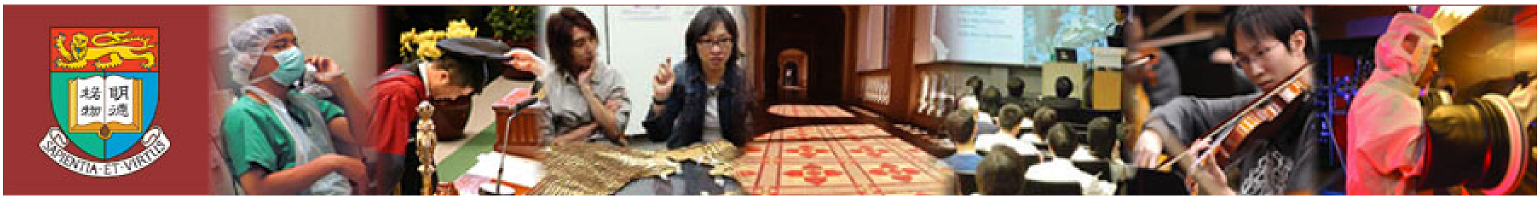

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REVIEW ARTICLE

Quantum superreplication of states and gates

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Although the no-cloning theorem forbids perfect replication of quantum information, it is sometimes possible to produce large numbers of replicas with vanishingly small error. This phenomenon, known as quantum superreplication, can occur for both quantum states and quantum gates. The aim of this paper is to review the central features of quantum superreplication and provide a unified view of existing results. The paper also includes new results. In particular, we show that when quantum superreplication can be achieved, it can be achieved through estimation up to an error of size $O(M/N^2)$, where N and M are the number of input and output copies, respectively. Quantum strategies still offer an advantage for superreplication in that they allow for exponentially faster reduction of the error. Using the relation with estimation, we provide *i)* an alternative proof of the optimality of Heisenberg scaling in quantum metrology, *ii)* a strategy for estimating arbitrary unitary gates with a mean square error scaling as $\log N/N^2$, and *iii)* a protocol that generates $O(N^2)$ nearly perfect copies of a generic pure state $U|0\rangle$ while using the corresponding gate U only N times. Finally, we point out that superreplication can be achieved using interactions among k systems, provided that k is large compared to M^2/N^2.

Keywords quantum cloning, quantum metrology, quantum superreplication, Heisenberg limit, quantum networks

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1 Introduction

The no-cloning theorem [1, 2] is one of the cornerstones of quantum information theory, with implications permeating the entire field [3, 4]. Most famously, the impossibility of copying non-orthogonal quantum states provides a working principle for quantum cryptography with applications to key distribution [5, 6], unforgeable banknotes [7], and secret sharing [8]. The no-cloning theorem forbids perfect cloning. A natural question, originally asked by Bužek and Hillery [9], is how well cloning can be approximated by the processes allowed by quantum mechanics. The question is relevant both to cryptographic applications and to the foundations of quantum theory, shedding light on the relationship between quantum and classical copy machines [10–15] and providing benchmarks that certify the advantages of quantum information processing over classical information processing [16–20]. Due to the broad spectrum of applications, the research into optimal cloning machines is still an active and fruitful line of investigation [21].

Among the cloning machines allowed by quantum mechanics, one can distinguish two types: deterministic and probabilistic machines. Deterministic machines produce approximate copies with certainty, whereas probabilistic machines sometimes produce a failure message indicating that the copying process has gone wrong. In general, probabilistic machines can produce more accurate copies at the price of a reduced probability of success. For ex-

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ample, Duan and Guo [22] showed that a set of non-orthogonal states can be cloned without error as long as they are linearly independent. More recently, Fiurášek [23] showed that probabilistic cloners can offer an advantage even for linearly dependent states, including, e.g., coherent states of harmonic oscillators with known amplitude. Ralph and Lund [24] proposed a concrete optical setup achieving noiseless probabilistic amplification and cloning of coherent states. The possibility of noiseless probabilistic amplification was later extended to the case of a Gaussian-distributed coherent state amplitude [19]. Although the probability of success vanishes as the accuracy increases, nearly perfect amplification of coherent states has been observed experimentally for small values of the amplitude [25–28].

The recent developments in probabilistic amplification and cloning of coherent states motivate the search for new scenarios where non-orthogonal states can be cloned with vanishingly small error. In [29], we considered the task of cloning *clock states*, that is, quantum states generated by time evolution under a known Hamiltonian. Here, the cloner is given N identical copies of the same clock state and attempts to produce $M = M(N)$ copies. Inspired by the asymptotic framework of information theory, we considered the scenario where N is large. Here, the main question is how fast $M(N)$ can grow under the condition that the copying process is *reliable*, meaning that the global error of all the output copies vanishes in the large-N limit. For deterministic cloners, we showed that the number of reliable copies has to scale as

$$M(N) = N\left[1 + f(N)\right] \quad \text{with} \quad \lim_{N\to\infty} f(N) = 0\,. \qquad (1)$$

That is, the number of extra copies produced by the cloner must be negligible compared to the number of input copies. This result can be seen as an *asymptotic no-cloning theorem*, which extends the original no-cloning theorem from perfect cloners to cloners that become perfect in the asymptotic limit. The asymptotic no-cloning theorem originates from the requirement that the error vanishes *globally* on all the clones. By using tomography, it is generally easy to produce copies that individually have an error vanishing on the order of $1/N$. However, this does not guarantee that the error vanishes at the global level. Indeed, the errors accumulate when multiple copies are examined jointly. Consequently, only a cloner satisfying Eq. (1) can have a vanishing error at the global level. For probabilistic cloners of clock states, the situation is different; the number of reliable copies produced by a probabilistic machine can scale as

$$M(N) = N[1+f(N)] \quad \text{with} \quad f(N) = \Theta\left(N^{\delta}\right), \delta < 1\,. \qquad (2)$$

When δ is zero, this means that the number of extra copies scales as N, evading the limitation posed by the asymptotic no-cloning theorem. When δ is larger than zero, the number of extra copies grows faster than N, and one can produce up to $M = \Theta(N^2)$ copies. In both cases, we refer to this phenomenon as *superreplication*, emphasizing that the number of extra copies grows beyond the limits imposed by the asymptotic no-cloning theorem. Note, however, that the number of output copies allowed by Eq. (2) can grow at most at a rate $M = \Theta\left(N^{2-\epsilon}\right)$, where ϵ is an arbitrarily small constant. In finite dimensions, the quadratic replication rate is the ultimate limit for superreplication of clock states, in close connection with the Heisenberg limit in quantum metrology [29, 30].

In addition to producing a large number of extra copies, probabilistic cloners exhibit better scaling of the error, which vanishes as $\exp[-cN^2/M(N)]$ for a suitable constant $c > 0$ depending on the particular set of clock states under consideration. In contrast, the error for deterministic machines can scale at most as $1/N^4$ [29]. The benefits of probabilistic cloners, however, are not without cost. The price to pay is a very small probability of success; precisely, the probability of superreplication has to be small compared to $\exp\left[-M/N\right]$. This fact severely limits the ability to observe superreplication, especially when M grows much faster than N. Nevertheless, the cloner can be devised in such a way that, when superreplication fails, the original input state is almost undisturbed. Precisely, Winter's gentle measurement lemma [31] implies that the error introduced by the failure of superreplication scales as $O\left(\sqrt{p_{\rm succ}}\right)$, where $p_{\rm succ}$ is the probability of success. For superreplication processes with $\delta > 0$, this means that the error introduced by the failure of superreplication vanishes faster than $\exp\left[-N^{\delta/2}\right]$. As a consequence, the state resulting from failed superreplication can still be used to achieve standard, deterministic cloning with asymptotically optimal performance; for large N, the error $\exp\left[-N^{\delta/2}\right]$ is covered by the error introduced by deterministic cloning. In summary, superreplication is a rare event, but trying to observe it does not prevent the application of standard deterministic cloning techniques.

Interestingly, the limitation on the probability of success is lifted if we consider the problem of cloning gates instead of states. Cloning a quantum gate means simulating M uses of it while actually using it only $N < M$ times. A no-cloning theorem for quantum gates was proven by D'Ariano, Perinotti, and one of the authors [32], who showed that no quantum protocol can perfectly clone a generic quantum gate. Recently, Dür *et al.* [33] analyzed the cloning of phase gates, i.e., unitary gates, by describing the time evolution with a known Hamiltonian. They devised a quantum network that simulates up to N^2 uses of an unknown phase gate by using it only N times. The network works deterministically and has vanishing error on average over all input states. We refer to this phenomenon as *gate superreplication*. The discovery by Dür *et al.* opened the question of whether superreplication can be achieved not only for phase gates, but also for arbitrary quantum gates. In Ref. [34], we answered the question in the affirmative, constructing a universal quantum network that replicates completely unknown unitary gates. The network has vanishing error on almost all input states, except for a vanishingly small fraction of the Hilbert space. These "bad states" can be characterized explicitly, making it easy to identify applications where gate superreplication can be safely employed.

In this paper, we review the key facts about superreplication of states and gates, unifying the ideas presented in the literature and emphasizing the connections between superreplication and other tasks, such as estimating quantum gates and generating states using gates as oracles. In addition to this review, the paper contains a number of new results:

1) We show that whenever quantum superreplication is achievable, it can also be achieved through estimation. However, the error for estimation-based strategies will vanish according to a power law, whereas the error for genuine quantum strategies vanishes faster than any polynomial.
2) We establish an equivalence between the optimality of Heisenberg scaling in quantum metrology and the ultimate limit on the rate of superreplication, which is encapsulated in Eq. (2). On the one hand, Heisenberg scaling implies the impossibility of producing more than $M = O(N^2)$ copies of a clock state with vanishing error [29]. On the other hand, here we prove the converse result, showing that the limit on the replication rate set by Eq. (2) implies the optimality of Heisenberg scaling.
3) We explore the possibility of achieving superreplication without acting globally on all N input copies and on $M-N$ additional blank copies. Specifically, we consider strategies where the N input copies are divided into subgroups and each subgroup generates new copies, mimicking a scenario in which groups of cells fuse together and then split into approximate clones. For replication strategies of this form, we show that interactions among groups of $O(M^2/N^{2-\epsilon})$ particles are necessary and sufficient to achieve superreplication. With respect to the original superreplication protocol, the modified protocol reduces the size of the interactions by a factor $M/N^{2-\epsilon}$, which is asymptotically large for all the replication rates allowed by Eq. (2).
4) We construct a simple protocol that estimates an unknown unitary gate with a mean square error of $\log N/N^2$. The protocol uses gate superreplication to produce $M = \Theta(N^2/\log N)$ copies and gate tomography to estimate the gate within a mean square error of size $1/M$. The resulting scaling of the error is close to the optimal scaling $1/N^2$ [35–38].
5) We analyze the task of generating M copies of a quantum state $U|0\rangle$ given N uses of a completely unknown unitary gate U [34]. In this setting we show that $M = \Theta(N^{2-\epsilon})$ copies of the state can be generated with fidelity going to one in the large-N limit for every $\epsilon > 0$.
6) We examine replication of quantum gates in the worst case over all possible input states. In the worst-case scenario, we establish an *asymptotic no-cloning theorem for quantum gates*, which states that the number of reliable copies of the gate scales as $M(N) = N[1 + f(N)]$, where $f(N)$ vanishes in the large-N limit.

