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CLT for eigenvalue statistics of large-dimensional general Fisher matrices with applications

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Random Fisher matrices arise naturally in multivariate statistical analysis and understanding the properties of its eigenvalues is of primary importance for many hypothesis testing problems like testing the equality between two covariance matrices, or testing the independence between sub-groups of a multivariate random vector. Most of the existing work on random Fisher matrices deals with a particular situation where the population covariance matrices are equal. In this paper, we consider general Fisher matrices with arbitrary population covariance matrices and develop their spectral properties when the dimensions are proportionally large compared to the sample size. The paper has two main contributions: first the limiting distribution of the eigenvalues of a general Fisher matrix is found and second, a central limit theorem is established for a wide class of functionals of these eigenvalues. Applications of the main results are also developed for testing hypotheses on high-dimensional covariance matrices.

Keywords: central limit theorem; equality of covariance matrices; large-dimensional covariance matrices; large-dimensional Fisher matrix; linear spectral statistics

1. Introduction

For testing the equality of variances from two populations, a well-known statistic is the Fisher statistic defined as the ratio of two sample variances. Its multivariate counter-part is a random *Fisher matrix* defined by

$$\mathbf{F} := \mathbf{B}_1 \mathbf{B}_2^{-1}, \tag{1.1}$$

where $\mathbf{B}_1$ and $\mathbf{B}_2$ are p-dimensional sample covariance matrices from two independent samples, say $\{\boldsymbol{\xi}_k, 1 \le k \le n_1\}$ and $\{\boldsymbol{\eta}_\ell, 1 \le \ell \le n_2\}$ with population covariance matrices $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$, respectively. Fisher matrices, especially their eigenvalues, arise in many hypothesis testing problems in multivariate analysis. Examples include the test of the equality hypothesis $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$ where the likelihood ratio (LR) statistic can be written as a functional of the eigenvalues of a Fisher matrix, see Bai *et al.* [2]. In multivariate analysis of variance (MANOVA), the test on the equality of means is reduced to a statistic depending on a Fisher matrix which is a functional of the "between" sum of squares and the "within" sum of squares (Anderson [1], page 346). In multivariate linear regression, the likelihood ratio criterion for testing linear hypotheses about

regression coefficients is expressed as a functional of the eigenvalues of a Fisher matrix (Anderson [1], page 298). To test the independence between sub-groups of a multivariate population, the LR statistic is a functional of a Fisher matrix defined by sub-matrices of sample covariance matrices (Anderson [1], page 381). Fisher matrices appear also in the canonical correlation analysis, see Yang and Pan [26] for a recent account.

This paper concerns the high-dimensional situation where the population dimension p is large compared to the sample sizes n_1 and n_2. It is now well understood that classical procedures as those presented in Anderson [1] become impracticable or dramatically lose efficiency in high-dimensional data. For example, the deficiency of the Hotelling's T^2 statistic has been reported in Dempster [12] and Bai and Saranadasa [3]. Regarding hypothesis testing on one population high-dimensional covariance matrix, many recent works appeared in the literature, see Ledoit and Wolf [13], Srivastava [23], Srivastava *et al.* [24] and Schott [18,19], among others. About the equality of two population covariance matrices, please see Bai *et al.* [2], Cai, Liu and Xia [10], Li and Chen [14] and Zheng, Bai and Yao [29] among others. For two-sample tests on covariance matrices, Cai, Liu and Xia [10] and Li and Chen [14] use some *ad-hoc* distance measures to avoid the corresponding Fisher matrix, particularly when $p > n$ where the Fisher matrix is indeed undefined. However, all these works are not scale invariant and require some regularities of the population covariance matrix, for example, Cai, Liu and Xia [10] requires the sparsity condition, and Li and Chen [14] requires that the eigenvalues of the population covariance matrices are not dominated by a few of them. Therefore, tests defined by Fisher matrices are still important ones in multivariate analysis. Furthermore, Zheng, Bai and Yao [29] presented an example showing that tests defined by F matrices has larger powers than the method of Li and Chen [14] in a few cases, after corrections using random matrix theory.

In the literature from random matrix theory and assuming that the dimension grows to infinity proportionally to sample sizes, the convergence of the eigenvalues of a Fisher matrix to a limiting distribution has been studied under the equality of two population covariance matrices by several authors, see e.g. Wachter [25], Bai [5], Bai *et al.* [9], Pillai [16], Pillai and Flury [17], Silverstein [20] and Yin *et al.* [27]. As an important contribution of the paper, we establish a limiting distribution for the eigenvalues of a general Fisher matrix with arbitrary population covariance matrices. Regarding central limit theorems for *linear spectral statistics* (or LSS), Chatterjee [11] establishes the existence of a Gaussian limit assuming that the populations are Gaussian. However, he does not provide explicit formulae for the asymptotic mean and asymptotic covariances of the Gaussian limit. A closely related work by Bai and Silverstein [7] establishes a CLT for spectral statistics of a general sample covariance matrices of form $\mathbf{B}_1\mathbf{T}_p$ where $\mathbf{B}_1$ is a sample covariance matrix and $\mathbf{T}_p$ is a *non-random* Hermitian matrix. This CLT is later refined in Pan and Zhou [15] where the original restriction on the values of the fourth moments of the population components is removed. However, the CLT in Bai and Silverstein [7] cannot cover spectral statistics of a Fisher matrix through simply replacing $\mathbf{T}_p$ by $\mathbf{B}_2^{-1}$ for the reason that the centering term of this CLT would thus become a random term without an explicit expression. To overcome this difficulty, Zheng [28] establishes a CLT for LSS of a Fisher matrix which has a non-random and explicit centering term. In particular, the components of the observations $\{\boldsymbol{\xi}_i\}$ and $\{\boldsymbol{\eta}_j\}$ can have an arbitrary fourth moment. To our best knowledge, this is the only CLT reported in the literature for LSS of a Fisher matrix. However, this CLT has a severe limitation with the assumption that the population covariance matrices are equal that is, $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$. Although the derivation

of this CLT is complex and highly nontrivial, it has a small impact on the statistical problems mentioned above where the population covariance matrices $\mathbf{\Sigma}_i$ can be arbitrary and not necessarily equal. Specifically, for the test of the equality hypothesis "$\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$", this CLT enables us to find the distribution of the LR statistic under the null hypothesis, but not under the alternative "$\mathbf{\Sigma}_1 \neq \mathbf{\Sigma}_2$", that is, the asymptotic test size of the test can be found by this CLT but not the power function.

The main contribution of the paper is the establishment of a central limit theorem for linear spectral statistics $\{W_{\mathbf{n}}\}$ of a general Fisher matrix where the population covariance matrices $\mathbf{\Sigma}_i$ are arbitrary. Under this scheme and as a preparatory step, we also establish a limiting distribution for their eigenvalues and give an explicit equation satisfied by its Stieltjes transform. Due to the fact that the population covariance matrices are arbitrary, the proofs of these results have required several new techniques compared to the existing literature on the central limit theory although the general scheme remains similar to the one used in Bai and Silverstein [7], Zheng [28]. A significantly different tool used here is another CLT reported in Zheng, Bai and Yao [30] for random matrix of type $\mathbf{S}^{-1}\mathbf{T}_p$ where $\mathbf{S}$ is a standard sample covariance matrix (with i.i.d. standardised components) and $\mathbf{T}_p$ a nonnegative definite and deterministic Hermitian matrix. These two papers are related to each other but focus on different random matrices.

The paper is organized as follows. In Section 2, we first introduce the asymptotic scheme and the technical assumptions used, and then establish the limiting spectral distribution of the eigenvalues. Section 3 presents the CLT for linear spectral statistics of general Fisher matrices which is the main result of the paper. Section 4 gives two algorithms for approximation of the limiting spectral density, the asymptotic mean and covariance functions involved in the new CLT. In Section 5, applications of the main results are proposed for hypothesis testing and confidence intervals about high-dimensional covariance matrices. Technical lemmas and proofs are postponed to the Appendix B.

2. Limiting spectral distribution of large dimensional general *F*-matrices

Following Bai and Silverstein [7] and Zheng [28], we will impose the following structure on the observation model. Assume that the samples can be expressed as

$$\boldsymbol{\xi}_k = \mathbf{Q}_1 \mathbf{X}_{\cdot k}, \qquad 1 \leq k \leq n_1; \qquad \boldsymbol{\eta}_\ell = \mathbf{Q}_2 \mathbf{Y}_{\cdot \ell}, \qquad 1 \leq \ell \leq n_2,$$

where the observation matrices

$$\mathbf{X} := (\mathbf{X}_{\cdot 1}, \ldots, \mathbf{X}_{\cdot n_1}) = (X_{jk} : 1 \leq j \leq p, 1 \leq k \leq n_1),$$
$$\mathbf{Y} := (\mathbf{Y}_{\cdot 1}, \ldots, \mathbf{Y}_{\cdot n_2}) = (Y_{j\ell} : 1 \leq j \leq p, 1 \leq \ell \leq n_2),$$

are the upper-left corners, of size $p \times n_1$ and $p \times n_2$, of two independent arrays of independent random variables $\{X_{jk}, j, k = 1, 2, \ldots\}$ and $\{Y_{jk}, j, k = 1, 2, \ldots\}$, respectively, and $\mathbf{Q}_i$ is any square matrix such that $\mathbf{Q}_i \mathbf{Q}_i^* = \mathbf{\Sigma}_i$, $i = 1, 2$ where $*$ denotes the (complex) adjoint of a matrix

(i.e., transpose and conjugate). The corresponding sample covariance matrices become

$$\mathbf{B}_1 = \frac{1}{n_1}\sum_{k=1}^{n_1} \boldsymbol{\xi}_k \boldsymbol{\xi}_k^* = \mathbf{Q}_1 \mathbf{S}_1 \mathbf{Q}_1^*, \qquad \text{with } \mathbf{S}_1 = \frac{1}{n_1}\sum_{k=1}^{n_1} \mathbf{X}_{\cdot k}\mathbf{X}_{\cdot k}^*, \tag{2.1}$$

$$\mathbf{B}_2 = \frac{1}{n_2}\sum_{\ell=1}^{n_2} \boldsymbol{\eta}_\ell \boldsymbol{\eta}_\ell^* = \mathbf{Q}_2 \mathbf{S}_2 \mathbf{Q}_2^*, \qquad \text{with } \mathbf{S}_2 = \frac{1}{n_2}\sum_{\ell=1}^{n_2} \mathbf{Y}_{\cdot \ell}\mathbf{Y}_{\cdot \ell}^*. \tag{2.2}$$

Because $\mathbf{F} = \mathbf{B}_1\mathbf{B}_2^{-1}$ has the same eigenvalues as $\mathbf{S}_1\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1}\mathbf{T}_{p,1/2}$ where $\mathbf{T}_{p,1/2} = \mathbf{Q}_2^{-1}\mathbf{Q}_1$, we can define as well the Fisher matrix to be $\mathbf{F} := \mathbf{S}_1\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1}\mathbf{T}_{p,1/2}$. It is also noticed that obviously, the matrix $\mathbf{S}_2$ should be invertible (almost surely) so that in our asymptotic analysis, we will assume $n_2 > p$ for large p and n_2.

Throughout the paper, the *empirical spectral distribution* (or ESD) of a complex $p \times p$ matrix $\mathbf{A}$ is the probability measure $F_n^{\mathbf{A}}(x) = p^{-1}\sum_{j=1}^p \delta_{\lambda_j}$ where $\{\lambda_j\}_{j=1}^p$ are the eigenvalues of $\mathbf{A}$, and δ_a denotes the Dirac mass at a point a. We consider the following assumptions.

Assumption [A]. *The two double arrays* $\{X_{ki}, i, k = 1, 2, \ldots\}$ *and* $\{Y_{ki}, i, k = 1, 2, \ldots\}$ *consist of independent but not necessarily identically distributed random variables with mean* 0 *and variance* 1.

Assumption [B1]. *For any fixed* $\eta > 0$ *and when* $n_1, n_2, p \to \infty$,

$$\frac{1}{n_1 p}\sum_{j=1}^{p}\sum_{k=1}^{n_1} \mathrm{E}\big[|X_{jk}|^2 I_{\{|X_{jk}| \geq \eta\sqrt{n_1}\}}\big] \to 0, \qquad \frac{1}{n_2 p}\sum_{j=1}^{p}\sum_{k=1}^{n_2} \mathrm{E}\big[|Y_{jk}|^2 I_{\{|Y_{jk}| \geq \eta\sqrt{n_2}\}}\big] \to 0. \tag{2.3}$$

Assumption [B2]. *If the two arrays are either both real, we then set the indicator* $\kappa = 2$; *or both complex, we then set* $\kappa = 1$, *with homogeneous 4th moments*: $\mathrm{E}|X_{jk}|^4 = 1 + \kappa + \beta_x + o(1)$, $\mathrm{E}|Y_{jk}|^4 = 1 + \kappa + \beta_y + o(1)$. *Moreover, for any fixed* $\eta > 0$ *when* $n_1, n_2, p \to \infty$,

$$\frac{1}{n_1 p}\sum_{j=1}^{p}\sum_{k=1}^{n_1} \mathrm{E}\big[|X_{jk}|^4 I_{\{|X_{jk}| \geq \eta\sqrt{n_1}\}}\big] \to 0, \qquad \frac{1}{n_2 p}\sum_{j=1}^{p}\sum_{k=1}^{n_2} \mathrm{E}\big[|Y_{jk}|^4 I_{\{|Y_{jk}| \geq \eta\sqrt{n_2}\}}\big] \to 0. \tag{2.4}$$

In addition, $\mathrm{E}X_{jk}^2 = o(n_1^{-1})$, $\mathrm{E}Y_{jk}^2 = o(n_2^{-1})$ when both arrays $\{X_{jk}\}$ and $\{Y_{jk}\}$ are complex.

Assumption [C]. *The sample sizes* n_1, n_2 *and the dimension* p *grow to infinity in such a way that*

$$y_{n_1} := p/n_1 \to y_1 \in (0, +\infty), \qquad y_{n_2} := p/n_2 \to y_2 \in (0, 1), \tag{2.5}$$

where $\mathbf{S}_2$ *has the inverse matrix* $\mathbf{S}_2^{-1}$.

Assumption [D]. *The matrices* $\mathbf{T}_p = \mathbf{T}_{p,1/2}\mathbf{T}_{p,1/2}^*$ *are non-random and nonnegative definite Hermitian matrices and the sequence* $\{\mathbf{T}_p\}$ *is bounded in spectral norm. Moreover, the ESD* H_p *of* $\mathbf{T}_p$ *tends to a proper probability measure* H *when* $p \to \infty$.

Table 1. Notations for distributions, Stieltjes transforms (S.T.) and linear spectral statistics (LSS) of random matrices

Matrix	ESD/S.T.	LSD/S.T.	LSS
$\mathbf{F}=\mathbf{S}_1\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1}\mathbf{T}_{p,1/2}$	$U_{\mathbf{n}}(x)/m_{\mathbf{n}}(z)$	$U_{\mathbf{y}}(x)/m_{\mathbf{y}}(z)$	$U_{\mathbf{n}}(f)=\int f(x)\,dU_{\mathbf{n}}(x)$ $U_{\mathbf{y}}(f)=\int f(x)\,dU_{\mathbf{y}}(x)$
$\mathbf{X}^*\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1}\mathbf{T}_{p,1/2}\mathbf{X}/n_1$	$\underline{U}_{\mathbf{n}}(x)/\underline{m}_{\mathbf{n}}(z)$	$\underline{U}_{\mathbf{y}}(x)/\underline{m}_{\mathbf{y}}(z)$	

Here $m_{\mathbf{n}}(z)=\int\frac{1}{\lambda-z}\,dU_{\mathbf{n}}(x)$, $m_{\mathbf{y}}(z)=\int\frac{1}{\lambda-z}\,dU_{\mathbf{y}}(x)$, $\underline{m}_{\mathbf{n}}(z)=\int\frac{1}{\lambda-z}\,d\underline{U}_{\mathbf{n}}(x)$, $\underline{m}_{\mathbf{y}}(z)=\int\frac{1}{\lambda-z}\,d\underline{U}_{\mathbf{y}}(x)$, f is an analytic function, $U_{\mathbf{n}}(f)=p^{-1}\sum_{j=1}^{p}f(\lambda_j^{\mathbf{F}})$ with the eigenvalues $\{\lambda_j^{\mathbf{F}}\}_{j=1}^{p}$ of $\mathbf{F}$ and $U_{\mathbf{y}_{\mathbf{n}}}(x)$ is obtained by substituting $\mathbf{y}_{\mathbf{n}}=(y_{n_1},y_{n_2})$ for $\mathbf{y}=(y_1,y_2)$ in $U_{\mathbf{y}}(x)$.

The assumptions (2.3) and (2.4) are standard Lindeberg type conditions which are necessary for the existence of the limiting spectral distribution for $\mathbf{F}$, and for the CLT for LSS of $\mathbf{F}$, respectively. Moreover, under these conditions, the variables $\{X_{ik}\}$ and $\{Y_{ik}\}$ can be truncated at size $\eta_p\sqrt{p}$ ($\eta_p\downarrow 0$) without altering asymptotic results.

The following notations are used throughout the paper:

$$\mathbf{n}=(n_1,n_2),\qquad \mathbf{y}_{\mathbf{n}}=(y_{n_1},y_{n_2}),\qquad \mathbf{y}=(y_1,y_2),\qquad h^2=y_1+y_2-y_1y_2.$$

In the sequel, the limiting results will be investigated under the regime (2.5) that will be simply referred as $\mathbf{n}\to\infty$. Some useful concepts are now recalled. The Stieltjes transform of a positive Borel measure G on the real line is defined by

$$m_G(z)\equiv\int\frac{1}{\lambda-z}\,dG(\lambda),\qquad z\in\mathbb{C}^+=\{z:z\in\mathbb{C},\Im(z)>0\}. \tag{2.6}$$

This transform has a natural extension to the lower-half plane by the formula

$$m_G(z)=\overline{m}_G(\bar{z})\qquad \text{for } z\in\mathbb{C}^-=\{z:z\in\mathbb{C},\Im(z)<0\}.$$

In addition to $\mathbf{F}$, we will also need several other matrices. Table 1 contains the notations used in the sequel for the characteristics of these matrices: ESD, LSD, the associated Stieltjes transforms and LSS.

The matrices $\mathbf{F}$ and $\mathbf{X}^*\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1}\mathbf{T}_{p,1/2}\mathbf{X}/n_1$ are companion matrices each other sharing same nonnull eigenvalues so that we have

$$\underline{m}_{\mathbf{n}}(z)=-\frac{1-y_{n_1}}{z}+y_{n_1}m_{\mathbf{n}}(z), \tag{2.7}$$

$$\underline{m}_{\mathbf{y}}(z)=-\frac{1-y_1}{z}+y_1m_{\mathbf{y}}(z). \tag{2.8}$$

Furthermore, when $\mathbf{\Sigma}_1=\mathbf{\Sigma}_2$, that is, $\mathbf{T}_p=\mathbf{I}_p$, it is well-known that the LSD $U_{\mathbf{y}}$ of $\mathbf{F}$ and its Stieltjes transform $m_{\mathbf{y}}(z)$ can be found on page 79 of Bai and Silverstein [4]. As a first result of the paper, we prove the existence of $U_{\mathbf{y}}$ and one of its characteristics for general Fisher matrix $\mathbf{F}$ where $\mathbf{T}_p$ is a Hermitian matrix.

Theorem 2.1. *Under Assumptions* [A], [B1], [C] *and* [D], *The Fisher matrix*

$$\mathbf{F} = \mathbf{S}_1 \mathbf{T}_{p,1/2}^{*} \mathbf{S}_2^{-1} \mathbf{T}_{p,1/2}$$

has a non-random LSD $U_{\mathbf{y}}$. *Moreover,* $U_{\mathbf{y}}$ *is characterized by the fact that the Stieltjes transform* $\underline{m}_{\mathbf{y}}(z)$ *of its companion measure* $\underline{U}_{\mathbf{y}}$ *is the unique solution to the equation*

$$z = \frac{h^2 m_0(z)}{y_2(-1 + y_2 \int \frac{m_0(z)\,dH(t)}{t + m_0(z)})} + \frac{y_1}{y_2} m_0(z), \qquad z \in \mathbb{C}^+, \tag{2.9}$$

where $m_0(z) = \underline{m}_{y_2}(-\underline{m}_{\mathbf{y}}(z))$ *and* $\underline{m}_{y_2}(z)$ *satisfies the equation* $z = -(\underline{m}_{y_2}(z))^{-1} + y_2 \int (t + \underline{m}_{y_2}(z))^{-1}\,dH(t)$.

The proof of this theorem is given in Appendix B.1.

Remark 2.1. If the Assumption [B1] in Theorem 2.1 is strengthened to [B2], following Bai and Yin [8], after truncation, one can show that with probability 1, $\mathbf{S}_2$ is invertible. However, under the Assumption [B1], there may be a small portion of eigenvalues of $\mathbf{B}_2$ equal or tending to zero. In this case, Theorem 2.1 remains true if $\mathbf{S}_2^{-1}$ is understood as the Moore–Penrose generalized inverse of $\mathbf{S}_2$.

Remark 2.2. For a given $z \in \mathbb{C}^+$, the equation (2.9) has a unique solution m_0 such that $\Im(m_0) < 0$.

In fact by Silverstein [21], $\underline{m}_{\mathbf{y}}(z)$ is the unique solution to the equation

$$z = -\frac{1}{\underline{m}_{\mathbf{y}}(z)} + y_1 \int \frac{x\,dG_{y_2}(x)}{1 + x\underline{m}_{\mathbf{y}}(z)}, \tag{2.10}$$

subject to the condition that $\Im(\frac{1-y_1}{z} + \underline{m}_{\mathbf{y}}(z))\Im(z) > 0$. Here, G_{y_2} denotes the limiting spectral distribution of the random matrix $\mathbf{T}_{p,1/2}^{*} \mathbf{S}_2^{-1} \mathbf{T}_{p,1/2}$.

In the sequel, for brevity, the notations $m_{\mathbf{y}}(z)$ and $\underline{m}_{\mathbf{y}}(z)$ will be simplified to $m(z)$ and $\underline{m}(z)$, or even to m and $\underline{m}$, respectively, if no confusion would be possible. We will use the notation $G_{y_{n_2}}$ that is obtained by substituting $y_{n_2} = p/n_2$ for y_2 in G_{y_2}.

3. CLT for LSS of large dimensional general Fisher matrices

We consider a centered version of the LSS

$$p\big[U_{\mathbf{n}}(f) - U_{\mathbf{y_n}}(f)\big] = \sum_{j=1}^{p} f\big(\lambda_j^{\mathbf{F}}\big) - p \int f(x)\,dU_{\mathbf{y_n}}(x) \tag{3.1}$$

with the eigenvalues $\{\lambda_j^{\mathbf{F}}\}_{j=1}^{p}$ of $\mathbf{F}$. Note that (3.1) is translation-invariant in f, we may assume $f(0) = 0$. Due to the exact separation theorem (see Bai and Silverstein [6]), with probability

one, for large enough n_j and p, the possible point mass at the origin of $U_{\mathbf{n}}$ will coincide exactly with that of $U_{\mathbf{y_n}}$. By Theorem 1.1 of Silverstein and Choi [22], $U_{\mathbf{y}}$ has the continuous density $u_{\mathbf{y}}$ except $x=0$. Therefore, we can restrict the integral (3.1) to their continuous components on $(0,\infty)$, that is,

$$p\big[U_{\mathbf{n}}(f)-U_{\mathbf{y_n}}(f)\big]=\sum_{j=1}^{p} f\big(\lambda_j^{\mathbf{F}}\big)I_{(\lambda_j^{\mathbf{F}}>0)}-p\int f(x)u_{\mathbf{y_n}}(x)\,dx, \tag{3.2}$$

where $u_{\mathbf{y_n}}(x)$ is the density of $U_{\mathbf{y_n}}(x)$.

