. \tag{4.4}$$

We now apply a quadratic polynomial $f(z) = z^2 - 1$ to $H_f(x_n)$ and construct a complex rational weighting function $H(z)$ in the form of

$$H(z) = \frac{F(z)}{2(1+z^2)T(z)}, \quad F(z) = (3\rho-4)z^6 - 5(\rho+8)z^4 + (\rho-20)z^2 + \rho, \tag{4.5}$$

where $T(z) = z^4(\rho-1) - 2(7+\rho)z^2 + \rho - 1$ and $\rho = d - \mathtt{H} - 4m + g + 3$.

It is interesting to investigate the complex dynamics of the rational iterative map $R_p$ of the form

$$z_{n+1} = R_p(z_n) = z_n - \frac{p(z_n)}{p'(z_n)}H(z_n), \tag{4.6}$$

in connection with the basins of attraction for a variety of polynomials $p(z_n)$. Indeed, $R_p(z)$ represents the classical Newton's method with weighing function $H(z)$ and may possess its fixed points as zeros of $p(z)$ or extraneous fixed points associated with $H(z)$. As a result, basins of attraction for the fixed points or the extraneous fixed points as well as their attracting periodic orbits may make an impact on the complicated and chaotic complex dynamics whose visual description for various polynomials will be shown in the latter part of Section 5.

To continue a further analysis on $H(z)$, we observe that the coefficients of $H(z)$ are expressed in terms of only a new single parameter $\rho(d,\mathtt{H},m,g)$ being independent of $c$. We wish to determine real values of $\rho$ in (4.5) such that the all the roots (other than zero $\alpha$ of $f$) of $H$ lie on the imaginary axis. Once a real value of $\rho$ is determined, then we can select three free real parameters among four $m, g, d, \mathtt{H}$ to simplify the explicit form of $K_f(s,u)$.

We now closely look at the numerator $F(z)$ of $H$ to position all the roots of $H$ on the imaginary axis. Suppose that all zeros $\xi$ of $F(z)$ are written as $\xi = i\cdot\lambda, i = \sqrt{-1}, \lambda \neq 0$. Since $F(\xi) = 0$ yields the following sextic equation with real coefficients:

$$-(3\rho-4)\lambda^6 - 5(\rho+8)\lambda^4 - (\rho-20)\lambda^2 + \rho = 0. \tag{4.7}$$

By letting $t = \lambda^2 > 0$, Eq. (4.7) reduces to the following cubic equation:

$$(4-3\rho)t^3 - 5(\rho+8)t^2 - (\rho-20)t + \rho = 0. \tag{4.8}$$

Hence by taking two square roots of each of all three positive real roots of (4.8), we equivalently find all corresponding six imaginary roots of $F(z)$ in (4.5).

It still remains to investigate the conditions on $\rho$ for (4.8) to have all distinct positive real roots. If $\rho = 0$, then $t = 0$ is a root of (4.8). Therefore we restrict $\rho \neq 0$. If $\rho \neq 0$, divide both sides of (4.8) by $\rho$ and simplify to obtain after rearrangement with $\omega = 1/\rho$:

$$(3-4\omega)t^3 + 5(1+8\omega)t^2 + (1-20\omega)t = 1. \tag{4.9}$$

Let $y_1(t) = (3-4\omega)t^3 + 5(1+8\omega)t^2 + (1-20\omega)t$ and $y_2(t) = 1$. Then the problem of locating three distinct positive real roots of (4.8) reduces to that of counting the number of positive crossing points of the cubic polynomial $y_1(t)$ with the horizontal line $y_2(t) = 1$, as $\omega$ varies. Suppose that the leading coefficient $3-4\omega < 0$ or $\omega > 3/4$ in $y_1(t)$, then $y_1(-1) = 1 + 64\omega > 1$. By the continuity of $y_1(t)$, it must intersect the horizontal line $y_2(t) = 1$ at a negative crossing point. As a result, we must restrict $\omega < 3/4$ for (4.9) to possess three distinct positive real roots. A typical sketch of graphs for both $y_1(t)$ and $y_2(t)$ is shown in Fig. 1. To find the extremal points, we set $y_1'(t) = 3t^2(3-4\omega) - 20\omega + 10t(1+8\omega) + 1 = 0$ which yields two extremal points $t_1$ and $t_2$:

$$t_1 = \frac{5+40\omega+4D}{3(4\omega-3)}, \quad t_2 = \frac{5+40\omega-4D}{3(4\omega-3)} \tag{4.10}$$

with $D = \sqrt{1+37\omega+85\omega^2}$, provided that $1+37\omega+85\omega^2 > 0$ yielding conditions on $\omega$:

$$\omega < \frac{(-37-7\sqrt{21})}{170} \approx -0.406341 \text{ or } \frac{(-37+7\sqrt{21})}{170} \approx -0.0289528 < \omega < \frac{3}{4}. \tag{4.11}$$

Since $t_1 < t_2$ for $\omega < \frac{3}{4}$, we find that $y_1(t)$ has a local maximum and minimum at $t_1$ and $t_2$, respectively. We also require both $t_1 > 0$ and $t_2 > 0$, which amounts to $D > 0$ and $t_1 + t_2 = -\frac{10(1+8\omega)}{3(3-4\omega)} > 0$ and $\frac{1-20\omega}{3(3-4\omega)} > 0$. Consequently, we find that $\omega < \frac{(-37-7\sqrt{21})}{170} \approx -0.406341$ for positive extremal points. Finally, we require that $y_1(t_2) < 1 < y_1(t_1)$, which yields $\omega = \frac{1}{\rho} < \frac{1}{75}\left(-20 - \frac{2\times10^{2/3}}{(-215+21\sqrt{105})^{1/3}} + (-2150+210\sqrt{105})^{1/3}\right) \approx -0.467119$. Hence, we find the constraints

$$-2.14078101258 < \rho < 0 \tag{4.12}$$

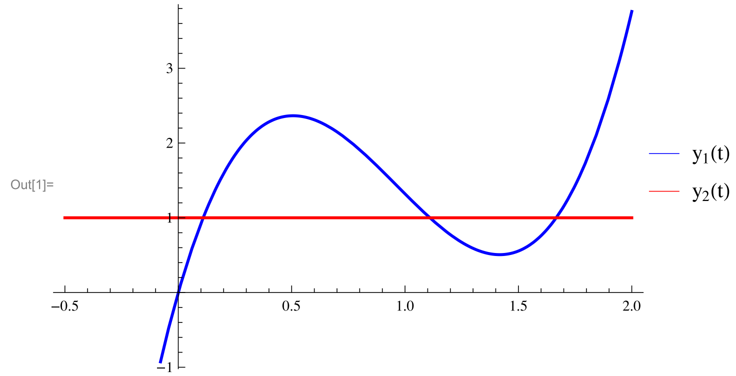


**Fig. 1.** Three distinct positive crossing points.

for (4.8) to have all three distinct positive real roots. Via direct computation, we also check that these values of $\rho$ in (4.12) do not satisfy the zeros of $T(z)$ in (4.5). If $z = \pm i$ is a zero of $F(z)$, then $F(\pm i) = -8(2+\rho) = 0$ would hold. Hence, $\rho = -2$ would yield $(1+z^2)$ as a factor of $F(z)$ and cause only four extraneous fixed points from the degeneracy of $H_f(z) = \frac{1+10z^2+5z^4}{(3+z^2)(1+3z^2)}$. Since $\rho = d - \mathrm{H} - 4m + g + 3$, we are free to choose 3 parameters among four parameters $d, H, m, g$, once $\rho$ is properly chosen to satisfy (4.12). Table 1 lists some interesting choices of parameters $\rho, d, \mathrm{H}, m, g$ for all purely imaginary extraneous fixed points $\xi$ and simplified forms of $H_f(z)$ as well as $K_f(s,u)$.