The paper is organized as follows. Section 2 introduces the task of asymptotic cloning and precisely analyses the phenomenon of quantum state superreplication. Section 3 analyses the relationship between state superreplication and quantum metrology. Section 4 discusses the scale of the interactions needed for state superreplication. In Section 5, we review the existing results for superreplication of quantum gates. Several applications of gate superreplication are presented in Section 6. Finally, conclusions are drawn in Section 7.

2 Quantum state superreplication

Deterministic cloners. Although perfect cloning of non-orthogonal quantum states is forbidden by the no-cloning theorem [1, 2], approximate cloning can be realized to various degrees of accuracy, depending on the set of states to be cloned [10–15]. Consider a scenario in which one is given N identical systems, each prepared in the same state $|\psi_x\rangle$, and the goal is to generate M systems in a state close to M perfect copies of the state $|\psi_x\rangle$. For the moment, we assume the cloning process to be deterministic, meaning that it produces approximate clones with unit probability. Mathematically, a deterministic cloner is described by a completely positive trace-preserving linear map $\mathcal{C}$, a.k.a. a quantum channel, sending states on the Hilbert space $\mathcal{H}^{\otimes N}$ to states on the Hilbert space $\mathcal{H}^{\otimes M}$, where $\mathcal{H}$ is the Hilbert space of a single copy. The cloning accuracy can be quantified as the average fidelity between the ideal state of M perfect copies and the actual output state of the cloner. Explicitly, the average fidelity is given by

$$F_{\rm det}[N\to M]=\sum_x p_x {\rm Tr}\left[|\psi_x\rangle\langle\psi_x|^{\otimes M}\mathcal{C}\left(|\psi_x\rangle\langle\psi_x|^{\otimes N}\right)\right], \quad (3)$$

where p_x is the prior probability of the state $|\psi_x\rangle$. The quantum channel that maximizes the fidelity is the optimal N-to-M cloner for the ensemble $\{|\psi_x\rangle, p_x\}$. In general, the form of the optimal channel varies for different ensembles, depending on both the states and their prior probabilities.

Example: the equatorial qubit cloner. The general settings of optimal cloning can be nicely illustrated in the example of equatorial qubit states [39–41]. Here, the ensemble consists of states of the form $|\psi_t\rangle = (|0\rangle + {\rm e}^{-{\rm i}t}|1\rangle)/\sqrt{2}$, where θ is drawn uniformly at random from the interval $[0, 2\pi)$. The input state can be expanded as

$$|\psi_t\rangle^{\otimes N} = \frac{1}{2^{N/2}}\sum_{n=0}^{N}\sqrt{\binom{N}{n}}{\rm e}^{-{\rm i}nt}|N,n\rangle,$$

where $\{|N,n\rangle \mid n = 0,\ldots,N\}$ is the orthonormal basis of the Dicke states, which are defined as

$$|N,n\rangle := \frac{1}{\sqrt{N!\,n!\,(N-n)!}}\sum_{\pi\in\mathsf{S}_N} U_\pi\,|0\rangle^{\otimes(N-n)}\,|1\rangle^{\otimes n},$$

where π is a permutation of N qubits, U_π is the unitary operator that implements the permutation π, and the sum running over the symmetric group is S_N.

For convenience of discussion, we focus on the case where both N and M are even. In this case, the optimal cloning channel has the simple form $\mathcal{C}(\rho) = V\rho V^\dagger$, where V is the isometry, which is defined as [41]

$$V|N,\frac{N}{2}+m\rangle = |M,\frac{M}{2}+m\rangle, \quad m\in\left[-\frac{N}{2},\frac{N}{2}\right].$$

Plugging the above relation into the definition of the average fidelity (3), one obtains the optimal value

$$\begin{aligned}F_{\rm det}[N\to M] &= \frac{1}{2^{N+M}}\left[\sum_{m=-N/2}^{N/2}\sqrt{\binom{N}{\frac{N}{2}+m}\binom{M}{\frac{M}{2}+m}}\right]^2\\ &\approx \frac{2\sqrt{MN}}{M+N} \quad N,M\gg 1.\end{aligned} \quad (4)$$

The asymptotic no-cloning theorem. Inspired by the asymptotic framework of information theory, we now focus on quantum cloners in the limit $N\to\infty$. We model the cloning process as a sequence of cloners $(\mathcal{C}_N)_{N\in\mathbb{N}}$, where $\mathcal{C}_N$ transforms N copies into $M(N)$ approximate copies for a given function $M(N)$. We call the cloning process *reliable* if the cloning fidelity goes to one in the asymptotic limit, namely,

$$\lim_{N\to\infty} F_{\rm det}[N\to M(N)] = 1.$$

The key question here is how many extra copies can be produced reliably. Can we produce N extra copies or more? To provide a rigorous answer, we define the *rate* of a cloning process as

$$\alpha := \liminf_{N\to\infty}\frac{\log[M(N)-N]}{\log N}.$$

That is, if the rate is α, the number of output copies grows as $M(N) = N + \Theta(N^\alpha)$. We say that a cloning rate α is *achievable* if and only if there exists a reliable cloning process that has a rate equal to α. For a given set of states, the task is to find the maximum achievable rate over all cloning processes. For deterministic processes, the following theorem tightly limits the number of copies that can be produced reliably.

Theorem 1 (Asymptotic no-cloning theorem for quantum states [29]). *No deterministic process can reliably clone a continuous set of quantum states at a rate* $\alpha \geqslant 1$.

The intuition for the proof comes from the standard quantum limit of metrology [30], which limits the precision of deterministic estimation of a parameter t encoded into N product copies of a state $|\psi_t\rangle$. The standard quantum limit states that the mean square error vanishes as c/N, where c is a constant that depends on the encoding $t\mapsto|\psi_t\rangle$. Intuitively, a deterministic cloner that violates Theorem 1 would contradict the standard quantum limit, because one could increase the precision by first cloning the probe states and then measuring them.

The argument based on the standard quantum limit of metrology is partly heuristic. A complete argument can be made for *clock states*, i.e., quantum states of the form

$$|\psi_t\rangle = {\rm e}^{-{\rm i}tH}|\psi\rangle, \quad t\in\mathbb{R}, H^\dagger = H.$$

The argument is conceptually independent of the standard quantum limit and allows one to prove a strong converse of the asymptotic no-cloning theorem.

Theorem 2 (Strong converse of the asymptotic no-cloning theorem [29]). *Every deterministic process that clones clock states at a rate* $\alpha > 1$ *will necessarily have a vanishing fidelity in the large-*N *limit.*

The asymptotic no-cloning theorem implies that a deterministic cloning process cannot produce more than a negligible number of extra replicas in the asymptotic limit. Let us check a few examples. The first example is the cloning of equatorial qubit states introduced earlier in the paper. In the large-N limit, the cloning fidelity is given by $F \approx 2\sqrt{MN}/(M+N)$; cf. Eq. (4). It is straightforward to see that any achievable cloning rate must be smaller than one. Another example is the universal cloning of pure states. For d-dimensional quantum systems, the optimal fidelity is $F = \binom{d+N-1}{N}/\binom{d+M-1}{M}$ [42]. For large N and M, the fidelity scales as $(N/M)^{d-1}$ and converges to one if and only if the cloning rate satisfies the condition $\alpha < 1$. The example of universal cloning shows that one can always find a deterministic process that reliably produces $M = N + \Theta(N^\delta)$ copies for every desired exponent $\delta > 0$.

It is important to stress that the asymptotic no-cloning theorem holds for *continuous* sets of states but does not place any restriction on the cloning rate of discrete sets of states. For example, every finite set of quantum states can be cloned with vanishing error in the large-N limit, no matter how large M is [15]. Hence, for finite sets of states, every rate is achievable.

Probabilistic cloners. The asymptotic no-cloning theorem poses a stringent limit on the ability to replicate information. In the following we will see that the limit can be broken by allowing the cloner to be probabilistic.

To analyze probabilistic cloners, it is useful to introduce the notion of a quantum instrument [43, 44]. A quantum instrument consists of an indexed set of completely positive, trace non-increasing maps (a.k.a. quantum operations), $\{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$. When an input state ρ is fed into the quantum instrument, the quantum operation $\mathcal{M}_i$ occurs with probability $p_i = \mathrm{Tr}[\mathcal{M}_i(\rho)]$, and the instrument outputs a quantum system in the state $\rho' = \mathcal{M}_i(\rho)/p_i$. In the context of our cloning problem, we consider a quantum instrument consisting of two quantum operations, $\mathcal{M}_{\rm yes}$ and $\mathcal{M}_{\rm no}$, where $\mathcal{M}_{\rm yes}$ describes the realization of a successful cloning process. When acting on the N-copy input state $|\psi_x\rangle^{\otimes N}$, the probabilistic cloner succeeds with probability

$$p(\mathrm{yes}|x) = \mathrm{Tr}\left[\mathcal{M}_{\rm yes}(|\psi_x\rangle\langle\psi_x|^{\otimes N})\right], \tag{5}$$

in which case it produces the M-copy output state

$$\rho'_x = \mathcal{M}_{\rm yes}(|\psi_x\rangle\langle\psi_x|^{\otimes N})/p(\mathrm{yes}|x). \tag{6}$$

Conditional on the success of the cloning process, the average fidelity is equal to

$$F_{\rm prob}[N\to M] = \sum_x p(x|\mathrm{yes})\,\mathrm{Tr}\left[|\psi_x\rangle\langle\psi_x|^{\otimes M}\rho'_x\right], \tag{7}$$

where $p(x|\mathrm{yes})$ is the conditional probability given by Bayes' rule. Inserting Eqs. (5) and (6) into the above expression, we obtain the explicit formula

$$F_{\rm prob}[N\to M] = \frac{\sum_x p_x \mathrm{Tr}\left[|\psi_x\rangle\langle\psi_x|^{\otimes M}\mathcal{M}_{\rm yes}(|\psi_x\rangle\langle\psi_x|^{\otimes N})\right]}{\sum_y p_y \mathrm{Tr}\left[\mathcal{M}_{\rm yes}(|\psi_y\rangle\langle\psi_y|^{\otimes N})\right]}. \tag{8}$$

The maximum fidelity over all possible quantum instruments represents the ultimate performance allowed by quantum mechanics, even if the probability of success is arbitrarily small. In the following, we will focus on the maximum fidelity in the asymptotic limit of large N and M.

