On Polynomial Kernelization of $\mathcal{H}$-free Edge Deletion

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Abstract. For a set of graphs $\mathcal{H}$, the $\mathcal{H}$-free Edge Deletion problem asks to find whether there exist at most k edges in the input graph whose deletion results in a graph without any induced copy of $H \in \mathcal{H}$. In [3], it is shown that the problem is fixed-parameter tractable if $\mathcal{H}$ is of finite cardinality. However, it is proved in [4] that if $\mathcal{H}$ is a singleton set containing H, for a large class of H, there exists no polynomial kernel unless $coNP \subseteq NP/poly$. In this paper, we present a polynomial kernel for this problem for any fixed finite set $\mathcal{H}$ of connected graphs and when the input graphs are of bounded degree. We note that there are $\mathcal{H}$-free Edge Deletion problems which remain NP-complete even for the bounded degree input graphs, for example Triangle-free Edge Deletion [2] and Custer Edge Deletion(P_3-free Edge Deletion) [15]. When $\mathcal{H}$ contains $K_{1,s}$, we obtain a stronger result - a polynomial kernel for K_t-free input graphs (for any fixed $t > 2$). We note that for $s > 9$, there is an incompressibility result for $K_{1,s}$-free Edge Deletion for general graphs [5]. Our result provides first polynomial kernels for Claw-free Edge Deletion and Line Edge Deletion for K_t-free input graphs which are NP-complete even for K_4-free graphs [23] and were raised as open problems in [4, 19].

1 Introduction

For a graph property Π, the Π Edge Deletion problem asks whether there exist at most k edges such that deleting them from the input graph results in a graph with property Π. Numerous studies have been done on edge deletion problems from 1970s onwards dealing with various aspects such as hardness [1, 2, 7–9, 14, 20–23], polynomial-time algorithms [13, 21, 22], approximability [1, 21, 22], fixed-parameter tractability [3, 10], polynomial problem kernels [2, 10–12] and incompressibility [4, 5, 16].

There are not many generalized results on the NP-completeness of edge deletion problems. This is in contrast with the classical result by Lewis and Yannakakis [18] on the vertex counterparts which says that

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Π Vertex Deletion problems are NP-complete if Π is non-trivial and hereditary on induced subgraphs. By a result of Cai [3], the Π Edge Deletion problem is fixed-parameter tractable for any hereditary property Π that is characterized by a finite set of forbidden induced subgraphs. We observe that polynomial problem kernels have been found only for a few parameterized Π Edge Deletion problems.

In this paper, we study a subset of Π Edge Deletion problems known as $\mathcal{H}$-free Edge Deletion problems where $\mathcal{H}$ is a set of graphs. The objective is to find whether there exist at most k edges in the input graph such that deleting them results in a graph with no induced copy of $H \in \mathcal{H}$. In the natural parameterization of this problem, the parameter is k. In this paper, we give a polynomial problem kernel for parameterized version of $\mathcal{H}$-free Edge Deletion where $\mathcal{H}$ is any fixed finite set of connected graphs and when the input graphs are of bounded degree. In this context, we note that Triangle-free Edge Deletion [2] and Custer Edge Deletion(P_3-free Edge Deletion) [15] are NP-complete even for bounded degree input graphs. We also note that, under the complexity theoretic assumption $coNP \not\subseteq NP/poly$, there exist no polynomial problem kernels for the H-free Edge Deletion problems when H is 3-connected but not complete, or when H is a path or cycle of at least 4 edges [4]. When the input graph has maximum degree at most Δ and if the maximum diameter of graphs in $\mathcal{H}$ is D, then the number of vertices in the kernel we obtain is at most $2\Delta^{2D+1} \cdot k^{pD+1}$ where $p = \log_{\frac{2\Delta}{2\Delta-1}} \Delta$. Our kernelization consists of a single rule which removes vertices of the input graph that are 'far enough' from all induced $H \in \mathcal{H}$ in G.

When $\mathcal{H}$ contains $K_{1,s}$, we obtain a stronger result - a polynomial kernel for K_t-free input graphs (for any fixed $t > 2$). Let $s > 1$ be the least integer such that $K_{1,s} \in \mathcal{H}$. Then the number of vertices in the kernel we obtain is at most $8d^{3D+1} \cdot k^{pD+1}$ where $d = R(s, t-1) - 1$, $R(s, t-1)$ is the Ramsey number and $p = \log_{\frac{2d}{2d-1}} d$. We note that Claw-free Edge Deletion and Line Edge Deletion are NP-complete even for K_4-free input graphs [23]. As a corollary of our result, we obtain the first polynomial kernels for these problems when the input graphs are K_t-free for any fixed $t > 2$. The existence of a polynomial kernel for Claw-free Edge Deletion and Line Edge Deletion were raised as open problems in [4, 19]. We note that for $s > 9$, there is an incompressibility result for $K_{1,s}$-free Edge Deletion for general graphs [5].

1.1 Related Work

Here, we give an overview of various results on edge deletion problems.

