Consensus for Nonlinear Monotone Networks with Unilateral Interactions

S. Manfredi* and D. Angeli$^{\diamond}$

Abstract— **This paper deals with an extended framework of the distributed asymptotic agreement problem by allowing the presence of unilateral interactions (optimistic or pessimistic) in place of bilateral ones, for a large class of nonlinear monotone time-varying networks. In this original setup we firstly introduce notions of unilateral optimistic and/or pessimistic interaction, of associated bicolored edge in the interaction graph and a suitable graph-theoretical connectedness property. Secondly, we formulate a new assumption of integral connectivity and show that it is sufficient to guarantee exponential convergence towards the agreement subspace. Finally, we remark that the proposed conditions are also necessary for consensuability. Theoretical advances are emphasized through illustrative examples given both to support the discussion and to highlight how the proposed framework extends all existing conditions for consensus of monotone networks.**

I. Introduction

In the last decade the scientific community has devoted considerable attention to the agreement problem (see [1], [3] just to cite a few) formulating several criteria to assess consensus both in discrete and continuous time (e.g. [6]), under different classes of both nonlinear time invariant and switching/time varying networks [5], [13], [2] or by adopting several remarkable new instantaneous, averaged or integrated notions of connectivity [9], [12], [11], [6], [8]. All the above contributions assume the presence of *bilateral* agent interactions, in the sense that each node with a certain state value may feel the influence of the neighbooring agents regardless of whether their current state is higher or lower than its own (precise definitions to be given later). Recent works extend the standard diffusive-type interaction scenario by considering signed graph networks in which the edges may assume also negative weights ([14]), or introducing protocols where each agent is affected by the influence of the neighbooring nodes with maximum and minimum state value ([4]), however the interaction between agents is still assumed to be bilateral. In some applications, however, node interactions are likely to be unilateral or specifically designed to be such. For example, in the opinion dynamics setting, an agent may be willing to update its own belief on the basis of the input from a certain neighbour only if its opinion happens to be "optimistic" [and/or "pessimistic"] (that is larger or smaller) when compared to his current state value. Additionally, in many gossip and sampling algorithms, nodes update laws are designed so that they are influenced just by agents carrying out higher or lower estimated or measured values. Therefore, both from a theoretical and practical point of view, it is of interest to qualify structural conditions on the network topology and agents interactions which may guarantee exponential convergence to an agreement state in the extended scenario of unilateral node interactions.

A. Paper Contribution

In this paper we develop original tools for the study of the agreement problem under minimal connectivity assumptions for nonlinear time-varying monotone networks of agents in the presence of unilateral interactions. This scenario is challenging to be dealt with due to the presence of unilateral (intrinsically nonlinear) node interactions and the fact that connectivity becomes both state-dependent and time-dependent: while the interplay of these two factors often lends itself to conditions for consensus that need to be tested along solutions of the system, our approach aims at avoiding such a limitation. In this respect our contributions are manifold. Firstly we introduce new notions of directed state-frozen agents interaction, that extends the previous ones being intrinsically unilateral and thus nonlinear. The notion models separately (and possibly simultaneously) "optimistic" and "pessimistic" type influences on a node from neighbouring agents. Accordingly new notions of bicolored interaction graphs and suitable associated connectivity definitions are provided to capture the nature of information flows allowed by unilateral interactions. Interestingly, the above notions extend the ones for monocolored graphs and boil down to the former in the case of standard bilateral interactions. Secondly, based on the previous graph-theoretical concepts, we propose novel integral connectivity conditions which characterize exponential convergence to consensus *from arbitrary initial conditions of individual agents* for monotone time-varying networks under unilateral interactions, thus significantly extending (and encompassing as a special case) existing consensus criteria for (linear and nonlinear) monotone networks. Thirdly, we highlight how our approach avoids the circular argument by which solutions depend on the connectivity and the latter is in turn influenced by state evolutions. This type of circular argument normally makes up for conditions that can hardly be tested in the case of time-varying nonlinear agent dynamics. Moreover, for the special case of unilateral interactions, this gives rise to a combinatorial explosion of possible "active" interaction graphs as influenced by all possible orderings of state variables