Superreplication of equatorial qubit states. Let us start from the simple example of equatorial qubit states. In this case, the optimal probabilistic cloner is given by a quantum instrument with successful operation $\mathcal{M}_{\rm yes}(\rho) = Q\rho Q^\dagger$ given by

$$Q\,|N, \frac{N}{2}+m\rangle = \sqrt{\frac{\binom{M}{\frac{M}{2}+m}}{\binom{M}{\frac{M+N}{2}}\binom{N}{\frac{N}{2}+m}}}\,|M, \frac{M}{2}+m\rangle,$$
$$m \in \left[-\frac{N}{2}, \frac{N}{2}\right] \tag{9}$$

(recall that we are assuming that N and M are even for simplicity). Inserting the expression for the optimal cloner into the fidelity (8), we obtain the optimal value

$$F_{\rm prob}[N\to M] = \frac{1}{2^M}\sum_{m=-N/2}^{N/2}\binom{M}{\frac{M}{2}+m} \geqslant 1 - 2\exp\left(-\frac{N^2}{2M}\right), \tag{10}$$

where we have used Hoeffding's inequality. One can immediately see that any cloning rate $\alpha < 2$ is achievable; in this case, the cloning error vanishes faster than the inverse of any polynomial in N. Interestingly, such rapid decay of the error is impossible for deterministic cloners, whose error can vanish at most as $1/N^4$ [29]. In summary, allowing the cloner to claim failure yields two advantages: i) the cloning rate can be boosted beyond the limits of the asymptotic no-cloning theorem, and ii) the cloning error vanishes as N at a speed that could not be achieved by deterministic cloners. The contrast between deterministic and probabilistic cloners can also be seen in the non-asymptotic setting; for example, when $N = 20$ and $M = 120$, the probabilistic fidelity maintains the high value of 94.52%, whereas the deterministic fidelity drops to 69.39%.

The optimal probabilistic cloner dramatically outperforms all deterministic cloners in terms of rate and accuracy. However, these advantages have a cost; the price to pay is that the probability of success vanishes in the large-N limit. For example, the probability of success for the qubit cloner of Eq. (9) is given by

$$p_{\rm yes}[N \to M] = \frac{\left|\langle\psi_t|^{\otimes M} Q|\psi_t\rangle^{\otimes N}\right|^2}{|\langle\psi_t|^{\otimes N} Q^\dagger Q|\psi_t\rangle^{\otimes N}|^2} = \frac{1}{2^N \binom{M}{\frac{N+M}{2}}} \sum_{m=-N/2}^{N/2} \binom{M}{\frac{M}{2}+m}, \quad \forall t \in [0, 2\pi) .$$

Using Stirling's approximation, one can see that the above probability satisfies the asymptotic equality

$$p_{\rm yes}[N \to M] \approx {\rm e}^{-N\left(\ln 2 - \frac{N}{M}\right)}, \quad N \ll M .$$

This example indicates a trade-off between the cloning rate and the success probability. In the following, we will see that such a trade-off is a generic feature of state superreplication.

General case: superreplication of quantum clocks. Superreplication of equatorial qubit states can be generalized to the broad family of clock states of the form $|\psi_t\rangle = {\rm e}^{-{\rm i}tH}|\psi\rangle$ with $t \in \mathbb{R}$ and $H^\dagger = H$. For every family of clock states, one can find a probabilistic N-to-M cloner that achieves a fidelity [29]

$$F_{\rm prob}[N \to M] \geqslant 1 - 2K \exp\left(-\frac{2p_{\rm min}^2 N^2}{M} + \frac{4N}{MK}\right)$$

independently of t. Here, K is the number of distinct energy levels of the Hamiltonian H, and $p_{\rm min}$ is the smallest non-zero probability in the probability distribution of the energy for the state $|\psi\rangle$. It is immediately obvious that every cloning rate $\alpha < 2$ is achievable, and thus that clock states can be superreplicated. Again, the cloning error vanishes with N faster than the inverse of every polynomial—a rapid scaling that could not be achieved by deterministic cloners. In other words, the probabilistic cloner produces not only *more* but also *better* replicas.

The exponential decay of the success probability is a necessary condition for superreplication of clock states; indeed, every reliable superreplication process must have a success probability that is vanishingly small compared to $\exp[-M/N]$. In other words, as soon as a cloning process violates the asymptotic no-cloning theorem, the probability of the process must vanish in the large-N limit. A strong converse result can also be proven [29]:

Theorem 3 *Every cloning process with rate $\alpha \geqslant 1$ and success probability of order $\exp[-M(N)/N]$ or larger will necessarily have vanishing fidelity.*

On the other hand, it is possible to show that superreplication *can* be achieved with a success probability $p_{\rm yes} > \exp\left[-M/N^{1-\epsilon}\right]$ for every $\epsilon > 0$.

Informally, we can think of superreplication as a lottery where one invests the initial copies of the state in the hope of obtaining a much larger number of approximate copies. Despite the low probability of success, the superreplication lottery has almost no risk of loss; indeed, it is possible to guarantee that when replication fails, the cloner returns the N input copies, up to an error that is vanishingly small compared to $\exp[-M/(2N)]$. This result follows from the gentle measurement lemma [31]. Because the error is small, the input copies can still be used for deterministic cloning. Recall that the cloning error of deterministic cloners vanishes at most as $1/N^4$. Hence, the error introduced by the failed attempt at superreplication is negligible compared to the cloning error. In summary, one can achieve the same asymptotic performance as the optimal deterministic cloning while still allowing for the chance to observe the exotic superreplication phenomenon.

Building on this observation, we can construct a cloner that makes a series of attempts to copy clock states probabilistically, where every failed attempt results in slight deterioration of the input copies [45]. This cloner increases the probability of success, at the price of a performance that decreases with the number of attempts. In Fig. 1, we show a plot of the fidelity as a function of the total probability of success for equatorial qubit states with $N = 50$ and $M = 1000$. The plot shows a high fidelity/low probability region corresponding to the first attempts and a rapid decrease in the fidelity with increasing number of attempts.

The Heisenberg limit for superreplication. We

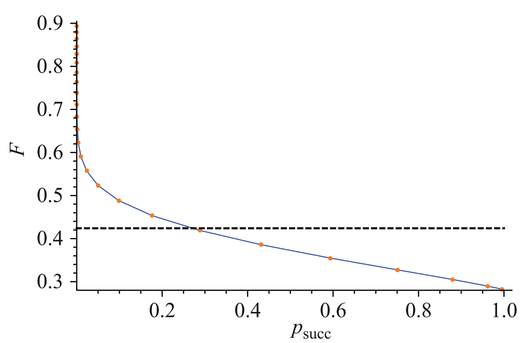


Fig. 1 Replication of equatorial qubit states through successive attempts of probabilistic cloning. The figure shows the tradeoff between fidelity and success probability for $N = 50$ and $M = 1000$. The points on the blue line represent successive attempts to generate M copies through the optimal probabilistic process. The black dashed line represents the fidelity of the optimal deterministic cloner. In this example, the errors introduced by failed attempts cause the fidelity to fall below the optimal deterministic fidelity when the probability of success becomes larger than 20%.

have seen that clock states can be superreplicated for every cloning rate smaller than two. Higher cloning rates are forbidden by the following theorem.

Theorem 4 (Heisenberg limit for superreplication [29]) *In finite dimensions, no physical process can achieve a cloning rate larger than two for a set of states containing clock states.*

The limit on the cloning rate originates in the Heisenberg limit in quantum metrology [30], which can be extended to probabilistic strategies in the case of finite-dimensional systems [29]. The intuitive idea is the following: the Heisenberg limit implies that the probabilistic estimation of t from the state $|\psi_t\rangle^{\otimes N}$ has a minimum mean square error scaling as N^{-2}. If one could produce $M = \Theta(N^\alpha)$ clones with sufficiently small error, these clones could be used to estimate t. Performing individual measurements of the clones and collecting the statistics, one could make the mean square error as small as $O(1/N^\alpha)$. Clearly, the Heisenberg limit implies $\alpha \leqslant 2$.

The argument based on the Heisenberg limit is partially heuristic because is relies on the assumption that the replication error vanishes sufficiently rapidly. However, it is possible to make a complete argument based on direct optimization of the probabilistic cloner. This argument is conceptually independent of the Heisenberg limit and makes it possible to prove a strong converse [29].

Theorem 5 (Strong converse of the Heisenberg limit for superreplication) *Every physical process that replicates clock states with replication rate $\alpha > 2$ will necessarily have vanishing fidelity, no matter how small its probability of success is.*

Note, however, that the restriction to finite-dimensional systems is essential for the above conclusions. For infinite-dimensional systems, the probabilistic Heisenberg bound can easily become trivial. For example, consider the set of all coherent states with fixed amplitude $r > 0$, which can be seen as an infinite-dimensional example of clock states:

$$|\psi_t\rangle = \mathrm{e}^{-\mathrm{i}t\, a^\dagger a}\,|\psi_0\rangle\,, \quad |\psi_0\rangle = \mathrm{e}^{-r^2/2}\sum_{n=0}^{\infty}\frac{r^n}{n!}\,|n\rangle\,,$$

where $a\,|n\rangle = \sqrt{n}\,|n-1\rangle$, $\forall n \in \mathbb{N}$. Now, coherent states with known amplitude can be probabilistically cloned with a fidelity arbitrarily close to 1 [23] by using Ralph and Lund's noiseless probabilistic amplifier [24]. This means that we can pick every function $M(N)$ and build a probabilistic cloner that transforms N input copies into $M(N)$ approximate copies with an error smaller than, say, $1/N$. The cloning process constructed in this way is reliable and can have *every* desired rate α, thus breaching the Heisenberg limit $\alpha < 2$. The catch is, of course, that the probability of success will decay faster for processes with a higher rate. Pandey *et al.* provide a general bound [46], implying that a replication process with rate α must satisfy the relation

$$p_{\mathrm{yes}}[N \to M] \leqslant \exp\left[-(\alpha-1)r^2 N^\alpha \ln N\right].$$

State superreplication beyond clock states. Superreplication of quantum clocks implies superreplication of other relevant families of states, including the multiphase covariant states

$$|\psi_{\boldsymbol{\theta}}\rangle = \sqrt{p_0}\,|0\rangle + \sum_{j=1}^{d-1}\sqrt{p_j}\mathrm{e}^{\mathrm{i}\theta_j}\,|j\rangle, \quad \theta_j \in [0,2\pi)\,,$$
$$\forall j = 1,\ldots,d-1\,.$$

To see that these states can be superreplicated, it is enough to consider the family of clock states $|\psi_t\rangle = \mathrm{e}^{-\mathrm{i}tH}\,|\psi\rangle$, where $|\psi\rangle = \sum_{j=0}^{d-1}\sqrt{p_j}\,|j\rangle$, $H = \sum_{j=0}^{d-1}\sqrt{n_j}\,|j\rangle\langle j|$, and n_j is the j-th prime number. With this choice, the closure of the set $\{|\psi_t\rangle \mid t \in \mathbb{R}\}$ coincides with the full set of multiphase covariant states, $\{|\psi_{\boldsymbol{\theta}}\rangle \mid \boldsymbol{\theta} \in [0,2\pi)^{\times d-1}\}$. On the other hand, an N-to-M cloner for the clock states will achieve a fidelity of

$$F_{\mathrm{prob}}[N \to M] \geqslant 1 - 2d\exp\left(-\frac{2p_{\min}^2 N^2}{M} + \frac{4N}{Md}\right),$$
$$p_{\min} \equiv \min_j p_j$$

independently of t. By taking limits, we then obtain that every multiphase covariant state can be cloned with the same fidelity.