At this point, we now wish to compare the dynamical behavior of (4.6) with that of another complex rational iterative map associated with an existing method (1.5) suggested by Neta [20] in 1979. By repeating a similar analysis that we have done so far, iterative method (1.5) can be put in the form:

$$x_{n+1} = \mathcal{R}_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}\mathcal{H}_f(x_n), \tag{4.13}$$

where $\mathcal{H}_f(x_n) = 1 + \frac{u(1+\beta u)}{1+(\beta-2)u} + \frac{v(1-u)}{1-3u}$, $v = \frac{f(z_n)}{f(x_n)}$ and $z_n = y_n - \frac{f(x_n)}{f'(x_n)}\frac{u(1+\beta u)}{1+(\beta-2)u}$.

Like rational iterative map $R_p$ (4.6), the complex rational iterative map $\mathcal{R}_p$ associated with $\mathcal{R}_f$ can be written as

$$z_{n+1} = \mathcal{R}_p(z_n) = z_n - \frac{p(z_n)}{p'(z_n)}\mathcal{H}(z_n). \tag{4.14}$$

In addition, the rational weighting function $\mathcal{H}(z)$ associated with $\mathcal{H}_f(x_n)$ for $f(z) = z^2 - 1$ is found to be:

$$\mathcal{H}(z) = \frac{\mathcal{F}(z)}{64z^6(3+z^2)[2-\beta+(\beta+2)z^2]^2}, \tag{4.15}$$

where $\mathcal{F}(z) = -\beta^2 + 2z^2\beta(-4+5\beta) + z^4(-16+72\beta-51\beta^2) + 4z^6(96-148\beta+63\beta^2) + z^8(1760-624\beta-279\beta^2) + z^{10}(1536+728\beta-38\beta^2) + z^{12}(432+424\beta+107\beta^2)$.

We wish again to locate all the roots of $\mathcal{H}(z)$ on the imaginary axis. It is, however, expected that some roots may easily escape from the imaginary axis, since all the coefficients of $\mathcal{F}(z)$ of degree 12 depend upon only a single parameter $\beta$. In fact, if $\beta = 0$ or $\beta = 1$, then $\mathcal{H}(z)$ degenerates to an eighth-order rational function due to a common divisor $z^4$ or $(1+3z^2)^2$; this fact, however, yields only 2 imaginary roots $\pm i/\sqrt{3}$ or $\pm 1.35684i$, while remaining six roots are all complex. Thus we are not interested in $\beta = 0$ or $\beta = 1$ and assume that $\beta \neq 0$ and $\beta \neq 1$. Consequently, our aim suffices to locate as many roots on the imaginary axis as possible, based on an appropriate selection of parameter $\beta$. Suppose that all zeros $\xi$ of $\mathcal{F}(z)$ are written as $\xi = i\cdot\lambda, i = \sqrt{-1}, \lambda \neq 0$. Since $\mathcal{F}(\xi) = 0$ yields the following polynomial equation of degree 12 with real coefficients:

**Table 1**
Extraneous fixed points $\xi, H(z)$ and $K_f$ for selected parameters with $\rho = d + g - \mathrm{H} - 4m + 3$.

| $\rho$ | $d$ | H | $m$ | $g$ | $c$ | $\xi$ | $H(z)$ | $K_f(s,u)$ |
|---|---|---|---|---|---|---|---|---|
| $-2$ | $-5$ | 0 | 0 | 0 | 0 | $\pm 0.32492i, \pm 1.37638i$ | $\frac{1+10z^2+5z^4}{(3+z^2)(1+3z^2)}$ | $\frac{1+2(s-1)+4u^2}{1+2(s-1)-\frac{7}{4}(s-1)^2-5(s-1)u}$ |
| $-\frac{3}{2}$ | 0 | 0 | $\frac{11}{8}$ | 1 | 0 | $\pm\sqrt{3}i, \pm 0.281085i, \pm 0.862856i$ | $\frac{(3+z^2)(1+14z^2+17z^4)}{2(1+z^2)(5+22z^2+5z^4)}$ | $\frac{1+2(s-1)+\frac{11}{8}(s-1)^2+(s-1)u}{1+2(s-1\frac{5}{8}(s-1)^2}$ |
| $-1$ | 2 | 6 | 0 | 0 | 2 | $\pm 0.228243i, \pm 0.797473i, \pm 2.07652i$ | $\frac{1+21z^2+35z^4+7z^6}{4(1+z^2)(1+6z^2+z^4)}$ | $\frac{1+3(s-1)+2u+3u^2}{1+3(s-1)+2u+2(s-1)u+6u^2}$ |
| $-1$ | $\frac{5}{2}$ | 3 | 1 | $\frac{1}{2}$ | 2 | $\pm 0.228243i, \pm 0.797473i, \pm 2.07652i$ | $\frac{1+21z^2+35z^4+7z^6}{4(1+z^2)(1+6z^2+z^4)}$ | $\frac{1+3(s-1)+(s-1)^2+2u+\frac{1}{2}(s-1)u}{1+3(s-1)+(s-1)^2+2u+\frac{5}{2}(s-1)u+3u^2}$ |
| $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{5}{8}$ | 0 | 0 | $\pm 0.159947i, \pm 0.755758i, \pm 2.49428i$ | $\frac{1+41z^2+75z^4+11z^6}{2(1+z^2)(3+26z^2+3z^4)}$ | $\frac{1+2(s-1)+\frac{5}{8}(s-1)^2}{1+2(s-1)-\frac{1}{2}(s-1)u+\frac{1}{2}u^2}$ |

$$- \beta^2 - 2\beta(-4+5\beta)\lambda^2 + (-16+72\beta-51\beta^2)\lambda^4 - 4(96-148\beta+63\beta^2)\lambda^6 + (1760-624\beta-279\beta^2)\lambda^8 - (1536+728\beta-38\beta^2)\lambda^{10} + (432+424\beta+107\beta^2)\lambda^{12} = 0. \tag{4.16}$$