Regarding the central limit theory on linear spectral statistics of random matrices, it has been well known (Bai and Silverstein [7], Pan and Zhou [15], Zheng [28]) that the mean and covariance parameters of the limiting Gaussian distribution depend on the values of the fourth moments of $\{X_{jk}\}$ and $\{Y_{jk}\}$. When these moments match the Gaussian case, that is, $\beta_x=0$ or $\beta_y=0$ in Assumption [B2], the limiting parameters have a simpler expression. Otherwise, they have a more involved expression that depends on other limiting functionals of sample covariance matrices. More specifically, if $\beta_x\neq 0$, we will need the existence of the following limits

$$\begin{aligned}&\frac{1}{p}\sum_{i=1}^{p}\mathrm{E}\big[\mathbf{e}_i'\mathbf{T}_{p,1/2}^{*}\mathbf{S}_2^{-1/2}\mathbf{D}_1^{-1}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\\&\quad\times\mathbf{e}_i'\mathbf{T}_{p,1/2}^{*}\mathbf{S}_2^{-1/2}\mathbf{D}_1^{-1}\big(\underline{m}(z)\mathbf{S}_2^{-1/2}\mathbf{T}_p\mathbf{S}_2^{-1/2}+\mathbf{I}_p\big)^{-1}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i|\mathbf{S}_2\big]\longrightarrow h_{m1}(z),\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\frac{1}{n_1p}\sum_{j=1}^{n_1}\sum_{i=1}^{p}\mathbf{e}_i'\mathbf{T}_{p,1/2}^{*}\mathbf{S}_2^{-1/2}\big[\mathrm{E}_j\mathbf{D}_{1j}^{-1}(z_1)\big]\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\\&\quad\times\mathbf{e}_i'\mathbf{T}_{p,1/2}^{*}\mathbf{S}_2^{-1/2}\big[\mathrm{E}_j\mathbf{D}_{1j}^{-1}(z_2)\big]\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\xrightarrow{\text{i.p.}}h_{v1}(z_1,z_2),\end{aligned} \tag{3.4}$$

where E_j denotes the conditional expectation given $\mathbf{X}_{\cdot 1},\dots,\mathbf{X}_{\cdot j}$ and $\mathbf{S}_2$, and if $\beta_y\neq 0$, we will need the existence of the limits

$$\begin{aligned}&\frac{1}{p}\sum_{i=1}^{p}\mathrm{E}\mathbf{e}_i'\big(\underline{m}(z)\mathbf{T}_p+\mathbf{S}_2\big)^{-1}\mathbf{e}_i\cdot\mathbf{e}_i'\big(\underline{m}(z)\mathbf{T}_p+\mathbf{S}_2\big)^{-1}\mathbf{T}_p\big(\underline{m}(z)\mathbf{T}_p+\underline{m}(z)m_0(z)\mathbf{I}_p\big)^{-1}\mathbf{e}_i\\&\quad\longrightarrow h_{m2}\big(-\underline{m}^{-1}(z)\big),\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\frac{1}{n_2p}\sum_{j=1}^{n_2}\sum_{i=1}^{p}\mathbf{e}_i'E_j\big(\underline{m}(z_1)\mathbf{T}_p+\mathbf{S}_{2,j}\big)^{-1}\mathbf{e}_i\cdot\mathbf{e}_i'E_j\big(\underline{m}(z_2)\mathbf{T}_p+\mathbf{S}_{2,j}\big)^{-1}\mathbf{e}_i\\&\quad\xrightarrow{\text{i.p.}}h_{v2}\big(-\underline{m}^{-1}(z_1),-\underline{m}^{-1}(z_2)\big),\end{aligned} \tag{3.6}$$

where E_j denotes the conditional expectation given $\mathbf{Y}_{\cdot 1},\dots,\mathbf{Y}_{\cdot j}$. Here

$$\mathbf{S}_{2,j}=\mathbf{S}_2-\boldsymbol{\eta}_j\boldsymbol{\eta}_j^{*},\qquad \mathbf{D}_1(z)=\sum_{i=1}^{n_1}\boldsymbol{\gamma}_i\boldsymbol{\gamma}_i^{*}-z\mathbf{I}_p,\qquad \mathbf{D}_{1j}(z)=\mathbf{D}_1(z)-\boldsymbol{\gamma}_j\boldsymbol{\gamma}_j^{*},$$

and $\mathbf{e}_i$ denotes the ith vector of the canonical basis of $\mathbb{C}^p$ where $\boldsymbol{\eta}_j = n_2^{-1/2}\mathbf{Y}_{\cdot j}$ and $\boldsymbol{\gamma}_j = n_1^{-1/2}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{X}_{\cdot j}$.

The following CLT is the main result of the paper.

Theorem 3.1. *Under the Assumptions* [A], [B2], [C] *and* [D], *assume that the limits* (3.3)–(3.4) *exist whenever* $\beta_x \neq 0$, *and the limits* (3.5)–(3.6) *exist whenever* $\beta_y \neq 0$. *Let* $f_1, \ldots, f_s$ *be* s *functions analytic in an open domain of the complex plane that enclosed the support interval* $[c_1, c_2]$ *of the continuous component of the LSD* $U_{\mathbf{y}}$. *Then, as* $\mathbf{n} \to \infty$, *the random vector*

$$\big\{p\big[U_{\mathbf{n}}(f_j) - U_{\mathbf{y}_\mathbf{n}}(f_j)\big], 1 \le j \le s\big\},$$

converges weakly to a Gaussian vector $(X_{f_1}, \ldots, X_{f_s})$ *with the mean function*

$$\begin{aligned}
\mathrm{E}X_f = {} & \frac{\kappa - 1}{4\pi i} \oint_{\mathcal{C}} f(z)\, d\log\left(\frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{(1 - y_2 \int \frac{m_0(z)}{t+m_0(z)}\, dH(t))^2}{1 - y_2 \int \frac{m_0^2(z)}{(t+m_0(z))^2}\, dH(t)}\right) \\
& - \frac{\beta_x y_1}{2\pi i} \cdot \oint_{\mathcal{C}} f(z) \frac{z^2 \underline{m}^3(z) \cdot h_{m1}(z)}{\frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{(1 - \int \frac{y_2 m_0(z)}{t+m_0(z)}\, dH(t))^2}{1 - \int \frac{y_2 m_0^2(z)}{(t+m_0(z))^2}\, dH(t)}}\, dz \\
& + \frac{\kappa - 1}{4\pi i} \oint_{\mathcal{C}} f(z)\, d\log\left(1 - y_2 \int \frac{m_0^2(z)\, dH(t)}{(t + m_0(z))^2}\right) \\
& + \frac{\beta_y y_2}{2\pi i} \cdot \oint f(z) \underline{m}'(z) \frac{\underline{m}^3(z) m_0^3(z) h_{m2}(-\underline{m}^{-1}(z))}{1 - y_2 \int \frac{m_0^2(z)}{(t+m_0(z))^2}\, dH(t)}\, dz,
\end{aligned} \tag{3.7}$$

and the covariance function

$$\begin{aligned}
& \mathrm{Cov}\,(X_{f_i}, X_{f_j}) \\
& = -\frac{\beta_x y_1}{4\pi^2} \cdot \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} f_i(z_1) f_j(z_2) \frac{\partial^2 [z_1 z_2 \underline{m}(z_1) \underline{m}(z_2) h_{v1}(z_1, z_2)]}{\partial z_1\, \partial z_2}\, dz_1\, dz_2 \\
& \quad - \frac{\kappa}{4\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{f_i(z_1) f_j(z_2)}{(m_0(z_1) - m_0(z_2))^2}\, dm_0(z_1)\, dm_0(z_2) \\
& \quad - \frac{\beta_y y_2}{4\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} f_i(z_1) f_j(z_2) \\
& \qquad \times \frac{\partial^2 [\underline{m}(z_1) m_0(z_1) \underline{m}(z_2) m_0(z_2) h_{v_2}(-\underline{m}^{-1}(z_1), -\underline{m}^{-1}(z_2))]}{\partial z_1\, \partial z_2}\, dz_1\, dz_2,
\end{aligned} \tag{3.8}$$

where the contours $\mathcal{C}$, $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *all enclose the support of* $U_{\mathbf{y}}$, *and* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are disjoint.*

Similar to CLT's developed in Bai and Silverstein [7], Zheng [28], all the limiting parameters depend on contour integrals using the associated Stieltjes transforms. Some specific examples of calculations of such integrals can be found in these references.

When $\mathbf{T}_p$'s are diagonal, we find explicit expressions for the limiting functions $h_{mj}(z)$ and $h_{vj}(z_1, z_2)$, $j = 1, 2$. This in turn simplifies the expressions of limiting mean and covariance functions in the CLT. Two propositions about $h_{mj}(z)$ and $h_{vj}(z_1, z_2)$ are given in Appendix A. In this case, the limiting mean and covariance function in Theorem 3.1 are simplified as follows.

Theorem 3.2. *Under the conditions of Theorem 3.1 and assuming that the matrices $\mathbf{T}_p$'s are diagonal, we obtain that as $\mathbf{n} \to \infty$, the random vector*

$$\big\{p\big[U_{\mathbf{n}}(f_j) - U_{\mathbf{y_n}}(f_j)\big], 1 \le j \le s\big\},$$

converges weakly to a Gaussian vector $(X_{f_1}, \ldots, X_{f_s})$ with the mean function

$$\begin{aligned}
\mathrm{E}X_f = {} & \frac{\kappa - 1}{4\pi i} \oint_{\mathcal{C}} f(z)\, d \log\left(\frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{(1 - y_2 \int \frac{m_0(z)}{t + m_0(z)}\, dH(t))^2}{1 - y_2 \int \frac{m_0^2(z)}{(t + m_0(z))^2}\, dH(t)} \right) \\
& + \frac{\kappa - 1}{4\pi i} \oint_{\mathcal{C}} f(z)\, d \log\left(1 - y_2 \int \frac{m_0^2(z)\, dH(t)}{(t + m_0(z))^2} \right) \\
& + \frac{\beta_x y_1}{2\pi i} \oint_{\mathcal{C}} f(z) \left[\int \frac{t^2}{(t + m_0(z))^3}\, dH(t) \right] dm_0(z) \\
& + \frac{\beta_y}{4\pi i} \oint_{\mathcal{C}} f(z) \left(1 - y_2 \int \frac{m_0^2(z)\, dH(t)}{(t + m_0(z))^2} \right) d \log\left(1 - y_2 \int \frac{m_0^2(z)\, dH(t)}{(t + m_0(z))^2} \right)
\end{aligned}$$

and the covariance function

$$\begin{aligned}
& \mathrm{Cov}\,(X_{f_i}, X_{f_j}) \\
& = -\frac{\kappa}{4\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{f_i(z_1) f_j(z_2)}{(m_0(z_1) - m_0(z_2))^2}\, dm_0(z_1)\, dm_0(z_2) \\
& \quad - \frac{\beta_x y_1 + \beta_y y_2}{4\pi^2} \\
& \quad \times \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} f_i(z_1) f_j(z_2) \left[\int \frac{t^2\, dH(t)}{(t + m_0(z_1))^2 (t + m_0(z_2))^2} \right] dm_0(z_1)\, dm_0(z_2),
\end{aligned}$$

where the contours $\mathcal{C}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ all enclose the support of $U_{\mathbf{y}}$, and $\mathcal{C}_1$ and $\mathcal{C}_2$ are disjoint.

Remark 3.1. The contours in Theorem 3.1 and Propositions A.1–A.2 are taken in the z space. In this case, the contours can be arbitrary provided that they enclose the support of the LSD $U_{\mathbf{y}}$. Since the integrands are functions of m_0, thus the integrals can be taken in the m_0 space using the change of variable $z \mapsto m_0(z)$.

Remark 3.2. When $\mathbf{T}_p$ is an identity matrix, (A.3) and (A.4) are the same as (3.6) and (3.7) in Zheng [28]. That is, Theorem 3.2 in Zheng [28] is a special case of Theorem 3.2 in this paper when $\mathbf{T}_p = \mathbf{I}_p$.

4. Evaluation of the asymptotic parameters EX_f, $\mathrm{Cov}(X_{f_i}, X_{f_j})$ and the limiting density $u_{\mathbf{y}}(x)$

Notice that practical applications of Theorem 3.1 or Proposition A.1 require the knowledge of the limiting spectral density $u_{\mathbf{y}}(x)$, the asymptotic mean EX_f and covariance function $\mathrm{Cov}(X_{f_i}, X_{f_j})$. In particular, the last two functions depend on some non trivial contour integrals. In the simple case where $\mathbf{T}_p = \mathbf{I}_p$ and for simple functions like $f(x) = x^j$ or $f(x) = \log(x)$, analytical results can be found exactly, see Zheng [28]. However, this is a very particular case and for general population matrices or more complex functions f, such exact results are not available. In this section, we introduce some numerical procedures to approximate these asymptotic parameters while deliberately place ourselves in the context of practical applications with real data sets. In such a situation, the sample sizes and dimension of data (n_1, n_2, p) are given and the empirical spectral distribution H_p of $\mathbf{T}_p = \mathbf{T}_{p,1/2}\mathbf{T}_{p,1/2}^*$ is known. In this section, we denote the eigenvalues of $\mathbf{T}_p$ simply by $\{\lambda_j^0\}_{j=1}^p$ so that $H_p(t) = p^{-1}\sum_{j=1}^p I_{(\lambda_j^0 \le t)}$. However in such a concrete application situation, the LSD H is never known and we need an estimate of H. A very reasonable and widely used estimate of H is indeed just H_p.

Notice that the parameters $u_{\mathbf{y}}(x)$, $\mathrm{E}X_f$ and $\mathrm{Cov}(X_{f_i}, X_{f_j})$ all depend on the Stieltjes transform $m_0(z)$. By (2.9), we have

$$z = \frac{h^2 m_0(z)}{y_2(-1 + y_2 \int \frac{m_0(z)\,dH(t)}{t+m_0(z)})} + \frac{y_1}{y_2} m_0(z).$$

That is,

$$m_0(z) = z\left(\frac{h^2}{y_2(-1 + y_2 \int \frac{m_0(z)\,dH(t)}{t+m_0(z)})} + \frac{y_1}{y_2}\right)^{-1},$$

which leads to numerical approximations of $m_0(z)$. Choose $s_0(z)$ be the initial values of $m_0(z)$ and iterate for $k \ge 0$ the above mapping

$$s_{k+1}(z) = z\left(\frac{h^2}{y_2(-1 + y_2 \int \frac{s_k(z)\,dH(t)}{t+s_k(z)})} + \frac{y_1}{y_2}\right)^{-1} \tag{4.1}$$

until convergence and let $s_k(z)$ be the final values. Define the approximate value for $m_0(z)$ be $s_k(z)$.

Next, the limiting spectral density $u_{\mathbf{y}}(x)$ can be approximated as follows.

Remark 4.1. By Theorem B.10 of Bai and Silverstein [4], we have

$$\underline{m}(z) = \frac{1}{m_0(z)} - y_2 \int \frac{dH(t)}{t + m_0(z)} \approx \frac{1}{m_0(z)} - y_2 p^{-1} \sum_{j=1}^{p} \frac{1}{\lambda_j^0 + m_0(z)} \tag{4.2}$$

and

$$u_{\mathbf{y}}(x) = \frac{1}{\pi y_1} \lim_{\varepsilon \to o_+} \Im\big(\underline{m}(x + \varepsilon \mathbf{i})\big), \tag{4.3}$$

where $\{\lambda_j^0\}_{j=1}^p$ are the eigenvalues of $\mathbf{T}_p$.

The following remark will give a simplified form of the asymptotic mean function $\mathrm{E}X_f$ and asymptotic covariance function $\mathrm{Cov}(X_{f_i}, X_{f_j})$.

Remark 4.2. In Proposition A.1, assuming that $\mathbf{T}_p$'s are diagonal, then by (A.3)–(A.4) and (1.7) in Bai and Silverstein [7], the mean and covariance functions have the alternative expressions

$$\begin{aligned}
\mathrm{E}X_f = &-\frac{\kappa - 1}{4\pi i} \oint_{\mathcal{C}} f'(z) \log\left(\frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{(1 - y_2 \int \frac{m_0(z)}{t + m_0(z)} \, dH(t))^2}{1 - y_2 \int \frac{m_0^2(z)}{(t + m_0(z))^2} \, dH(t)} \right) dz \\
&- \frac{\kappa - 1}{4\pi i} \oint_{\mathcal{C}} f'(z) \log\left(1 - y_2 \int \frac{m_0^2(z) \, dH(t)}{(t + m_0(z))^2} \right) dz \\
&- \frac{\beta_x y_1}{2\pi i} \oint_{\mathcal{C}} f(z) \frac{\int \frac{t^2}{(t + m_0(z))^3} \, dH(t) \cdot (-1 + y_2 \int \frac{m_0(z)}{t + m_0(z)} \, dH(t))^2}{\left(\frac{h^2}{y_2} - \frac{y_1}{y_2} \frac{\underline{m}_y^2(z) m_0^2(z)}{1 - y_2 \int \frac{m_0^2(z)}{(t + m_0(z))^2} \, dH(t)} \right) \left(1 - y_2 \int \frac{m_0^2(z)}{(t + m_0(z))^2} \, dH(t) \right)} \, dz \\
&+ \frac{\beta_y y_2}{2\pi i} \oint_{\mathcal{C}} f(z) \frac{m_0(z) \int \frac{t \, dH(t)}{(t + m_0)^3}}{1 - y_2 \int \frac{m_0^2 \, dH(t)}{(t + m_0)^2}} \frac{(-1 + y_2 \int \frac{m_0(z)}{t + m_0(z)} \, dH(t))^2}{\left(\frac{h^2}{y_2} - \frac{y_1}{y_2} \frac{\underline{m}_y^2(z) m_0^2(z)}{1 - y_2 \int \frac{m_0^2(z)}{(t + m_0(z))^2} \, dH(t)} \right)} \, dz
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
&\mathrm{Cov}(X_{f_i}, X_{f_j}) \\
&= -\frac{\kappa}{2\pi^2} \int \int f_i'(x) f_j'(y) \log \left| \frac{\underline{m}(x) - \overline{\underline{m}(y)}}{\underline{m}(x) - \underline{m}(y)} \right| dx \, dy \\
&\quad - \frac{\beta_x y_1 + \beta_y y_2}{4\pi^2} \oint \oint f_i'(z_1) f_j'(z_2) \left[\int \frac{t^2 \, dH(t)}{(t + m_0(z_1))(t + m_0(z_2))} \right] dz_1 \, dz_2.
\end{aligned} \tag{4.5}$$

Combining Remarks 4.1 and 4.2, we now describe the general procedure to approximate the limiting spectral density $u_{\mathbf{y}}(x)$, the asymptotic mean and covariance functions.

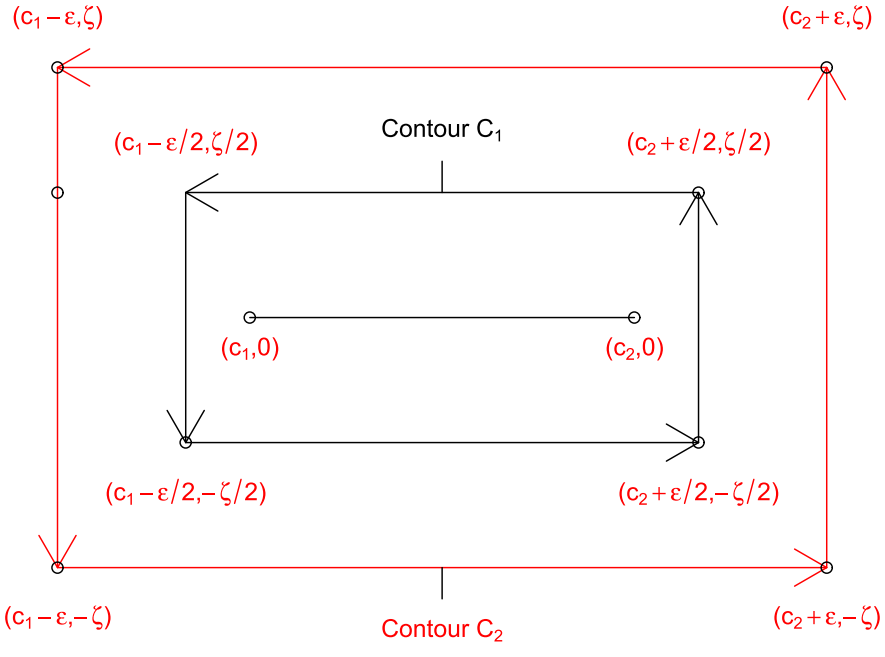

Figure 1. Contours used to approximate the asymptotic mean and covariance.

Algorithm 1: *Approximating the limiting spectral density* $u_{\mathbf{y}}(x)$. Cut the support set $[c_1, c_2]$ of the LSD of the Fisher matrix $\mathbf{F}$ into a mesh set as

$$\mathcal{A} = \left\{ z_j = x_j + \varepsilon \mathbf{i}, x_j = c_1 + \frac{(c_2 - c_1)j}{K}, j = 0, \ldots, K \right\},$$

where ε is a small step size, e.g. 10^{-3}. By (4.1), we obtain $m_0(z_j)$ with $z_j \in \mathcal{A}$. By (4.2), we obtain $\underline{m}(z_j)$ with $z_j \in \mathcal{A}$. Then by (4.3). let

$$u_{\mathbf{y}}(x_j) \simeq \frac{1}{\pi y_1} \Im\big(\underline{m}(z_j)\big), \tag{4.6}$$

which is an approximation of the density $u_{\mathbf{y}}(x_j)$. Then we have

$$\int f(x) u_{\mathbf{y}}(x)\, dx \approx \frac{c_2 - c_1}{K} \sum_{l=1}^{K} f(x_l) u_{\mathbf{y}}(x_l). \tag{4.7}$$

Algorithm 2: *Approximating the asymptotic mean function* (4.4) *and covariance function* (4.5).

Step 1. Choose two disjoint contours $\mathcal{C}_1$ and $\mathcal{C}_2$ both enclosing the support $[c_1, c_2]$ of $u_{\mathbf{y}}$ as depicted on Figure 1 where ε and ζ are small numbers, for example, $\varepsilon = \zeta = 10^{-3}$.