*S. Manfredi is with the Department of Electrical Engineering and Information Technology, University of Naples Federico II, Italy and with the Control and Power Group, Electrical and Electronic Engineering Department, Imperial College, London. Email: sabato.manfredi@unina.it

$^{\diamond}$D. Angeli is with the Control and Power Group, Electrical and Electronic Engineering Department, Imperial College, London and with the Dip. di Ingegneria dell'Informazione, University of Florence, Italy, University of Florence, Italy. Email: d.angeli@imperial.ac.uk

which would most likely be hard to treat for large networks. Remarkably, our conditions only entail integrals performed in time on frozen state variables and can be related to a seemingly weaker notion (later called *Equilibrium Integral Connectivity*) which may be tested by considering consensus states alone. We emphasize that the proposed convergence results guarantee exponential consensus for any assignment of initial conditions, despite adoption of unilateral interaction as well as nonlinearity and time variance of the network. Finally, we prove necessity of the conditions on a specific class of models which serves as a paradigm of network with unilateral interactions.

II. Graph theoretical preliminaries

Abstracting the nature of information flows in networks with unilateral interactions requires the introduction of specific graph-theoretical concepts. Since these are new, to the best of our knowledge, and possibly deserve attention of their own, regardless of the specific results reported in this paper, we devote an independent Section for their detailed explanation.

Recall that a directed graph (digraph) $G(N,E)$ is a pair of sets N, E, with $N=\{1,2,\dots,n\}$ the set of *nodes* and $E\subseteq N\times N$ the set of *edges*. A node j is reachable from node i if there exists a path in a directed graph connecting nodes i and j, namely there is a finite sequence $n_1,n_2,...,n_k$ of distinct nodes such that $(n_i,n_{i+1})\in E$ for $i=1,\dots,k-1$ with $n_1=i$ and $n_k=j$. A digraph is *quasi-strongly connected* if there exists a node (often called the 'root') from which every other node is reachable. Equivalently, using the notions of subgraphs and trees, this condition can be formulated by asking that the digraph admits a *spanning tree*. A digraph is *strongly connected* if every node is reachable from any other node.

Next, we define the concept of bicolored graph to deal with the presence of two kinds of influences in the network (optimistic and pessimistic).

Definition 1 (*Bicolored graph*) *We say that $G(N,E_o,E_p)$ is a directed bicolored graph if $N=\{1,2,\dots,n\}$ is the set of nodes and $E_o\subseteq N\times N$, $E_p\subseteq N\times N$ are two set of edges with, possibly, $E_o\cap E_p\neq\emptyset$.*

Notice that this notion is different from the usual notion of bicolored graph ([17]) in that a single edge is allowed to take both "colors" at the same time, namely an edge

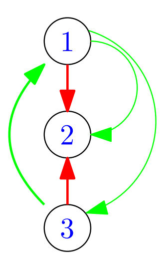

Fig. 1. An example of Bicolored graph

(i,j) may belong both to E_o and E_p. This is needed in order to accommodate for standard bilateral interactions within this generalized framework. Graphically, a bicolored graph can be represented by using circles to represent nodes and arrows (oriented arcs) joining the nodes to represent the elements of E_o and E_p by using two different colors. A simple example of bicolored graph is shown in Fig. 1, where the convention has been adopted to use green for edges in E_o and red for edges in E_p. Notice, in this respect, that arc $(1,2)$ is both an element of E_o and E_p.