Letting $\lambda^2 = t > 0$ in (4.16) leads us to the following sextic equation in $\lambda$ with real coefficients:

$$- \beta^2 + t(4-5\beta)\beta + t^2(-16+72\beta-51\beta^2) - 4t^3(96-148\beta+63\beta^2) + t^4(1760-624\beta-279\beta^2) + 2t^5(-768-364\beta+19\beta^2) + t^6(432+424\beta+107\beta^2) = 0. \tag{4.17}$$

Let $\phi(t)$ denote the left side of (4.17). Since $\phi(-1) = 4096 > 0, \phi(0) = -\beta^2 < 0$ and $\phi(2) = 1359(\beta - \frac{184}{453})^2 + \frac{497664}{151} > 0$, there exist a negative root $t_0$ satisfying $-1 < t_0 < 0$ and a positive root $t_1$ satisfying $0 < t_1 < 2$ due to the continuity of $\phi$. Hence, $\phi$ has at most 5 positive roots including $t_1$. We find that the highest-order term $t^6(432+424\beta+107\beta^2)$ dominates $\phi(t)$ for sufficient large $\beta$, say, for $|\beta| > 4$ when $t > 2$ due to the large coefficient $432+424\beta+107\beta^2 = 107(\beta+\frac{212}{107})^2 + \frac{1280}{107}$; this tells us that $\phi$ has no positive roots for $t \geqslant 2$.

To find further possible positive roots of $\phi$, we favorably rely on the graphical analysis by plotting $\phi$ for $0 < t < 2$ with a variety of selected parameters ranging $0 < |\beta| \leqslant 4$. We especially pay attention to an interval $0 < \beta < 1$ whose endpoints degenerates $\mathcal{H}$ to an eighth-order polynomial equation. Indeed, we have found three positive roots in an interval $\frac{2}{5} \leqslant \beta < 1$ as shown in Fig. 2. Table 2 lists imaginary extraneous fixed points and $\mathcal{H}_f$ for selected three values of parameter $\beta$. In remaining intervals other than $0 < \beta < 1$, we have found at most one positive root. As another choice for $\beta$, we list the case for $0 < \beta = 3 - 2\sqrt{2} < 1$ in Table 2 based on the analysis for the fourth-order King's method done by Neta et al. [22]. It turned out that most of corresponding extraneous fixed points with this $\beta$ are located near the imaginary axis.

The latter part of the next section will discuss complex dynamics as well as chaotic behavior of both rational iterative maps (4.6) and (4.14) when applied to various polynomials, based on visual description of their basins of attraction along with comparison of their dynamic properties and characteristics.

## 5. Numerical experiments and concluding remarks

This first part of this section deals with computational characteristics of proposed method (1.6) for a variety of test functions in comparison with other existing methods. In the second part we discuss the complex dynamics of two rational iterative maps (4.6) and (4.14) along with concluding remarks.

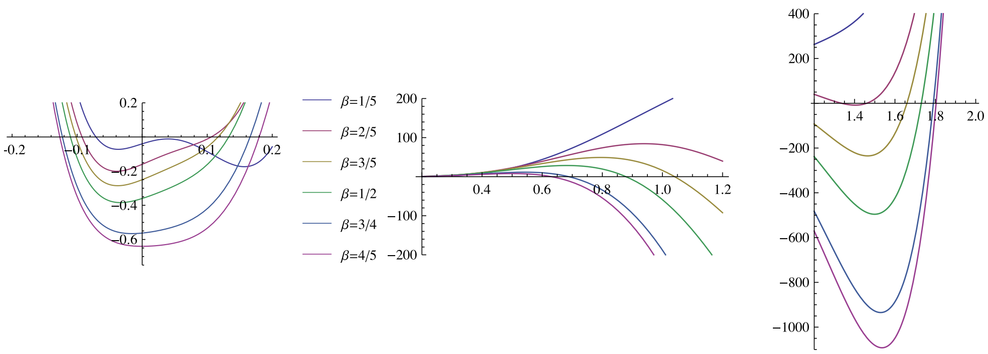


**Fig. 2.** Three positive roots of $\phi(t)$ for $0 < t < 2$.

**Table 2**
Extraneous fixed points $\xi, \mathcal{H}(z)$ and $G_f(u)$ for selected parameters $\beta$.

| $\beta$ | $\xi$ | $\mathcal{H}(z)$ | $G_f(u)$ |
|---|---|---|---|
| $\frac{2}{5}$ | $\pm0.325611i, \pm1.15557i, \pm1.20827i$ | $\frac{-1-10Z^2+29z^4+1172z^6+9161z^8+11382z^{10}+3867z^{12}}{256z^6(3+z^2)(2+3z^2)^2}$ | $\frac{u(5+2u)}{5-8u}$ |
| | $\pm0.304113, \pm0.156573 \pm 0.302977i$ | | |
| $\frac{1}{2}$ | $\pm0.341541i, \pm1.02013i, \pm1.28811i$ | $\frac{-1-6z^2+29z^4+604z^6+5513z^8+7562z^{10}+2683z^{12}}{64z^6(3+z^2)(3+5z^2)^2}$ | $\frac{u(2+u)}{2-3u}$ |
| | $\pm0.318902, \pm0.203793 \pm 0.305548i$ | | |
| $\frac{4}{5}$ | $\pm0.424138i, \pm0.794406i, \pm1.34307i$ | $\frac{-1+14z^4+112z^6+1691z^8+3272z^{10}+1312z^{12}}{16z^6(3+z^2)(3+7z^2)^2}$ | $\frac{u(5+4u)}{5-6u}$ |
| | $\pm0.355921, \pm0.290282 \pm 0.295202i$ | | |
| $3-2\sqrt{2}$ | $\pm0.259813, \pm0.19545 \pm 1.24831i$ | $\frac{-0.0294373-1.07821z^2-5.14805z^4+289.847z^6+1644.73z^8+1659.79z^{10}+507.897z^{12}}{64z^6(-1.82843-2.17157z^2)^2(3+z^2)}$ | $\frac{u(1+(3-2\sqrt{2})u)}{1+(1-2\sqrt{2})u}$ |
| | $\pm0.476847i, \pm0.0475696 \pm 0.190337i$ | | |