Another multiparameter family of states that can be superreplicated is the family of all maximally entangled states. The superreplication of maximally entangled states will be discussed in detail in Section 5.

It is important to note that not all quantum states can be superreplicated. First, the symmetry of the input ensemble can sometimes inhibit superreplication. For instance, if one tries to clone an arbitrary unknown finite-dimensional state, the probabilistic cloner will yield the same fidelity as the optimal deterministic cloner [29]. The lack of probabilistic advantages for universal cloning is an appealing feature from the viewpoint of the many-worlds interpretation of quantum mechanics [47] and can even be viewed as an axiom:

Axiom 1 (Many-Worlds Fairness) *The maximum rate at which a completely unknown state can be cloned is the same in all possible worlds.*

Interestingly, Many-Worlds Fairness rules out the variant of quantum mechanics on real Hilbert spaces originally considered by Stueckelberg [48] and recently analyzed by Hardy and Wootters [49, 50]. The argument for excluding quantum mechanics on real Hilbert spaces is the following: if the wavefunction had only real amplitudes, then one branch of the wavefunction would allow one to superreplicate every state of a quantum bit, whereas the other branches would not, in violation of Many-Worlds Fairness. The fact that ordinary quantum mechanics allows for complex amplitudes results in the Many-World Fairness property and in the impossibility of superreplicating completely unknown pure states.

An interesting open question is what are the probability distributions on the Bloch sphere that allow for superreplication, interpolating between the uniform distribution (for which superreplication is impossible) and a distribution concentrated on the equator (for which superreplication is possible). Similarly, it is interesting to wonder whether there exists a family of optimal cloners that interpolate between the optimal universal cloner and the optimal probabilistic cloner that achieves superreplication. Such interpolation between universal and non-universal cloning was known in the *deterministic* case [51], while the probabilistic case is still open.

Another limitation on superreplication can be observed for coherent states of a harmonic oscillator parametrized as

$$|z\rangle = \mathrm{e}^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{n!} |n\rangle , \quad z \in \mathbb{C} .$$

When z is chosen at random from a Gaussian distribution $p(z) = \lambda/\pi \, \mathrm{e}^{-\lambda |z|^2}$, $\lambda > 0$, the optimal probabilistic fidelity is given by [19]

$$F_{\text{prob}}[N \to M] = \begin{cases} \frac{(1+\lambda)N}{M}, & \lambda < \frac{M}{N} - 1, \\ 1, & \lambda \geqslant \frac{M}{N} - 1 . \end{cases} \tag{11}$$

From the above expression, we see that the fidelity vanishes for all replication rates $\alpha > 1$. This fact is in stark contrast to the situation for coherent states with *known* amplitude [23, 24], for which every rate is achievable.

Eq. (11) implies that superreplication with a rate $\alpha = 1$ is still possible, provided that λ is sufficiently large. For example, for $\lambda \geqslant 1$, one can probabilistically transform N copies into $M = 2N$ copies, thus violating the asymptotic no-cloning theorem. Here it is important to stress the role of prior information; in its absence ($\lambda = 0$), the fidelity of the optimal probabilistic cloner is equal to the fidelity of the optimal deterministic cloner originally proposed by Braunstein *et al.* [52].

3 The relation between state superreplication and state estimation

Superreplication via state estimation. A fundamental fact about quantum cloning is its asymptotic equivalence with state estimation [10–15]; when the number of output copies becomes large, the optimal cloning performance can be achieved by a classical cloning strategy that consists of measuring the input copies and using the outcome of the measurement as an instruction to prepare the output copies. In the standard setting, one fixes the number of input copies N and lets the number of output copies M go to infinity. For probabilistic cloning, this scenario has been analyzed by Gendra *et al.* [53], who proved the asymptotic equivalence between probabilistic cloning and probabilistic state estimation. For clock states that are dense in an f-dimensional manifold, Gendra *et al.* showed that the probabilistic fidelity decays as $F_{\text{prob}}[N \to M] \propto (N^2/M)^{f/2}$ for $M \gg N^2$, whereas the deterministic fidelity decays as $F_{\text{det}}[N \to M] \propto (N/M)^{f/2}$. An open question is whether probabilistic estimation also makes it possible to achieve superreplication. We now show that this is indeed the case.

Consider a classical cloning strategy for the states $\{|\psi_x\rangle \mid x \in X\}$ based on probabilistic estimation of the parameter x and on preparation of the state $|\psi_{\hat{x}}\rangle^{\otimes M}$, where $\hat{x}$ is the estimate. This is not the most general classical strategy, but it will be sufficient to establish the possibility of superreplication. The successful quantum operation is given by

$$\mathcal{M}_{\text{yes}}(\rho) = \sum_{\hat{x} \in X} |\psi_{\hat{x}}\rangle\langle\psi_{\hat{x}}|^{\otimes M} \, \mathrm{Tr}[P_{\hat{x}} \, \rho] , \tag{12}$$

where $\{P_x\}_{x \in X}$ are positive operators satisfying the normalization condition

$$\sum_{x \in X} P_x + P_? = I$$

for some operator $P_? \geqslant 0$ associated with the unsuccessful outcome of the estimation strategy. The fidelity of the strategy can be computed using Eq. (7), which gives

$$\begin{aligned} F_{\text{prob}}[N \to M] &= \sum_{x,\hat{x}} p(x|\text{yes}) \, p(\hat{x}|x) \, |\langle\psi_{\hat{x}}|\psi_x\rangle|^{2M} \\ &= \mathbb{E}(|\langle\psi_x|\psi_{\hat{x}}\rangle|^{2M}) , \end{aligned}$$

where $\mathbb{E}$ denotes the expectation with respect to the probability distribution $p(x, \hat{x}|\text{yes}) := p(x|\text{yes}) \, p(\hat{x}|x)$. Using the convexity of the function $f(y) = y^M$, we then obtain the inequality

$$\mathbb{E}(|\langle\psi_x|\psi_{\hat{x}}\rangle|^{2M}) \geqslant \left[\mathbb{E}(|\langle\psi_x|\psi_{\hat{x}}\rangle|^2)\right]^M ,$$

or equivalently

$$F_{\text{prob}}[N \to M] \geqslant (F_{\text{prob}}[N \to 1])^M ,$$

which means that the M-copy fidelity of our measure-and-prepare strategy is lower-bounded by the M-th power of the single-copy fidelity. Now, suppose that the probabilistic single-copy fidelity approaches one as $1/N^\beta$ for some $\beta > 0$, namely,

$$F_{\text{prob}}[N \to 1] \geqslant 1 - O\left(\frac{1}{N^\beta}\right) .$$

Then, Bernoulli's inequality yields the bound

$$F_{\text{prob}}[N \to M] \geqslant 1 - O\left(\frac{M}{N^\beta}\right) . \tag{13}$$

From the bound, it is clear that every replication rate $\alpha < \beta$ can be achieved; the fidelity converges to one as long as $M(N)/N^\beta$ vanishes in the large-N limit. In summary, we have proven the following.

Theorem 6 *Let $\mathsf{S} = \{U_x \mid x \in \mathsf{X}\}$ be a continuous set of gates. A probabilistic strategy that estimates the gates in S with fidelity larger than $1 - O(1/N^\beta)$ can be used to replicate the states in S at every rate $\alpha < \beta$ via a measure-and-prepare strategy.*

Theorem 6 applies to arbitrary sets of states and to probabilistic strategies with arbitrary constraints on the probability of success. For clock states, assuming no constraint on the probability of success, the single-copy fidelity of phase estimation exhibits the Heisenberg scaling $F_{\text{prob}}[N \to 1] = 1 - O\left(1/N^2\right)$ [54]. Hence, the classical strategy described above achieves superreplication for every replication rate $\alpha < 2$. Note, however, that the classical superreplication strategy has an error that vanishes with the power law $1/N^{2-\alpha}$, whereas the quantum superreplication strategy has an error that vanishes faster than every polynomial.

Deriving precision limits from superreplication. We have seen that the replication rate for clock states is determined by the Heisenberg limit in the probabilistic case and by the standard quantum limit in the deterministic case. On the other hand, we have also seen strong converse results (Theorems 2 and 5) that prove the optimality of the replication rates without invoking the precision limits of quantum metrology. In fact, the quantum metrology limits can be *derived* from our bounds on the replication rates. The argument proceeds by contradiction:

1) Suppose that one could violate the standard quantum limit using N copies of the clock state $|\psi_t\rangle$ to estimate the parameter t deterministically with an error scaling as $1/N^{1+\epsilon}$, $\epsilon > 0$. Then, Eq. (13) would imply that one can produce M clones with $M = \Theta(N^{1+\epsilon/2})$, thus violating Theorem 2.
2) Suppose that one could violate the Heisenberg limit of probabilistic metrology using N copies of the clock state $|\psi_t\rangle$ to estimate the parameter t probabilistically with an error scaling as $1/N^{2+\epsilon}$, $\epsilon > 0$. Then, Eq. (13) would imply that one can produce M clones with $M = \Theta(N^{2+\epsilon/2})$, thus violating Theorem 5.