Step 2. Let K_1, K_2 be large integers, e.g. 10^3. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are cut into a grid set as

$$\begin{aligned}
\mathcal{A}_1 = \Bigg\{ & z_k = c_1 - \varepsilon + \left(\zeta - \frac{2\zeta k}{K_1}\right)\mathbf{i}, k = 0, \ldots, K_1, \\
& z_{K_1+j} = c_1 - \varepsilon + \frac{(c_2 - c_1 + 2\varepsilon) j}{K_2} - \zeta\mathbf{i}, j = 0, \ldots, K_2, \\
& z_{K_1+K_2+k} = c_2 + \varepsilon + \left(-\zeta + \frac{2\zeta k}{K_1}\right)\mathbf{i}, k = 0, \ldots, K_1, \\
& z_{2K_1+K_2+j} = c_2 + \varepsilon - \frac{(c_2 - c_1 + 2\varepsilon) j}{K_2} + \zeta\mathbf{i}, j = 0, \ldots, K_2 \Bigg\}, \\
\mathcal{A}_2 = \Bigg\{ & z_k = c_1 - \frac{\varepsilon}{2} + \left(\frac{\zeta}{2} - \frac{\zeta k}{K_1}\right)\mathbf{i}, k = 0, \ldots, K_1, \\
& z_{K_1+j} = c_1 - \frac{\varepsilon}{2} + \frac{(c_2 - c_1 + \varepsilon) j}{K_2} - \frac{\zeta}{2}\mathbf{i}, j = 0, \ldots, K_2, \\
& z_{K_1+K_2+k} = c_2 + \frac{\varepsilon}{2} + \left(-\frac{\zeta}{2} + \frac{\zeta k}{K_1}\right)\mathbf{i}, k = 0, \ldots, K_1, \\
& z_{2K_1+K_2+j} = c_2 + \frac{\varepsilon}{2} - \frac{(c_2 - c_1 + \varepsilon) j}{K_2} + \frac{\zeta}{2}\mathbf{i}, j = 0, \ldots, K_2 \Bigg\}.
\end{aligned}$$

Step 3. By (4.1), we obtain $m_0(z_j)$. By (4.2), we obtain $\underline{m}(z_j)$. Then the mean and covariance functions are approximated by

$$\begin{aligned}
\mathrm{E}X_f \approx & -\frac{\kappa - 1}{4\pi} \sum_{j=0}^{2K_1+2K_2-1} \Im\left[f'(z_j) \log\left(\frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{(1 - y_2 \int \frac{m_0(z_j)}{t + m_0(z_j)} \, dH(t))^2}{1 - y_2 \int \frac{m_0^2(z_j)}{(t + m_0(z_j))^2} \, dH(t)} \right) (z_{j+1} - z_j) \right] \\
& - \frac{\kappa - 1}{4\pi} \\
& \times \sum_{j=0}^{2K_1+2K_2-1} \Im\left[f'(z_j) \log\left(1 - y_2 \int \frac{m_0^2(z_j) \, dH(t)}{(t + m_0(z_j))^2} \right) (z_{j+1} - z_j) \right] \\
& - \frac{\beta_x y_1}{2\pi} \\
& \times \sum_{j=0}^{2K_1+2K_2-1} \Im\left[f(z_j) \frac{\int \frac{t^2}{(t + m_0(z_j))^3} \, dH(t) \cdot (-1 + y_2 \int \frac{m_0(z_j)}{t + m_0(z_j)} \, dH(t))^2 (z_{j+1} - z_j)}{\left(\frac{h^2}{y_2} - \frac{y_1}{y_2} \frac{\underline{m}_y^2(z_j) m_0^2(z_j)}{1 - y_2 \int \frac{m_0^2(z_j)}{(t + m_0(z_j))^2} \, dH(t)} \right) \left(1 - y_2 \int \frac{m_0^2(z_j)}{(t + m_0(z_j))^2} \, dH(t) \right)} \right]
\end{aligned} \tag{4.8}$$

$$+\frac{\beta_y y_2}{2\pi}$$
$$\times \sum_{j=0}^{2K_1+2K_2-1} \Im\Bigg[f(z_j) \frac{m_0(z_j)\int \frac{t\,dH(t)}{(t+m_0(z_j))^3}}{1-y_2\int \frac{m_0^2(z_j)\,dH(t)}{(t+m_0(z_j))^2}} \frac{(-1+y_2\int \frac{m_0(z_j)}{t+m_0(z_j)}\,dH(t))^2(z_{j+1}-z_j)}{\left(\frac{h^2}{y_2}-\frac{y_1}{y_2}\frac{\underline{m}^2(z_j)m_0^2(z_j)}{1-y_2\int \frac{m_0^2(z_j)}{(t+m_0(z_j))^2}\,dH(t)}\right)} \Bigg]$$

and

$$\begin{aligned}
&\operatorname{Cov}(X_{f_i}, X_{f_j}) \\
&\approx -\frac{\kappa}{2\pi^2}\sum_{k,l=0}^{K_2} I_{(k\neq l)}\Re\left(f_i'(z_{K_1+k}) f_j'(z_{K_1+l}) \log\left|\frac{\underline{m}(x)-\overline{\underline{m}(y)}}{\underline{m}(z_k)-\underline{m}(z_l)}\right|\right)(c_2-c_1)^2/K_1^2 \qquad (4.9)\\
&\quad -\frac{\beta_x y_1+\beta_y y_2}{4\pi^2}\sum_{j,k=0}^{2K_1+2K_2-1}\Re\left[f_i'(z_{1k}) f_j'(z_{2l})\int \frac{t^2\,dH(t)(z_{1,k+1}-z_{1k})(z_{2,l+1}-z_{2l})}{(t+m_0(z_{1k}))(t+m_0(z_{2k}))}\right],
\end{aligned}$$

where $z_{K_1+k}\in\mathcal{A}_1$, $z_{K_1+l}\in\mathcal{A}_2$, $z_{1k}\in\mathcal{A}_1$ and $z_{2l}\in\mathcal{A}_2$ and $I_{(\cdot)}=1$ is an indicator function.

5. Applications to high-dimensional statistical analysis

In this section, we introduce two applications of the theory developed in the paper to two high-dimensional statistical problems.

5.1. Power function for testing the equality of two high-dimensional covariance matrices

First, we consider the two-sample test of the hypothesis that two high-dimensional covariance matrices are equal, that is,

$$H_0:\mathbf{\Sigma}_1=\mathbf{\Sigma}_2 \quad \text{vs.} \quad H_1:\mathbf{\Sigma}_1\neq\mathbf{\Sigma}_2. \qquad (5.1)$$

By Zheng, Bai and Yao [29], the likelihood ratio test statistic for (5.1) is

$$-T_{\mathbf{n}}=\sum_{i=1}^{p}\log(y_{N_1}+y_{N_2}\lambda_i)-\sum_{i=1}^{p}\frac{y_{N_2}}{y_{N_1}+y_{N_2}}\log\lambda_i-\log(y_{N_1}+y_{N_2}),$$

where λ_i's are eigenvalues of a Fisher matrix $\mathbf{AB}^{-1}$ with

$$\mathbf{A}=\frac{1}{N_1}\sum_{k=1}^{n_1}\mathbf{\Sigma}_1^{1/2}(\mathbf{X}_{\cdot k}-\bar{\mathbf{X}})(\mathbf{X}_{\cdot k}-\bar{\mathbf{X}})^T\mathbf{\Sigma}_1^{1/2}, \qquad \mathbf{B}=\frac{1}{N_2}\sum_{k=1}^{n_2}\mathbf{\Sigma}_2^{1/2}(\mathbf{Y}_{\cdot k}-\bar{\mathbf{Y}})(\mathbf{Y}_{\cdot k}-\bar{\mathbf{Y}})^T\mathbf{\Sigma}_2^{1/2},$$

where $N_i = n_i - 1$ and $y_{N_i} = p/N_i$ for $i = 1, 2$. Under H_0 and as $\mathbf{n} \to \infty$, we have

$$\widetilde{T}_{\mathbf{n}} = \upsilon(f)^{-1/2}\big[T_{\mathbf{n}} - p \cdot F_{y_{N_1}, y_{N_2}}(f) - m(f)\big] \overset{H_0}{\Rightarrow} N(0, 1), \tag{5.2}$$

where $f(x) = \log(y_{N_1} + y_{N_2}x) - \frac{y_{N_2}}{y_{N_1}+y_{N_2}} \log x$, $F_{y_{N_1}, y_{N_2}}(f)$, $m(f)$ and $\upsilon(f)$ are

$$\begin{aligned}
F_{y_{N_1}, y_{N_2}}(f) &= \frac{h^2}{y_{N_1} y_{N_2}} \log \frac{y_{N_1} y_{N_2}}{h^2} + \frac{y_{N_2}(1 - y_{N_1})}{y_{N_1}(y_{N_1} + y_{N_2})} \log(1 - y_{N_1}) \\
&\quad + \frac{y_{N_1}(1 - y_{N_2})}{y_{N_2}(y_{N_1} + y_{N_2})} \log(1 - y_{N_2}), \\
m(f) &= \frac{1}{2}\bigg[\log \frac{h^2}{y_{N_1} + y_{N_2}} - \frac{y_{N_2} \log(1 - y_{N_1})}{y_{N_1} + y_{N_2}} - \frac{y_{N_1} \log(1 - y_{N_2})}{y_{N_1} + y_{N_2}} \\
&\quad + \frac{y_{N_1} y_{N_2}(\beta_x y_{N_2} + \beta_y y_{N_1})}{(y_{N_1} + y_{N_2})^2}\bigg], \\
\upsilon(f) &= -\frac{2y_{N_1}^2}{(y_{N_1} + y_{N_2})^2} \log(1 - y_{N_2}) - \frac{2y_{N_2}^2}{(y_{N_1} + y_{N_2})^2} \log(1 - y_{N_1}) + 2 \log \frac{h^2}{y_{N_1} + y_{N_2}}.
\end{aligned}$$

Then the critical region at the test size $\alpha = 0.05$ is

$$\big\{T_{\mathbf{n}} > 1.64\upsilon(f)^{1/2} + p \cdot F_{y_{N_1}, y_{N_2}}(f) + m(f)\big\}.$$

By Theorem 3.1 in this paper, under H_1 we have

$$\upsilon^1(f)^{-1/2}\big[T_{\mathbf{n}} - p \cdot F^1_{y_{N_1}, y_{N_2}}(f) - m^1(f)\big] \overset{H_1}{\Rightarrow} N(0, 1),$$

where $m^1(f)$ and $\upsilon^1(f)$ can be approximated by (4.8) and (4.9), and $F^1_{y_{N_1}, y_{N_2}}(f)$ by

$$F^1_{y_{N_1}, y_{N_2}}(f) = \int_{c_1}^{c_2} f(x) u_{\mathbf{y}}(x)\, dx \approx \frac{c_2 - c_1}{10^3} \sum_{j=1}^{10^3} f(x_j) u_{\mathbf{y}}(x_j), \qquad x_j = c_1 + \frac{(c_2 - c_1) j}{10^3}$$

and $u_{\mathbf{y}}(x_j)$ is computed by (4.6). Since

$$\begin{aligned}
&T_{\mathbf{n}} \geq 1.64\upsilon(f)^{1/2} + p \cdot F_{y_{N_1}, y_{N_2}}(f) + m(f) \\
&\quad \Leftrightarrow \quad \upsilon^1(f)^{-1/2}\big[T_{\mathbf{n}} - p \cdot F^1_{y_{N_1}, y_{N_2}}(f) - m^1(f)\big] \\
&\geq \upsilon^1(f)^{-1/2}\big[1.64\upsilon(f)^{1/2} + p \cdot F_{y_{N_1}, y_{N_2}}(f) + m(f) - p \cdot F^1_{y_{N_1}, y_{N_2}}(f) - m^1(f)\big],
\end{aligned}$$

then the power function of the test is

$$\begin{aligned}
1 - \Phi\big(&\upsilon^1(f)^{-1/2}\big[1.64\upsilon(f)^{1/2} + p \cdot F_{y_{N_1}, y_{N_2}}(f) + m(f) \\
&- p \cdot F^1_{y_{N_1}, y_{N_2}}(f) - m^1(f)\big]\big),
\end{aligned} \tag{5.3}$$

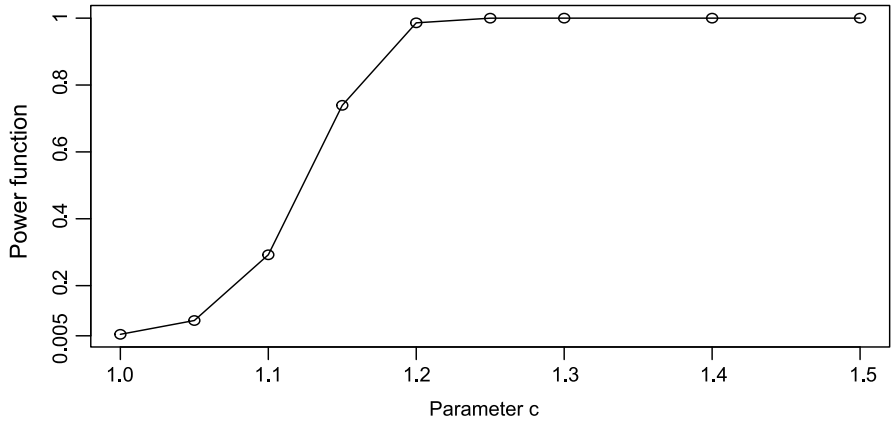

Figure 2. Power function is for (5.3) in the testing problem (5.1) by numerical methods (4.7), (4.8) and (4.9).

where $\Phi(\cdot)$ is the standardized normal distribution function.

To show the feasibility of the proposed numerical methods (4.7)–(4.8)–(4.9), we plot the power function (5.3) for the testing problem (5.1) in the Gaussian case where $p = 300, n_1 = 600, n_2 = 6000$, $\mathbf{\Sigma}_1$ is the identity matrix and $\mathbf{\Sigma}_2 = \text{diag}(1, \ldots, 1, c, \ldots, c)$ with the parameter $c = 1.00$, 1.05, 1.10, 1.15, 1.20, 1.25, 1.30, 1.40, 1.50 and the number of c in $\mathbf{\Sigma}_2$ being $p/2$. The numerical power function (5.3) for the testing problem (5.1) is plotted in Figure 2. Moreover, we compare the true values with numerical values of $F_{y_{N_1}, y_{N_2}}(f), m(f), \text{Cov}(f)$ in Table 2 in the case of $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$. These values are very close each other and the proposed numerical methods perform well.

Table 2. True values and numerical values of three parameters $F_{y_{N_1}, y_{N_2}(f)}, m(f), \upsilon(f)$ where a-3e $= a \cdot 10^{-3}$

	True values of			Numerical values of		
	$F_{y_{N_1}, y_{N_2}(f)}$	$m(f)$	$\upsilon(f)$	$F_{y_{N_1}, y_{N_2}(f)}$	$m(f)$	$\upsilon(f)$
$y_1 = 0.2, y_2 = 0.02$	9.775-3e	10.111-3e	0.376-3e	9.772-3e	10.151-3e	0.382-3e
$y_1 = 0.3, y_2 = 0.03$	15.269-3e	16.151-3e	0.918-3e	15.264-3e	16.232-3e	0.938-3e
$y_1 = 0.4, y_2 = 0.04$	21.282-3e	23.101-3e	1.790-3e	21.274-3e	23.254-3e	1.835-3e
$y_1 = 0.5, y_2 = 0.05$	27.998-3e	31.657-3e	3.121-3e	27.931-3e	31.562-3e	3.199-3e
$y_1 = 0.6, y_2 = 0.06$	35.449-3e	41.186-3e	5.050-3e	35.429-3e	41.730-3e	5.239-3e
$y_1 = 0.5, y_2 = 0.5$	170.085-3e	199.063-3e	110.863-3e	169.899-3e	202.7326-3e	117.783-3e

5.2. Confidence interval of θ in $\mathbf{T}_p(\theta)$

As the second application, we consider $\mathbf{T}_p = \mathbf{T}_p(\theta)$, that is, $\mathbf{T}_p$ is determined by the parameter θ which takes values in an interval $[a, b]$. We are interested in the confidence interval for the parameter θ. Then using the fact

$$\upsilon^{\theta}(f)^{-1/2}\big[T_{\mathbf{n}} - p \cdot F^{\theta}_{y_{N_1}, y_{N_2}}(f) - m^{\theta}(f)\big] \overset{H_1}{\Rightarrow} N(0, 1),$$

we will give a method to determine the confidence interval of the parameter θ.

First, cut $[a, b]$ as $\mathcal{A}_3 = \{\theta_j = a + \frac{(b-a)j}{N}, j = 0, \ldots, N\}$ where N is a large integer, for example, 10^3. Giving θ_j, that is, $\mathbf{T}_p = \mathbf{T}_p(\theta_j)$ and using Algorithms 1–2, we obtain $m^{\theta_j}(f) = EX_f$, $\upsilon^{\theta_j}(f) = \mathrm{Cov}(X_f, X_f)$ and $F^{\theta_j}_{y_{N_1}, y_{N_2}}(f)$, $j = 0, \ldots, N$. Then the confidence interval of θ is $[\theta_L, \theta_U]$ where

$$\theta_L = \min\big\{\theta_j : \upsilon^{\theta_j}(f)^{-1/2}\big(T_{\mathbf{n}} - p \cdot F^{\theta_j}_{y_{N_1}, y_{N_2}}(f) - m^{\theta_j}(f)\big) \le 1.64\big\}$$

and

$$\theta_U = \max\big\{\theta_j : \upsilon^{\theta_j}(f)^{-1/2}\big(T_{\mathbf{n}} - p \cdot F^{\theta_j}_{y_{N_1}, y_{N_2}}(f) - m^{\theta_j}(f)\big) \le 1.64\big\}.$$

6. Concluding remarks

In this paper, we have considered a general Fisher matrix $\mathbf{F}$ where the (high-dimensional) population covariance matrices $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ can be arbitrary and not necessarily equal. First the limiting distribution of its eigenvalues has been found. Next and more importantly, we establish a CLT for its linear spectral statistics. This CLT is unavoidable in many two-sample statistical problems with high-dimensional data. Besides, this CLT extends and covers the CLT of Zheng [28] which is related to standard Fisher matrices.

An important and unsolved issue on the developed theory is about the evaluation of the limiting mean and covariance function in the CLT. These functions have a very complex structure depending on non-trivial contour integrals. In the special case where the matrices $\mathbf{\Sigma}_2^{-1}\mathbf{\Sigma}_1$ are diagonal, we have proposed some simplifications though the obtained results are still complex. In Section 4, we have devised some numerical procedures to approximate numerically these asymptotic parameters. The advantage of these procedures is that they depend on the observed data only. However, the accuracy of these procedure is currently unknown. A precise analysis of these procedures or finding other more accurate procedures for the approximation are certainly a valuable and challenging question in future research.

Appendix A: Two propositions about $h_{mj}(z)$ and $h_{v_1,v_2}(z_1, z_2)$ when $\mathbf{T}_p$'s are diagonal

Here, we develop an important special example where the matrices $\mathbf{T}_p$'s are diagonal. In this case, the explicit expressions for the limiting functions $h_{mj}(z)$ and $h_{vj}(z_1, z_2)$, $j = 1, 2$ are given.

Proposition A.1. *In addition to the assumptions of Theorem 3.1, assume that the matrices* $\mathbf{T}_p$*'s are diagonal. Then, the limits (3.5) and (3.6) exist and equal to*

$$h_{m2}\big(-\underline{m}^{-1}(z)\big) = -\frac{1}{\underline{m}^3(z)} \int \frac{t}{(t+\underline{m}_0(z))^3}\, dH(t), \tag{A.1}$$

$$\begin{aligned} &h_{v2}\big(-\underline{m}^{-1}(z_1), -\underline{m}^{-1}(z_2)\big) \\ &\quad = \frac{1}{\underline{m}(z_1)\underline{m}(z_2)} \int \frac{1}{(t+\underline{m}_0(z_1))(t+\underline{m}_0(z_2))}\, dH(t). \end{aligned} \tag{A.2}$$

Consequently, the same conclusions as in Theorem 3.1 hold where the last term of $\mathrm{E}X_f$ *in (3.7) is simplified to*

$$\frac{\beta_y}{4\pi i} \oint_{\mathcal{C}} f(z) \left(1 - y_2 \int \frac{m_0^2(z)\, dH(t)}{(t+m_0(z))^2}\right) d\log\left(1 - y_2 \int \frac{m_0^2(z)\, dH(t)}{(t+m_0(z))^2}\right) \tag{A.3}$$

and the last term of $\mathrm{Cov}(X_{f_i}, X_{f_j})$ *in (3.8) is simplified to*

$$-\frac{\beta_y y_2}{4\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} f_i(z_1) f_j(z_2) \left[\int \frac{t^2\, dH(t)}{(t+m_0(z_1))^2 (t+m_0(z_2))^2}\right] dm_0(z_1)\, dm_0(z_2), \tag{A.4}$$

where each of the contours $\mathcal{C}$, $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *encloses the support of* $U_{\mathbf{y}}$, $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are disjoint and* $m_0(z) = \underline{m}_{y_2}(-\underline{m}(z))$.