We propose the following original notion of connectivity for bicolored graphs:

Definition 2 (*Quasi-strongly connected bicolored graph*) *We say that the bicolored graph $G(N,E_o,E_p)$ is quasi-strongly connected if for any ordered pair $(i,j)\in N^2$ with $i\neq j$ there exist at least one node $r_{ij}\in N$, with a path from r_{ij} to i in $G(N,E_p)$ and a path from r_{ij} to j in $G(N,E_o)$ (with either path possibly being of 0 length when $r_{ij}=i$ or $r_{ij}=j$).*

Notice that, in the example considered in Fig. 1, there are 6 possible choices of ordered pairs (i,j), namely $(1,2),(1,3),(2,1),(2,3),(3,1)$, and $(3,2)$, and for each such choice the corresponding "root" node r_{ij} and associated paths in $G(N,E_p)$ and $G(N,E_o)$ can be selected as in the table below:

(i,j)	r_{ij}	path in E_p	path in E_o
$(1,2)$	1	$\emptyset$	$1,2$
$(1,3)$	1	$\emptyset$	$1,3$
$(2,1)$	1	$1,2$	$\emptyset$
$(2,3)$	3	$3,2$	$\emptyset$
$(3,1)$	3	$\emptyset$	$3,1$
$(3,2)$	3	$\emptyset$	$3,1,2$

It is worth pointing out that for a given pair (i,j) the choice of the root node r_{ij} and associated optimistic and pessimistic paths need not be unique. For instance, in the example above, one could alternatively take $r_{21}=3$, with pessimistic path $3,2$ and optimistic path $3,1$.

A. Relation to standard notions of graph connectivity

In the following we will highlight how the proposed notions of connectivity for bicolored digraphs relate to standard ones in some special cases.

1) Optimistic only networks: $E_p=\emptyset$ (or $E_o=\emptyset$): In this case of $E_p=\emptyset$ (resp. $E_o=\emptyset$), the Definiton 2 trivially overlaps with the standard definition of strong connectivity of the (monocolored) graph $G(N,E_o)$ (resp. $G(N,E_p)$) [16]. Indeed, due to the absence of pessimistic edges ($E_p=\emptyset$), the only option is to choose $r_{ij}=i$ so that the path of length 0 connects r_{ij} to i in $G(N,E_p)$. Because of this, then, a path of optimistic edges exists between r_{ij} and j in $G(N,E_o)$ iff there is a path between i and j in $G(N,E_o)$ for all possible pairs of nodes. As already remarked this property is usually referred to as strong connectivity of $G(N,E_o)$.

2) Bilateral interactions: $E_p = E_o$: As it will be clearer in the following, when interactions between agents are of bilateral nature, viz. not conditioned by the reciprocal ordering of the agents states, (i.e. bilateral interactions), every edge belongs to both E_o and E_p. It is therefore of interest to highlight which graph-theoretical concept underlies the notion of bicolored quasi-strong connectedness in the special case in which the two edge sets E_p, E_o coincide. This is carried out in the Lemma given below:

Lemma 1 *The bicolored quasi-strong connectivity of $G(N, E, E)$ is equivalent to quasi-strong connectivity of $G(N, E)$.*

The proof is omitted for space constraints. A simpler interpretation of quasi-strong connectivity of bicolored graphs can be gained by considering the *mirroring* operation defined below.

Definition 3 (*Mirrored Edge set*) *We say that $\overleftarrow{E} \subseteq N \times N$ is the mirrored edge set of E if $\overleftarrow{E} := \{(j, i) : (i, j) \in E\}$.*

The same operation can be applied to the edge sets of a bicolored graph as detailed below:

Definition 4 (*Mirrored Graph*) *We say that $\overleftarrow{G}$ is a pessimistic mirrored graph of $G(N, E_o, E_p)$ if $\overleftarrow{G} = G(N, E_o, \overleftarrow{E}_p)$ (resp. $\overleftarrow{G} = G(N, \overleftarrow{E}_o, E_p)$).*

We similarly defined the optimistic mirrored graph, replacing E_o by $\overleftarrow{E}_o$ in the previous definition.