**Table 3**
Convergence for sample test functions $F_1(x) - F_6(x)$ with methods **KY1** – **KY3**.

| $\mathbf{KY_i}$ | $F(x_n)$ | $n$ | $x_n$ | $\lvert F(x_n)\rvert$ | $\lvert e_n\rvert$ | $\left\lvert\frac{e_n}{e_{n-1}^6}\right\rvert$ | $\eta$ |
|---|---|---|---|---|---|---|---|
| $\mathbf{KY_1}$ | $F_1$ | 0 | −0.95 | 0.0524792 | 0.0500000 | | |
| | | 1 | −0.999999864300288 | $1.357 \times 10^{-7}$ | $1.357 \times 10^{-7}$ | 8.684781593 | 15.16666667 |
| | | 2 | −1.00000000000000 | $9.470 \times 10^{-41}$ | $9.470 \times 10^{-41}$ | 15.16664250 | |
| | | 3 | −1.00000000000000 | $0.0 \times 10^{-100}$ | $0.0 \times 10^{100}$ | | |
| | $F_2$ | 0 | 0.75 | 2.12413 | 0.136227 | | |
| | | 1 | 0.859748914260037 | 0.2002 | 0.02648 | 4142.930343 | 46.17549177 |
| | | 2 | 0.886226916675717 | $6.223 \times 10^{-8}$ | $8.777 \times 10^{-9}$ | 25.47041020 | |
| | | 3 | 0.886226925452758 | $1.497 \times 10^{-46}$ | $2.111 \times 10^{-47}$ | 46.17548430 | |
| | | 4 | 0.886226925452758 | $0.0 \times 10^{-100}$ | $0.0 \times 10^{-100}$ | | |
| $\mathbf{KY_2}$ | $F_3$ | 0 | $\begin{pmatrix}-0.95\\2.15\end{pmatrix}$[a] | 19.2659 | 0.0995374 | | |
| | | 1 | $\begin{pmatrix}-0.999997479564086\\2.23607016675128\end{pmatrix}$ | 0.0006925 | $3.338 \times 10^{-6}$ | 3.33848 | 2.154207065 |
| | | 2 | $\begin{pmatrix}-1.00000000000000\\2.23606797749979\end{pmatrix}$ | $6.186 \times 10^{-31}$ | $2.982 \times 10^{-33}$ | 2.154214741 | |
| | | 3 | $\begin{pmatrix}-1.00000000000000\\2.23606797749979\end{pmatrix}$ | $0.0 \times 10^{-97}$ | $0.0 \times 10^{-99}$ | | |
| | $F_4$ | 0 | 0.1 | 0.400623 | 0.100000 | | |
| | | 1 | $6.812 \times 10^{-11}$ | $2.725 \times 10^{-10}$ | $6.813 \times 10^{-11}$ | 0.00006812682743 | 0.0008102972659 |
| | | 2 | $-8.101 \times 10^{-65}$ | $3.241 \times 10^{-64}$ | $8.101 \times 10^{-65}$ | 0.0008102972653 | |
| | | 3 | $0.0 \times 10^{-229}$ | $0.0 \times 10^{-228}$ | $0.0 \times 10^{-229}$ | | |
| $\mathbf{KY_3}$ | $F_5$ | 0 | 0.91 | 0.889859 | 0.0645898 | | |
| | | 1 | 0.974590839309637 | 0.00001478 | $9.948 \times 10^{-7}$ | 13.70124208 | 10.52644713 |
| | | 2 | 0.974589844487655 | $1.516 \times 10^{-34}$ | $1.020 \times 10^{-35}$ | 10.52640383 | |
| | | 3 | 0.974589844487655 | $0.0 \times 10^{-101}$ | $0.0 \times 10^{-99}$ | | |
| | $F_6$ | 0 | −1.48 | 0.344055 | 0.0907963 | | |
| | | 1 | −1.57079632759000 | $2.942 \times 10^{-9}$ | $7.951 \times 10^{-10}$ | 0.001419094785 | 0.0009648572271 |
| | | 2 | −1.57079632679490 | $9.021 \times 10^{-58}$ | $2.438 \times 10^{-58}$ | 0.0009648572240 | |
| | | 3 | −1.57079632679490 | $0.0 \times 10^{-99}$ | $0.0 \times 10^{-99}$ | | |

[a] $\begin{pmatrix}-0.95\\2.15\end{pmatrix} = -0.95 + 2.15i$.

In many real-life root-finding problems under normal circumstances of computations, it is quite common to find their numerical results accurate up to approximately 6 or 7 significant decimal digits with second-order Newton-like methods using common programming languages *Fortran* or *C*. In such programming languages, empirically 15 or 16 decimal working-precision digits are adopted for numerical results with 6 or 7 significant decimal digits. Likewise, about 48 decimal working-precision digits would be reasonable for approximately 21 significant decimal digits with sixth-order numerical methods. Computing asymptotic error constants $\eta = \lim_{n\to\infty} \frac{|e_n|}{|e_{n-1}|^p}$ with several significant digits of accuracy would encounter extreme calculations due to the indeterminate form of a small-number division near the root $\alpha$. We, therefore, need to increase the number of working-precision digits much more for numerical results with moderate number of significant decimal digits.

During the current numerical experiments with programming language *Mathematica* (Version 7), all computations have been done with 100 working-precision digits, which minimize round-off errors and let us clearly observe the computed asymptotic error constants requiring small-number divisions. In addition, the error bound $\epsilon = \frac{1}{2} \times 10^{-80}$ was assigned. The initial guesses $x_0$ were selected close to $\alpha$ to guarantee the convergence of the iterative methods. Only 15 significant digits of approximated roots $x_n$ are displayed in Tables 3–5 due to the limited paper space, although 80 significant digits are available. When exact root is not available, it is best approximated with 150 digits of precision to hold sufficient number of significant digits of $x_n - \alpha$. Numerical experiments have been carried out on a personal computer equipped with an AMD 3.1 Ghz dual-core processor and 64-bit Windows 7 operating system.

Iterative methods (1.6) with (3.4), (3.8), (3.12) were respectively identified by **KY1**, **KY2**, **KY3** and have shown successful results for the following test functions:

$$\begin{cases} \textit{Method } \mathbf{KY1} : F_1(x) = \sin(x+1) + (x+1)^2,\ \alpha = -1. \\ \textit{Method } \mathbf{KY1} : F_2(x) = \sin(2x^2) - \log(1+4x^2-\pi) - 1, \alpha = \sqrt{\pi/4}. \\ \textit{Method } \mathbf{KY2} : F_3(x) = x^5 + 2x^2 - 64x + 20 + 88i\sqrt{5},\ \alpha = -1 + i\sqrt{5}, i = \sqrt{-1}. \\ \textit{Method } \mathbf{KY2} : F_4(x) = \sin(x^3) + 2 - (2x+1)\log(e^2+x^2),\ \alpha = 0. \\ \textit{Method } \mathbf{KY3} : F_5(x) = x^5 - x^4 + e^{2x} - 7,\ \alpha \approx 0.974589844487655. \\ \textit{Method } \mathbf{KY3} : F_6(x) = 2x - \log(e^2 + 4x^2 - \pi^2) + \pi + 2,\ \alpha = -\pi/2. \end{cases} \tag{5.1}$$

Methods **KY1, KY2, KY3** in Table 3 clearly confirms sixth-order convergence based on results of $|\frac{e_n}{e_{n-1}^6}|$. Table 3 lists iteration indexes $n$, approximate zeros $x_n$, residual errors $|f(x_n)|$, errors $|e_n| = |x_n - \alpha|$ and $|\frac{e_n}{e_{n-1}^6}|$ as well as the theoretical asymptotic error constant $\eta$. The values of $|\frac{e_n}{e_{n-1}^6}|$ agree up to 9 significant digits with $\eta$.