Proposition A.2. *In addition to the assumptions of Theorem 3.1, assume that the matrices* $\mathbf{T}_p$*'s are diagonal. Then (B.64) and (B.66) exist and equal to*

$$h_{m1}(z) = -\frac{\frac{m_0^2(z)}{z^2\underline{m}(z)} \int \frac{t^2}{(t+m_0(z))^3}\, dH(t)}{1 - y_2 \oint \frac{m_0^2(z)}{(t+m_0(z))^2}\, dH(t)} \tag{A.5}$$

and

$$h_{v1}(z_1, z_2) = \frac{1}{z_1 z_2 \underline{m}(z_1)\underline{m}(z_2)} \int \frac{t^2}{(t+m_0(z_1))(t+m_0(z_2))}\, dH(t). \tag{A.6}$$

Consequently, the same conclusions as in Theorem 3.1 hold where

$$\begin{aligned} &-\frac{\beta_x y_1}{2\pi i} \oint_{\mathcal{C}} f(z) \frac{z^2 \underline{m}^3(z) \cdot h_{m1}(z)}{\frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{(1 - \int \frac{y_2 m_0(z)}{t+m_0(z)}\, dH(t))^2}{1 - \int \frac{y_2 m_0^2(z)}{(t+m_0(z))^2}\, dH(t)}}\, dz \\ &\quad = \frac{\beta_x y_1}{2\pi i} \oint_{\mathcal{C}} f(z) \left[\int \frac{t^2}{(t+m_0(z))^3}\, dH(t)\right] dm_0(z) \end{aligned} \tag{A.7}$$

and

$$
\begin{aligned}
&-\frac{\beta_x y_1}{4\pi^2}\cdot\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_2} f_i(z_1)f_j(z_2)\frac{\partial^2[z_1z_2\underline{m}(z_1)\underline{m}(z_2)h_{v1}(z_1,z_2)]}{\partial z_1\,\partial z_2}\,dz_1\,dz_2\\
&\quad=-\frac{\beta_x y_1}{4\pi^2}\cdot\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_2} f_i(z_1)f_j(z_2)\left[\int\frac{t^2\,dH(t)}{(t+m_0(z_1))^2(t+m_0(z_2))^2}\right]dm_0(z_1)\,dm_0(z_2).
\end{aligned}
\tag{A.8}
$$

Appendix B: Proofs

B.1. Proof of Theorem 2.1

Let

$$
s_{n_2}(z)=\int_0^\infty\frac{1}{t-z}\,dG_{n_2}(t),\qquad s_{y_2}(z)=\int_0^\infty\frac{1}{t-z}\,dG_{y_2}(t),
$$

be the Stieltjes transforms of the ESD $G_{n_2}(t)$ and LSD $G_{y_2}(t)$ of random matrix $\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1}\times\mathbf{T}_{p,1/2}$, respectively. Let

$$
m_{y_2}(z)=\int_0^\infty\frac{t}{1-tz}\,dG_{y_2}(t),\tag{B.1}
$$

which is the Stieltjes transform of the image measure of G_{y_2} by the reciprocal transformation $\lambda\mapsto 1/\lambda$ on $(0,\infty)$. It is easily checked that the Stieltjes transforms are

$$
m_{y_2}(z)=-\frac{1}{z}-\frac{1}{z^2}s_{y_2}(1/z).\tag{B.2}
$$

Similarly, consider the image measure and the associated Stieltjes transform

$$
m_{n_2}(z)=-\frac{1}{z}-\frac{1}{z^2}s_{n_2}(1/z).\tag{B.3}
$$

Let

$$
\underline{m}_{y_2}(z)=-\frac{1-y_2}{z}+y_2m_{y_2}(z),\tag{B.4}
$$

then by Theorem 2.1 of Zheng, Bai and Yao [30], we have

$$
z=-\frac{1}{\underline{m}_{y_2}(z)}+y_2\int\frac{dH(t)}{t+\underline{m}_{y_2}(z)},\tag{B.5}
$$

where $H(t)$ is the LSD of $\mathbf{T}_p$. In fact, we have

$$
\underline{m}_{y_2}(z)=-\frac{1}{z}-\frac{y_2}{z^2}s_{y_2}(1/z)\quad\text{or}\quad-\frac{1}{z}\underline{m}_{y_2}\left(\frac{1}{z}\right)=1+y_2zs_{y_2}(z).
$$

By Silverstein and Choi [22], we have

$$z = -\frac{1}{\underline{m}(z)} + y_1 \int \frac{t\,dG_{y_2}(t)}{1+t\underline{m}(z)} = -\frac{1}{\underline{m}(z)} + y_1 m_{y_2}\big(-\underline{m}(z)\big). \tag{B.6}$$

So by (B.4) the above equation reduces to

$$z = -\frac{h^2}{\underline{m}(z)\cdot y_2} + \frac{y_1}{y_2}\underline{m}_{y_2}\big(-\underline{m}(z)\big), \tag{B.7}$$

where $h^2 = y_1 + y_2 - y_1 y_2$. Write $m_0(z) = \underline{m}_{y_2}(-\underline{m}(z)) = \frac{1-y_2}{\underline{m}(z)} + y_2 \int \frac{t\,dG_{y_2}(t)}{1+t\underline{m}(z)}$. Replacing z by $-\underline{m}(z)$, equation (B.5) becomes

$$-\underline{m}(z) = -\frac{1}{m_0(z)} + y_2 \int \frac{dH(t)}{t+m_0(z)}. \tag{B.8}$$

Therefore, equation (B.7) reduces to

$$z = \frac{h^2 m_0(z)}{y_2(-1+y_2\int \frac{m_0(z)\,dH(t)}{t+m_0(z)})} + \frac{y_1}{y_2} m_0(z). \tag{B.9}$$

The proof of Theorem 2.1 is then completed.

B.2. Uniqueness of the solution to equation (2.9)

Rewrite (2.9) as

$$\frac{h^2}{y_2 z - y_1 m_0(z)} = -\frac{1}{m_0(z)} + y_2 \int \frac{dH(t)}{t+m_0(z)}, \tag{B.10}$$

and denote $m_0(z) = m_r(z) + \mathbf{i}m_i(z)$, where $m_r(z)$ and $m_i(z)$ are real. Comparing the imaginary parts of both sides of (B.10) and then dividing m_i, we obtain

$$-\frac{h^2(y_2 v - y_1 m_i)}{m_i|y_2 z - y_1 m_0|^2} = \frac{1}{|m_0|^2} - y_2 \int \frac{dH(t)}{|t+m_0|^2}. \tag{B.11}$$

Noting the fact that $m_i < 0$, we have

$$\frac{1}{|m_0|^2} - y_2 \int \frac{dH(t)}{|t+m_0|^2} > \frac{h^2 y_1}{|y_2 z - y_1 m_0|^2}. \tag{B.12}$$

Now suppose that $m_1 \neq m_2$ are two roots to the equation (B.10) with negative imaginary parts. Making difference of (B.10) with m_1 and m_2, we obtain

$$\frac{h^2 y_1}{(y_2 z - y_1 m_1)(y_2 z - y_1 m_2)} = \frac{1}{m_1 m_2} - y_2 \int \frac{dH(t)}{(t+m_1)(t+m_2)}. \tag{B.13}$$

Applying Cauchy–Schwarz to (B.13), then we obtain

$$\begin{aligned}
&\left|\frac{1}{m_1m_2}-y_2\int\frac{dH(t)}{(t+m_1)(t+m_2)}\right|\\
&\quad\geq\frac{1}{|m_1m_2|}-\left(y_2^2\int\frac{dH(t)}{|t+m_1|^2}\int\frac{dH(t)}{|t+m_2|^2}\right)^{1/2}\\
&\quad\geq\left(\frac{1}{|m_1|^2}-y_2\int\frac{dH(t)}{|t+m_1|^2}\right)^{1/2}\left(\frac{1}{|m_2|^2}-y_2\int\frac{dH(t)}{|t+m_2|^2}\right)^{1/2}\\
&\quad>\frac{h^2y_1}{|(y_2z-y_1m_1)(y_2z-y_1m_2)|}.
\end{aligned}$$

This contradicts the equation (B.13).

The proof of the uniqueness is done.

B.3. Some useful identities

Lemma B.1. *Let $m_0(z)=\underline{m}_{y_2}(-\underline{m}(z))$ where $\underline{m}(z)$ is the solution of (B.6), then we have the following identities*

$$1-y_1\int\frac{\underline{m}^2(z)x^2\,dG_{y_2}(x)}{(1+x\underline{m}(z))^2}=\frac{h^2}{y_2}-\frac{y_1}{y_2}\cdot\frac{(1-\int\frac{y_2m_0}{t+m_0}\,dH(t))^2}{1-\int\frac{y_2m_0^2}{(t+m_0)^2}\,dH(t)},\tag{B.14}$$

$$\left[\log\left(\frac{h^2}{y_2}-\frac{y_1}{y_2}\cdot\frac{(1-\int\frac{y_2m_0}{t+m_0}\,dH(t))^2}{1-\int\frac{y_2m_0^2}{(t+m_0)^2}\,dH(t)}\right)\right]'=\frac{-2y_1\int\frac{\underline{m}^3(z)(z)x^2\,dG_{y_2}(x)}{(1+x\underline{m}(z))^3}}{[1-y_1\int\frac{\underline{m}^2(z)x^2\,dG_{y_2}(x)}{(1+x\underline{m}(z))^2}]^2},\tag{B.15}$$

$$\left(\frac{h^2}{y_2}-\frac{y_1}{y_2}\cdot\frac{(1-\int\frac{y_2m_0}{t+m_0}\,dH(t))^2}{1-\int\frac{y_2m_0^2}{(t+m_0)^2}\,dH(t)}\right)'=\frac{-2y_1\int\frac{\underline{m}^3(z)(z)x^2\,dG_{y_2}(x)}{(1+x\underline{m}(z))^3}}{1-y_1\int\frac{\underline{m}^2(z)x^2\,dG_{y_2}(x)}{(1+x\underline{m}(z))^2}},\tag{B.16}$$

$$\left[\log\left(1-y_2\int\frac{m_0^2\,dH(t)}{(t+m_0)^2}\right)\right]'=\frac{2\underline{m}'(z)y_2\int\frac{tm_0^3\,dH(t)}{(t+m_0)^3}}{(1-y_2\int\frac{m_0^2\,dH(t)}{(t+m_0)^2})^2},\tag{B.17}$$

$$m_0(z)=\frac{1}{\underline{m}(z)}\left(1-\frac{y_2}{\underline{m}(z)}s_{y_2}\left(-\frac{1}{\underline{m}(z)}\right)\right),\qquad m_0'=\frac{-\underline{m}'m_0^2}{1-y_2\int\frac{m_0^2\,dH(t)}{(t+m_0)^2}},\tag{B.18}$$

$$1-y_1\int\frac{(\underline{m}(z))^2x^2\,dG_{y_2}(x)}{(1+x\underline{m}(z))^2}=\frac{(\underline{m}(z))^2}{\underline{m}'(z)},\tag{B.19}$$

$$\left(1-y_2\int\frac{m_0^2(z)\,dH(t)}{(t+m_0(z))^2}\right)'=2\underline{m}'(z)\frac{y_2\int\frac{m_0^3(z)t}{(t+m_0(z))^3}\,dH(t)}{1-y_2\int\frac{m_0^2(z)\,dH(t)}{(t+m_0(z))^2}},\tag{B.20}$$

where $m_0'(z)=\frac{d}{dz}m_0(z)$ *and* $\underline{m}'(z)=\frac{d}{dz}\underline{m}(z)$.

Proof. By (B.1), we have $m'_{y_2}(z)=\int_0^\infty\frac{x^2\,dG_{y_2}(x)}{(1-xz)^2}$ where $'$ denotes the derivative with respect to z. So by (B.4) we have

$$\int\frac{x^2\,dG_{y_2}(x)}{(1+x\underline{m}(z))^2}=m'_{y_2}\big(-\underline{m}(z)\big)=-\frac{1-y_2}{y_2}\cdot\frac{1}{(\underline{m}(z))^2}+\frac{1}{y_2}\cdot\underline{m}'_{y_2}\big(-\underline{m}(z)\big),\tag{B.21}$$

where $\underline{m}'_{y_2}(-\underline{m}(z))=\frac{d}{d\xi}\underline{m}_{y_2}(\xi)_{|\xi=-\underline{m}(z)}$ instead of $\frac{d}{dz}\underline{m}_{y_2}(-\underline{m}(z))$. By (B.21), we have

$$1-y_1\int\frac{(\underline{m}(z))^2x^2\,dG_{y_2}(x)}{(1+x\underline{m}(z))^2}=\frac{h^2}{y_2}-\frac{y_1(\underline{m}(z))^2\underline{m}'_{y_2}(-\underline{m}(z))}{y_2}.\tag{B.22}$$

Differentiating both sides of (B.5) and then replacing z by $-\underline{m}$, we obtain

$$1=\left(\frac{1}{m_0^2}-y_2\int\frac{dH(t)}{(t+m_0)^2}\right)m'_{y_2}(-\underline{m}).\tag{B.23}$$

This equation, together with (B.8), (B.22) and (B.23) imply that

$$1-y_1\int\frac{\underline{m}^2(z)x^2\,dG_{y_2}(x)}{(1+x\underline{m}(z))^2}=\frac{h^2}{y_2}-\frac{y_1}{y_2}\cdot\frac{(1-\int\frac{y_2m_0}{t+m_0}\,dH(t))^2}{1-\int\frac{y_2m_0^2}{(t+m_0)^2}\,dH(t)}.\tag{B.24}$$

Differentiating both sides of (B.7) with respect to z, we obtain

$$1=\frac{h^2}{y_2(\underline{m}(z))^2}\underline{m}'(z)-\frac{y_1}{y_2}\underline{m}'_{y_2}\big(-\underline{m}(z)\big)\underline{m}'(z).$$

This implies that

$$\underline{m}'(z)=\frac{y_2(\underline{m}(z))^2}{h^2-y_1(\underline{m}(z))^2\underline{m}'_{y_2}(-\underline{m}(z))},$$

or equivalently

$$y_1\big(\underline{m}(z)\big)^2\underline{m}'_{y_2}\big(-\underline{m}(z)\big)=h^2-\frac{y_2(\underline{m}(z))^2}{\underline{m}'(z)}.\tag{B.25}$$

So by (B.22) and (B.25), we have

$$1-y_1\int\frac{(\underline{m}(z))^2x^2\,dG_{y_2}(x)}{(1+x\underline{m}(z))^2}=\frac{(\underline{m}(z))^2}{\underline{m}'(z)}.\tag{B.26}$$

Differentiating both sides of (B.8), we have

$$m_0' = \frac{-\underline{m}' m_0^2}{1 - y_2 \int \frac{m_0^2 \, dH(t)}{(t+m_0)^2}}. \tag{B.27}$$

So we have

$$\left(1 - y_2 \int \frac{m_0^2(z) \, dH(t)}{(t + m_0(z))^2}\right)' = 2\underline{m}'(z) \frac{y_2 \int \frac{m_0^3(z)t}{(t+m_0(z))^3} \, dH(t)}{1 - y_2 \int \frac{m_0^2(z) \, dH(t)}{(t+m_0(z))^2}}.$$

So by (B.24) and (B.26), we obtain (B.15). By (B.27), we have the conclusion (B.17). By (B.2) and (B.4), we have

$$m_0(z) = \frac{1}{\underline{m}(z)}\left(1 - \frac{y_2}{\underline{m}(z)} s_{y_2}\left(-\frac{1}{\underline{m}(z)}\right)\right).$$

The proof of the lemma is completed. □

In the sequel, for brevity, $s_{y_2}(z)$ will be denoted as $s(z)$ if no confusion would be possible.

B.4. Proof of Theorem 3.1

B.4.1. Deriving CLT of general Fisher matrix

Following the same techniques of truncation and normalization given in Bai and Silverstein [7] (see lines -9 to -6 from the bottom of page 559), the assumptions can be simplified as the following:

- $|X_{jk}| < \eta_p \sqrt{p}$, $|Y_{jk}| < \eta_p \sqrt{p}$, for some $\eta_p \to 0$, as $p \to \infty$;
- $EX_{jk} = 0$, $EY_{jk} = 0$ and $E|X_{jk}|^2 = 1$, $E|Y_{jk}|^2 = 1$;
- $E|X_{jk}|^4 = 1 + \kappa + \beta_x + o(1)$ and $E|Y_{jk}|^4 = 1 + \kappa + \beta_y + o(1)$;
- For the complex case, $EX_{jk}^2 = o(n_1^{-1})$ and $EY_{jk}^2 = o(n_2^{-1})$.

We have

$$n_1\big[\underline{m}_{\mathbf{n}}(z) - \underline{m}_{\mathbf{y_n}}(z)\big] = n_1\big[\underline{m}_{\mathbf{n}}(z) - \underline{m}^{\{y_{n_1}, G_{n_2}\}}(z)\big] + n_1\big[\underline{m}^{\{y_{n_1}, G_{n_2}\}}(z) - \underline{m}_{\mathbf{y_n}}(z)\big],$$

where $\underline{m}^{\{y_{n_1}, G_{n_2}\}}(z)$ and $\underline{m}_{\mathbf{y_n}}(z)$ are the unique roots with imaginary parts having the same signs as that of z to the following equations, see equation (2.10),

$$z = -\frac{1}{\underline{m}^{\{y_{n_1}, G_{n_2}\}}} + y_{n_1} \cdot \int \frac{t \, dG_{n_2}(t)}{1 + t\underline{m}^{\{y_{n_1}, G_{n_2}\}}} \quad \text{and} \quad z = -\frac{1}{\underline{m}_{\mathbf{y_n}}} + y_{n_1} \cdot \int \frac{t \, dG_{y_{n_2}}(t)}{1 + t\underline{m}_{\mathbf{y_n}}}.$$

The proof follows two steps and we unify the real and complex cases with the indicator notation κ.

Step 1. Consider the conditional distribution of

$$n_1\big[\underline{m}_{\mathbf{n}}(z) - \underline{m}^{\{y_{n_1}, G_{n_2}\}}(z)\big] \tag{B.28}$$

given $\mathscr{S}_2 = \{\text{all } \mathbf{S}_2\}$. In the proof of Theorem 2.1, we have proved that G_{n_2} converges to G_{y_2}. Using Lemma 1.1 of Bai and Silverstein [7], we conclude that the conditional distribution of

$$n_1\big[\underline{m}_{\mathbf{n}}(z) - \underline{m}^{\{y_{n_1}, G_{n_2}\}}(z)\big] = p\big[m_{\mathbf{n}}(z) - m^{\{y_{n_1}, G_{n_2}\}}(z)\big]$$

given $\mathscr{S}_2$ converges to a Gaussian process $M_1(z)$ on the contour $\mathcal{C}$ enclosing the support $[a, b]$ of the LSD $U_{\mathbf{y}}$ of Fisher matrix. Moreover, its mean function equals

$$\begin{aligned}\mathrm{E}\big(M_1(z)|\mathscr{S}_2\big) &= (\kappa - 1)\frac{y_1 \int \underline{m}(z)^3 x^2 [1 + x\underline{m}(z)]^{-3}\, dG_{y_2}(x)}{[1 - y_1 \int \underline{m}^2(z) x^2 (1 + x\underline{m}(z))^{-2}\, dG_{y_2}(x)]^2} \\ &\quad + \beta_x \frac{y_1 z^2 \underline{m}^3(z) \cdot h_{m1}(z)}{[1 - y_1 \int \frac{x^2 \underline{m}^2(z)}{\{1 + x\underline{m}(z)\}^2}\, dG_{y_2}(x)]},\end{aligned} \tag{B.29}$$

where $h_{m1}(z)$ is the limit of

$$\begin{aligned}&\frac{1}{p}\sum_{i=1}^{p} \mathrm{E}\big[\mathbf{e}_i' \big\{\mathbf{S}_2^{-1/2}\mathbf{T}_p^{1/2}\big\}^* \mathbf{D}_1^{-1} \big\{\mathbf{S}_2^{-1/2}\mathbf{T}_p^{1/2}\big\}\mathbf{e}_i \\ &\quad \times \mathbf{e}_i' \big\{\mathbf{S}_2^{-1/2}\mathbf{T}_p^{1/2}\big\}^* \mathbf{D}_1^{-1} \big(\underline{m}(z)\big\{\mathbf{T}_p^{1/2}\big\}^* \mathbf{S}_2^{-1}\mathbf{T}_p^{1/2} + \mathbf{I}_p\big)\big\{\mathbf{S}_2^{-1/2}\mathbf{T}_p^{1/2}\big\}\mathbf{e}_i |\mathbf{S}_2\big],\end{aligned}$$

which is obtained through replacing $\mathbf{S}_2^{-1/2}$ by $\mathbf{S}_2^{-1/2}\mathbf{T}_p^{1/2}$ in (6.40) of Zheng [28]. The conditional covariance function of the process $M_1(z)$ equals

$$\begin{aligned}&\mathrm{Cov}\big(M_1(z_1), M_1(z_2)|\mathscr{S}_2\big) \\ &= \kappa \cdot \left(\frac{\underline{m}'(z_1) \cdot \underline{m}'(z_2)}{(\underline{m}(z_1) - \underline{m}(z_2))^2} - \frac{1}{(z_1 - z_2)^2}\right) \\ &\quad - \frac{\beta_x y_1}{4\pi^2} \cdot \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} f_i(z_1) f_j(z_2) \frac{\partial^2 [z_1 z_2 \underline{m}(z_1)\underline{m}(z_2) h_{v1}(z_1, z_2)]}{\partial z_1\, \partial z_2}\, dz_1\, dz_2,\end{aligned}$$

where $h_{v1}(z_1, z_2)$ is obtained through replacing $\mathbf{S}_2^{-1/2}$ by $\mathbf{S}_2^{-1/2}\mathbf{T}_p^{1/2}$ in (6.41) of Zheng [28].

It is remarkable that these limiting functions are independent of the conditioning $\mathscr{S}_2$, which shows that the limiting process $M_1(z)$ is independent of the limit of the second part below.

Step 2. Now, we consider the limiting process of

$$n_1\big[\underline{m}^{\{y_{n_1}, G_{n_2}\}}(z) - \underline{m}_{\mathbf{y_n}}(z)\big] = p\big[m^{\{y_{n_1}, G_{n_2}\}}(z) - m_{\mathbf{y_n}}(z)\big]. \tag{B.30}$$

By (B.1), we have

$$z = -\frac{1}{\underline{m}_{\mathbf{y_n}}} + y_{n_1} \int \frac{t}{1 + t \cdot \underline{m}_{\mathbf{y_n}}}\, dG_{y_{n_2}}(t) = -\frac{1}{\underline{m}_{\mathbf{y_n}}} + y_{n_1} \cdot m_{y_{n_2}}\big(-\underline{m}_{\mathbf{y_n}}(z)\big). \tag{B.31}$$

On the other hand, $\underline{m}^{\{y_{n_1},G_{n_2}\}}$ is the solution to the equation

$$z=-\frac{1}{\underline{m}^{\{y_{n_1},G_{n_2}\}}}+y_{n_1}\int\frac{t\cdot dG_{n_2}(t)}{1+t\cdot\underline{m}^{\{y_{n_1},G_{n_2}\}}}$$

and

$$\begin{aligned}z&=-\frac{1}{\underline{m}^{\{y_{n_1},G_{n_2}\}}}+y_{n_1}\int\frac{t\,dG_{n_2}(t)}{1+t\cdot\underline{m}^{\{y_{n_1},G_{n_2}\}}}\\&=-\frac{1}{\underline{m}^{\{y_{n_1},G_{n_2}\}}}+y_{n_1}\int\left\{\frac{t\,dG_{n_2}(t)}{1+t\underline{m}^{\{y_{n_1},G_{n_2}\}}}-\frac{t\,dG_{n_2}(t)}{1+t\underline{m}_{\mathbf{y}_n}}\right\}+y_{n_1}\int\frac{t\,dG_{n_2}(t)}{1+t\underline{m}_{\mathbf{y}_n}},\end{aligned}\tag{B.32}$$

where

$$\int\frac{t}{1+t\cdot\underline{m}_{\mathbf{y_n}}(z)}\,dG_{n_2}(t)=m_{n_2}\big(-\underline{m}_{\mathbf{y_n}}(z)\big).$$

Taking the difference of (B.31) and (B.32) yields

$$\begin{aligned}0=&-\frac{1}{\underline{m}^{\{y_{n_1},G_{n_2}\}}}+\frac{1}{\underline{m}_{\mathbf{y_n}}}y_{n_1}\int\left\{\frac{t\,dG_{n_2}(t)}{1+t\underline{m}^{\{y_{n_1},G_{n_2}\}}}-\frac{t\,dG_{n_2}(t)}{1+t\underline{m}_{\mathbf{y}_n}}\right\}\\&+y_{n_1}\cdot\int\frac{t\cdot dG_{n_2}(t)}{1+t\cdot\underline{m}_{y_n}}-y_{n_1}\cdot\int\frac{t}{1+t\cdot\underline{m}_{\mathbf{y_n}}}\,dG_{y_{n_2}}(t).\end{aligned}$$