The following is an alternative formulation of quasi-strong connectivity that makes use of the mirroring operation:

Definition 5 (*Single switch Strong connectedness*) *We say that the pessimistic (resp. optimistic) mirrored graph $\overleftarrow{G} = G(N, E_o, \overleftarrow{E}_p)$ (resp. $\overleftarrow{G} = G(N, \overleftarrow{E}_o, E_p)$) is strongly connected with a single color switch if for any pair $(i, j) \in N^2$ with $i \neq j$, there exists a path between i and j in the pessimistic (resp. optimistic) mirrored graph $G(N, E_o, \overleftarrow{E}_p)$ (resp. $G(N, \overleftarrow{E}_o, E_p)$), of the following kind: $i = n_1, n_2, \ldots, n_m = j$ are distinct nodes such that $(n_q, n_{q+1}) \in \overleftarrow{E}_p$ (resp. E_p) for $q = 1, \ldots, k-1$ while $(n_q, n_{q+1}) \in E_o$ (resp. $\overleftarrow{E}_o$) for $q = k, \ldots, m-1$ with $1 \leq k \leq m$ (so as to allow "single-colored" paths as a special case).*

Equivalence to the previous definition is straightforward when considering that 'color' switch happens exactly at the node r_{ij}, viz. $n_k = r_{ij}$. For illustration purposes we display in Fig. 2 the pessimistic mirrored graph $\overleftarrow{G} = G(N, E_o, \overleftarrow{E}_p)$ relative to the example in Fig. 1.

III. Problem formulation and main result

We are now ready to state our consensus problem, delineating the considered class of networks equations as well as the translation of dynamics into suitable interaction graphs, for the analysis of consensuability. In the following, all vectors are assumed to be column vectors and we write $x = (x_1, \ldots, x_n)$ for the column vector $x \in \mathbb{R}^n$ while the symbol $|x|$ denotes its Euclidean norm. $\mathcal{K} \subset \mathbb{R}^n$ is a compact set, $\mathbf{1}$ is the vector of all ones and e_j is the j-th element of the canonical basis of $\mathbb{R}^n$, where n should normally be clear from the context. We recall that for a function $f(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, piecewise continuous in t and locally Lipschitz continuous with respect to x, the associated system of differential equations $\dot{x}(t) = f(t, x(t))$, is said to be *cooperative* if for any $i \in \{1, 2, \ldots, n\}$, $f_i(t, x)$ is non-decreasing with respect to x_j for all $j \neq i$. Notice that this condition implies (and is in fact equivalent) to monotonicity of the flow $\phi(t, x_0)$ with respect to initial conditions, namely, for all $t \geq 0$, it holds $\phi(t, x_1) \geq \phi(t, x_2)$ if $x_1 \geq x_2$ (where "$\geq$" is meant componentwise), [15].

A network is described by a nonlinear dynamical system:

$$\dot{x}(t) = f(t, x(t)) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $t \in \mathbb{R}_+$ is the time variable and f is the vector field $\mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$. We denote with $F(t, x) = [F_{ij}(t, x)]$ the Jacobian matrix, when this can be defined and with $F_{ij}^+(t, x)$ (resp. $F_{ij}^-(t, x)$) the right (resp. left) partial derivative of $f_i(t, x)$ with respect to x_j at x.
We assume: f is locally Lipschitz continuous with respect to x uniformly in time, viz. for all compacts $\mathcal{K} \in \mathbb{R}^n$ there exists $L_{\mathcal{K}} > 0$, such that, for all $x_a, x_b \in \mathcal{K}$ and all $t \geq 0$ it holds $|f(t, x_a) - f(t, x_b)| \leq L_{\mathcal{K}} |x_a - x_b|$;[1]
The assumptions on f, imply the local existence and the uniqueness of the system's solution on some maximally extended open time interval. Additionally in the following we will assume nonlinear networks (1) to be cooperative in the sense stated above. For any vector x, we define the following quantities: $x_M = \max_{k \in N}\{x_k\}$; $x_m = \min_{k \in N}\{x_k\}$. We start making the following assumption to guarantee that consensus configurations are equilibrium states of network (1).