NP-completeness: It has been proved that Π Edge Deletion problems are NP-complete if Π is one of the following properties: without cycle of any fixed length $l \geq 3$, without any cycle of length at most l for any fixed $l \geq 4$, connected with maximum degree r for every fixed $r \geq 2$, outerplanar, line graph, bipartite, comparability [23], claw-free (implicit in the proof of NP-completeness of the Line Edge Deletion problem in [23]), P_l-free for any fixed $l \geq 3$ [7], circular-arc, chordal, chain, perfect, split, AT-free [21], interval [9], threshold [20] and complete [14].

Fixed-parameter Tractability and Kernelization: Cai proved in [3] that parameterized Π Edge Deletion problem is fixed-parameter tractable if Π is a hereditary property characterized by a finite set of forbidden induced subgraphs. Hence $\mathcal{H}$-free Edge Deletion is fixed-parameter tractable for any finite set of graphs $\mathcal{H}$. Polynomial problem kernels are known for chain, split, threshold [12], triangle-free [2], cograph [11] and cluster [10] edge deletions. It is proved in [4] that for 3-connected H, H-free Edge Deletion admits no polynomial kernel if and only if H is not a complete graph, under the assumption $coNP \not\subseteq NP/poly$. Under the same assumption, it is proved in [4] that for H being a path or cycle, H-free Edge Deletion admits no polynomial kernel if and only if H has at least 4 edges. Unless $NP \subseteq coNP/poly$, H-free Edge Deletion admits no polynomial kernel if H is $K_1 \times \ (2K_1 \cup 2K_2)$ [16].

2 Preliminaries and Basic Results

We consider only simple graphs. For a set of graphs $\mathcal{H}$, a graph G is $\mathcal{H}$-free if there is no induced copy of $H \in \mathcal{H}$ in G. For $V' \subseteq V(G)$, $G \setminus V'$ denotes the graph $(V(G) \setminus V', E(G) \setminus E')$ where $E' \subseteq E(G)$ is the set of edges incident to vertices in V'. Similarly, for $E' \subseteq E(G)$, $G \setminus E'$ denotes the graph $(V(G), E(G) \setminus E')$. For any edge set $E' \subseteq E(G)$, $V_{E'}$ denotes the set of vertices incident to the edges in E'. For any $V' \subseteq V(G)$, the closed neighbourhood of V', $N_G[V'] = \{v : v \in V' \text{ or } (u,v) \in E(G) \text{ for some } u \in V'\}$. In a graph G, distance from a vertex v to a set of vertices V' is the shortest among the distances from v to the vertices in V'.

A parameterized problem is *fixed-parameter tractable*(FPT) if there exists an algorithm to solve it which runs in time $O(f(k)n^c)$ where f is a computable function, n is the input size, c is a constant and k is the parameter. The idea is to solve the problem efficiently for small parameter values. A related notion is *polynomial kernelization* where the parameterized problem instance is reduced in polynomial (in $n+k$) time to a polynomial (in k) sized instance of the same problem called *problem kernel* such that the original instance is a yes-instance if and only if the problem kernel is a yes-instance. We refer to [6] for an exhaustive treatment on these topics. A kernelization rule is *safe* if the answer to the problem instance does not change after the application of the rule.

In this paper, we consider $\mathcal{H}$-FREE EDGE DELETION[1] which is defined as given below.

$\mathcal{H}$-FREE EDGE DELETION **Instance**: A graph G and a positive integer k. **Problem**: Does there exist $E' \subseteq E(G)$ with $\|E'\| \leq k$ such that $G \setminus E'$ does not contain $H \in \mathcal{H}$ as an induced subgraph. **Parameter**: k

We define an $\mathcal{H}$ *deletion set (HDS)* of a graph G as a set $M \subseteq E(G)$ such that $G \setminus M$ is $\mathcal{H}$-free. The *minimum $\mathcal{H}$ deletion set (MHDS)* is an HDS with smallest cardinality. We define a partition of an MHDS M of G as follows.

$M_1 = \{e : e \in M$ and e is part of an induced $H \in \mathcal{H}$ in $G\}$.

$M_j = \{e : e \in M \setminus \bigcup_{i=1}^{i=j-1} M_i$ and e is part of an induced $H \in \mathcal{H}$ in $G \setminus \bigcup_{i=1}^{i=j-1} M_i\}$, for $j > 1$.

We define the *depth* of an MHDS M of G, denoted by l_M, as the least integer such that $|M_i| > 0$ for all $1 \leq i \leq l_M$ and $|M_i| = 0$ for all $i > l_M$. Proposition 1 shows that this notion is well defined.

Proposition 1. *1. $\{M_j\}$ forms a partition of M.*
2. There exists $l_M \geq 0$ such that $|M_i| > 0$ for $1 \leq i \leq l_M$ and $|M_i| = 0$ for $i > l_M$.