That is,

$$\begin{aligned}0=&\frac{\underline{m}^{\{y_{n_1},G_{n_2}\}}-\underline{m}_{\mathbf{y_n}}}{\underline{m}_{\mathbf{y_n}}\cdot\underline{m}^{\{y_{n_1},G_{n_2}\}}}-y_{n_1}\int\frac{(\underline{m}^{\{y_{n_1},G_{n_2}\}}-\underline{m}_{\mathbf{y_n}})t^2\,dG_{n_2}(t)}{(1+t\underline{m}^{\{y_{n_1},G_{n_2}\}})(1+t\underline{m}_{\mathbf{y_n}})}\\&+y_{n_1}\big\{m_{n_2}(-\underline{m}_{\mathbf{y_n}})-m_{y_{n_2}}(-\underline{m}_{\mathbf{y_n}})\big\}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&n_1\cdot\big[\underline{m}^{\{y_{n_1},G_{n_2}\}}(z)-\underline{m}_{\mathbf{y_n}}(z)\big]\\&=-y_{n_1}\cdot\underline{m}^{\{y_{n_1},G_{n_2}\}}\underline{m}_{\mathbf{y_n}}\cdot\frac{n_1[m_{n_2}(-\underline{m}_{\mathbf{y_n}})-m_{y_{n_2}}(-\underline{m}_{\mathbf{y_n}})]}{1-y_{n_1}\cdot\int\frac{\underline{m}_{\mathbf{y_n}}\cdot\underline{m}^{\{y_{n_1},G_{n_2}\}}t^2\,dG_{n_2}(t)}{(1+t\underline{m}_{\mathbf{y_n}})\cdot(1+t\underline{m}^{\{y_{n_1},G_{n_2}\}})}}\\&=-\underline{m}^{\{y_{n_1},G_{n_2}\}}\underline{m}_{\mathbf{y_n}}\cdot\frac{p[m_{n_2}(-\underline{m}_{\mathbf{y_n}})-m_{y_{n_2}}(-\underline{m}_{\mathbf{y_n}})]}{1-y_{n_1}\cdot\int\frac{\underline{m}_{\mathbf{y_n}}\cdot\underline{m}^{\{y_{n_1},G_{n_2}\}}t^2\,dG_{n_2}(t)}{(1+t\underline{m}_{\mathbf{y_n}})\cdot(1+t\underline{m}^{\{y_{n_1},G_{n_2}\}})}}.\end{aligned}\tag{B.33}$$

We then consider the limiting process of

$$p\big[m_{n_2}\big(-\underline{m}_{\mathbf{y_n}}(z)\big)-m_{y_{n_2}}\big(-\underline{m}_{\mathbf{y_n}}(z)\big)\big]=-\frac{p}{(\underline{m}_{\mathbf{y_n}}(z))^2}\left[s_{n_2}\left(\frac{-1}{\underline{m}_{\mathbf{y_n}}(z)}\right)-s_{y_{n_2}}\left(\frac{-1}{\underline{m}_{\mathbf{y_n}}(z)}\right)\right]$$

by (B.3). Noticing that for any $z \in \mathbb{C} \setminus \mathbb{R}$, $\underline{m}_{\mathbf{y_n}}(z) \to \underline{m}(z)$, the limiting distribution of

$$-\frac{p}{z^2}\left[s_{n_2}\left(\frac{-1}{\underline{m}_{\mathbf{y_n}}(z)}\right) - s_{y_{n_2}}\left(\frac{-1}{\underline{m}_{\mathbf{y_n}}(z)}\right)\right]$$

is the same as that of

$$-\frac{p}{(\underline{m}_{\mathbf{y_n}}(z))^2}\left[s_{n_2}\left(\frac{-1}{\underline{m}(z)}\right) - s_{y_{n_2}}\left(\frac{-1}{\underline{m}(z)}\right)\right].$$

From now on, we use the notation $g(z) = -1/\underline{m}(z)$. By Theorem 2.2 of Zheng, Bai and Yao [30], we conclude that

$$-pg^2(z)\big[s_{n_2}\big(g(z)\big) - s_{y_{n_2}}\big(g(z)\big)\big],$$

converges weakly to a Gaussian process $M_2(\cdot)$ on $z \in \mathcal{C}$ with the mean function

$$\mathrm{E}\big(M_2(z)\big) = (\kappa - 1) \cdot \frac{y_2 \int \frac{t[1+y_2 g(z)s(g(z))]^3\, dH(t)}{[-t\underline{m}(z)-1-y_2 g(z)s(g(z))]^3}}{(1 - y_2 \int \frac{[1+y_2 g(z)s(g(z))]^2\, dH(t)}{[-t\underline{m}(z)-1-y_2 g(z)s(g(z))]^2})^2} \tag{B.34}$$

$$+ \frac{\beta_y y_2 [1 + y_2 g(z)s(g(z))]^3 h_{m2}(g(z))}{1 - y_2 \int \frac{[1+y_2 g(z)s(g(z))]^2\, dH(t)}{[-t\underline{m}(z)-1-y_2 g(z)s(g(z))]^2}}, \tag{B.35}$$

and covariance function $\mathrm{Cov}(M_2(z_1), M_2(z_2))$ equaling

$$\begin{aligned}
&\kappa g^2(z_1)g^2(z_2)\left(\frac{\frac{\partial\{g(z_1)[1+y_2 g(z_1)s(g(z_1))]\}}{\partial\{-1/\underline{m}(z_1)\}}\frac{\partial\{g(z_2)[1+y_2 g(z_2)s(g(z_2))]\}}{\partial\{-1/\underline{m}(z_1)\}}}{\{g(z_1)[1 + y_2 g(z_1)s(g(z_1))] - g(z_2)[1 + y_2 g(z_2)s(g(z_2))]\}^2}\right.\\
&\quad\left. - \frac{1}{[g(z_1) - g(z_2)]^2}\right)\\
&\quad + \beta_y y_2 g^2(z_1)g^2(z_2)\frac{\partial^2[(1 + y_2 g(z_1)s(g(z_1)))(1 + y_2 g(z_2)s(g(z_2)))h_{v2}(g(z_1), g(z_2))]}{\partial(-1/\underline{m}(z_1))\,\partial(-1/\underline{m}(z_2))}
\end{aligned} \tag{B.36}$$

for $z_1, z_2 \in \mathcal{C}$, where $H(t)$ is the LSD of $\mathbf{T}_p$. Here we have used the fact that the limits $h_{m2}(z)$ and $h_{v2}(z_1, z_2)$ in (3.5)–(3.6) exist whenever $\beta_y \neq 0$. Since

$$1 - y_{n_1} \cdot \int \frac{\underline{m}_{\mathbf{y_n}}(z) \cdot \underline{m}^{\{y_{n_1}, G_{n_2}\}} t^2\, dG_{n_2}(t)}{(1 + t\underline{m}_{\mathbf{y_n}}(z))(1 + t\underline{m}^{\{y_{n_1}, G_{n_2}\}})} \longrightarrow 1 - y_1 \int \frac{t^2 \underline{m}^2(z)\, dG_{y_2}(t)}{[1 + t\underline{m}(z)]^2},$$

almost surely, this limit equals $\frac{m^2}{\underline{m}'}$ by (B.26). Then by (B.33) we obtain that

$$n_1\big[\underline{m}^{\{y_{n_1}, G_{n_2}\}}(z) - \underline{m}_{\mathbf{y_n}}(z)\big],$$

converges weakly to a Gaussian process

$$M_3(z) = -\underline{m}'(z)M_2(z),$$

with the mean function $\mathrm{E}(M_3(z)) = -\underline{m}'(z)\mathrm{E}M_2(z)$ and covariance functions

$$\mathrm{Cov}\big(M_3(z_1), M_3(z_2)\big) = \underline{m}'(z_1)\underline{m}'(z_2)\,\mathrm{Cov}\big(M_2(z_1), M_2(z_2)\big).$$

Since the limit process $M_1(z)$ of

$$n_1 \cdot \big[\underline{m}_{\mathbf{n}}(z) - \underline{m}^{\{y_{n_1}, G_{n_2}\}}(z)\big]$$

is independent of the ESD of S_{n_2}, we know that

$$\big\{n_1 \cdot \big[\underline{m}_{\mathbf{n}}(z) - \underline{m}^{\{y_{n_1}, G_{n_2}\}}(z)\big], n_1 \cdot \big[\underline{m}^{\{y_{n_1}, G_{n_2}\}}(z) - \underline{m}_{\mathbf{y_n}}(z)\big]\big\}$$

converge to a two-dimensional Gaussian process $(M_1(z), M_3(z))$ with independent components. Consequently, $n_1 \cdot [\underline{m}_{\mathbf{n}}(z) - \underline{m}_{\mathbf{y_n}}(z)]$ converges weakly to $M_4(z)$, a Gaussian process with mean function

$$\mathrm{E}\big(M_4(z)\big) = (\kappa - 1) \cdot \frac{y_1 \int \underline{m}^3(z)x^2[1 + x\underline{m}(z)]^{-3}\,dG_{y_2}(x)}{[1 - y_1 \int \underline{m}^2(z)x^2(1 + x\underline{m}(z))^{-2}\,dG_{y_2}(x)]^2} \tag{B.37}$$

$$+ \beta_x \cdot \frac{y_1 z^2 \underline{m}^3(z) \cdot h_{m1}(z)}{1 - y_1 \int \frac{x^2\underline{m}^2(z)}{[1+x\underline{m}(z)]^2}\,dG_{y_2}(x)} \tag{B.38}$$

$$- (\kappa - 1)\underline{m}'(z) \cdot \frac{y_2 \int \frac{t[1+y_2 g(z)s(g(z))]^3\,dH(t)}{[-t\underline{m}(z)-1-y_2 g(z)s(g(z))]^3}}{(1 - y_2 \int \frac{[1+y_2 g(z)s(g(z))]^2\,dH(t)}{[-t\underline{m}(z)-1-y_2 g(z)s(g(z))]^2})^2} \tag{B.39}$$

$$- \beta_y \cdot \underline{m}'(z)\frac{y_2[1 + y_2 g(z)s(g(z))]^3 h_{m2}(g(z))}{1 - y_2 \int \frac{[1+y_2 g(z)s(g(z))]^2\,dH(t)}{[-t\underline{m}(z)-1-y_2 g(z)s(g(z))]^2}}, \tag{B.40}$$

and covariance function

$$\begin{aligned}
&\mathrm{Cov}\big(M_4(z_1), M_4(z_2)\big)\\
&= \kappa \cdot \left(\frac{\underline{m}'(z_1) \cdot \underline{m}'(z_2)}{(\underline{m}(z_1) - \underline{m}(z_2))^2} - \frac{1}{(z_1 - z_2)^2}\right) + \beta_x y_1 \cdot \frac{\partial^2[z_1 z_2 \underline{m}(z_1)\underline{m}(z_2)h_{v1}(z_1, z_2)]}{\partial z_1\,\partial z_2}\\
&\quad + \kappa g'(z_1)g'(z_2)\frac{\frac{\partial\{g(z_1)[1+y_2 g(z_1)s(g(z_1))]\}}{\partial\{-1/\underline{m}(z_1)\}}\frac{\partial\{g(z_2)[1+y_2 g(z_2)s(g(z_2))]\}}{\partial\{-1/\underline{m}(z_1)\}}}{\{g(z_1)[1 + y_2 g(z_1)s(g(z_1))] - g(z_2)[1 + y_2 g(z_2)s(g(z_2))]\}^2}\\
&\quad - \kappa g'(z_1)g'(z_2)\frac{1}{[g(z_1) - g(z_2)]^2}\\
&\quad + \beta_y y_2 g'(z_1)g'(z_2)\frac{\partial^2[(1 + y_2 g(z_1)s(g(z_1)))(1 + y_2 g(z_2)s(g(z_2)))h_{v2}(g(z_1), g(z_2))]}{\partial(-1/\underline{m}(z_1))\,\partial(-1/\underline{m}(z_2))}
\end{aligned}$$

$$= -\kappa \cdot \frac{1}{(z_1 - z_2)^2} + \beta_x y_1 \cdot \frac{\partial^2 [z_1 z_2 \underline{m}(z_1)\underline{m}(z_2) h_{v1}(z_1, z_2)]}{\partial z_1 \, \partial z_2} \tag{B.41}$$

$$+ \kappa g'(z_1) g'(z_2) \frac{\frac{\partial\{g(z_1)[1+y_2 g(z_1)s(g(z_1))]\}}{\partial\{-1/\underline{m}(z_1)\}} \frac{\partial\{g(z_2)[1+y_2 g(z_2)s(g(z_2))]\}}{\partial\{-1/\underline{m}(z_1)\}}}{\{g(z_1)[1 + y_2 g(z_1)s(g(z_1))] - g(z_2)[1 + y_2 g(z_2)s(g(z_2))]\}^2} \tag{B.42}$$

$$+ \beta_y y_2 g'(z_1) g'(z_2) \times \frac{\partial^2 [(1 + y_2 g(z_1)s(g(z_1)))(1 + y_2 g(z_2)s(g(z_2)))h_{v2}(g(z_1), g(z_2))]}{\partial(-1/\underline{m}(z_1)) \, \partial(-1/\underline{m}(z_2))}. \tag{B.43}$$

B.4.2. *Simplifying the mean expressions* (B.37) *to* (B.40) *and the covariance expressions* (B.42)–(B.43)

Recall that $m_0(z) = \underline{m}_{y_2}(-\underline{m}(z))$. By (B.15), we have

$$(\text{B.37}) = \frac{(\kappa - 1) \cdot y_1 \int \frac{\underline{m}^3(z)x^2}{[1+x\underline{m}(z)]^3} \, dG_{y_2}(x)}{[1 - y_1 \int \frac{\underline{m}^2(z)x^2}{(1+x\underline{m}(z))^2} \, dG_{y_2}(x)]^2} = \frac{-(\kappa - 1)}{2} \frac{d \log(\frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{(1 - y_2 \int \frac{m_0(z)}{t+m_0(z)} \, dH(t))^2}{1 - y_2 \int \frac{m_0^2(z)}{(t+m_0(z))^2} \, dH(t)})}{dz}.$$

By (B.14), we have

$$(\text{B.38}) = \beta_x \frac{h_{m1}(z)}{[y_1 z^2 \underline{m}^3(z)]^{-1} \cdot [1 - y_1 \int \frac{x^2 \underline{m}^2(z)}{\{1+x\underline{m}(z)\}^2} \, dG_{y_2}(x)]} = \beta_x \frac{y_1 z^2 \underline{m}^3(z) \cdot h_{m1}(z)}{\frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{(1 - \int \frac{y_2 m_0}{t+m_0} \, dH(t))^2}{1 - \int \frac{y_2 m_0^2}{(t+m_0)^2} \, dH(t)}}.$$

By (B.17) and (B.18), we have

$$(\text{B.39}) = -\frac{(\kappa - 1)\underline{m}'(z) \cdot y_2 \int \frac{t[1+y_2 g(z)s(g(z))]^3 \, dH(t)}{[-t\underline{m}(z) - 1 - y_2 g(z)s(g(z))]^3}}{(1 - y_2 \int \frac{[1+y_2 g(z)s(g(z))]^2 \, dH(t)}{[-t\underline{m}(z) - 1 - y_2 g(z)s(g(z))]^2})^2} = -\frac{\kappa - 1}{2} \frac{d \log(1 - y_2 \int \frac{m_0^2(z) \, dH(t)}{(t+m_0(z))^2})}{dz}.$$

We have

$$(\text{B.40}) = -\beta_y \underline{m}'(z) \frac{y_2[1 + y_2 g(z)s(g(z))]^3 h_{m2}(g(z))}{1 - y_2 \int \frac{[1+y_2 g(z)s(g(z))]^2 \, dH(t)}{[-t\underline{m}(z) - 1 - y_2 g(z)s(g(z))]^2}} = -\beta_y \underline{m}'(z) \frac{y_2 \underline{m}^3(z) m_0^3(z) h_{m2}(g(z))}{1 - y_2 \int \frac{m_0^2(z)}{(t+m_0(z))^2} \, dH(t)}.$$

By (B.18) we have

$$(\text{B.42}) = \kappa g'(z_1) g'(z_2) \frac{\frac{\partial\{g(z_1)[1+y_2 g(z_1)s(g(z_1))]\}}{\partial\{-1/\underline{m}(z_1)\}} \frac{\partial\{g(z_2)[1+y_2 g(z_2)s(g(z_2))]\}}{\partial\{-1/\underline{m}(z_1)\}}}{\{g(z_1)[1 + y_2 g(z_1)s(g(z_1))] - g(z_2)[1 + y_2 g(z_2)s(g(z_2))]\}^2}$$

$$= \kappa \frac{1}{(m_0(z_1) - m_0(z_2))^2} \frac{\partial m_0(z_1)}{\partial z_1} \frac{\partial m_0(z_2)}{\partial z_2}$$

and

$$(\text{B.43}) = \beta_y y_2 g'(z_1) g'(z_2) \frac{\partial^2 [\underline{m}(z_1) m_0(z_1) \underline{m}(z_2) m_0(z_2) h_{v2}(g(z_1), g(z_2))]}{\partial(-1/\underline{m}(z_1))\, \partial(-1/\underline{m}(z_2))}.$$

So we obtain

$$\begin{aligned}
&\frac{-1}{2\pi i} \oint_{\mathcal{C}} f_i(z) \cdot (\text{B.37})\, dz \\
&\quad = \frac{\kappa - 1}{4\pi i} \oint_{\mathcal{C}} f_i(z)\, d \log\left(\frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{(1 - y_2 \int \frac{m_0}{t+m_0}\, dH(t))^2}{1 - y_2 \int \frac{m_0^2}{(t+m_0)^2}\, dH(t)} \right) \qquad (\text{B.44}) \\
&\quad = -\frac{\kappa - 1}{4\pi i} \oint_{\mathcal{C}} f'(z) \log\left(\frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{(1 - y_2 \int \frac{m_0(z)}{t+m_0(z)}\, dH(t))^2}{1 - y_2 \int \frac{m_0^2(z)}{(t+m_0(z))^2}\, dH(t)} \right) dz, \\
&-\frac{1}{2\pi i} \oint_{\mathcal{C}} f_i(z) \cdot (\text{B.38})\, dz = -\frac{\beta_x}{2\pi i} \oint f_i(z) \frac{y_1 z^2 \underline{m}^3(z) \cdot h_{m1}(z)}{\frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{(1 - \int \frac{y_2 m_0}{t+m_0}\, dH(t))^2}{1 - \int \frac{y_2 m_0^2}{(t+m_0)^2}\, dH(t)}}\, dz, \qquad (\text{B.45}) \\
&-\frac{1}{2\pi i} \oint_{\mathcal{C}} f_i(z) \cdot (\text{B.39})\, dz = \frac{\kappa - 1}{4\pi i} \oint_{\mathcal{C}} f_i(z)\, d \log\left(1 - y_2 \int \frac{m_0^2(z)\, dH(t)}{(t + m_0(z))^2} \right) \\
&\qquad\qquad - \frac{\kappa - 1}{4\pi i} \oint_{\mathcal{C}} f'(z) \log\left(1 - y_2 \int \frac{m_0^2(z)\, dH(t)}{(t + m_0(z))^2} \right) dz, \qquad (\text{B.46}) \\
&-\frac{1}{2\pi i} \oint_{\mathcal{C}} f_i(z) \cdot (\text{B.40})\, dz = \frac{\beta_y}{2\pi i} \cdot \oint f_i(z) \underline{m}'(z) \frac{y_2 \underline{m}^3(z) m_0^3(z) h_{m2}(g(z))}{1 - y_2 \int \frac{m_0^2(z)}{(t+m_0(z))^2}\, dH(t)}\, dz, \qquad (\text{B.47})
\end{aligned}$$

for the mean function and

$$\begin{aligned}
&-\frac{1}{4\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} f_i(z_1) f_j(z_2) \cdot (\text{B.41})\, dz \\
&\quad = -\frac{\beta_x y_1}{4\pi^2} \oint \oint f_i(z_1) f_j(z_2) \frac{\partial^2 [z_1 z_2 \underline{m}(z_1) \underline{m}(z_2) h_{v1}(z_1, z_2)]}{\partial z_1\, \partial z_2}\, dz_1\, dz_2 \qquad (\text{B.48}) \\
&\quad = -\frac{\beta_x y_1}{4\pi^2} \oint \oint f_i'(z_1) f_j'(z_2) \cdot \left[z_1 z_2 \underline{m}(z_1) \underline{m}(z_2) \cdot h_{v1}(z_1, z_2) \right] dz_1\, dz_2, \\
&-\frac{1}{4\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} f_i(z_1) f_j(z_2) \cdot (\text{B.42})\, dz \\
&\quad = -\frac{\kappa}{4\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{f_i(z_1) f_j(z_2)}{(m_0(z_1) - m_0(z_2))^2}\, dm_0(z_1)\, dm_0(z_2) \qquad (\text{B.49})
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\kappa}{4\pi^2}\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_2} f_i'(z_1)f_j'(z_2)\log\bigl(m_0(z_1)-m_0(z_2)\bigr)\,dz_1\,dz_2,\\
&-\frac{1}{4\pi^2}\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_2} f_i(z_1)f_j(z_2)\cdot(\text{B.43})\,dz\\
&= \frac{-\beta_y y_2}{4\pi^2}\oint\oint f_i(z_1)f_j(z_2)\frac{\partial^2[\underline{m}(z_1)m_0(z_1)\underline{m}(z_2)m_0(z_2)h_{v2}(g(z_1),g(z_2))]}{\partial z_1\,\partial z_2}\,dz_1\,dz_2
\end{aligned}
\tag{B.50}
$$

for the covariance function where $h^2 = y_1 + y_2 - y_1y_2$. The respective sums lead to the mean and covariance functions of the theorem.

B.5. Proofs of Propositions A.1 and A.2

B.5.1. Some lemmas

Write $\boldsymbol{\eta}_j = \frac{1}{\sqrt{n_2}}\mathbf{y}_{\cdot j}$ and $\boldsymbol{\gamma}_j = \frac{1}{\sqrt{n_1}}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{X}_{\cdot j}$, then

$$
\mathbf{D}_1(z) = \sum_{j=2}^{n_1}\boldsymbol{\gamma}_j\boldsymbol{\gamma}_j^* - z\mathbf{I}_p, \qquad \mathbf{D}_2(z) = \sum_{j=1}^{n_2}\boldsymbol{\eta}_j\boldsymbol{\eta}_j^* + z\mathbf{T}_p.
$$

Let $\mathbf{S}_{1j} = \mathbf{S}_1 - \boldsymbol{\gamma}_j\boldsymbol{\gamma}_j^*$, $\mathbf{S}_{2j} = \mathbf{S}_2 - \boldsymbol{\eta}_j\boldsymbol{\eta}_j^*$, $\mathbf{D}_{1j}(z) = \mathbf{D}_1(z) - \boldsymbol{\gamma}_j\boldsymbol{\gamma}_j^*$, $\mathbf{D}_{2j}(z) = \mathbf{D}_2(z) - \boldsymbol{\eta}_j\boldsymbol{\eta}_j^*$, $\mathbf{D}_{1jk}(z) = \mathbf{D}_{1j}(z) - \boldsymbol{\gamma}_k\boldsymbol{\gamma}_k^*$, and $\mathbf{D}_{2jk}(z) = \mathbf{D}_{2j}(z) - \boldsymbol{\eta}_k\boldsymbol{\eta}_k^*$. Moreover, Let $\beta_{1j}(z) = \frac{1}{1+\boldsymbol{\gamma}_j^*\mathbf{D}_{1j}^{-1}(z)\boldsymbol{\gamma}_j}$, $\beta_{2j}(z) = \frac{1}{1+\boldsymbol{\eta}_j^*\mathbf{D}_{2j}^{-1}(z)\boldsymbol{\eta}_j}$, $\beta_{1j(k)}(z) = \frac{1}{1+\boldsymbol{\gamma}_j^*\mathbf{D}_{1jk}^{-1}(z)\boldsymbol{\gamma}_j}$ and $\beta_{2j(k)}(z) = \frac{1}{1+\boldsymbol{\eta}_j^*\mathbf{D}_{2jk}^{-1}(z)\boldsymbol{\eta}_j}$.