Assumption 1 *We assume a cooperative nonlinear network (1) with f that admits an agreement equilibrium set, that is:*

$$\mathcal{E} := span_{\mathbb{R}}\{\mathbf{1}\} \subseteq \{x \in \mathbb{R}^n : f(t, x) = 0 \; \forall t \in \mathbb{R}_+\}.$$

[1]This holds, for instance, if the Jacobian is uniformly bounded as a function of time.

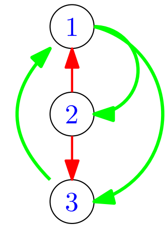

Fig. 2. Pessimistically mirrored graph

We associate a given network (1) to a bicolored digraph $G(N, E_p, E_o)$ by letting $N = \{1, 2, \ldots, n\}$ (with n the number of agents) and specifying edge sets E_p and E_o according to the following definitions:

Definition 6 (*Pessimistic edge*) *We say that $(j, i) \in E_p \subset N^2$ is a pessimistic edge connecting j to i for network* (1)*, if for all compacts $\mathcal{K} \subseteq \mathbb{R}$, there exist $\varepsilon_{\mathcal{K}} > 0$ and sufficiently large $T_{\mathcal{K}} > 0$ so that for any $t \geq 0$, for all pairs $x_i > x_j \in \mathcal{K}^2$ it holds:*

$$\int_t^{t+T_{\mathcal{K}}} f_i(\tau, x_i \mathbf{1} + e_j(x_j - x_i))\, d\tau \leq -\varepsilon_{\mathcal{K}}(x_i - x_j). \quad (2)$$

Definition 7 (*Optimistic edge*) *We say that $(j, i) \in E_o \subset N^2$ is an optimistic edge connecting j to i for network* (1)*, if for all compacts $\mathcal{K} \subseteq \mathbb{R}$, there exist $\varepsilon_{\mathcal{K}} > 0$ and sufficiently large $T_{\mathcal{K}} > 0$ so that for any $t \geq 0$, for all pairs $x_i < x_j \in \mathcal{K}^2$ it holds:*

$$\int_t^{t+T_{\mathcal{K}}} f_i(\tau, x_i \mathbf{1} + e_j(x_j - x_i))\, d\tau \geq \varepsilon_{\mathcal{K}}(x_j - x_i). \quad (3)$$

Notice that in the above definitions x_i and x_j are local variables, being arguments of universal quantifiers. Therefore, the subscripts i and j are merely typographical symbols. It is worth pointing out that inequalities (2) and (3) are an integrated measure of how much a displaced j-th agent (with respect to an agreement configuration where all agents are equal to x_i) can influence, either from below (in the pessimistic case) or above (in the optimistic one) the current state of agent i. Notice also that integration is performed on frozen state variables and not along solutions of (1). This is a generalization of the conditions introduced in [2], [9], the novelty being that influences from below and above are separately accounted for.

Definition 8 (*Bicolored interaction graph*) *We say that $G(N, E_o, E_p)$ is a bicolored interaction graph for* (1)*, where $E_o \subseteq N \times N$ is the set of optimistic edges* (3)*, and $E_p \subseteq N \times N$ is the set of pessimistic edges* (2) *with, possibly, $E_o \cap E_p \neq \emptyset$.*

To illustrate how the new definition applies to nonlinear networks consider the following network of 3 agents connected both unilaterally and bilaterally:

$$\begin{aligned} \dot{x}_1 &= \max\{0, x_3 - x_1\} \quad (4) \\ \dot{x}_2 &= (x_1 - x_2) + \min\{0, x_3 - x_2\} \\ \dot{x}_3 &= \max\{0, x_1 - x_3\} \end{aligned}$$