In conclusion, we have shown a complete equivalence between the limits on the scaling of the error in quantum metrology and the limits on the replication rate set by Theorems 2 and 5.

4 Superreplication with reduced interaction size

The divide-and-clone approach. To realize the optimal N-to-M probabilistic cloner, a global interaction involving at least M systems is required. This can be seen in the example of the equatorial qubit cloner, which is defined as an evolution acting coherently on the basis of the Dicke states; cf. Eq. (9). Of course, the need for interactions among large numbers of systems makes it challenging to implement cloning. Here we investigate the possibility of reducing the scale of the interactions.

A simple strategy to reduce the interaction scale is a "divide-and-clone" strategy in which one divides the input copies into groups and performs optimal cloning on each group. Suppose that we divide the N input copies into groups of $N' := \Theta(N^\beta)$ copies. Applying the optimal N'-to-M' probabilistic cloner to each individual group, we then achieve a fidelity

$$F[N' \to M'] \geqslant 1 - 2K \exp\left[-\frac{2p_{\min}^2 (N')^2}{M'} + \frac{4N'}{M'K}\right]$$

for each group. The value of M' can be chosen in order to reach the desired replication rate; to obtain M copies overall, one needs M' to satisfy the condition $M = M'\,N/N'$. For a replication process producing $M = \Theta(N^{1+\delta})$ copies, the condition yields $M' = \Theta(N^{\delta+\beta})$.

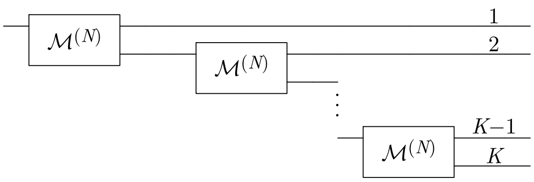
Because there are N/N' groups, the overall fidelity of this strategy is

$$\begin{aligned} F[N \to M] &= (F[N' \to M'])^{N/N'} \\ &\geqslant 1 - \Theta(N^{1-\beta}) \exp\left[-\Theta\left(N^{\beta-\delta}\right)\right] . \end{aligned}$$

From the above bound, it is immediately clear that the fidelity converges to one whenever $\beta > \delta$. Hence, quantum clocks can be superreplicated with interactions among $M' = O\left(N^{2\delta+\epsilon}\right) = O\left(M^2/N^{2-\epsilon}\right)$ particles for every desired $\epsilon > 0$. Now, recall that the original superreplication strategy [29] requires interactions among $O(M)$ particles. This means that the modified strategy reduces the interaction scale by a factor of $N^{2-\epsilon}/M$.

On the other hand, the modified strategy does not make it possible to achieve superreplication with interactions among less than $\Theta(N^\delta)$ systems. Indeed, superreplication can be achieved only if the fidelity of the individual processes approaches one. By the Heisenberg limit, this condition is satisfied only if N' and M' satisfy the asymptotic relation $M' \ll (N')^2$. Recalling that M' has to scale as $N^{\beta+\delta}$ and that N' scales as N^β, we then obtain that $\beta > \delta$ is a necessary condition. In summary, the strategy of dividing the input copies into non-interacting groups achieves superreplication with $M = \Theta(N^{1+\delta})$ output copies if and only if the size of each group grows faster than $N^\delta = M/N$.

A sequential cloning approach. Another approach to reducing the scale of interactions is to generate clones by local mechanisms. For example, we can imagine a cloning process that involves repeated interactions among $O(N)$ systems. Let us analyze this idea for equatorial qubit states; here we consider a sequence of cloners that take N to $2N$ copies arranged in the following circuit:



Each wire in the circuit represents a composite system of N qubits. At each step, the quantum operation $\mathcal{M}^{(N)}$ performs the optimal N-to-$2N$ probabilistic cloning given by the map

$$|N, \frac{N}{2}+m\rangle \longrightarrow \sqrt{\frac{\binom{2N}{N+m}}{\binom{2N}{\frac{3N}{2}}\binom{N}{\frac{N}{2}+m}}}\ |2N, N+m\rangle\ ,$$

$$\forall m \in \left[-\frac{N}{2}, \frac{N}{2}\right]\ ,$$

as one can see by replacing M with $2N$ in Eq. (9). In the following, it will be convenient to express the map as

$$|N, n\rangle \longrightarrow \sqrt{\frac{\binom{2N}{N/2+n}}{\binom{2N}{\frac{3N}{2}}\binom{N}{n}}}\,|2N, \frac{N}{2}+n\rangle\,, \quad \forall n \in [0, N]\ .$$

We now analyze the performance of the sequential cloner after $K-1$ steps, which result in the generation of KN approximate copies. At the first step, the input state is the N-copy state

$$|\Psi_{t,0}\rangle = \frac{1}{2^{N/2}}\sum_{n=0}^{N}\sqrt{\binom{N}{n}}\mathrm{e}^{-int}\ |N, n\rangle_1\ ,$$

where the subscript 1 indicates the first group of N qubits. The first cloner transforms $|\Psi_{t,0}\rangle$ into a state on the first and second group, yielding $2N$ qubits in the state

$$\begin{aligned}|\Psi_{t,1}\rangle &= N_1\sum_{n=0}^{N}\sqrt{\binom{2N}{N/2+n}}\ \mathrm{e}^{-int}\,|2N, \frac{N}{2}+n\rangle_{1,2}\\ &= N_1\sum_{n=0}^{N}\mathrm{e}^{-int}\sum_{x_1\in\mathsf{S}_{n,1}}\sqrt{\binom{N}{n-x_1}\binom{N}{N/2+x_1}}\\ &\quad\cdot|N, n-x_1\rangle_1|N, \frac{N}{2}+x_1\rangle_2,\end{aligned}$$

where N_1 is a normalization constant, and the set $\mathsf{S}_{n,1}$ is defined via the relation

$$\mathsf{S}_{n,K} = \left\{x := (x_1, \ldots, x_k)\ \middle|\ x_j \in \left[-\frac{N}{2}, \frac{N}{2}\right],\right.$$
$$\left.\sum_{l=1}^{j} x_l \in [n-N, n]\,, \quad \forall j \in \{1, \ldots, K\}\right\}. \qquad (14)$$

Iterating the cloning process for $K-1$ steps, we finally obtain K groups of qubits in the joint state

$$\begin{aligned}|\Psi_{t,K-1}\rangle &= N_{K-1}\sum_{n=0}^{N}\mathrm{e}^{-int}\sum_{\mathsf{x}\in\mathsf{S}_{n,K-1}} c_{n,x}\\ &\cdot|N, n-\sum_{i=1}^{K-1}x_i\rangle_1|N, \frac{N}{2}+x_1\rangle_2\cdots|N, \frac{N}{2}+x_{K-1}\rangle_K\ ,\end{aligned}$$

where $c_{n,x}$ is a coefficient given by

$$c_{n,x} := \sqrt{\binom{N}{n-\sum_{i=1}^{K-1}x_i}\prod_{j=1}^{K-1}\binom{N}{N/2+x_j}}.$$

The overall fidelity of the cloner with the state of NK perfect copies does not depend on t and is given by

$$\begin{aligned}F_{\mathrm{seq}}[N\to KN] &= \frac{1}{2^{NK}}\sum_{n=0}^{N}\sum_{x\in\mathsf{S}_{n,K-1}}\binom{N}{n-\sum_{i=1}^{K-1}x_i}\\ &\cdot\left[\prod_{j=1}^{K-1}\binom{N}{N/2+x_j}\right].\end{aligned} \qquad (15)$$

For $K > 2$, the cloning fidelity (15) is strictly smaller than the fidelity (8) of the optimal probabilistic cloner. Fig. 2 compares the sequential cloner, optimal

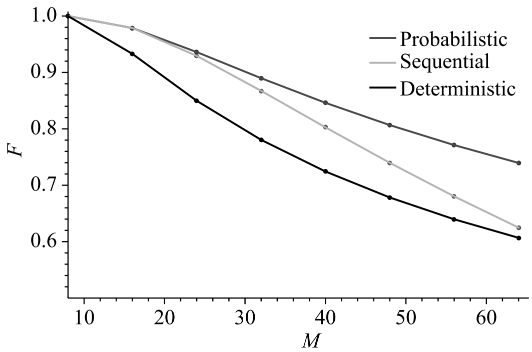


Fig. 2 Comparison between the three cloners. In this figure we compare the performance of the network cloner to the one of the probabilistic cloner as well as the one of the deterministic cloner. The fidelity is plotted as a function of the output number, fixing the input number to $N = 8$. The green line (with numerics represented by red dots) represents the fidelity-output curve for the network cloner. The blue line (with numerics represented by red dots) stands for the fidelity-output curve for the probabilistic cloner, and the black line (with numerics represented by black dots) stands for the fidelity-output curve for the deterministic cloner.

probabilistic cloner, and optimal deterministic cloner for $N = 8$. Note that the fidelity of the sequential cloner decays quickly with the number of cloning steps. In our example, the fidelity of the sequential cloner approaches that of the deterministic cloner as the number of output copies approaches $M = N^2 = 64$. This numerical result suggests that it may not be possible to achieve superreplication sequentially. However, it is an open question whether superreplication can be achieved by the the sequential cloner or by similar mechanisms. For example, one could construct a quantum cellular automaton for cloning, as proposed by D'Ariano *et al.* [55]. They considered a tree-shaped network of deterministic 1-to-2 cloners. To achieve superreplication, one would have to extend the model, allowing for probabilistic cloners. Moreover, symmetry arguments imply that probabilistic 1-to-2 cloners do not offer any advantage over their deterministic counterparts [29]. Hence, one has to consider local cloners with more input copies. An interesting possibility is to construct a probabilistic cellular automaton where the building blocks are the optimal k-to-$2k$ probabilistic cloners with $k > 1$.

5 Quantum gate superreplication

We have seen that state superreplication necessarily has a vanishing success probability in the asymptotic limit. Here we analyze the task of replicating quantum gates, where superreplication can be achieved with unit probability of success on average over all input states.