Lemma B.2. *Under the conditions of Theorem 3.1, in h_{m1} and h_{v1}, the matrix $\mathbf{D}_1^{-1}(z)$ can be replaced by $-z^{-1}\{\underline{m}(z)\mathbf{S}_2^{-1/2}\mathbf{T}_p\mathbf{S}_2^{-1/2} + \mathbf{I}_p\}^{-1}$.*

Proof. Let $\mathbf{K} = -\frac{z\underline{m}(z)(n_1-1)}{n_1}\mathbf{S}_2^{-1/2}\mathbf{T}_p\mathbf{S}_2^{-1/2}$ and let $\mathbf{e}_i$ denote the ith vector of the canonical basis of $\mathbb{C}^p$. For any non-random $\mathbf{M}$ ($p\times p$) with bound operation norm ($\mathbf{M}$ will be $\mathbf{I}_p$ and $\underline{m}(z)\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1}\mathbf{T}_{p,1/2} + \mathbf{I}_p$ in application), we have $(\mathbf{K}-z\mathbf{I}_p)^{-1} - \mathbf{D}_1^{-1} = \sum_{j=2}^{n_1} d_j$ where $d_j = d_j^1 + d_j^2 + d_j^3$ and

$$
\begin{aligned}
d_j^1 &= (\mathbf{K}-z\mathbf{I}_p)^{-1}\boldsymbol{\gamma}_j\boldsymbol{\gamma}_j^*\mathbf{D}_{1j}^{-1}\bigl(\beta_j + z\underline{m}(z)\bigr),\\
d_j^2 &= -z\underline{m}(z)(\mathbf{K}-z\mathbf{I}_p)^{-1}\biggl(\boldsymbol{\gamma}_j\boldsymbol{\gamma}_j^* - \frac{1}{n_1}\mathbf{S}_2^{-1/2}\mathbf{T}_p\mathbf{S}_2^{-1/2}\biggr)\mathbf{D}_{1j}^{-1},\\
d_j^3 &= \frac{1}{n_1-1}(\mathbf{K}-z\mathbf{I}_p)^{-1}\mathbf{K}\bigl(\mathbf{D}_{1j}^{-1} - \mathbf{D}_1^{-1}\bigr).
\end{aligned}
$$

Then, we have

$$\begin{aligned}&\mathrm{E}\left|\sum_{j=2}^{n_1}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}d_j^1\mathbf{M}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\right|^2\\&\quad\le\mathrm{E}\left|\sum_{j=2}^{n_1}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}(\mathbf{K}-z\mathbf{I}_p)^{-1}\boldsymbol{\gamma}_j\boldsymbol{\gamma}_j^*\mathbf{D}_{1j}^{-1}\big(\beta_j+z\underline{m}(z)\big)\mathbf{M}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\right|^2=o(1)\end{aligned}\tag{B.51}$$

because, with $\mathbf{S}_{11}=\sum_{j=2}^{n_1}\boldsymbol{\gamma}_j\boldsymbol{\gamma}_j^*$,

$$\begin{aligned}&\mathrm{E}\sum_{j=2}^{n_1}\big|\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}(\mathbf{K}-z\mathbf{I}_p)^{-1}\boldsymbol{\gamma}_j\big|^2\\&\quad=\mathrm{E}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}(\mathbf{K}-z\mathbf{I}_p)^{-1*}\mathbf{S}_{11}(\mathbf{K}-\bar{z}\mathbf{I}_p)^{-1}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\\&\quad\le Kv^{-2}\big|\mathbf{e}_i'\big(\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1}\mathbf{T}_{p,1/2}\big)^2\mathbf{e}_i\big|=O(1)\end{aligned}$$

and since $\beta_j+z\underline{m}(z)=o_p(1)$ and by the control convergence theorem, we have

$$\begin{aligned}&\mathrm{E}\sum_{j=2}^{n_1}\big|\boldsymbol{\gamma}_j^*\mathbf{D}_{1j}^{-1}\mathbf{M}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\big(\beta_j+z\underline{m}(z)\big)\big|^2\\&\quad=\mathrm{E}\sum_{j=2}^{n_1}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}\mathbf{M}^*\mathbf{D}_{1j}^{-1*}\boldsymbol{\gamma}_j\boldsymbol{\gamma}_j^*\mathbf{D}_{1j}^{-1}\mathbf{M}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\times o_p(1)\\&\quad=v^{-2}\big(\mathrm{E}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}\mathbf{M}^*\mathbf{S}_{11}\mathbf{M}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\big)\times o(1)=o(1).\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&\mathrm{E}\sum_{j=2}^{n_1}\big|\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}d_j^2\mathbf{M}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\big|^2\\&\quad=\sum_{j=2}^{n_1}\mathrm{E}\big|\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}(\mathbf{K}-z\mathbf{I}_p)^{-1}\big(\boldsymbol{\gamma}_j\boldsymbol{\gamma}_j^*-n_1^{-1}\mathbf{S}_2^{-1/2}\mathbf{T}_p\mathbf{S}_2^{-1/2}\big)\mathbf{D}_{1j}^{-1}\mathbf{M}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\big|^2\\&\quad\le\frac{K}{n_1^2}\sum_{j=2}^{n_1}\mathrm{E}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}(\mathbf{K}-z\mathbf{I}_p)^{-1}\mathbf{S}_2^{-1/2}\mathbf{T}_p\mathbf{S}_2^{-1/2}\big(\mathbf{K}^*-\bar{z}\mathbf{I}\big)^{-1}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\\&\qquad\times\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}\mathbf{M}^*\mathbf{D}_{1j}^{-1*}\mathbf{S}_2^{-1/2}\mathbf{T}_p\mathbf{S}_2^{-1/2}\mathbf{D}_{1j}^{-1}\mathbf{M}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i=O\big(n_1^{-1}\big)\end{aligned}\tag{B.52}$$

and

$$\begin{aligned}
&\left|\sum_{j_1\neq j_2}^{n_1}\mathrm{E}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}d_{j_1}^2\mathbf{M}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}\mathbf{M}^*d_{j_2}^{2*}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\right| \\
&\quad=\left|\sum_{j_1\neq j_2}^{n_1}\mathrm{E}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}(\mathbf{K}-z\mathbf{I}_p)^{-1}\biggl(\boldsymbol{\gamma}_{j_1}\boldsymbol{\gamma}_{j_1}^*-\frac{1}{n_1}\mathbf{S}_2^{-1/2}\mathbf{T}_p\mathbf{S}_2^{-1/2}\biggr)\mathbf{D}_{1j_1j_2}^{-1}\boldsymbol{\gamma}_{j_2}\right. \\
&\qquad\times\boldsymbol{\gamma}_{j_2}^*\mathbf{D}_{1j_1j_2}^{-1}\beta_{1j_2(j_1)}\mathbf{M}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}\mathbf{M}^*\beta_{1j_1(j_2)}\mathbf{D}_{1j_1j_2}^{-1}\boldsymbol{\gamma}_{j_1} \\
&\qquad\left.\times\boldsymbol{\gamma}_{j_1}^*\mathbf{D}_{1j_1j_2}^{-1}\biggl(\boldsymbol{\gamma}_{j_2}\boldsymbol{\gamma}_{j_2}^*-\frac{1}{n_1}\mathbf{S}_2^{-1/2}\mathbf{T}_p\mathbf{S}_2^{-1/2}\biggr)\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\right|\cdot\bigl|z\underline{m}(z)\bigr|^2=O\bigl(n_1^{-1}\bigr)
\end{aligned}\tag{B.53}$$

because

$$\begin{aligned}
&\mathrm{E}\left|\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}(\mathbf{K}-z\mathbf{I}_p)^{-1}\biggl(\boldsymbol{\gamma}_{j_1}\boldsymbol{\gamma}_{j_1}^*-\frac{1}{n_1}\mathbf{S}_2^{-1/2}\mathbf{T}_p\mathbf{S}_2^{-1/2}\biggr)\mathbf{D}_{1j_1j_2}^{-1}\right. \\
&\quad\left.\times\boldsymbol{\gamma}_{j_2}\boldsymbol{\gamma}_{j_2}^*\mathbf{D}_{1j_1j_2}^{-1}\beta_{1j_2(j_1)}\mathbf{M}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\right|^2 \\
&\quad\le\frac{K}{n_1^2}\mathrm{E}\bigl\|\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}(\mathbf{K}-z\mathbf{I}_p)^{-1}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\bigr\|^2 \\
&\qquad\times\bigl\|\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}\mathbf{D}_{1j_1j_2}^{-1}\boldsymbol{\gamma}_{j_2}\boldsymbol{\gamma}_{j_2}^*\mathbf{D}_{1j_1j_2}^{-1}\beta_{1j_2(j_1)}\mathbf{M}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\bigr\|^2 \\
&\quad\le\frac{K}{n_1^2}\mathrm{E}\bigl\|\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}\mathbf{D}_{1j_1j_2}^{-1}\boldsymbol{\gamma}_{j_2}\boldsymbol{\gamma}_{j_2}^*\mathbf{D}_{1j_1j_2}^{-1}\beta_{1j_2(j_1)}\mathbf{M}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\bigr\|^2 \\
&\quad\le\frac{K}{n_1^4}\mathrm{E}\operatorname{tr}\bigl(\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}\mathbf{D}_{1j_1j_2}^{-1}\mathbf{S}_2^{-1/2}\mathbf{T}_p\mathbf{S}_2^{-1/2}\mathbf{D}_{1j_1j_2}^{-1/2*}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\bigr)=O\bigl(n_1^{-3}\bigr).
\end{aligned}$$

The two estimates above imply that

$$\sum_{j=2}^{n_1}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}d_j^2\mathbf{M}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i=O_p\bigl(n_1^{-1/2}\bigr).\tag{B.54}$$

We can similarly prove that

$$\begin{aligned}
&\mathrm{E}\left|\sum_{j=2}^{n_1}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}d_j^3\mathbf{M}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\right|^2 \\
&\quad\le\frac{1}{n_1^2}\mathrm{E}\left|\sum_{j=2}^{n_1}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}(\mathbf{K}-z\mathbf{I}_p)^{-1}\mathbf{K}\mathbf{D}_{1j}^{-1}\boldsymbol{\gamma}_j\boldsymbol{\gamma}_j^*\mathbf{D}_{1j}^{-1}\mathbf{M}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\beta_{1j}\right|^2 \\
&\quad=O\bigl(n_1^{-2}\bigr).
\end{aligned}\tag{B.55}$$

Therefore, the matrix $\mathbf{D}_1^{-1}(z)$ can be replaced by

$$(\mathbf{K}-z\mathbf{I}_p)^{-1}=-z^{-1}\big(\underline{m}(z)^*\mathbf{S}_2^{-1/2}\mathbf{T}_p\mathbf{S}_2^{-1/2}+\mathbf{I}_p\big)^{-1}$$

in h_{m1} and h_{v1}.

The proof is completed. □

Define $m_{\mathbf{I}}$ and $m_{\mathbf{T}}$ as follows

$$m_{\mathbf{I}}(-z)=\int\frac{1}{\frac{1}{1+y_2m_{\mathbf{I}}(-z)}+zt}\,dH(t),\qquad m_{\mathbf{T}}(-z)=\int\frac{t}{\frac{1}{1+y_2m_{\mathbf{I}}(-z)}+zt}\,dH(t).\tag{B.56}$$

Lemma B.3. *For any $\Im(z)\neq 0$, the equation (B.56) has a unique solution $m_{\mathbf{M}}(-z)$ such that $\Im(m_{\mathbf{M}}(-z))\Im(z)<0$ where $\mathbf{M}=\mathbf{I}$ or $\mathbf{M}=\mathbf{T}$.*

Proof. We only need to prove the lemma for $\mathbf{M}=\mathbf{I}$. Without loss of generality, assume that $\Im(z)=v>0$. At first, comparing the imaginary parts of both sides, we obtain

$$m_{I2}(-z)=\int\frac{-tv+\frac{y_2m_{I2}(-z)}{|1+y_2m_I(-z)|^2}}{|tz+\frac{1}{1+y_2m_I(-z)}|^2}\,dH(t)<\int\frac{\frac{y_2m_{I2}(-z)}{|1+y_2m_I(-z)|^2}}{|tz+\frac{1}{1+y_2m_I(-z)}|^2}\,dH(t),$$

where $m_{I2}(-z)<0$ is the imaginary part of $m_I(-z)$. Therefore, we have

$$\int\frac{\frac{y_2}{|1+y_2m_I(-z)|^2}}{|tz+\frac{1}{1+y_2m_I(-z)}|^2}\,dH(t)<1.\tag{B.57}$$

Suppose that the equation (B.56) has two solutions $m_1\neq m_2$ with $\Im(m_k)<0$, $k=1,2$. Then making difference of (B.56) and dividing both sides by m_1-m_2, we obtain

$$1=\int\frac{\frac{y_2}{(1+y_2m_1)(1+y_2m_2)}}{(tz+\frac{1}{1+y_2m_1})(tz+\frac{1}{1+y_2m_2})}\,dH(t).$$

Using Cauchy–Schwarz to the above equation and then using (B.57), we obtain a contradiction.

The proof of the lemma is completed. □

Lemma B.4. *Define $\underline{m}_{y_2}(z)=y_2m_T(z)-\frac{1-y_2}{z}$. Then, we have*

$$\frac{1}{1+y_2m_I(z)}=-y_2zm_T(z)+(1-y_2)=-z\underline{m}_{y_2}(z).\tag{B.58}$$

Moreover, $\underline{m}_{y_2}(z)$ is the same as that defined in Theorem 2.1.

Proof. By the second equation of (B.56), we obtain

$$m_T(-z) = \int \frac{t}{tz + \frac{1}{1+y_2 m_I(-z)}}\, dH(t)$$
$$= \frac{1}{z} - \frac{1}{z(1+y_2 m_I(-z))} \int \frac{1}{tz + \frac{1}{1+y_2 m_I(-z)}}\, dH(t) = \frac{1}{z} - \frac{1}{z}\frac{m_I(-z)}{1+y_2 m_I(-z)}.$$

Then we have $\frac{1}{1+y_2 m_I(-z)} = y_2 z m_T(-z) + (1-y_2)$. Thus, we obtain

$$\frac{1}{1+y_2 m_I(z)} = -y_2 z m_T(z) + (1-y_2) = -z\underline{m}_{y_2}(z).$$

This implies (B.58). Moreover, replacing (B.58) to (B.56), we obtain

$$z = -\frac{1}{\underline{m}_{y_2}(z)} + y_2 \int \frac{dH(t)}{t + \underline{m}_{y_2}(z)},$$

where the equation has the unique solution. Then $\underline{m}_{y_2}(z)$ is the same as that defined in Theorem 2.1.

The proof is completed.

Lemma B.5. *Defining* $\tilde{\mathbf{K}}(z) = \frac{\mathbf{I}_p}{1+y_2 m_{\mathbf{I}}(-z)}$, *then we have*

$$\big(\tilde{\mathbf{K}}(z) + z\mathbf{T}_p\big)^{-1} - (\mathbf{S}_2 + z\mathbf{T}_p)^{-1} = \sum_{j=1}^{n_2} \tilde{d}_j(z) = \sum_{j=1}^{n_2} \breve{d}_j(z), \tag{B.59}$$

where $\tilde{d}_j(z) = \tilde{d}_j^1(z) + \tilde{d}_j^2(z) + \tilde{d}_j^3(z)$, *and by symmetry,* $\breve{d}_j(z) = \breve{d}_j^1(z) + \breve{d}_j^2(z) + \breve{d}_j^3(z)$

$$\tilde{d}_j^1(z) = \big(\tilde{\mathbf{K}}(z) + z\mathbf{T}_p\big)^{-1} \boldsymbol{\eta}_j \boldsymbol{\eta}_j^* \mathbf{D}_{2j}^{-1}(z) \bigg(\beta_{2j}(z) - \frac{1}{1+y_2 m_{\mathbf{I}}(-z)}\bigg),$$
$$\tilde{d}_j^2(z) = \frac{1}{1+y_2 m_{\mathbf{I}}(-z)} \big(\tilde{\mathbf{K}}(z) + z\mathbf{T}_p\big)^{-1} \bigg(\boldsymbol{\eta}_j \boldsymbol{\eta}_j^* - \frac{1}{n_2}\mathbf{I}_p\bigg) \mathbf{D}_{2j}^{-1}(z),$$
$$\tilde{d}_j^3(z) = n_2^{-1} \big(\tilde{\mathbf{K}}(z) + z\mathbf{T}_p\big)^{-1} \tilde{\mathbf{K}}(z) \big(\mathbf{D}_{2j}^{-1}(z) - \mathbf{D}_2^{-1}(z)\big)$$

and

$$\breve{d}_j^1(z) = \mathbf{D}_{2j}^{-1}(z) \boldsymbol{\eta}_j \boldsymbol{\eta}_j^* \big(\tilde{\mathbf{K}}(z) + z\mathbf{T}_p\big)^{-1} \bigg(\beta_{2j}(z) - \frac{1}{1+y_2 m_{\mathbf{I}}(-z)}\bigg),$$
$$\breve{d}_j^2(z) = \frac{1}{1+y_2 m_{\mathbf{I}}(-z)} \mathbf{D}_{2j}^{-1}(z) \bigg(\boldsymbol{\eta}_j \boldsymbol{\eta}_j^* - \frac{1}{n_2}\mathbf{I}_p\bigg) \big(\tilde{\mathbf{K}}(z) + z\mathbf{T}_p\big)^{-1},$$
$$\breve{d}_j^3(z) = n_2^{-1} \big(\mathbf{D}_{2j}^{-1}(z) - \mathbf{D}_2^{-1}(z)\big) \tilde{\mathbf{K}}(z) \big(\tilde{\mathbf{K}}(z) + z\mathbf{T}_p\big)^{-1}.$$

Assuming that $\mathbf{T}_p$*'s are diagonal, we have*

$$\sum_{j=1}^{n_2} \mathbf{e}_i' \mathbf{T}_{p,1/2}^* \tilde{d}_j^k(z) \mathbf{T}_{p,1/2} \mathbf{e}_i \to 0, \qquad a.s.\ k = 1, 2, 3, \tag{B.60}$$

for any random matrix $\mathbf{A}$ *with bounded non-random norm*

$$\sum_{j=1}^{n_2} \mathbf{e}_i' \mathbf{T}_{p,1/2}^* \mathbf{A} \mathbf{T}_p \tilde{d}_j^k(z) \mathbf{T}_{p,1/2} \mathbf{e}_i \to 0, \qquad a.s.\ k = 1, 3, \tag{B.61}$$

and for any non-random nonnegative definite matrix $\mathbf{B}$ *with bounded norm*

$$\sum_{j=1}^{n_2} \mathbf{e}_i' \mathbf{T}_{p,1/2}^* \mathbf{B} \mathbf{T}_p \tilde{d}_j^2(z) \mathbf{T}_{p,1/2} \mathbf{e}_i \to 0, \qquad a.s. \tag{B.62}$$

Proof. We have $\mathbf{D}_2^{-1}(z) = (z\mathbf{T}_p + \mathbf{S}_2)^{-1}$ and $\tilde{\mathbf{K}}(z) = \frac{1}{1+y_2 m_{\mathbf{I}}(-z)} \mathbf{I}_p$. We write

$$\mathbf{D}_2(z) - \big(z\mathbf{T}_p + \tilde{\mathbf{K}}(z)\big) = \sum_{j=1}^{n_2} \boldsymbol{\eta}_j \boldsymbol{\eta}_j^* - \tilde{\mathbf{K}}(z).$$

Multiplying by $(z\mathbf{T}_p + \tilde{\mathbf{K}}(z))^{-1}$ on the left, $\mathbf{D}_2^{-1}$ on the right and using

$$\boldsymbol{\eta}_j^* \mathbf{D}_2^{-1}(z) = \boldsymbol{\eta}_j^* \mathbf{D}_{2j}^{-1}(z) \beta_{2j},$$

we obtain the first equation of (B.59). And similarly we get the second equation of (B.59) by multiplying $(z\mathbf{T}_p + \tilde{\mathbf{K}}(z))^{-1}$ on the right, and $\mathbf{D}_2^{-1}$ on the left. At first, we derive a limit of $\frac{1}{p}\operatorname{tr}(D_2^{-1}(z)\mathbf{M})$, where $\mathbf{M}$ is a non-random $p \times p$ n.n.d. matrix of bounded operation norm and $= \mathbf{I}_p$ or $\mathbf{T}_p$ in accordance of application. For fixed $z = x + iv$ with $v \neq 0$ and any n_2, we show that $\frac{1}{p}\operatorname{tr}(\mathbf{D}_2^{-1}\mathbf{M})$ is bounded. In fact, let t_j and $\mathbf{x}_j$ denote the eigenvalues and eigenvectors of $\mathbf{S}_2^{-1/2}\mathbf{T}_p\mathbf{S}_2^{-1/2}$ and $u_j^2 = \mathbf{x}_j^*\mathbf{S}_2^{-1/2}\mathbf{M}\mathbf{S}_2^{-1/2}\mathbf{x}_j$, we have

$$\begin{aligned} \left| \frac{1}{p} \operatorname{tr}\big(\mathbf{D}_2^{-1}\mathbf{M}\big) \right| &= \left| \frac{1}{p} \sum_{j=1}^{p} \frac{|u_j|^2}{zt_j + 1} \right| \\ &\leq \frac{1}{|z|} \frac{1}{p} \sum_{j=1}^{p} \frac{|u_j^2|}{|\Im(t_j + z^{-1})|} \leq \frac{|z|}{|v|} \operatorname{tr}\big(\mathbf{S}_2^{-1}\mathbf{M}\big) \leq K < \infty, \end{aligned}$$

for some constant K.

Therefore, for any subsequence $\{n_2'\}$ of $\{n_2\}$, there is a convergent subsequence $\{n_2''\}$ such that $\frac{1}{p}\operatorname{tr}(\mathbf{D}_2^{-1}\mathbf{T}_p)$ converges to some limit $m_{y_2}(-z)$ when n_2 runs to infinity along $\{n_2''\}$. Thus, to

show that $\frac{1}{p}\operatorname{tr}(\mathbf{D}_2^{-1}\mathbf{M})$ converges to $m_M(-z)$, we only need to derive an equation that $m_M(-z)$ satisfies and that the equation has a unique solution with $\Im(m_M(-z))\Im(-z) < 0$. Without loss of generality, we assume that $\frac{1}{p}\operatorname{tr}(\mathbf{D}_2^{-1}\mathbf{M}) \to m_M(-z)$ as $n_2 \to \infty$ subject to $p/n_2 \to y_2 \in (0,1)$. Therefore, we have $\beta_{2j}(z) \to \frac{1}{1+y_2 m_{\mathbf{I}}(-z)}$.

Similar to the proofs of (B.51)–(B.55), one can prove that for any $i = 1, 2, 3$,

$$\frac{1}{p}\sum_{j=1}^{n_2}\operatorname{tr}\big(\tilde{d}_j^i\mathbf{M}\big) = o_{\text{a.s.}}(1).$$

Therefore, from (B.59), one concludes that

$$m_{\mathbf{I}}(-z) = \int \frac{1}{\frac{1}{1+y_2 m_{\mathbf{I}}(-z)} + zt}\,dH(t), \qquad m_{\mathbf{T}}(-z) = \int \frac{t}{\frac{1}{1+y_2 m_{\mathbf{I}}(-z)} + zt}\,dH(t). \quad \text{(B.63)}$$

By Lemma B.3, the equation above has a unique solution of $m_{\mathbf{I}}(-z)$ subject to

$$\Im\big(m_{\mathbf{I}}(-z)\big)\Im(-z) < 0.$$

Hence, we have proved that $\frac{1}{p}\operatorname{tr}(\mathbf{D}_2^{-1}\mathbf{M}) \to m_{\mathbf{M}}(-z)$ with probability 1.