Throughout the paper an optimistic edge will be represented in Figures by an arc of green colour while pessimistic edges will be represented in red. From the networks equations we see that agent 1 is influenced by agent 3 provided this is supplying an 'optimistic' information. This translates into an edge from 3 to 1 in green color. Symmetrically agent 3 is influenced by agent 1 only when the latter is providing an 'optimistic' information. Again this is represented as a green arc, from 1 to 3. Agent 2, instead, is influenced by agent 1 regardless of the pessimistic or optimistic nature of the information provided. Hence, this can be modeled as two edges in green and red joining 1 to 2. Influence of agent 3 towards agent 2 is of pessimistic nature only, and this is therefore modeled as a red arc (see the scheme in Fig 1). Notice that, if not allowing pessimistic and optimistic influences to be accounted for separately, the above network's equation would only afford a single bilateral influence from node 1 to 2, (which is clearly insufficient for achieving consensus). In other words the actual graph of influences between neighboring agents would be heavily underestimated.

The following Assumption captures the appropriate connectivity property for guaranteeing solutions of a network to converge towards consensus regardless of their assigned initial conditions.

Assumption 2 (*Bicolored quasi-strong connectivity*) *We say that the network* (1) *fulfills quasi-strong connectivity if it admits an associated bicolored interaction graph $G(N, E_o, E_p)$ which is quasi-strongly connected, (or, equivalently, a single switch strongly connected mirrored graph).*

We are now ready to state our main result:

Theorem 1 *Consider the cooperative time-varying network modeled by equations* (1)*. If Assumption 2 holds, then the equilibrium set $\mathcal{E}$ is uniformly exponentially stable and $x(t)$ converges exponentially to an agreement equilibrium state.*

Additionally the following Corollary holds:

Corollary 1 *Consider the network modeled by equations* (1)*, if Assumption 2 holds, then the equilibrium set $\mathcal{E}$ is uniformly exponentially stable and $\lim_{t\to\infty} x_i(t) = \max_k x_k(0)$, $\forall i$ provided for all $i \neq j$ and all x with $x_j < x_i$ it holds:*

$$\frac{\partial f_i(t, x)}{\partial x_j} = 0 \quad (5)$$

The proof is omitted for space constraints.

Notice that condition (5) (resp. $\partial f_i / \partial x_j = 0$ for all $i \neq j$ with $x_j > x_i$) actually implies the absence of pessimistic (resp. optimistic) edges and Corollary 1 allows to assess the interesting equilibrum of maximum (resp. minimum) initial state consensus in the presence of unilateral interaction, thus extending preliminary work on the subject formulated for the max-min (i.e. bilateral) agents influence (i.e. [4]).

Furthermore, as in the case of bilateral interactions ([9]), it is useful to formulate the concept of Equilibrium interaction graph by imposing satisfaction of appropriate inequalities on the agreement subspace only. The derivations and the related results are here omitted for space constraints.

We point out that while in general Assumption 2 is not necessary for exponential consensus (counterexample may be provided, herein omitted for space constrain), it is necessary for time-periodic models. Details are herein omitted for space constraints and my be found in [10].

IV. Discussion and Representative Examples

In this Section we will discuss illustrative examples showing the merits of the proposed conditions as well as their relation to previously available approaches.

Consider first network (4) earlier introduced in Section III and composed of 3 agents connected both unilaterally and bilaterally (in a time-invariant fashion, for the sake of simplicity).

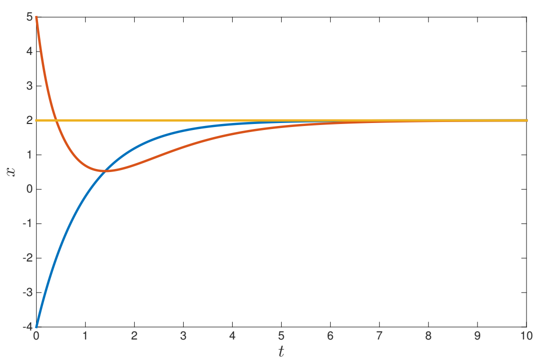

Fig. 3. Dynamic state evolution and convergence to the consensus equilibria of network (4)

The associated bicolored graph is shown in Fig. 1.