To further check the convergence behavior of proposed scheme (1.6), we list additional functions with roots and initial guesses in Table 4.

For the purpose of comparison, we first identify methods (1.1), (1.2), (1.3), (1.4), (1.5) by **DBN**, **CHU**, **PGU**, **WAN**, **NET** with ($\beta = -\frac{1}{2}$), respectively. Table 5 displays the values of $|x_n - \alpha|$ for methods **DBN**, **CHU**, **PGU**, **WAN**, **NET**, **KY1**, **KY2**, **KY3**. As can be seen in Table 5, proposed methods show favorable or equivalent performance as compared with existing met hods **DBN**, **CHU**, **PGU**, **WAN** and **NET**. It is well expected that method **DBN** displays the largest error of $|x_n - \alpha|$ due to its lower order of four, in comparison with the rest of the listed methods. In Table 5, italicized numbers refer to the least errors $|x_n - \alpha|$ within

**Table 4**
Additional test functions $f_i(x)$, roots $\alpha$ and initial guesses $x_0$.

| $i$ | $f_i(x)$ | $\alpha$ | $x_0$ |
|---|---|---|---|
| 1 | $\sin(\pi x) + (x-1)^2$ | 1 | 0.95 |
| 2 | $2\cos(x^2) - \log(1+4x^2-\pi) - \sqrt{2}$ | $\sqrt{\pi/4}$ | 1.0 |
| 3 | $\cos(x^2 + x + \frac{9}{16}) + 4x + 1 - i\sqrt{5}$ | $-\frac{1}{2} + i\frac{\sqrt{5}}{4}$ | $-0.45 + 0.5i$ |
| 4 | $\sin(x^3) - 3 + (x+1)\log(e^3 + x^2)$ | 0 | 0.1 |
| 5 | $x^5 + x^2 + xe^{2x} - 7$ | 0.906962092165271 | 0.85 |
| 6 | $2x - \pi - 2 + \log(e^2 + 4x^2 - \pi^2)$ | $\pi/2$ | 1.5 |
| 7 | $x^4 + x^2e^{1-x} - 2 + \sin(2 + x^3)$ | 0.926429193234728 | 0.9 |

Here $\log z (z \in \mathbb{C})$ represents a principal analytic branch with $-\pi \leqslant Im(\log z) < \pi$.

**Table 5**
Comparison of $|x_n - \alpha|$ for $f_1(x) - f_7(x)$ among listed methods.

| $f$ | $x_0$ | $\lvert x_n-\alpha\rvert$ | DBN | CHU | PGU | WAN | NET | KY1 | KY2 | KY3 |
|---|---|---|---|---|---|---|---|---|---|---|
| $f_1$ | 0.95 | $\lvert x_1-\alpha\rvert$ | 4.53e−8[a] | 6.35e−9 | 8.25e−9 | 7.67e−9 | 6.39e−9 | 8.11e−10 | 3.89e−9 | 2.43e−9 |
| | | $\lvert x_2-\alpha\rvert$ | 1.36e−31 | 5.63e−50 | 3.05e−49 | 3.07e−49 | 5.88e−50 | *1.18e−55* | 1.17e−51 | 2.91e−53 |
| | | $\lvert x_3-\alpha\rvert$ | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 |
| $f_2$ | 1.0 | $\lvert x_1-\alpha\rvert$ | 2.16e−4 | 9.54e−6 | 1.16e−4 | 4.51e−6 | 2.86e−5 | 4.43e−6 | 1.31e−5 | 2.06e−5 |
| | | $\lvert x_2-\alpha\rvert$ | 1.33e−14 | 4.52e−29 | 7.70e−22 | 1.11e−30 | 9.18e−26 | *2.25e−31* | 1.078e−28 | 4.48e−27 |
| | | $\lvert x_3-\alpha\rvert$ | 1.90e−55 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 |
| | | $\lvert x_4-\alpha\rvert$ | 0.0e−100 | | | | | | | |
| $f_3$ | −0.45 +0.5$i$ | $\lvert x_1-\alpha\rvert$ | 8.69e−8 | 1.87e−9 | 2.518e−9 | 1.53e−9 | 1.91e−9 | 7.55e−10 | 1.38e−9 | 9.58e−10 |
| | | $\lvert x_2-\alpha\rvert$ | 2.18e−31 | 5.64e−55 | 3.94e−54 | 1.87e−55 | 5.92e−55 | *1.40e−57* | 6.77e−56 | 5.24e−57 |
| | | $\lvert x_3-\alpha\rvert$ | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 |
| $f_4$ | 0.1 | $\lvert x_1-\alpha\rvert$ | 1.25e−8 | 7.15e−9 | 1.19e−8 | 8.52e−9 | 1.07e−8 | 1.21e−9 | 8.31e−9 | 5.87e−9 |
| | | $\lvert x_2-\alpha\rvert$ | 1.12e−37 | 1.87e−52 | 7.38e−51 | 7.79e−52 | 3.04e−51 | *3.68e−59* | 5.07e−52 | 4.20e−53 |
| | | $\lvert x_3-\alpha\rvert$ | 7.68e−154 | 0.0e−151 | 0.0e−149 | 0.0e−150 | 0.0e−202 | 0.0e−218 | 0.0e−204 | 0.0e−206 |
| $f_5$ | 0.85 | $\lvert x_1-\alpha\rvert$ | 3.38e−5 | 4.11e−8 | 1.79e−7 | 8.16e−6 | 9.37e−8 | 3.21e−6 | 3.59e−6 | 1.38e−6 |
| | | $\lvert x_2-\alpha\rvert$ | 3.76e−18 | *7.03e−45* | 1.35e−40 | 3.28e−29 | 1.48e−42 | 6.33e−32 | 1.34e−31 | 2.03e−34 |
| | | $\lvert x_3-\alpha\rvert$ | 6.71e−70 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 |
| | | $\lvert x_4-\alpha\rvert$ | 0.0e−100 | | | | | | | |
| $f_6$ | 1.5 | $\lvert x_1-\alpha\rvert$ | 4.32e−7 | 2.73e−10 | 2.60e−9 | 9.47e−10 | 1.69e−9 | 5.08e−10 | 3.16e−10 | 1.63e−10 |
| | | $\lvert x_2-\alpha\rvert$ | 5.08e−28 | 5.48e−61 | 5.07e−54 | 4.13e−57 | 2.17e−55 | 5.52e−59 | 1.99e−60 | *1.86e−62* |
| | | $\lvert x_3-\alpha\rvert$ | 0.0e−99 | 0.0e−99 | 0.0e−99 | 0.0e−99 | 0.0e−99 | 0.0e−99 | 0.0e−99 | 0.0e−99 |
| $f_7$ | 0.9 | $\lvert x_1-\alpha\rvert$ | 5.42e−8 | 1.40e−11 | 4.65e−10 | 1.72e−11 | 2.58e−10 | 9.64e−11 | 2.08e−11 | 2.88e−13 |
| | | $\lvert x_2-\alpha\rvert$ | 1.23e−30 | 1.44e−66 | 1.93e−56 | 2.20e−66 | 3.61e−58 | 2.20e−61 | 1.63e−65 | *7.37e−77* |
| | | $\lvert x_3-\alpha\rvert$ | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 | 0.0e−100 |

[a] 4.53e−8 denotes $4.53 \times 10^{-8}$.

the prescribed error bound. The method **DBN** requires four iterations to meet the error criterion for test functions $f_2$ and $f_5$, unlike the rest of other listed methods requiring only three. Even with the same order of convergence, one should note that the behavior of local convergence of $|x_n - \alpha|$ is dependent on $c_j$, namely $f(x)$ and $\alpha$.