Gate superreplication. In many applications, including quantum metrology and quantum algorithms, one is given access to a black box implementing an unknown quantum gate. In these applications, the uses of the gate are a resource; indeed, a no-cloning theorem for gates asserts that it is impossible to perfectly simulate two uses of an unknown gate by using it only once [32]. Still, a natural question is: how well can we simulate M uses of the unknown gate with $N < M$ uses? Here the problem is to engineer a quantum computational network that uses the unknown gate as a subroutine, as in Fig. 3. In analogy with superreplication of quantum states, we say that superreplication of quantum gates is possible if and only if one can find a sequence of networks with the property that the simulation error vanishes in the asymptotic limit and the number of extra copies of the input gate grows as N or faster. In the following, we will see that gate superreplication can be achieved deterministically for almost all input states. The first result of this type was discovered by Dür *et al.* [33] for phase gates, that is, gates of the form

$$U_t = \sum_{n=0}^{d-1} \mathrm{e}^{-\mathrm{i}\,nt}\,|n\rangle\langle n|\,, \quad \forall t \in [0, 2\pi)\,.$$

The performance of the simulation was quantified by the fidelity between the output state of the simulation and the output state of M perfect uses of the gate U_t, averaged over t and over all possible input states. Using a deterministic network, the authors showed how to simulate $M \ll N^2$ uses with asymptotically unit fidelity. This result can be extended from phase gates to arbitrary gates [34], as discussed in the following paragraphs.

Universal gate superreplication. Given N uses of a unitary gate, the goal is to simulate M parallel uses of the gate. Let us denote by $\mathcal{C}_U^{(N)}$ the quantum channel that results from inserting N uses of the unknown gate in the cloning network. With this notation, the average fidelity is given by

$$F_{\rm gate}[N \to M] = \int \mathrm{d}U \int \mathrm{d}\,\Psi \langle \Psi|(U^\dagger)^{\otimes M}\mathcal{C}_U^{(N)}\left(|\Psi\rangle\langle\Psi|\right)U^{\otimes M}|\Psi\rangle\,, \tag{16}$$

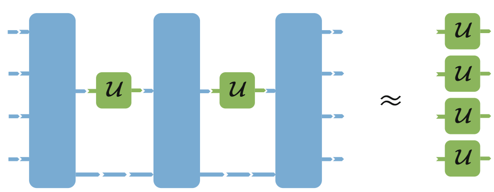


Fig. 3 Quantum gate cloning. A quantum network for gate cloning. Given N uses of an unknown unitary gate $\mathcal{U}$ (green boxes on the left), the network (blue boxes on the left) simulates $M > N$ uses of $\mathcal{U}$ (green boxes on the right).

where $\mathrm{d}U$ is the normalized Haar measure over the group of all unitary gates, and $\mathrm{d}\Psi$ is the normalized Haar measure over the manifold of pure M-partite states. By Markov's inequality, a simulation with gate fidelity $F_{\text{gate}}[N \to M] \geqslant 1 - \epsilon$ works with fidelity $F \geqslant 1 - \delta$ on all pure states except for a small fraction, whose probability is smaller than ϵ/δ.

A replication process is described as a sequence of networks in which the N-th network transforms N copies of the unknown gate into $M(N)$ approximate copies. We say that the replication process is *reliable* if and only if

$$\lim_{N\to\infty} F_{\text{gate}}[N \to M(N)] = 1 .$$

In other words, for a reliable replication process, one has $F[N \to M(N)] \geqslant 1 - \epsilon_N$ for some ϵ_N converging to zero. Choosing $\delta_N = \sqrt{\epsilon_N}$, we then have that the replication process simulates the desired gate with fidelity larger than $1 - \epsilon_N$ on all pure states except for a small fraction, whose probability is smaller than $\epsilon_N/\delta_N = \sqrt{\epsilon_N}$. As in the case of state replication, we say that a process has a *replication rate*

$$\alpha = \liminf_{N\to\infty} \frac{\log[M(N) - N]}{\log N} ,$$

and we say that the rate α is *achievable* if and only if there exists a reliable replication process with that rate.

In the following, we will see that every rate smaller than 2 is achievable. For simplicity, we discuss the case of qubit gates, although the protocol can be extended in a straightforward way to unitary gates in arbitrary finite-dimensional quantum systems.

The key to construction of the universal gate cloning network is decomposition of the Hilbert space of K qubits into orthogonal subspaces corresponding to different values of the total angular momentum. In the formula, one has

$$\mathcal{H}^{\otimes K} \simeq \bigoplus_{j=0}^{K/2} (\mathcal{R}_j \otimes \mathcal{M}_{jK}) ,$$

where j is the quantum number of the total angular momentum, $\mathcal{R}_j$ is a representation space, and $\mathcal{M}_j$ is a multiplicity space [56]. With respect to this decomposition, the action of K parallel uses of a generic qubit unitary U has the block-diagonal form

$$U^{\otimes K} \simeq \bigoplus_{j=0}^{K/2} (U_j \otimes I_{jK}) ,$$

where I_{jK} is the identity on the multiplicity space $\mathcal{M}_{jK}$.

Note that $U^{\otimes K}$ acts nontrivially only in the representation spaces. This means that essentially, the multiplicity spaces can be eliminated without losing any information about U. Likewise, they can be inflated by adding ancillas. The working principle for simulation of $U^{\otimes M}$ given $U^{\otimes N}$ will be to conveniently adjust the size of the multiplicity spaces, as in the following protocol.

Protocol 1 (Gate superreplication network [34]) *The replication network is constructed as follows:*

1) *Choose an ancilla A such that the rank of $I_{jN} \otimes I_A$ is no smaller than the rank of I_{jM} for every $j \in [0, N/2]$;*
2) *Compose the gate $U^{\otimes N}$ with the identity on A, thus obtaining the gate $U^{\otimes N} \otimes I_A = \bigoplus_{j=0}^{N/2}(U_j \otimes I_{jN} \otimes I_A)$;*
3) *Project the resulting gate $U^{\otimes N} \otimes I_A$ into the subspace*

$$\mathcal{H}_N = \bigoplus_{j=0}^{N/2} (\mathcal{R}_j \otimes \mathcal{M}_{jM}) \subset \mathcal{H}^{\otimes N} \otimes \mathcal{H}_A,$$

where the gate acts as

$$U' = \bigoplus_{j=0}^{N/2} (U_j \otimes I_{jM}) .$$

4) *Embed the subspace $\mathcal{H}_N$ into the Hilbert space $\mathcal{H}^{\otimes M}$. With a slight abuse of notation, we use $\mathcal{H}_N$ both for the subspace of $\mathcal{H}^{\otimes N} \otimes \mathcal{H}_A$ and for the corresponding subspace of $\mathcal{H}^{\otimes M}$.*
5) *Given an input state of M qubits, perform a projective measurement that projects the qubits either into the subspace $\mathcal{H}_N$ or into its orthogonal complement $\mathcal{H}_N^{\perp}$. If the qubits are projected into $\mathcal{H}_N$, then apply the gate U'. If the qubits are projected into $\mathcal{H}_N^{\perp}$, then perform the identity.*

The network described by Protocol 1 perfectly imitates $U^{\otimes M}$ for any state in the subspace $\mathcal{H}_N \subset \mathcal{H}^{\otimes M}$. Now, it is possible to show that a random pure state in $\mathcal{H}^{\otimes M}$ has a large overlap with the subspace $\mathcal{H}_N$ whenever $M \ll N^2$. Indeed, we have the following lemma.

Lemma 1 *Let F_N be the fidelity between the pure state $|\Psi\rangle \in \mathcal{H}^{\otimes M}$ and its projection $|\Psi_N\rangle = P_N|\Psi\rangle/\|P_N|\Psi\rangle\|$. The expectation value of the fidelity over the uniform measure is given by*

$$\begin{aligned}\mathbb{E}(F_N) &= \int \mathrm{d}\Psi \, \langle\Psi|P_N|\Psi\rangle \\ &= \frac{\mathrm{Tr}[P_N]}{2^M} \\ &= \sum_{j=0}^{N/2} \frac{d_j m_{jM}}{2^M}\end{aligned}$$

$$\geqslant 1 - (M+1)\ \exp\left[-\frac{N^2}{2M}\right].$$

In the above inequality, d_j and m_{jM} are the dimensions of the representation space $\mathcal{R}_j$ and the multiplicity space $\mathcal{M}_{jM}$, respectively. The detailed proof of the last inequality can be found in the supplemental material of Ref. [34]. Using the above lemma, it is easy to show that the fidelity is close to one whenever M is small compared to N^2. Indeed, the cloning fidelity (16) is lower-bounded as

$$\begin{aligned}
&F_{\mathrm{gate}}[N \to M] \\
&\geqslant \int \mathrm{d}U \int \mathrm{d}\,\Psi \langle \Psi | (U^\dagger)^{\otimes M} U' P_N | \Psi \rangle \langle \Psi | P_N U'^\dagger U^{\otimes M} | \Psi \rangle \\
&= \int \mathrm{d}\,\Psi\ (\langle \Psi | P_N | \Psi \rangle)^2 \\
&\geqslant \left(\int \mathrm{d}\,\Psi\ \langle \Psi | P_N | \Psi \rangle \right)^2 \\
&= [\mathbb{E}(F_N)]^2 \\
&\geqslant 1 - 2(M+1)\ \exp\left[-\frac{N^2}{2M}\right].
\end{aligned}$$

In conclusion, the gate fidelity converges to one for all replication processes with replication rate $\alpha < 2$, establishing the possibility of gate superreplication. Similar bounds can be found for arbitrary d-dimensional systems, in which case we have

$$F_{\mathrm{gate}}[N \to M] \geqslant 1 - 2(M+1)^{\frac{d(d-1)}{2}} \exp\left[-\frac{N^2}{2M}\right], \quad \forall d < \infty. \tag{17}$$

The inequality follows from the concentration of the Schur–Weyl measure [57, 58], which has applications in quantum estimation [59] and information compression [60, 61]. The bound (17) allows one to produce $M = \Theta(N^{2-\epsilon})$ copies with an error that vanishes superpolynomially fast. In addition, one can produce $M = \Theta[N^2/\log N]$ copies with an error vanishing faster than $1/N^k$ for every desired $k > 0$. For this purpose, it is enough to choose

$$M = \left\lfloor \frac{N^2}{2(d^2 - d + k)\ \log N} \right\rfloor$$

so that Eq. (17) becomes

$$F_{\mathrm{gate}}[N \to M] \geqslant 1 - O\left(\frac{1}{N^k\ \log N^{d(d-1)/2}} \right), \quad N \gg d. \tag{18}$$

Finally, we note that the size of the interactions used in the superreplication network can be reduced from $O(M)$ qubits to $O(M/N)$ qubits by dividing the N input gates into groups of $O(N^2/M)$ gates. This fact can be proven by the same argument as that used for quantum states.