Notice that the Assumption 2 is fulfilled (as already remarked in Section II) and exponential convergence of solutions towards consensus is guaranteed (see Fig. 3 for a simulation with initial condition $x(0) = [-4\ 5\ 2]^T$). Notice that only 1 arc is bilateral in the above example; therefore, a bilateral spanning tree does not exist for the considered network and this example does not fulfill the connectivity conditions required by previous criterions, such as [9]. Additionally, they may be of easier verification even for time-invariant network's scenarios as shown by the example introduced by system (4).

Notice that, if pursuing linear time-varying embedding approaches in order to study consensus for networks involving unilateral interactions, one will be faced with a challenging problem of interplay of state and time dependence of links. Let $\mathcal{P}(n)$ denote the set of permutations of the first n integers. For each permutation of the n agents, $p_1, p_2, \ldots, p_n \in \mathcal{P}(n)$, one may consider the associated subsets of state space:

$$X(p_1, p_2, \ldots, p_n) = \{x : x_{p_1} \geq x_{p_2} \geq \ldots \geq x_{p_n}\}. \quad (6)$$

All of these subset obviously partition the state space since $\bigcup_{p_1,\ldots,p_n \in \mathcal{P}(n)} X(p_1, p_2, \ldots, p_n) = \mathbb{R}^n$. For any permutation, moreover, we may induce an associated monocolored subgraph $G(p_1, p_2, \ldots, p_n) = \{N, E(p_1, \ldots, p_n)\}$ according to the following rule:

$$\begin{aligned} &E(p_1, p_2, \ldots, p_n) = \\ &\{(p_i, p_j) \in E_p : i > j\} \quad \cup \{(p_i, p_j) \in E_o : i < j\} \end{aligned} \quad (7)$$

In particular, whenever $x \in X(p_1, p_2, \ldots, p_n)$, the corresponding time-varying linear embedding will only exhibit the links provided in $G(p_1, p_2, \ldots, p_n)$. Next we would like to remark that the proposed graph-theoretical connectivity conditions are much weaker than asking the resulting network's embedding to exhibit a spanning tree for any permutation of initial conditions.

To show this feature, let consider a simple ring topology composed of just optimistic edges (green colour edge in the top-left Fig. 4) that satisfies the proposed Assumption 2.

Different orderings of state variables may induce different subgraphs of $G(N, E_o, E_p)$, as shown in the Fig. 4. Notice that, although the initial graph may fail to be connected, the solutions eventually will switch to a region where the topology admits a spanning tree. For instance starting from the disconnected graph $G(1,3,2)$ induced by the initial condition $x_1 \geq x_3 \geq x_2$, due to the presence of the optimistic edge between nodes 1 and 2, x_2 increases above x_3 thus providing the switch of the network graph to $G(1,2,3)$ that fulfills the conditions of [7] and guarantees consensus. Similar considerations arise starting from the all other disconnected graph configuration like $G(2,1,3)$ and $G(3,2,1)$. For these reasons, attempting to prove consensus on the basis of linear-like embeddings and exploiting criteria in [7], appears to be a difficult task in general for the considered set-up.

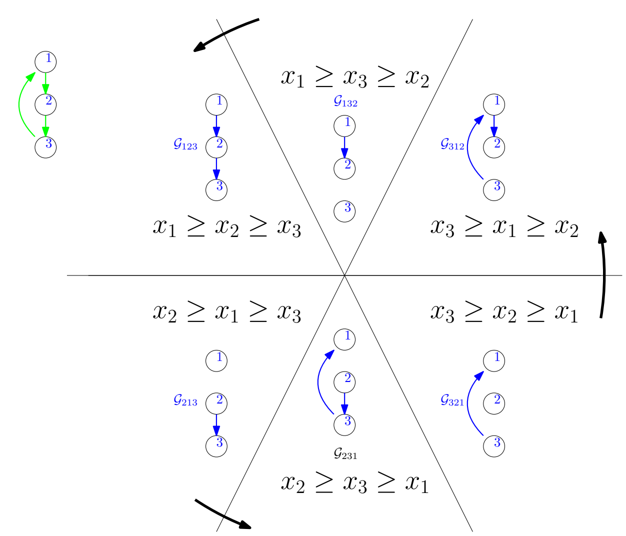

Fig. 4. Bicolored graphs $G_{p_1p_2p_3} = G(p_1, \ldots, p_3)$, induced by the initial state conditions $[x_{p_1}(t), x_{p_2}(t), x_{p_3}(t)]$ for any permutation of (p_1, p_2, p_3)