Although limited to the test functions chosen in these numerical experiments, based on the results after 2 iterations, **KY1** has shown best accuracy in $f_1, f_2, f_3, f_4$, while **CHU** in $f_5$, and **KY3** in $f_6, f_7$. Nevertheless, one should be aware that no iterative method always shows best accuracy for all the test functions. It is not too much to emphasize that computational accuracy is sensitively dependent on the structures of the iterative methods, the sought zeros and the test functions as well as good initial approximations. The corresponding efficiency index for the proposed family of methods (1.6) is found to be $6^{1/4}$, which is better than $4^{1/4}$ for the classical double-Newton method. The current analysis utilizing 2-point information will lead us to a new development of another family of higher-order root-finders.

We now are ready to discuss the complex dynamics of rational iterative maps (4.6) and (4.14) applied to various polynomials. To continue our discussion, let us first identify the three members of rational iterative map (4.6) by **GKN6m0H6**,**GKN6m1H3** and **GKN6m118H0** respectively with $m = 0, \mathtt{H} = 6, m = 1, \mathtt{H} = 3$ and $m = 11/8, \mathtt{H} = 0, g = 1, c = d = 0$ in Table 1. In addition, we identify one member of rational iterative map (4.14) by **Neta6** with $\beta = 3 - 2\sqrt{2}$ in Table 2. A variety of examples are shown here. Basins of attraction for both rational iterative maps (4.6) and (4.14) are illustrated by closely following the technique shown in [9].

**Example 1.** As a first example, we have taken a quadratic polynomial with all real roots:

$$p_1(z) = z^2 - 1. \tag{5.2}$$

Basins of attraction for iterative maps **GKN6m0H6**, **GKN6m1H3**, **GKN6m118H0** and **Neta6** with $p_1(z) = z^2 - 1$ are illustrated, respectively from left to right in Fig. 3. The darker a point of each basin gets, the slower it converges to a root. At a root or an extraneous fixed point its color is white, while getting darker for more iterations required for convergence within the iteration limit. At black points, we recognize that the corresponding iterative maps did not converge within the iteration limit of 40 currently prescribed in this experiment. Based on displayed results, we find that iterative map **Neta6** has performed better. Indeed, Table 6 shows average numbers of iterations to converge within the given error bound per point.

**Example 2.** As a second example, we have taken a cubic polynomial with one real and two complex roots:

$$p_2(z) = z^3 - 1. \tag{5.3}$$

Basins of attraction for **GKN6m0H6**, **GKN6m1H3**, **GKN6m118H0** and **Neta6** with $p_2(z) = z^3 - 1$ are illustrated, respectively from left to right in Fig. 4. **Neta6** performed best as in Example 1, but now it has more black points than before.

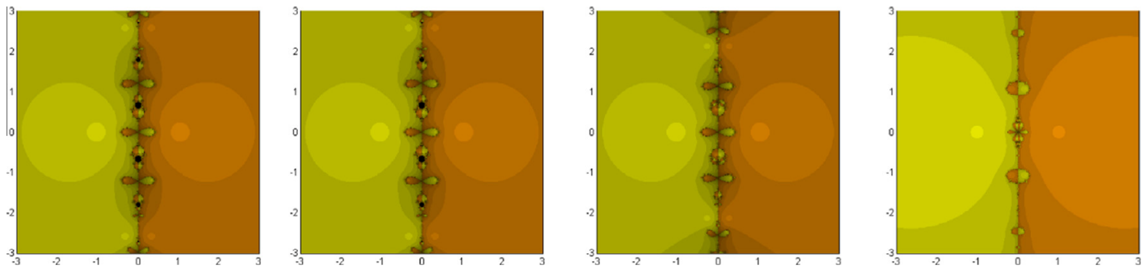

**Fig. 3.** Comparison of basins of attraction for $p_1(z) = z^2 - 1$.

**Table 6**
Average numbers of iterations for convergence per point.

| Examples | $p(z)$ | **GKNm0H6** | **GKNm1H3** | **GKN6m118H0** | **Neta6** |
|---|---|---|---|---|---|
| 1 | $z^2 - 1$ | 4.1380 | 4.1380 | 4.1298 | 2.5744 |
| 2 | $z^3 - 1$ | 5.7572 | 5.6097 | 5.4912 | 3.4305 |
| 3 | $z^3 - z$ | 5.1652 | 4.9327 | 5.0016 | 3.2130 |
| 4 | $z(z^2+1)(z^2+4)$ | 5.5559 | 5.2736 | 5.2439 | 4.0200 |
| 5 | $(z+1/2-i)(z+1/2-2i)(z+1)$ | 5.1273 | 5.1391 | 5.2274 | 3.2345 |
| 6 | $p_6(z)$[a] | 9.8523 | 7.7598 | 6.6478 | 19.7674 |
| | Averages | 7.1198 | 6.57058 | 6.34834 | 7.24796 |

[a] $p_6(z) = z^6 - \frac{1}{2}z^5 + \frac{11(1+i)}{4}z^4 - \frac{(19+3i)}{4}z^3 + \frac{(11+5i)}{4}z^2 - \frac{(11+i)}{4}z + \frac{3}{2} - 3i.$

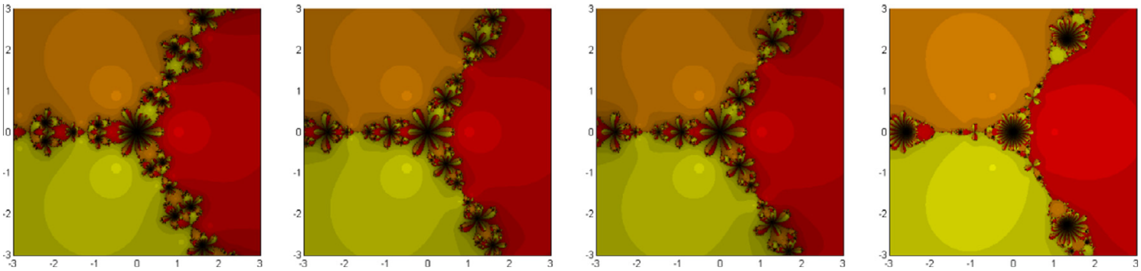

**Fig. 4.** Comparison of basins of attraction for $p_2(z) = z^3 - 1$.

**Example 3.** As a third example, we have taken a cubic polynomial with one real roots 0, −1,1:

$$p_3(z) = z^3 - z. \tag{5.4}$$

Basins of attraction for **GKN6m0H6**, **GKN6m1H3**, **GKN6m118H0** and **Neta6** with $p_3(z) = z^3 - z$ are illustrated, respectively from left to right in Fig. 5. Even though **Neta6** has less black points, the basin for the root $\alpha = 0$ is smaller. This means that there are points closer to this root that converge to one of the other two.