Furthermore, from (B.59), similar to the proofs of (B.51)–(B.55), one can prove that for $k = 1, 2, 3$,

$$\sum_{j=1}^{n_2}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\tilde{d}_j^k(z)\mathbf{T}_{p,1/2}\mathbf{e}_i \to 0, \qquad \text{a.s.}$$

and for any random matrix $\mathbf{A}$ with bounded non-random norm $k = 1, 3$,

$$\sum_{j=1}^{n_2}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{A}\mathbf{T}_p\tilde{d}_j^k(z)\mathbf{T}_{p,1/2}\mathbf{e}_i \to 0, \qquad \text{a.s.}$$

and for any non-random and matrix $\mathbf{B}$ with bounded norm

$$\sum_{j=1}^{n_2}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{B}\mathbf{T}_p\tilde{d}_j^2(z)\mathbf{T}_{p,1/2}\mathbf{e}_i \to 0, \qquad \text{a.s.}$$

The proof is completed. □

B.5.2. *Proof of Proposition* A.1

By Lemmas B.4–B.5, we have

$$-\underline{m}(z)\frac{1}{p}\sum_{j=1}^{p}\mathbf{e}_j'\big(\underline{m}(z)\mathbf{T}_p + \mathbf{S}_{2k}\big)^{-1}\mathbf{e}_j \cdot \mathbf{e}_j'\big(\underline{m}(z)\mathbf{T}_p + \mathbf{S}_{2k}\big)^{-1}\mathbf{T}_p\big(\mathbf{T}_p + \underline{m}_{y_2}\big(-\underline{m}(z)\big)\mathbf{I}_p\big)^{-1}\mathbf{e}_j$$

$$= -\underline{m}^3(z)\int \frac{t}{(t + \underline{m}_{y_2}(-\underline{m}(z)))^3}\,dH(t) + o_{\text{a.s.}}(1)$$

and

$$\frac{1}{p}\sum_{j=1}^{p}\mathbf{e}_j'\big(\underline{m}(z_1)\mathbf{T}_p+\mathbf{S}_{2i}\big)^{-1}\mathbf{e}_j\cdot\mathbf{e}_j'\big(\underline{m}(z_2)\mathbf{T}_p-\mathbf{S}_{2i}\big)^{-1}\mathbf{e}_j$$
$$=\underline{m}(z_1)\underline{m}(z_2)\int\frac{1}{(t+\underline{m}_{y_2}(\underline{m}(z_1)))(t+\underline{m}_{y_2}(-\underline{m}(z_2)))}\,dH(t)+o_{\mathrm{a.s.}}(1).$$

That is, we have proved that

$$h_{m2}\big(g(z)\big)=h_{m2}\big(-\underline{m}^{-1}(z)\big)=-\frac{1}{\underline{m}^3(z)}\int\frac{t}{(t+m_0(z))^3}\,dH(t),$$
$$h_{v2}\big(g(z_1),g(z_2)\big)=h_{v2}\big(-\underline{m}^{-1}(z_1),\underline{m}^{-1}(z_2)\big)$$
$$=\frac{1}{\underline{m}(z_1)\underline{m}(z_2)}\int\frac{1}{(t+m_0(z_1))(t+m_0(z_2))}\,dH(t).$$

Then we obtain

$$\text{(B.40)}=-\beta_y\underline{m}'(z)\frac{y_2[1+y_2g(z)s(g(z))]^3h_{m2}(g(z))}{1-y_2\int\frac{[1+y_2g(z)s(g(z))]^2\,dH(t)}{[-t\underline{m}(z)-1-y_2g(z)s(g(z))]^2}}$$
$$=\frac{\beta_y\underline{m}'(z)y_2[1+y_2g(z)s(g(z))]^3}{1-y_2\int\frac{[1+y_2g(z)s(g(z))]^2\,dH(t)}{[-t\underline{m}(z)-1-y_2g(z)s(g(z))]^2}}\frac{1}{\underline{m}^3(z)}$$
$$\times\int\frac{t}{(t+m_0(z))^3}\,dH(t)$$
$$=-\beta_y\cdot\underline{m}'(z)\frac{y_2\int\frac{tm_0^3(z)}{(t+m_0(z))^3}\,dH(t)}{1-y_2\int\frac{m_0^2(z)\,dH(t)}{(t+m_0(z))^2}}$$
$$=\frac{\beta_x}{2}\bigg(1-y_2\int\frac{m_0^2\,dH(t)}{(t+m_0)^2}\bigg)\bigg[\log\bigg(1-y_2\int\frac{m_0^2\,dH(t)}{(t+m_0)^2}\bigg)\bigg]'$$

and

$$\text{(B.43)}=\beta_yy_2\frac{\partial^2[(1+y_2g(z_1)s(g(z_1)))(1+y_2g(z_2)s(g(z_2)))h_{v2}(g(z_1),g(z_2))]}{\partial z_1\,\partial z_2}$$
$$=\beta_yy_2\frac{\partial^2\int\frac{m_0(z_1)m_0(z_2)}{(t+m_0(z_1))(t+m_0(z_2))}\,dH(t)}{\partial z_1\,\partial z_2}$$
$$=\beta_yy_2\int\frac{t^2\,dH(t)}{(t+m_0(z_1))^2(t+m_0(z_2))^2}\frac{\partial^2m_0(z_1)m_0(z_2)}{\partial z_1\,\partial z_2}.$$

We have

$$\begin{aligned}(\text{B.47}) &= -\frac{1}{2\pi i}\oint f(z)\cdot(\text{B.40})\,dz\\ &= \frac{\beta_y}{4\pi i}\oint f(z)\biggl(1-y_2\int\frac{m_0^2(z)\,dH(t)}{(t+m_0(z))^2}\biggr)\,d\log\biggl(1-y_2\int\frac{m_0^2\,dH(t)}{(t+m_0)^2}\biggr)\end{aligned}$$

and

$$\begin{aligned}(\text{B.50}) &= -\frac{1}{4\pi^2}\oint\oint f_i(z_1)f_j(z_2)\cdot(\text{B.43})\,dz_1\,dz_2\\ &= -\frac{\beta_y y_2}{4\pi^2}\oint\oint f_i(z_1)f_j(z_2)\biggl[\int\frac{t^2\,dH(t)}{(t+m_0(z_1))^2(t+m_0(z_2))^2}\biggr]\,dm_0(z_1)\,dm_0(z_2).\end{aligned}$$

B.5.3. *Proof of Proposition* A.2

h_{m1} and h_{v1} are the limits of

$$\begin{aligned}&\frac{1}{p}\sum_{i=1}^{p}\mathrm{E}\bigl[\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}\mathbf{D}_1^{-1}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\\ &\quad\times\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}\mathbf{D}_1^{-1}\bigl(\underline{m}(z)\mathbf{S}_2^{-1/2}\mathbf{T}_p\mathbf{S}_2^{-1/2}+\mathbf{I}_p\bigr)^{-1}\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i|\mathbf{S}_2\bigr]\end{aligned}\tag{B.64}$$

and

$$\begin{aligned}&\frac{1}{n_1 p}\sum_{j=1}^{n_1}\sum_{i=1}^{p}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}\bigl[\mathrm{E}_j\mathbf{D}_{1j}^{-1}(z_1)\bigr]\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i\\ &\quad\times\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{S}_2^{-1/2}\bigl[\mathrm{E}_j\mathbf{D}_{1j}^{-1}(z_2)\bigr]\mathbf{S}_2^{-1/2}\mathbf{T}_{p,1/2}\mathbf{e}_i.\end{aligned}\tag{B.65}$$

By Lemmas B.2 and B.5, we have

$$\begin{aligned}h_{v1}(z_1,z_2) &= \frac{1}{z_1z_2}\lim\frac{1}{p}\sum_{i=1}^{p}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\bigl(\underline{m}(z_1)\mathbf{T}_p+\mathbf{S}_2\bigr)^{-1}\mathbf{T}_{p,1/2}\mathbf{e}_i\\ &\quad\times\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\bigl(\underline{m}(z_2)\mathbf{T}_p+\mathbf{S}_2\bigr)^{-1}\mathbf{T}_{p,1/2}\mathbf{e}_i\\ &= \frac{1}{z_1z_2\underline{m}(z_1)\underline{m}(z_2)}\\ &\quad\times\int\frac{t^2}{(m_0(z_1)+t)(m_0(z_2)+t)}\,dH(t).\end{aligned}$$

By Lemma B.2, (B.64) reduces to

$$\begin{aligned} h_{m1} &= \lim \frac{1}{p}\sum_{i=1}^{p}\frac{1}{z^2}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*(\underline{m}\mathbf{T}_p+\mathbf{S}_2)^{-1}\mathbf{T}_{p,1/2}\mathbf{e}_i \\ &\quad\times \mathbf{e}_i'\mathbf{T}_{p,1/2}^*(\underline{m}\mathbf{T}_p+\mathbf{S}_2)^{-1}\mathbf{S}_2(\underline{m}\mathbf{T}_p+\mathbf{S}_2)^{-1}\mathbf{T}_{p,1/2}\mathbf{e}_i \\ &= \lim \frac{1}{p}\sum_{i=1}^{p}\Bigg[\frac{1}{z^2}\big(\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\big(\underline{m}(z)\mathbf{T}_p+\mathbf{S}_2\big)^{-1}\mathbf{T}_{p,1/2}\mathbf{e}_i\big)^2 \\ &\quad-\frac{\underline{m}(z)}{z^2}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\big(\underline{m}(z)\mathbf{T}_p+\mathbf{S}_2\big)^{-1}\mathbf{T}_{p,1/2}\mathbf{e}_i \\ &\quad\times \mathbf{e}_i'\mathbf{T}_{p,1/2}^*\big(\underline{m}(z)\mathbf{T}_p+\mathbf{S}_2\big)^{-1}\mathbf{T}_p\big(\underline{m}(z)\mathbf{T}_p+\mathbf{S}_2\big)^{-1}\mathbf{T}_{p,1/2}\mathbf{e}_i\Bigg]. \end{aligned} \tag{B.66}$$

In order to obtain the limit $h_{m1}(z)$, we first prove

$$\sum_{j_1\neq j_2}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\breve{d}_{j_1}^2\big(\underline{m}(z)\big)\mathbf{T}_p\tilde{d}_{j_2}^2\big(\underline{m}(z)\big)\mathbf{T}_{p,1/2}\mathbf{e}_i = o_p(1). \tag{B.67}$$

Substituting the expressions of $\breve{d}_j^2$ and $\tilde{d}_j^2$ into the left-hand side (LHS) of (B.67), we have

$$\begin{aligned} \text{LHS of (B.67)} &= \sum_{j_1\neq j_2}\underline{m}(z)^2m_0^2(z)\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{D}_{2j_1}^{-1}\bigg(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\bigg)\big(\underline{m}(z)\mathbf{T}_p+\tilde{\mathbf{K}}\big)^{-2}\mathbf{T}_p \\ &\quad\times\bigg(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}_p\bigg)\mathbf{D}_{2j_2}^{-1}\mathbf{T}_{p,1/2}\mathbf{e}_i. \end{aligned}$$

Then, (B.67) is equivalent to

$$\begin{aligned} &\sum_{j_1\neq j_2}\mathbf{e}_i'\mathbf{T}_{p,1/2}^*\mathbf{D}_{2j_1}^{-1}\bigg(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\bigg)\big(\underline{m}(z)\mathbf{T}_p+\tilde{\mathbf{K}}\big)^{-2}\mathbf{T}_p\bigg(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}_p\bigg)\mathbf{D}_{2j_2}^{-1}\mathbf{T}_{p,1/2}\mathbf{e}_i \\ &\quad= o_p(1). \end{aligned} \tag{B.68}$$

Define $\boldsymbol{\alpha}=\mathbf{T}_{p,1/2}\mathbf{e}_i$, $\mathbf{B}=(\underline{m}(z)\mathbf{T}_p+\tilde{\mathbf{K}})^{-2}\mathbf{T}_p$, $\mathbf{D}_{2j_1j_2}=\mathbf{D}_{j_1}-\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*$ and $\beta_{2j_1(j_2)}=\frac{1}{1+\boldsymbol{\eta}_{j_1}^*\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\eta}_{j_1}}$. Noting that $\beta_{2j_1(j_2)}=\underline{m}(z)m_0(z)+O_p(n_2^{-1})$, we have for any given j_1

$$\begin{aligned} &\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*\mathbf{D}_{2j_1j_2}^{-1}\bigg(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{B}\bigg(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\alpha}\beta_{2j_1(j_2)}\beta_{2j_2(j_1)} \\ &\quad= O_p\big(n_2^{-5/2}\big), \end{aligned} \tag{B.69}$$

which follows from the easily shown facts that $\boldsymbol{\alpha}^*\boldsymbol{\eta}_j = O_p(n_2^{-1/2})$, $\boldsymbol{\eta}_{j_1}^*\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\alpha} = O_p(n_2^{-1})$, and $\boldsymbol{\eta}_{j_1}^*\mathbf{B}\boldsymbol{\eta}_{j_2} = O_p(n_2^{-1/2})$. From (B.69), we obtain that

$$\begin{aligned}&\sum_{j_1\neq j_2}\boldsymbol{\alpha}^*\big(\mathbf{D}_{2j_1}^{-1}-\mathbf{D}_{2j_1j_2}^{-1}\big)\bigg(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{B}\bigg(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}\bigg)\big(\mathbf{D}_{2j_2}^{-1}-\mathbf{D}_{2j_1j_2}^{-1}\big)\boldsymbol{\alpha}\\&\quad=O_p\big(n^{-1/2}\big).\end{aligned}\tag{B.70}$$

Furthermore, we have

$$\begin{aligned}&\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_2}^{-1}\bigg(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{B}\bigg(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\alpha}\beta_{2j_1(j_2)}\\&=\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_2}^{-1}\bigg(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{B}\bigg(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\alpha}\underline{m}(z)m_0(z)+O_p\big(n_2^{-5/2}\big)\\&=O_p\big(n_2^{-2}\big)\end{aligned}$$

and by similarly defined $\mathbf{D}_{2j_1j_2j_3}=\mathbf{D}_{2j_1j_2}-\boldsymbol{\eta}_{j_3}\boldsymbol{\eta}_{j_3}^*$,

$$\begin{aligned}&\mathrm{E}\bigg|\sum_{j_1\neq j_2}\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_2}^{-1}\bigg(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{B}\bigg(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\alpha}\bigg|^2\\&\quad=\sum_{\substack{j_1\neq j_2\\ j_3\neq j_4}}\mathrm{E}\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_2}^{-1}\bigg(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{B}\bigg(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\alpha}\\&\qquad\times\boldsymbol{\alpha}^*\mathbf{D}_{2j_3j_4}^{-1*}\boldsymbol{\eta}_{j_3}\boldsymbol{\eta}_{j_3}^*\mathbf{D}_{2j_3j_4}^{-1*}\bigg(\boldsymbol{\eta}_{j_4}\boldsymbol{\eta}_{j_4}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{B}^*\bigg(\boldsymbol{\eta}_{j_3}\boldsymbol{\eta}_{j_3}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{D}_{2j_3j_4}^{-1*}\mathbf{D}_{2j_3j_4}^{-1*}\boldsymbol{\alpha}\\&\quad=\sum_{\substack{j_1,j_2,j_3,j_4\\ \text{disctinct}}}\mathrm{E}\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_2}^{-1}\bigg(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{B}\bigg(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\alpha}\\&\qquad\times\boldsymbol{\alpha}^*\mathbf{D}_{2j_3j_4}^{-1*}\boldsymbol{\eta}_{j_3}\boldsymbol{\eta}_{j_3}^*\mathbf{D}_{2j_3j_4}^{-1*}\bigg(\boldsymbol{\eta}_{j_4}\boldsymbol{\eta}_{j_4}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{B}^*\bigg(\boldsymbol{\eta}_{j_3}\boldsymbol{\eta}_{j_3}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{D}_{2j_3j_4}^{-1*}\boldsymbol{\alpha}\\&\quad=\sum_{\substack{j_1,j_2,j_3,j_4\\ \text{disctinct}}}\mathrm{E}\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_2}^{-1}\bigg(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{B}\bigg(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\alpha}\\&\qquad\times\boldsymbol{\alpha}^*\mathbf{D}_{2j_2j_3j_4}^{-1*}\boldsymbol{\eta}_{j_3}\boldsymbol{\eta}_{j_3}^*\mathbf{D}_{2j_3j_4}^{-1*}\bigg(\boldsymbol{\eta}_{j_4}\boldsymbol{\eta}_{j_4}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{B}^*\bigg(\boldsymbol{\eta}_{j_3}\boldsymbol{\eta}_{j_3}^*-\frac{1}{n_2}\mathbf{I}\bigg)\mathbf{D}_{2j_3j_4}^{-1*}\boldsymbol{\alpha}+O\big(n^{-1/2}\big)\\&\quad=O\big(n^{-1/2}\big).\end{aligned}$$

Finally, noting that

$$\mathrm{E}\left|\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_2}^{-1}\left(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{B}\left(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\alpha}\right|^2=O_p\big(n^{-3}\big)$$

we obtain

$$\begin{aligned}&\mathrm{E}\left|\sum_{j_1\neq j_2}\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_2}^{-1}\left(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{B}\left(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\alpha}\right|^2\\&\quad=III_1+4III_2+III_3,\end{aligned}\tag{B.71}$$

where

$$III_1=\sum_{j_1\neq j_2}\mathrm{E}|\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_2}^{-1}\left(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{B}\left(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\alpha}|^2=O_(n_2^{-1}),$$

$$\begin{aligned}III_2&=\sum_{\substack{j_1,j_2,j_3\\ \text{distinct}}}\mathrm{E}\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_2}^{-1}\left(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{B}\left(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\alpha}\\&\quad\times\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_3}^{-1*}\left(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{B}^*\left(\boldsymbol{\eta}_{j_3}\boldsymbol{\eta}_{j_3}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{D}_{2j_1j_3}^{-1}\boldsymbol{\alpha}\\&=\sum_{\substack{j_1,j_2,j_3\\ \text{distinct}}}\mathrm{E}\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_2j_3}^{-1}\left(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{B}\left(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\alpha}\\&\quad\times\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_3}^{-1*}\left(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{B}^*\left(\boldsymbol{\eta}_{j_3}\boldsymbol{\eta}_{j_3}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{D}_{2j_1j_3}^{-1}\boldsymbol{\alpha}+O\big(n_2^{-1/2}\big)\\&=O\big(n_2^{-1/2}\big),\end{aligned}$$

and

$$\begin{aligned}III_3&=\sum_{\substack{j_1,j_2,j_3,j_4\\ \text{distinct}}}\mathrm{E}\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_2}^{-1}\left(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{B}\left(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\alpha}\\&\quad\times\boldsymbol{\alpha}^*\mathbf{D}_{2j_3j_4}^{-1*}\left(\boldsymbol{\eta}_{j_3}\boldsymbol{\eta}_{j_3}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{B}^*\left(\boldsymbol{\eta}_{j_4}\boldsymbol{\eta}_{j_4}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{D}_{2j_3j_4}^{-1}\boldsymbol{\alpha}\\&=\sum_{\substack{j_1,j_2,j_3,j_4\\ \text{distinct}}}\mathrm{E}\left(\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_2}^{-1}\left(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{B}\left(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\alpha}\right.\\&\quad\left.-\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_2j_3}^{-1}\left(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{B}\left(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^*-\frac{1}{n_2}\mathbf{I}\right)\mathbf{D}_{2j_1j_2j_3}^{-1}\boldsymbol{\alpha}\right)\end{aligned}$$

$$
\begin{aligned}
&\times \left(\boldsymbol{\alpha}^*\mathbf{D}_{2j_3j_4}^{-1*}\left(\boldsymbol{\eta}_{j_3}\boldsymbol{\eta}_{j_3}^* - \frac{1}{n_2}\mathbf{I}\right)\mathbf{B}^*\left(\boldsymbol{\eta}_{j_4}\boldsymbol{\eta}_{j_4}^* - \frac{1}{n_2}\mathbf{I}\right)\mathbf{D}_{2j_3j_4}^{-1}\boldsymbol{\alpha}\right.\\
&\left.-\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_3j_4}^{-1*}\left(\boldsymbol{\eta}_{j_3}\boldsymbol{\eta}_{j_3}^* - \frac{1}{n_2}\mathbf{I}\right)\mathbf{B}^*\left(\boldsymbol{\eta}_{j_4}\boldsymbol{\eta}_{j_4}^* - \frac{1}{n_2}\mathbf{I}\right)\mathbf{D}_{2j_1j_3j_4}^{-1}\boldsymbol{\alpha}\right)\\
&= \underline{m}^2(z)m_0^2(z)\sum_{\substack{j_1,j_2,j_3,j_4\\ \text{distinct}}}\mathrm{E}\left(\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_2j_3}^{-1}\boldsymbol{\eta}_{j_3}\boldsymbol{\eta}_{j_3}^*\mathbf{D}_{2j_1j_2j_3}^{-1}\left(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^* - \frac{1}{n_2}\mathbf{I}\right)\right.\\
&\times \mathbf{B}\left(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^* - \frac{1}{n_2}\mathbf{I}\right)\mathbf{D}_{2j_1j_2}^{-1}\boldsymbol{\alpha}\\
&+\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_2}^{-1}\left(\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^* - \frac{1}{n_2}\mathbf{I}\right)\mathbf{B}\left(\boldsymbol{\eta}_{j_2}\boldsymbol{\eta}_{j_2}^* - \frac{1}{n_2}\mathbf{I}\right)\mathbf{D}_{2j_1j_2j_3}^{-1}\boldsymbol{\eta}_{j_3}\boldsymbol{\eta}_{j_3}^*\mathbf{D}_{2j_1j_2j_3}^{-1}\boldsymbol{\alpha}\Big)\\
&\times \left(\boldsymbol{\alpha}^*\mathbf{D}_{2j_1j_3j_4}^{-1*}\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*\mathbf{D}_{2j_1j_3j_4}^{-1*}\left(\boldsymbol{\eta}_{j_3}\boldsymbol{\eta}_{j_3}^* - \frac{1}{n_2}\mathbf{I}\right)\mathbf{B}^*\left(\boldsymbol{\eta}_{j_4}\boldsymbol{\eta}_{j_4}^* - \frac{1}{n_2}\mathbf{I}\right)\mathbf{D}_{2j_3j_4}^{-1}\boldsymbol{\alpha}\right.\\
&\left.+\boldsymbol{\alpha}^*\mathbf{D}_{2j_3j_4}^{-1*}\left(\boldsymbol{\eta}_{j_3}\boldsymbol{\eta}_{j_3}^* - \frac{1}{n_2}\mathbf{I}\right)\mathbf{B}^*\left(\boldsymbol{\eta}_{j_4}\boldsymbol{\eta}_{j_4}^* - \frac{1}{n_2}\mathbf{I}\right)\mathbf{D}_{2j_1j_3j_4}^{-1}\boldsymbol{\eta}_{j_1}\boldsymbol{\eta}_{j_1}^*\mathbf{D}_{2j_1j_3j_4}^{-1*}\boldsymbol{\alpha}\right) + O\left(n_2^{-1/2}\right)\\
&= O\left(n_2^{-1/2}\right).
\end{aligned}
$$

In the derivation above, we have used the fact that once we change a $\mathbf{D}^{-1}$ with one more subscript, the order of the error will increase $O(n_2^{-1/2})$. Finally, when all $\mathbf{D}^{-1}$ factors have been changed to $\mathbf{D}_{2j_1j_2j_3j_4}^{-1}$, the expectation is zero.