6 Applications of gate superreplication

Superreplication of maximally entangled states. Inspired by gate superreplication, one can construct a protocol for superreplicating maximally entangled states that achieves any cloning rate $\alpha < 2$ [34]. For qubits, we can cast an arbitrary maximally entangled state in the form $|\Phi_U\rangle = (U \otimes I)|\Phi^+\rangle$, where $|\Phi^+\rangle$ is the Bell state, $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. The replication protocol works as follows: First, the unitary gate U can be extracted from the state via a gate teleportation protocol [62]. The success probability of a single gate extraction is 1/4, which implies that $U^{\otimes N}$ can be extracted from N copies of the maximally entangled state with probability $(1/4)^N$. The extracted gates are then used in the gate superreplication network, which simulates the gate $U^{\otimes M}$ with vanishing error whenever M grows as N^α, $\alpha < 2$. Finally, the approximation of $U^{\otimes M}$ is applied locally to M copies of the Bell state $|\Phi^+\rangle$, thus providing M approximate copies of the state $|\Phi_U\rangle$. The fidelity of the replicas can be bounded as

$$\begin{aligned}
F_{\mathrm{ent}}[N \to M] &= \int \mathrm{d}U\ \langle \Phi^+|^{\otimes M} (U^\dagger \otimes I)^{\otimes M} \\
&\cdot \left(\mathcal{C}_U^{(N)} \otimes \mathcal{I} \right) \left(|\Phi^+\rangle\langle \Phi^+|^{\otimes M} \right) (U \otimes I)^{\otimes M} |\Phi^+\rangle^{\otimes M} \\
&= \frac{(2^M + 1) F_{\mathrm{gate}}[N \to M] - 1}{2^M}
\end{aligned}$$

where the relationship between the entanglement fidelity and the gate fidelity derived by Horodecki *et al.* [63] was used. The above equality implies that the fidelity of the state cloning protocol goes to one if and only if the fidelity of gate superreplication goes to one. Therefore, the above protocol can replicate maximally entangled states at every rate $\alpha < 2$.

The idea of gate teleportation followed by gate superreplication can also be applied to achieve superreplication of other families of states. For instance, gate superreplication yields an alternative superreplication protocol for equatorial qubit states. Given N copies of an equatorial qubit state $(|0\rangle + \mathrm{e}^{-\mathrm{i}t}|1\rangle)/\sqrt{2}$, one can first generate N copies of the maximally entangled qubit state $(|00\rangle + \mathrm{e}^{-\mathrm{i}t}|11\rangle)/\sqrt{2}$ by applying a CNOT gate to each input copy. Then, the maximally entangled qubit state can be superreplicated. Finally, applying a CNOT gate to each output copy makes it possible to transform the (approximate) copies of the maximally entangled qubit state into (approximate) copies of the equatorial qubit state. The net result of this protocol is superreplication of equatorial qubit states.

Similar arguments apply to superreplication of maximally entangled states in arbitrary finite dimensions. Every maximally entangled state can be parametrized as $|\Phi_U\rangle = (U \otimes I)|\Phi^+\rangle$, where $|\Phi^+\rangle = 1/\sqrt{d}\sum_{n=0}^{d-1}|n\rangle|n\rangle$. Again, the unitary gate U can be probabilistically extracted from the state $|\Phi_U\rangle$ with probability $p = 1/d^2$. Hence, maximally entangled states can be superreplicated by *i)* simulating M copies of the gate U and *ii)* applying the simulated gates locally to M copies of the state $|\Phi^+\rangle$. The fidelity is then given by

$$\begin{aligned} F_{\mathrm{ent}}[N \to M] &= \int \mathrm{d}U \ \langle \Phi^+|^{\otimes M}(U^\dagger \otimes I)^{\otimes M} \\ &\cdot \left(\mathcal{C}_U^{(N)} \otimes \mathcal{I}\right)\left(|\Phi^+\rangle\langle\Phi^+|^{\otimes M}\right)(U \otimes I)^{\otimes M}|\Phi^+\rangle^{\otimes M} \\ &= \frac{(d^M+1)F_{\mathrm{gate}}[N \to M] - 1}{d^M}, \end{aligned} \tag{19}$$

where the relationship between the entanglement fidelity and the gate fidelity [63] is again used. In conclusion, the cloning fidelity for maximally entangled states approaches one if and only if the cloning fidelity for unitary gates approaches one. Hence, superreplication of quantum gates implies (probabilistic) superreplication of maximally entangled states. In turn, superreplication of maximally entangled states implies superreplication of the uniform-weight multiphase states

$$|\psi_{\boldsymbol{\theta}}\rangle = \frac{1}{\sqrt{d}}\left(|0\rangle + \sum_{j=1}^{d-1} \mathrm{e}^{\mathrm{i}\theta_j}|j\rangle\right),$$
$$\theta_j \in [0, 2\pi), \quad \forall j = 1, \ldots, d-1,$$

which are unitarily equivalent to the maximally entangled states $|\Phi_{\boldsymbol{\theta}}\rangle = \frac{1}{\sqrt{d}}\left(|0\rangle|0\rangle + \sum_{j=1}^{d-1}\mathrm{e}^{\mathrm{i}\theta_j}|j\rangle|j\rangle\right)$.

Upper bound on the rate of gate replication. The relationship between gate replication and replication of maximally entangled states can be used to derive the ultimate limit on the rates of gate replication in finite dimensions. The argument proceeds by contradiction: Suppose that it is possible to devise a network that simulates $\Theta(N^\alpha)$ uses of U for $\alpha \geqslant 2$ with vanishing error. Then we could use it to superreplicate maximally entangled states at a rate $\alpha \geqslant 2$. Now, the family of maximally entangled states contains the family of clock states

$$|\Phi_t\rangle = (U_t \otimes I)\,|\Phi_+\rangle, \quad U_t = \sum_n \mathrm{e}^{-\mathrm{i}nt}\,|n\rangle\langle n|.$$

Replication of these states at a rate $\alpha \geqslant 2$ would contradict Theorem 5, which implies that every cloner with a rate $\alpha \geqslant 2$ must have vanishing fidelity. Hence, the possibility of achieving gate superreplication at rates larger than quadratic is excluded. Note that the above argument applies not only to deterministic gate replication networks, but also to probabilistic networks. Using the correspondence between entanglement fidelity and gate fidelity, we obtain the following.

Theorem 7 *Every physical process that replicates phase gates with a replication rate $\alpha > 2$ will necessarily have vanishing gate fidelity (no matter how small the probability of success).*

An alternative optimality proof for the quadratic replication rate was presented by Sekatski *et al.* [64], who devised an argument to bound the replication fidelity based on the no-signaling principle.

Supergeneration of maximally entangled states. Gate superreplication can be achieved deterministically, whereas state superreplication can be achieved only with a vanishing probability. This sharp difference originates in the fact that states and gates are inequivalent resources; whereas gates can be used to deterministically generate states as $|\psi_U\rangle = U|0\rangle$, the converse process is forbidden by the no-programming theorem [65]. It is then useful to distinguish between the task of state cloning, where the input consists of N copies of the state $|\psi_U\rangle = U|0\rangle$, and the task of *state generation*, where the input consists of N copies of the gate U. Deterministic gate superreplication cannot be used to achieve deterministic state superreplication, but it can be used to achieve deterministic state *supergeneration*, that is, the generation of up to N^2 almost perfect copies of the quantum state $|\psi_U\rangle$ from N copies of the gate U. A general supergeneration protocol works as follows:

1) Use a gate superreplication protocol to simulate $M \ll N^2$ uses of the gate U.
2) Apply the simulated gates to the state $|0\rangle^{\otimes N}$.

In general, the protocol can be tailored to the specific set of states that one wants to generate. For example, one could have *i)* clock states, where U is of the form $U = e^{-itH}$, *ii)* maximally entangled states, where U is of the product form $U = V_A \otimes I_B$ with respect to some bipartition of the Hilbert space, and *iii)* arbitrary pure states, where U is a generic unitary gate.

The fidelity of supergeneration depends on the protocol used to replicate the gates. For example, the phase gate replication by Dür *et al.* makes it possible to supergenerate clock states [33], whereas our universal gate replication makes it possible to supergenerate maximally entangled states of bipartite systems [34]. Interestingly, universal gate replication does not work for the set of all pure states parametrized as

$$\{|\psi_U\rangle = U|0\rangle \mid U \in \mathsf{SU}(d)\,, |0\rangle \in \mathcal{H}\}\,.$$

In this case, the approach of simulating M uses of the gate U and applying it to the state $|0\rangle^{\otimes N}$ does not work because the state $|0\rangle^{\otimes N}$ lies in the subspace $\mathcal{H}_N^{\perp}$, where our gate superreplication network fails. However, we will see below that arbitrary pure states can be supergenerated via a suitable protocol based on gate estimation.

Gate estimation with quasi-Heisenberg scaling. Supergeneration of maximally entangled states has an elegant application to quantum metrology. Specifically, it allows for an easy proof of the fact that an unknown quantum gate can be estimated with an error scaling as $\log N/N^2$. Here the error is defined as $\langle e\rangle_N = 1 - F_{\rm est,gate}[N]$, where $F_{\rm est,gate}[N]$ is the fidelity of gate estimation with N copies, namely,

$$F_{\rm est,gate}[N] = \int {\rm d}U \int {\rm d}\hat{U}\, p_N(\hat{U}|U)\, F_{\rm gate}(\hat{U}, U)\,, \quad (20)$$

where $p_N(\hat{U}|U)$ is the probability distribution resulting from the estimation, and $F_{\rm gate}(\hat{U}, U)$ is the gate fidelity, which is defined as

$$F_{\rm gate}(\hat{U}, U) := \int {\rm d}\psi \left|\langle\psi|U^\dagger\hat{U}|\psi\rangle\right|^2 .$$

We now show that gate superreplication can be used to achieve estimation fidelity scaling as

$$F_{\rm est,gate}[N] \geqslant 1 - O\left(\frac{\log N}{N^2}\right) . \quad (21)$$

The proof goes as follows: given N copies of the gate U, we can produce $M = \Theta(N^2/\log N)$ copies of the maximally entangled state $|\Phi_U\rangle$, with an error vanishing faster than $1/N^k$ for every desired $k > 0$; cf. Eq. (18). In particular, we set $k = 2$. With this choice, the error vanishes faster than $1/M$, and the M approximate copies can be used for state estimation, resulting in an estimate of the maximally entangled state with fidelity

$$F_{\rm est,ent}[M] \geqslant 1 - O\left(\frac{1}{M}\right)$$

according to the central limit theorem. Here, the estimation fidelity is defined as

$$F_{\rm est,ent}[M] = \int {\rm d}U \int {\rm d}\hat{U}\, p_M(\Phi_{\hat{U}}|\Phi_U) \left|\langle\Phi_{\hat{U}}|\Phi_U\rangle\right|^2 ,$$

where $p_M(\Phi_{\hat{U}}|\Phi_U)$ is the probability distribution resulting from estimation of a maximally entangled state with M nearly perfect copies. Now, we regard estimation of the maximally entangled state $|\Phi_U\rangle$ with M copies as a particular strategy for estimation of the unitary gate U with N copies, meaning that we have

$$p_N(\hat{U}|U) \equiv p_M(\Phi_{\hat{U}}|\Phi_U), \quad M = \Theta(N^2/\log N)\,.$$

Then, the estimation fidelity for the maximally entangled state can be easily converted into the gate fidelity for the corresponding gate using the relation [63]

$$F_{\rm est,gate}[N] = \frac{(d+1)\,F_{\rm est,ent}[M] - 1}{d}\,.$$

Because the state estimation fidelity converges to one as $1/M$, the gate estimation fidelity will also converge to one as $1/M$. Recalling that M scales as $N^2/\log N$, this proves Eq. (21). The error scaling $\langle e\rangle = \log N/N^2$ beats the central limit scaling of classical statistics and is close to the optimal quantum scaling $1/N^2$, which was derived in Refs. [35–37] for qubits and in Ref. [38] for general d-dimensional systems. The usefulness of our new derivation is that the proof is much simpler than full optimization of the estimation strategy.