**Example 4.** As a fourth example, we have taken a quintic polynomial with one real and four complex roots:

$$p_4(z) = z(z^2 + 1)(z^2 + 4). \tag{5.5}$$

Basins of attraction for **GKN6m0H6**, **GKN6m1H3**, **GKN6m118H0** and **Neta6** with $p_4(z) = z(z^2 + 1)(z^2 + 4)$ are illustrated, respectively from left to right in Fig. 6. The phenomenon observed in the previous example is more pronounced here. **Neta 6** has much larger basins for the roots $\pm 2i$.

**Example 5.** As a fifth example, we have taken another cubic polynomial with one real and two complex roots that are not complex conjugate to each other. The roots are $-\frac{1}{2} + i, -\frac{1}{2} + 2i$ and $-1$:

$$p_5(z) = \left(z + \frac{1}{2} - i\right)\left(z + \frac{1}{2} - 2i\right)(z + 1). \tag{5.6}$$

Basins of attraction for **GKN6m0H6**, **GKN6m1H3**, **GKN6m118H0** and **Neta6** with (5.6) are illustrated, respectively from left to right in Fig. 7. Notice that the basin for **Neta6** for the root $-\frac{1}{2} + i$ is cut to two because points that should have been in that basin has converged to $-\frac{1}{2} + 2i$. This did not happen with the new methods.

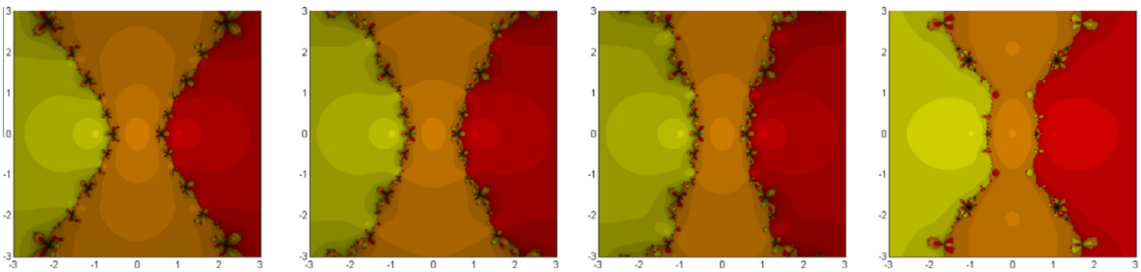

**Fig. 5.** Comparison of basins of attraction for $p_3(z) = z^3 - z$.

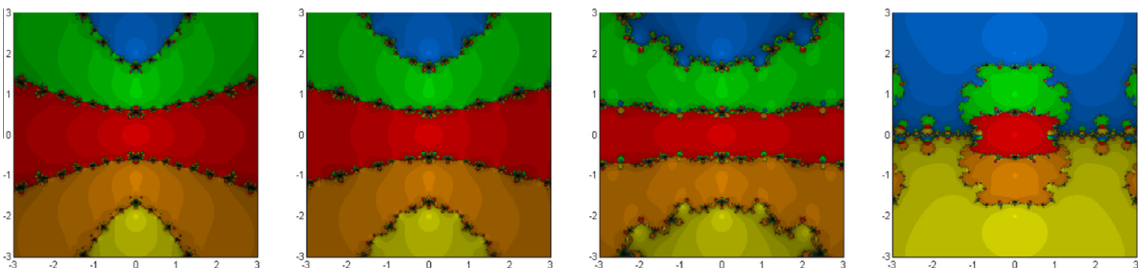

**Fig. 6.** Comparison of basins of attraction for $p_4(z) = z(z^2 + 1)(z^2 + 4)$.

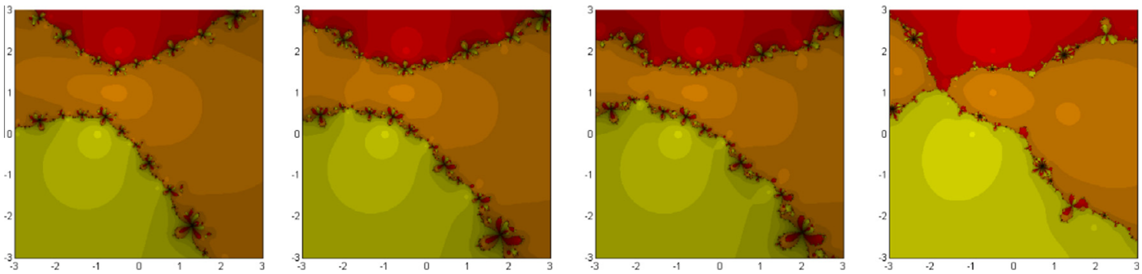

**Fig. 7.** Comparison of basins of attraction for $p_5(z) = (z+1)(z+\frac{1}{2}-i)(z+\frac{1}{2}-2i)$.

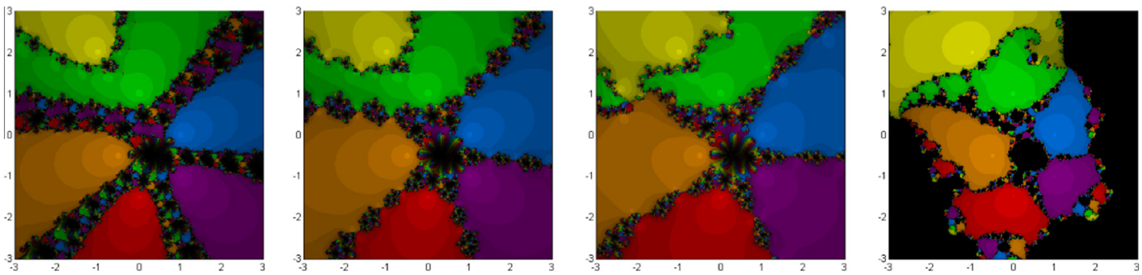

**Fig. 8.** Comparison of basins of attraction for $p_6(z) = z^6 - \frac{1}{2}z^5 + \frac{11(1+i)}{4}z^4 - \frac{(19+3i)}{4}z^3 + \frac{(11+5i)}{4}z^2 - \frac{(11+i)}{4}z + \frac{3}{2} - 3i$.