A. Max-Consensus

We consider the following network's equation (this could be easily extended to time-varying interconnections):

$$\begin{aligned} \dot{x}_1 &= \max\{0, x_3 - x_1\} + \max\{0, x_2 - x_1\} \\ \dot{x}_2 &= \max\{0, x_1 - x_2\} \\ \dot{x}_3 &= \max\{0, x_1 - x_3, x_2 - x_3\} \end{aligned} \tag{8}$$

Notice that terms in the equation for agent 1 are of additive nature, whereas agent 3 takes the maximum among neighbours influences. Both types of interactions can be easily accomodated, and the associated bicolored graph contains only optimistic edges: $E_o = \{(1,2),(1,3),(2,3),(3,1),(2,1)\}$. Moreover, the condition of Corollary 1 are fulfilled, therefore the network achieves exponential consensus on the maximum initial state (max-consensus). Indeed considering the same initial state condition of example (4) (i.e. $x(0) = [-4\ 5\ 2]^T$), the achieved consensus value is $\max_i x_i(0) = 5$ (see Fig. 5 for the simulation).

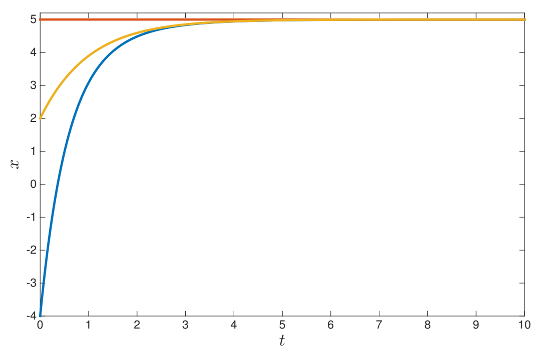

Fig. 5. Dynamic state evolution and convergence to the max (initial state) consensus equilibria of network (8)

B. Basin of Consensuability

An interesting open question that is not answered by our consensus criteria is how to asses the basin of attraction of the agreement manifold when the proposed connectivity assumption is not verified (in which case obviously this basin of attraction is the whole of $\mathbb{R}^n$). In other words, even if our conditions are violated there may be sets of initial conditions for which consensus is achieved. One promising direction for future investigations is in this case an exhaustive procedure to test standard weak conenctivity of $G(p_1, p_2, \ldots, p_n)$ for all permutations of initial condition. In general, however, $X(p_1, p_2, \ldots, p_n)$ regions are not invariant and may result in transitions to more than one neighbouring set, thus rendering the overall issue of estimating the basin of attraction much more involved. We find rather remarkable that consensus criteria from *arbitrary initial conditions* need not take into account such complex combinatorial approaches and yield simple conditions that can be tested a priori.

V. Conclusions

This paper presents conditions for exponential agreement suitable for nonlinear cooperative time-varying networks when the node interactions are unilateral. New notions of node pessimistic and optimistic unilateral interaction, bicolored and mirrored graph and associated connectivity are defined, thus extending to this nonlinear scenario, the standard graph theoretic notions well known in the literature. The proposed integral connectivity condition has the merit to be frozen in state variables to allow for a priori simpler verification. The exponential convergence to the agreement space is proved. A remark about how the proposed assumption can be proved to be equivalent to *Equilibrium* weak integral connectivity that only involves checking the condition on the agreement solution, a consideration on the necessity of the proposed conditions, and illustrative examples have given to highlight how the proposed framework extends all existing conditions for consensus of monotone networks.

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