Interestingly, for $d = 2$, an alternative estimation strategy achieving scaling by $\log N^2/N^2$ was proposed by Rudolph and Grover [66]. Their protocol is sequential and uses unentangled states to reach a quasi-Heisenberg scaling. It is an open question whether a similar protocol exists for $d > 2$.

Universal supergeneration of pure states. We now show that all pure states can be supergenerated. To this purpose, we parametrize the manifold of pure states as $\{|\psi_U\rangle = U\,|0\rangle \mid U \in \mathsf{SU}(d)\}$, where $|0\rangle$ is a fixed pure state. To achieve supergeneration, we use a classical strategy based on estimation of the unknown gate U and on preparation of the state $|\psi_{\hat{U}}\rangle^{\otimes M}$ conditional on the estimate $\hat{U}$. The fidelity of this strategy is given by

$$F_{\rm pure}[N \to M] = \int {\rm d}U \int {\rm d}\hat{U}\, p_N(\hat{U}|U)\, |\langle\psi_{\hat{U}}|\psi_U\rangle|^{2M} ,$$

where $p_N(\hat{U}|U)$ is the probability distribution resulting from gate estimation. The above choice of strategy implies that we have the bound

$$F_{\rm pure}[N \to M] \geqslant (F_{\rm pure}[N \to 1])^M . \quad (22)$$

Now, note that the single-copy fidelity satisfies the relation

$$\begin{aligned} &F_{\rm pure}[N \to 1] \\ &\quad = \int {\rm d}V \int {\rm d}U \int {\rm d}\hat{U}\, p_N(\hat{U}V|UV)\, |\langle\psi_{\hat{U}V}|\psi_{UV}\rangle|^2 \\ &\quad = \int {\rm d}V \int {\rm d}U \int {\rm d}\hat{U}\, p_N(\hat{U}|U)\, |\langle\psi_{\hat{U}V}|\psi_{UV}\rangle|^2 \\ &\quad = \int {\rm d}V \int {\rm d}U \int {\rm d}\hat{U}\, p_N(\hat{U}|U)\, |\langle\psi_V|\,\hat{U}^\dagger U\,|\psi_V\rangle|^2 \end{aligned}$$

$$= \int \mathrm{d}U \int \mathrm{d}\hat{U}\, p_N(\hat{U}|U)\, F_{\mathrm{gate}}(\hat{U}, U)$$
$$= F_{\mathrm{est,gate}}[N]\,, \tag{23}$$

where $F_{\mathrm{est,gate}}[N]$ is the gate estimation fidelity defined in Eq. (20), and in the second equality we used the fact that the optimal gate estimation strategy is covariant [67], i.e., it satisfies the condition $p_N(\hat{U}V, UV) = p_N(\hat{U}, U)$ for every $\hat{U}, U, V \in \mathsf{SU}(d)$.

Combining Eq. (23) with the bounds (22) and (21), we finally obtain

$$F_{\mathrm{pure}}[N \to M] \geqslant 1 - O\left(\frac{M \log N}{N^2}\right),$$

which means that arbitrary pure states can be supergenerated at every rate smaller than quadratic. Because the above protocol is based on estimation, the error vanishes only as a power law. It is an open question whether there exists a universal quantum supergeneration protocol whose error vanishes faster than any inverse polynomial.

Equivalence between gate superreplication and gate estimation. As in the case of states, superreplication can be achieved by gate estimation. The argument is the same as that used to derive Eq. (13); in short, one can show that every estimation strategy with fidelity scaling as $F_{\mathrm{est,gate}}[N] = 1 - O(1/N^{\beta})$ can be used to achieve gate replication with fidelity

$$F_{\mathrm{gate}}[N \to M] \geqslant 1 - O(M/N^{\beta})\,, \tag{24}$$

which means that every replication rate smaller than β can be achieved. In particular, we know from Eq. (21) that we can choose $\beta = 2 - \epsilon$. Hence, every replication rate smaller than quadratic can be achieved via state estimation. Note, however, that the replication error goes to zero only as a power law, whereas the quantum replication network has much better scaling of the error.

The connection between gate superreplication and estimation can be used to prove the Heisenberg limit [30]. The proof is by contradiction: Suppose that one could estimate the parameters of a unitary gate with a fidelity scaling as $F_{\mathrm{est,gate}}[N] = 1 - O(1/N^{2+\epsilon})$. Then, Eq. (24) would imply that one can simulate $M = O\left(N^{2+\epsilon/2}\right)$ copies of the gate, in contradiction to Theorem 7. In summary, the quadratic scaling of the Heisenberg limit can be derived solely from considerations regarding the optimal rates of gate superreplication.

Asymptotic no-cloning theorem for quantum gates in the worst-case scenario. We have seen that gate superreplication can be achieved on all input states except for a vanishingly small fraction. A natural question is whether one can find a superreplication protocol that works on *all* input states. The problem can be addressed by evaluating the worst-case fidelity

$$F_{\mathrm{gate,worst}}[N \to M] = \min_{|\Psi\rangle \in \mathcal{H}^{\otimes M}, \||\Psi\rangle\|=1} \int \mathrm{d}U\, \langle \Psi|(U^{\dagger})^{\otimes M} \mathcal{C}_U^{(N)}\left(|\Psi\rangle\langle\Psi|\right) U^{\otimes M}|\Psi\rangle\,, \tag{25}$$

where $\mathcal{C}_U^{(N)}$ is the channel implemented by the replication network. In this setting, we say that a gate replication process is *reliable in the worst case* if and only if

$$\lim_{N\to\infty} F_{\mathrm{gate,worst}}[N \to M(N)] = 1,$$

and we say that the replication rate α is *achievable in the worst case* if and only if there exists a replication protocol that has that rate and is reliable in the worst case. The supremum over all achievable rates is determined as follows.

Theorem 8 (Asymptotic no-cloning theorem for quantum gates) *For finite-dimensional quantum systems, no physical process can reliably replicate phase gates at a rate $\alpha \geqslant 1$ in the worst-case scenario.*

Here we give a heuristic proof based on optimal phase estimation. When an unknown phase gate is used N times, the optimal estimation strategy has a mean square error c/N^2 for some suitable constant $c > 0$ [68, 69]. The bound holds for both deterministic and probabilistic strategies [70], and the fact that the optimal estimation strategy can be achieved deterministically plays a crucial role in our argument. Now, suppose that one can produce M reliable replicas up to an error scaling as $1/M^{\beta}$ for some $\beta > 2$. In this case, one could use the replicas for phase estimation, reducing the error to

$$\langle e \rangle = \frac{c}{M^2} + O\left(\frac{1}{M^{\beta}}\right).$$

This result follows from the fact that the error of a deterministic estimation strategy is a continuous function of the input state. Because the above strategy cannot be better than the optimal strategy, we obtain the inequality

$$\frac{c}{N^2} \leqslant \frac{c}{M^2} + O\left(\frac{1}{M^{\beta}}\right), \tag{26}$$

which implies that M can grow at most as $M = N + \Theta(N^{\alpha})$ with $\alpha < 1$. In addition, by Taylor-expanding the r.h.s. of Eq. (26), we obtain the inequality $\beta < 3 - \alpha$, so the error can vanish at most as $1/N^3$. In summary, superreplication and exponentially vanishing errors are forbidden in the worst-case scenario.

7 Conclusion and outlook

In this paper, we reviewed the phenomenon of superreplication of quantum states and gates, emphasizing the applications and connections with other tasks in quantum information processing. In particular, we clarified the relation between superreplication and the precision limits of quantum metrology, showing that i) estimation can be used to achieve superreplication with an error vanishing as $O(M/N^2)$, probabilistically in the case of states and deterministically in the case of gates, ii) gate superreplication allows for a simpler proof of the quadratic precision enhancement in the estimation of an unknown gate, and iii) the optimality of Heisenberg scaling for estimation of unitary gates can be derived from the ultimate limit on the rate of superreplication. In addition, we showed that N uses of a completely unknown gate are sufficient to generate $O(N^2)$ approximate copies of the corresponding pure state with an error that vanishes in the large-N limit.

Among the future research directions, an important one is the study of replication processes for mixed states and noisy channels. We expect that in this case there is also an equivalence between replication and estimation. It is well known that in the presence of noise, Heisenberg scaling is often inhibited, leading to different sub-Heisenberg scalings [71–75]. We then expect that the replication rates will also have intermediate values ranging between $\alpha = 1$ and $\alpha = 2$, depending on the type of noise. The study of superreplication in the noisy case is also expected to shed light on the optimal strategies for estimation of noisy channels, the performance of which is sometimes hard to characterize analytically. Another interesting research direction is the study of replication processes that are subject to constraints, e.g., on the ability to perform joint operations on composite systems. Recently, Kumagai and Hayashi [76] investigated the problem of cloning bipartite quantum states using only *local operations* and classical communication. When the state to be cloned is perfectly known, they showed that the number of extra copies scales as $\sqrt{N}$. Note the contrast with the situation where the limitations of local operations and classical communication are not imposed, in which case one can produce $\Theta(N^\delta)$ extra copies for every $\delta < 1$. In addition to the constraints due to the locality of the operations, other physical constraints can arise from conservation laws, such as energy and angular momentum conservation [77–81]. The search for energy-preserving cloners was considered in our earlier work [45], where we identified the optimal operations for transformation of pure states. In this scenario, it is interesting to examine how the replication rates are affected by the presence of limited resources or, conversely, what resources are needed to achieve a desired replication rate.

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