Then, we begin to derive the limit h_{m1}. By (B.66), (B.60), (B.61) and Lemma B.4, we obtain

$$
\begin{aligned}
h_{m1}(z) &= \lim\frac{1}{p}\sum_{i=1}^{p}\left[\frac{1}{z^2}\left(\mathbf{e}_i^*\mathbf{T}_{p,1/2}^*\left(\underline{m}(z)\mathbf{T}_p + \mathbf{S}_2\right)^{-1}\mathbf{T}_{p,1/2}\mathbf{e}_i\right)^2\right.\\
&\quad - \frac{\underline{m}(z)}{z^2}\mathbf{e}_i^*\mathbf{T}_{p,1/2}^*\left(\underline{m}(z)\mathbf{T}_p + \mathbf{S}_2\right)^{-1}\mathbf{T}_{p,1/2}\mathbf{e}_i\\
&\quad \left.\times\, \mathbf{e}_i^*\mathbf{T}_{p,1/2}^*\left(\underline{m}(z)\mathbf{T}_p + \mathbf{S}_2\right)^{-1}\mathbf{T}_p\left(\underline{m}(z)\mathbf{T}_p + \mathbf{S}_2\right)^{-1}\mathbf{T}_{p,1/2}\mathbf{e}_i\right]\\
&= \lim\frac{1}{p}\sum_{i=1}^{p}\left[\frac{1}{z^2}\left(\mathbf{e}_i^*\mathbf{T}_{p,1/2}^*\left\{\tilde{K}(\underline{m}) + \underline{m}\mathbf{T}_p\right\}^{-1}\mathbf{T}_{p,1/2}\mathbf{e}_i\right)^2\right.\\
&\quad - \frac{\underline{m}(z)}{z^2}\mathbf{e}_i^*\mathbf{T}_{p,1/2}^*\left\{\tilde{K}(\underline{m}) + \underline{m}\mathbf{T}_p\right\}^{-1}\mathbf{T}_{p,1/2}\mathbf{e}_i\\
&\quad \left.\times\, \mathbf{e}_i^*\mathbf{T}_{p,1/2}^*\mathbf{D}^{-1}(\underline{m})\mathbf{T}_p\left\{\tilde{K}(\underline{m}) + \underline{m}\mathbf{T}_p\right\}^{-1}\mathbf{T}_{p,1/2}\mathbf{e}_i\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\underline{m}(z)}{pz^2} \lim \sum_{i=1}^{p} \mathbf{e}_i^* \mathbf{T}_{p,1/2}^* \mathbf{D}_2^{-1}\big(\underline{m}(z)\big) \mathbf{T}_{p,1/2} \mathbf{e}_i \\
&\times \sum_{j=1}^{n_2} \mathbf{e}_i^* \mathbf{T}_{p,1/2}^* \mathbf{D}_2^{-1}\big(\underline{m}(z)\big) \mathbf{T}_p \tilde{d}_j^2\big(\underline{m}(z)\big) \mathbf{T}_{p,1/2} \mathbf{e}_i \\
= \lim \frac{1}{p} \sum_{i=1}^{p} \Bigg[& \frac{1}{z^2} \big(\mathbf{e}_i^* \mathbf{T}_{p,1/2}^* \big\{ \tilde{K}(\underline{m}) + \underline{m} \mathbf{T}_p \big\}^{-1} \mathbf{T}_{p,1/2} \mathbf{e}_i \big)^2 \\
&- \frac{\underline{m}(z)}{z^2} \mathbf{e}_i^* \mathbf{T}_{p,1/2}^* \big\{ \tilde{K}(\underline{m}) + \underline{m} \mathbf{T}_p \big\}^{-1} \mathbf{T}_{p,1/2} \mathbf{e}_i \\
&\times \mathbf{e}_i^* \mathbf{T}_{p,1/2}^* \big\{ \tilde{K}(\underline{m}) + \underline{m} \mathbf{T}_p \big\}^{-1} \mathbf{T}_p \big\{ \tilde{K}(\underline{m}) + \underline{m} \mathbf{T}_p \big\}^{-1} \mathbf{T}_{p,1/2} \mathbf{e}_i \Bigg] \\
&- \frac{\underline{m}(z)}{pz^2} \lim \sum_{i=1}^{p} \mathbf{e}_i^* \mathbf{T}_{p,1/2}^* \mathbf{D}_2^{-1}\big(\underline{m}(z)\big) \mathbf{T}_{p,1/2} \mathbf{e}_i \\
&\times \sum_{j_1, j_2=1}^{n_2} \mathbf{e}_i^* \mathbf{T}_{p,1/2}^* \check{d}_{j_1}^2\big(\underline{m}(z)\big) \mathbf{T}_p \tilde{d}_{j_2}^2\big(\underline{m}(z)\big) \mathbf{T}_{p,1/2} \mathbf{e}_i \\
= \frac{m_0(z)}{z^2 \underline{m}^2(z)} \int & \frac{t^2 \, dH(t)}{(t + m_0(z))^3} \\
&- \frac{\underline{m}(z)}{pz^2} \lim \sum_{i=1}^{p} \mathbf{e}_i^* \mathbf{T}_{p,1/2}^* \mathbf{D}_2^{-1}\big(\underline{m}(z)\big) \mathbf{T}_{p,1/2} \mathbf{e}_i \\
&\times \sum_{j_1, j_2=1}^{n_2} \mathbf{e}_i^* \mathbf{T}_{p,1/2}^* \check{d}_{j_1}^2\big(\underline{m}(z)\big) \mathbf{T}_p \tilde{d}_{j_2}^2\big(\underline{m}(z)\big) \mathbf{T}_{p,1/2} \mathbf{e}_i \\
= \frac{m_0(z)}{z^2 \underline{m}^2(z)} \int & \frac{t^2 \, dH(t)}{(t + m_0(z))^3} - \frac{\underline{m}(z)}{pz^2} \lim \sum_{i=1}^{p} \mathbf{e}_i^* \mathbf{T}_{p,1/2}^* \mathbf{D}_2^{-1}\big(\underline{m}(z)\big) \mathbf{T}_{p,1/2} \mathbf{e}_i \\
&\times \underline{m}^2(z) m_0^2(z) \sum_{j=1}^{n_2} \boldsymbol{\eta}_j^* \big(\tilde{\mathbf{K}}\big(\underline{m}(z)\big) + \underline{m}(z) \mathbf{T}_p \big)^{-1} \mathbf{T}_p \big(\tilde{\mathbf{K}}\big(\underline{m}(z)\big) + \underline{m}(z) \mathbf{T}_p \big)^{-1} \boldsymbol{\eta}_j \\
&\times \mathbf{e}_i^* \mathbf{T}_{p,1/2}^* \mathbf{D}_{2j}^{-1}\big(\underline{m}(z)\big) \boldsymbol{\eta}_j \boldsymbol{\eta}_j^* \mathbf{D}_{2j}^{-1}\big(\underline{m}(z)\big) \mathbf{T}_{p,1/2} \mathbf{e}_i \\
= \frac{m_0(z)}{z^2 \underline{m}^2(z)} \int & \frac{t^2 \, dH(t)}{(t + m_0(z))^3} \\
&- \underline{m}^2(z) m_0^2(z) \frac{y_2 \underline{m}(z)}{z^2} \lim \frac{1}{pz^2} \operatorname{tr} \big(\tilde{\mathbf{K}}\big(\underline{m}(z)\big) + \underline{m}(z) \mathbf{T}_p \big)^{-1} \mathbf{T}_p \big(\tilde{\mathbf{K}}\big(\underline{m}(z)\big) + \underline{m}(z) \mathbf{T}_p \big)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{1}{p}\sum_{i=1}^{p}\mathbf{e}_i^*\mathbf{T}_{p,1/2}^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\mathbf{T}_{p,1/2}\mathbf{e}_i\cdot\mathbf{e}_i^*\mathbf{T}_{p,1/2}^*\mathbf{D}_{2j}^{-1}\big(\underline{m}(z)\big)\mathbf{S}_2\mathbf{D}_{2j}^{-1}\big(\underline{m}(z)\big)\mathbf{T}_{p,1/2}\mathbf{e}_i\\
&=\frac{m_0(z)}{z^2\underline{m}^2(z)}\int\frac{t^2\,dH(t)}{(t+m_0(z))^3}-\underline{m}^2(z)m_0^2(z)\frac{y_2}{\underline{m}(z)}\oint\frac{t}{(t+m_0(z))^2}\,dH(t)\\
&\quad\times\frac{1}{pz^2}\lim\sum_{i=1}^{p}\mathbf{e}_i^*\mathbf{T}_{p,1/2}^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\mathbf{T}_{p,1/2}\mathbf{e}_i\\
&\quad\times\mathbf{e}_i^*\mathbf{T}_{p,1/2}^*\mathbf{D}_{2j}^{-1}\big(\underline{m}(z)\big)\mathbf{S}_2\mathbf{D}_{2j}^{-1}\big(\underline{m}(z)\big)\mathbf{T}_{p,1/2}\mathbf{e}_i\\
&=\frac{m_0(z)}{z^2\underline{m}^2(z)}\int\frac{t^2\,dH(t)}{(t+m_0(z))^3}-\frac{y_2}{\underline{m}(z)}\oint\frac{t}{(t+m_0(z))^2}\,dH(t)\\
&\quad\times\frac{1}{pz^2}\lim\sum_{i=1}^{p}\mathbf{e}_i^*\mathbf{T}_{p,1/2}^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\mathbf{T}_{p,1/2}\mathbf{e}_i\\
&\quad\times\mathbf{e}_i^*\mathbf{T}_{p,1/2}^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\mathbf{S}_2\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\mathbf{T}_{p,1/2}\mathbf{e}_i\\
&=\frac{m_0(z)}{z^2\underline{m}^2(z)}\int\frac{t^2\,dH(t)}{(t+m_0(z))^3}-h_{m1}\frac{y_2}{\underline{m}(z)}\oint\frac{t}{(t+m_0(z))^2}\,dH(t),
\end{aligned}
$$

where

$$
\begin{aligned}
&\sum_{j=1}^{n_2}\mathbf{e}_i\mathbf{T}_{p,1/2}^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\boldsymbol{\eta}_j\boldsymbol{\eta}_j^*\mathbf{D}_{2j}^{-1}\big(\underline{m}(z)\big)\mathbf{T}_{p,1/2}\mathbf{e}_i\\
&\quad=\sum_{j=1}^{n_2}\mathbf{e}_i\mathbf{T}_{p,1/2}^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\boldsymbol{\eta}_j\boldsymbol{\eta}_j^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\mathbf{T}_{p,1/2}\mathbf{e}_i\\
&\qquad+\sum_{j=1}^{n_2}\mathbf{e}_i\mathbf{T}_{1/2}^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\boldsymbol{\eta}_j\boldsymbol{\eta}_j^*\mathbf{D}_{2j}^{-1}\big(\underline{m}(z)\big)\mathbf{T}_{1/2}\mathbf{e}_i\cdot\frac{\boldsymbol{\eta}_j^*\mathbf{D}_{2j}^{-1}(\underline{m}(z))\boldsymbol{\eta}_j}{1+y_2m_{\mathbf{I}}(-\underline{m}(z))}\\
&\quad=\sum_{j=1}^{n_2}\mathbf{e}_i\mathbf{T}_{p,1/2}^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\boldsymbol{\eta}_j\boldsymbol{\eta}_j^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\mathbf{T}_{p,1/2}\mathbf{e}_i\\
&\qquad+\sum_{j=1}^{n_2}\mathbf{e}_i\mathbf{T}_{1/2}^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\boldsymbol{\eta}_j\boldsymbol{\eta}_j^*\mathbf{D}_{2j}^{-1}\big(\underline{m}(z)\big)\mathbf{T}_{1/2}\mathbf{e}_i\cdot\frac{y_2m_{\mathbf{I}}(-\underline{m}(z))}{1+y_2m_{\mathbf{I}}(-\underline{m}(z))}+o_p(1)\\
&\quad=\sum_{j=1}^{n_2}\mathbf{e}_i\mathbf{T}_{p,1/2}^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\boldsymbol{\eta}_j\boldsymbol{\eta}_j^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\mathbf{T}_{p,1/2}\mathbf{e}_i
\end{aligned}
$$

$$+\sum_{j=1}^{n_2}\mathbf{e}_i\mathbf{T}_{1/2}^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\boldsymbol{\eta}_j\boldsymbol{\eta}_j^*\mathbf{D}_{2j}^{-1}\big(\underline{m}(z)\big)\mathbf{T}_{1/2}\mathbf{e}_i\cdot\frac{y_2m_{\mathbf{I}}(-\underline{m}(z))}{1+y_2m_{\mathbf{I}}(-\underline{m}(z))}+o_p(1)$$

$$=\frac{1}{\underline{m}(z)m_0(z)}\sum_{j=1}^{n_2}\mathbf{e}_i\mathbf{T}_{p,1/2}^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\boldsymbol{\eta}_j\boldsymbol{\eta}_j^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\mathbf{T}_{p,1/2}\mathbf{e}_i+o_p(1)$$

and similarly we have

$$\sum_{j=1}^{n_2}\mathbf{e}_i\mathbf{T}_{p,1/2}^*\mathbf{D}_{2j}^{-1}\big(\underline{m}(z)\big)\boldsymbol{\eta}_j\boldsymbol{\eta}_j^*\mathbf{D}_{2j}^{-1}\big(\underline{m}(z)\big)\mathbf{T}_{p,1/2}\mathbf{e}_i$$

$$=\frac{1}{\underline{m}^2(z)m_0^2(z)}\sum_{j=1}^{n_2}\mathbf{e}_i\mathbf{T}_{p,1/2}^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\boldsymbol{\eta}_j\boldsymbol{\eta}_j^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\mathbf{T}_{p,1/2}\mathbf{e}_i+o_p(1)$$

$$=\frac{1}{\underline{m}^2(z)m_0^2(z)}\mathbf{e}_i\mathbf{T}_{p,1/2}^*\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\mathbf{S}_2\mathbf{D}_2^{-1}\big(\underline{m}(z)\big)\mathbf{T}_{p,1/2}\mathbf{e}_i+o_p(1).$$

Then we have

$$h_{m1}=\frac{\frac{m_0(z)}{z^2\underline{m}^2(z)}\int\frac{t^2\,dH(t)}{(t+m_0(z))^3}}{1+\frac{y_2}{\underline{m}(z)}\oint\frac{t}{(t+m_0(z))^2}\,dH(t)}=\frac{\frac{m_0^2(z)}{z^2\underline{m}(z)}\int\frac{t^2}{(t+m_0(z))^3}\,dH(t)}{1-y_2\oint\frac{m_0^2(z)}{(t+m_0(z))^2}\,dH(t)},\tag{B.72}$$

where

$$\begin{aligned}&1+\frac{y_2}{\underline{m}(z)}\int\frac{t}{(t+m_0(z))^2}\,dH(t)\\&\quad=\frac{1}{\underline{m}(z)m_0(z)}\bigg[\underline{m}(z)m_0(z)+y_2\int\frac{m_0(z)}{t+m_0(z)}\,dH(t)\\&\qquad-y_2\int\frac{m_0^2(z)}{(t+m_0(z))^2}\,dH(t)\bigg]\\&\quad=\frac{1}{\underline{m}(z)m_0(z)}\bigg[1-y_2\int\frac{m_0^2(z)}{(t+m_0(z))^2}\,dH(t)\bigg].\end{aligned}$$

Because

$$\underline{m}(z)=-\frac{1}{m_0(z)}+y_2\int\frac{dH(t)}{m_0(z)+t},$$

then we have

$$\underline{m}'(z)=\frac{1}{m_0^2(z)}\bigg(1-y_2\int\frac{m_0^2(z)}{(t+\underline{m}_0(z))^2}\,dH(t)\bigg)m_0'(z).$$

Moreover, we have

$$z = \frac{h^2}{y_2\underline{m}(z)} + \frac{y_1}{y_2}m_0(z) \tag{B.73}$$

$$= -\frac{h^2 m_0(z)}{y_2(1-y_2+y_2\int \frac{t}{t+m_0(z)}\,dH(t))} + \frac{y_1 m_0(z)}{y_2}. \tag{B.74}$$

Therefore, by (B.73), we have

$$\begin{aligned} 1 &= -\frac{h^2}{y_2\underline{m}^2(z)}\underline{m}'(z) + \frac{y_1}{y_2}\underline{m}_0'(z) \\ &= -\frac{(1-y_2\int \frac{m_0^2(z)}{(t+m_0(z))^2}\,dH(t))}{\underline{m}^2(z)m_0^2(z)}\left(\frac{h^2}{y_2} - \frac{y_1}{y_2}\frac{\underline{m}^2(z)m_0^2(z)}{1-y_2\int \frac{m_0^2(z)}{(t+m_0(z))^2}\,dH(t)}\right)m_0'(z). \end{aligned}$$

That is,

$$\begin{aligned} m_0'(z) &= -\frac{\underline{m}^2(z)m_0^2(z)}{(\frac{h^2}{y_2} - \frac{y_1}{y_2}\frac{\underline{m}^2(z)m_0^2(z)}{1-y_2\int \frac{m_0^2(z)}{(t+m_0(z))^2}\,dH(t)})(1-y_2\int \frac{m_0^2(z)}{(t+m_0(z))^2}\,dH(t))} \\ &= -\frac{(-1+y_2\int \frac{m_0(z)}{t+m_0(z)}\,dH(t))^2}{(\frac{h^2}{y_2} - \frac{y_1}{y_2}\frac{\underline{m}^2(z)m_0^2(z)}{1-y_2\int \frac{m_0^2(z)}{(t+m_0(z))^2}\,dH(t)})(1-y_2\int \frac{m_0^2(z)}{(t+m_0(z))^2}\,dH(t))}. \end{aligned} \tag{B.75}$$

Then by (B.72) and (B.75), we obtain

$$\begin{aligned} &-\frac{\beta_x y_1}{2\pi i}\cdot\oint_{\mathcal{C}} f(z)\frac{z^2\underline{m}^3(z)\cdot h_{m1}(z)}{\frac{h^2}{y_2} - \frac{y_1}{y_2}\cdot\frac{(1-\int \frac{y_2 m_0(z)}{t+m_0(z)}\,dH(t))^2}{1-\int \frac{y_2 m_0^2(z)}{(t+m_0(z))^2}\,dH(t)}}\,dz \\ &= \frac{\beta_x y_1}{2\pi i}\cdot\oint_{\mathcal{C}} f(z)\frac{\underline{m}^2(z)m_0^2(z)\cdot\int \frac{t^2}{(t+m_0(z))^3}\,dH(t)}{(\frac{h^2}{y_2} - \frac{y_1}{y_2}\cdot\frac{(1-\int \frac{y_2 m_0(z)}{t+m_0(z)}\,dH(t))^2}{1-\int \frac{y_2 m_0^2(z)}{(t+m_0(z))^2}\,dH(t)})(1-y_2\oint \frac{m_0^2(z)}{(t+m_0(z))^2}\,dH(t))}\,dz \\ &= \frac{\beta_x y_1}{2\pi i}\cdot\oint_{\mathcal{C}} f(z)\left[\int \frac{t^2}{(t+m_0(z))^3}\,dH(t)\right]dm_0(z). \end{aligned}$$

The proof is completed. □

B.6. Proofs of Remark 4.2

We have

$$\frac{\beta_x y_1}{2\pi i}\oint_{\mathcal{C}} f(z)\left[\int \frac{t^2}{(t+m_0(z))^3}\,dH(t)\right]dm_0(z)$$
$$=-\frac{\beta_x y_1}{2\pi i}\oint_{\mathcal{C}} f(z)\frac{\int \frac{t^2}{(t+m_0(z))^3}\,dH(t)\cdot(-1+y_2\int \frac{m_0(z)}{t+m_0(z)}\,dH(t))^2}{(\frac{h^2}{y_2}-\frac{y_1}{y_2}\frac{\underline{m}_y^2(z)m_0^2(z)}{1-y_2\int \frac{m_0^2(z)}{(t+m_0(z))^2}\,dH(t)})(1-y_2\int \frac{m_0^2(z)}{(t+m_0(z))^2}\,dH(t))}\,dz.$$

Because

$$m_0'(z)=\frac{-\underline{m}'(z)m_0^2(z)}{1-y_2\int \frac{m_0^2(z)\,dH(t)}{(t+m_0(z))^2}},$$

then we have

$$\underline{m}'(z)=\frac{(-1+y_2\int \frac{m_0(z)}{t+m_0(z)}\,dH(t))^2}{m_0^2(z)(\frac{h^2}{y_2}-\frac{y_1}{y_2}\frac{\underline{m}_y^2(z)m_0^2(z)}{1-y_2\int \frac{m_0^2(z)}{(t+m_0(z))^2}\,dH(t)})}.$$

Thus we obtain

$$\left(1-y_2\int \frac{m_0^2(z)\,dH(t)}{(t+m_0(z))^2}\right)\left[\log\left(1-y_2\int \frac{m_0^2\,dH(t)}{(t+m_0)^2}\right)\right]'$$
$$=\left(1-y_2\int \frac{m_0^2(z)\,dH(t)}{(t+m_0(z))^2}\right)\frac{2\underline{m}'(z)y_2\int \frac{tm_0^3\,dH(t)}{(t+m_0)^3}}{(1-y_2\int \frac{m_0^2\,dH(t)}{(t+m_0)^2})^2}$$
$$=\frac{2y_2m_0(z)\int \frac{t\,dH(t)}{(t+m_0)^3}}{1-y_2\int \frac{m_0^2\,dH(t)}{(t+m_0)^2}}\frac{(-1+y_2\int \frac{m_0(z)}{t+m_0(z)}\,dH(t))^2}{(\frac{h^2}{y_2}-\frac{y_1}{y_2}\frac{\underline{m}_y^2(z)m_0^2(z)}{1-y_2\int \frac{m_0^2(z)}{(t+m_0(z))^2}\,dH(t)})}.$$

Then we have

$$\frac{\beta_y}{4\pi i}\oint_{\mathcal{C}} f(z)\left(1-y_2\int \frac{m_0^2(z)\,dH(t)}{(t+m_0(z))^2}\right)d\log\left(1-y_2\int \frac{m_0^2(z)\,dH(t)}{(t+m_0(z))^2}\right)$$
$$=\frac{\beta_y}{2\pi i}\oint_{\mathcal{C}} f(z)\frac{y_2m_0(z)\int \frac{t\,dH(t)}{(t+m_0)^3}}{1-y_2\int \frac{m_0^2\,dH(t)}{(t+m_0)^2}}\frac{(-1+y_2\int \frac{m_0(z)}{t+m_0(z)}\,dH(t))^2}{(\frac{h^2}{y_2}-\frac{y_1}{y_2}\frac{\underline{m}_y^2(z)m_0^2(z)}{1-y_2\int \frac{m_0^2(z)}{(t+m_0(z))^2}\,dH(t)})}\,dz. \tag{B.76}$$

So the proof of Remark 4.2 is completed.

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