**Example 6.** As a sixth example, we have taken a sextic polynomial with real and complex roots $-1+2i, -\frac{1}{2}-\frac{1}{2}i, -1.5i, i, 1$ and $1-i$:

$$p_6(z) = z^6 - \frac{1}{2}z^5 + \frac{11(1+i)}{4}z^4 - \frac{(19+3i)}{4}z^3 + \frac{(11+5i)}{4}z^2 - \frac{(11+i)}{4}z + \frac{3}{2} - 3i. \tag{5.7}$$

Basins of attraction for **GKN6m0H6**, **GKN6m1H3**, **GKN6m118H0** and **Neta6** with (5.7) are illustrated, respectively from left to right in Fig. 8. Now **Neta6** has large black regions and in fact the average number of iterations per point is very large as can be seen in Table 6. The difference between this example and the previous one is that the coefficients are no longer real. It is possible that this is the reason for the large number of points from which the method did not converge.

Even though in the first 3 examples **Neta6** performed batter than our new methods, the last 3 examples shows the robustness of our new methods relative to **Neta6**. The last example was the toughest for all methods, but was worse for **Neta6** as can be seen in Table 6.

We have shown a technique of selecting parameters of the weighting function of a proposed iterative method. One such technique is given by positioning the extraneous fixed points of the corresponding rational iterative map applied to a well-known quadratic polynomial $p(z) = z^2 - 1$ on the imaginary axis to get better basins of attraction. In view of the fact that the imaginary axis is the boundary of basins of attraction for classical Newton's method when applied to the polynomial $p(z) = z^2 - 1$, it is worth to position the extraneous fixed points on the imaginary axis for improving chance of obtaining better basins of attraction. In our future work for the development of a new family of iterative methods, our current approach will play an important role in selecting parameters of the relevant weight functions to enhance basins of attraction of the corresponding rational iterative map arising from the proposed iterative method.

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## References

[1] L.V. Ahlfors, Complex Analysis, McGraw-Hill Book Inc, 1979.
[2] Carlos Andreu, Noelia Cambil, Alicia Cordero, Juan R. Torregrosa, A class of optimal eighth-order derivative-free methods for solving the Danchick–Gauss problem, Appl. Math. Comput. 232 (2014) 237–246.
[3] I.K. Argyros, D. Chen, Q. Qian, The Jarratt method in Banach space setting, J. Comput. Appl. Math. 51 (1994) 103–106.
[4] W. Bi, Q. Wu, H. Ren, A new family of eighth-order iterative methods for solving nonlinear equations, Appl. Math. Comput. 214 (4) (2009) 236–245.
[5] C. Chun, Some improvements of Jarratt's method with sixth-order convergence, Appl. Math. Comput. 190 (2007) 1432–1437.
[6] C. Chun, B. Neta, Basins of attraction for several optimal fourth order methods for multiple roots, Math. Comput. Simul. 103 (2014) 39–59.
[7] C. Chun, B. Neta, Basins of attraction for Zhou-Chen-Song fourth order family of methods for multiple roots, Math. Comput. Simul. 109 (2015) 74–91.

[8] C. Chun, B. Neta, Sujin Kim, On Jarratt's family of optimal fourth-order iterative methods and their dynamics, Fractals, in press. http://dx.doi.org/10.1142/S0218348X1450013.
[9] C. Chun, B. Neta, J. Kozdon, M. Scott, Choosing weight functions in iterative methods for simple roots, Appl. Math. Comput. 227 (2014) 788–800.
[10] A. Cordero, J. Garcia-Maimo, J.R. Torregrosa, M.P. Vassileva, P. Vindel, Chaos in King's iterative family, Appl. Math. Lett. 26 (2013) 842–848.
[11] L. Fanga, T. Chen, L. Tian, L. Sun, B. Chen, A modified Newton-type method with sixth-order convergence for solving nonlinear equations, Proc. Eng. 15 (2011) 3124–3128.
[12] B. Kalantari, Y. Jin, On extraneous fixed-points of the basic family of iteration functions, BIT Numer. Math. 43 (2003) 453–458.
[13] Y.H. Geum, Y.I. Kim, A biparametric family of four-step sixteenth-order root-finding methods with the optimal efficiency index, Appl. Math. Lett. 24 (2011) 1336–1342.
[14] P. Jarratt, Multipoint iterative methods for solving certain equations, Comput. J. 8 (4) (1966) 398–400.
[15] S.K. Khattri, I.K. Argyros, How to develop fourth and seventh order iterative methods, Novi Sad J. Math. 40 (2) (2010) 61–67.
[16] J. Kou, Y. Li, X. Wang, Fourth-order iterative methods free from second derivative, Appl. Math. Comput. 184 (2007) 880–885.
[17] H.T. Kung, J.F. Traub, Optimal order of one-point and multipoint iteration, J. Assoc. Comput. Mach. 21 (1974) 643–651.
[18] Á. Alberto Magreñán, Different anomalies in a Jarratt family of iterative root-finding methods, Appl. Math. Comput. 233 (2014) 29–38.
[19] Á. Alberto Magreñán, Alicia Cordero, José M. Gutiérrez, Juan R. Torregrosa, Real qualitative behavior of a fourth-order family of iterative methods by using the convergence plane, Math. Comput. Simul. 105 (2014) 49–61.
[20] B. Neta, A sixth-order family of methods for nonlinear equations, Int. J. Comput. Math. 7 (1979) 157–161.
[21] B. Neta, C. Chun, M. Scott, Basins of attraction for optimal eighth order methods to find simple roots of nonlinear equations, Appl. Math. Comput. 227 (2014) 567–592.
[22] B. Neta, M. Scott, C. Chun, Basins of attraction for several methods to find simple roots of nonlinear equations, Appl. Math. Comput. 218 (2012) 10548–10556.
[23] S.K. Parhi, D.K. Gupta, A sixth order method for nonlinear equations, Appl. Math. Comput. 203 (2008) 50–55.
[24] Y. Peng, H. Feng, Q. Li, X. Zhang, A fourth-order derivative-free algorithm for nonlinear equations, J. Comput. Appl. Math. 235 (2011) 2551–2559.
[25] M.S. Petković, B. Neta, L.D. Petković, J. Džunić, Multipoint Methods for Solving Nonlinear Equations, Elsevier, 2012.
[26] B.V. Shabat, Introduction to Complex Analysis Part II, Functions of Several Variables, American Mathematical Society, 1992.
[27] F. Soleymani, Regarding the accuracy of optimal eighth-order methods, Math. Comput. Model. 53 (5–6) (2011) 1351–1357.
[28] B.D. Stewart, Attractor basins of various root-finding methods (M.S. thesis), Naval Postgraduate School, Department of Applied Mathematics, Monterey, CA, 2001.
[29] J.F. Traub, Iterative Methods for the Solution of Equations, Chelsea Publishing Company, 1982.
[30] E.R. Vrscay, W.J. Gilbert, Extraneous fixed points, basin boundaries and chaotic dynamics for Shröder and König rational iteration functions, Numer. Math. 52 (1988) 1–16.
[31] X. Wang, L. Liu, Two new families of sixth-order methods for solving non-linear equations, Appl. Math. Comput. 213 (2009) 73–78.
[32] S. Wolfram, The Mathematica Book, fifth ed., Wolfram Media, 2003.