Proof. If $i \neq j$ and M_i and M_j are nonempty, then $M_i \cap M_j = \emptyset$. For $i \geq 1$, $M_i \subseteq M$. Assume there is an edge $e \in M$ and $e \notin \bigcup M_j$. Delete all edges in $\bigcup M_j$ from G. What remains is an $\mathcal{H}$-free graph. As M is an MHDS, there can not exist such an edge e. Now let j be the smallest integer such that M_j is empty. Then from definition, for all $i > j$, $|M_i| = 0$. Therefore $l_M = j - 1$.

□

We observe that for an $\mathcal{H}$-free graph, the only MHDS M is $\emptyset$ and hence $l_M = 0$. For an MHDS M of G with a depth l_M, we define the following terms.

[1] we leave the prefix ‘parameterized’ henceforth as it is evident from the context

$S_j = \bigcup_{i=j}^{i=l_M} M_j$ for $1 \leq j \leq l_M + 1$.
$T_j = M \setminus S_{j+1}$ for $0 \leq j \leq l_M$.
$V_{\mathcal{H}}(G)$ is the set of all vertices part of some induced $H \in \mathcal{H}$ in G.

We observe that $S_1 = T_{l_M} = M$, $S_{l_M} = M_{l_M}$, $T_1 = M_1$ and $S_{l_M+1} = T_0 = \emptyset$.

Proposition 2. *For a graph G, let $E' \subseteq E(G)$ such that at least one edge in every induced $H \in \mathcal{H}$ in G is in E'. Then, at least one vertex in every induced $H \in \mathcal{H}$ in $G \setminus E'$ is in $V_{E'}$.*

Proof. Assume that there exists an induced $H \in \mathcal{H}$ in $G \setminus E'$ with the vertex set V'. For a contradiction, assume that $|V' \cap V_{E'}| = 0$. Then, V' induces a copy of H in G. Hence, E' must contain some of its edges.

□

Lemma 1. *Let G be the input graph of an $\mathcal{H}$-free Edge Deletion problem instance where $\mathcal{H}$ is a set of connected graphs with diameter at most D. Let M be an MHDS of G. Then, every vertex in V_M is at a distance at most $(l_M - 1)D$ from $V_{\mathcal{H}}(G)$ in G .*

Proof. For $2 \leq j \leq l_M$, from definition, at least one edge in every induced $H \in \mathcal{H}$ in $G \setminus T_{j-2}$ is in M_{j-1}. Hence by Proposition 2, at least one vertex in every induced $H \in \mathcal{H}$ in $G \setminus T_{j-1}$ is in $V_{M_{j-1}}$. By definition, every vertex in V_{M_j} is part of some induced $H \in \mathcal{H}$ in $G \setminus T_{j-1}$. This implies every vertex in V_{M_j} is at a distance at most D from $V_{M_{j-1}}$. Hence every vertex in $V_{M_{l_M}}$ is at a distance at most $(l_M - 1)D$ from V_{M_1}. By definition, $V_{M_1} \subseteq V_{\mathcal{H}}(G)$. Hence the proof.

□

Lemma 2. *Let G be a graph with maximum degree at most Δ and M be an MHDS of G. Then, for $1 \leq j \leq l_M$, $(2\Delta - 1) \cdot |M_j| \geq |S_{j+1}|$.*

Proof. For $1 \leq j \leq l_M$, from definition, M_j has at least one edge from every induced $H \in \mathcal{H}$ in $G \setminus T_{j-1}$. Let M'_j be the set of edges incident to vertices in V_{M_j} in $G \setminus T_{j-1}$. We observe that $(G \setminus T_{j-1}) \setminus M'_j$ is $\mathcal{H}$-free and hence $|T_{j-1} \cup M'_j|$ is an HDS of G. Clearly, $|M'_j| \leq \Delta |V_{M_j}| \leq 2\Delta |M_j|$. Since M is an MHDS, $|T_{j-1} \cup M'_j| = |T_{j-1}| + |M'_j| \geq |M| = |T_{j-1}| + |S_j|$. Therefore $|M'_j| \geq |S_j|$. Hence, $2\Delta |M_j| \geq |S_j| = |M_j| + |S_{j+1}|$.

□

Now we give an upper bound for the depth of an MHDS in terms of its size and maximum degree of the graph.

Lemma 3. *Let M be an MHDS of G. If the maximum degree of G is at most $\Delta > 0$, then $l_M \leq 1 + \log_{\frac{2\Delta}{2\Delta - 1}} |M|$.*

Proof. The statement is clearly true when $l_M \leq 1$. Hence assume that $l_M \geq 2$. The result follows from repeated application of Lemma 2.

$$\begin{aligned} |M| = |S_1| = |M_1| + |S_2| &\geq \frac{|S_2|}{2\Delta - 1} + |S_2| \\ &\geq |S_{l_M}| \left(\frac{2\Delta}{2\Delta - 1}\right)^{l_M - 1} \\ &\geq \left(\frac{2\Delta}{2\Delta - 1}\right)^{l_M - 1} \qquad [\because |S_{l_M}| \geq 1] \end{aligned}$$

□

Corollary 1. *Let* (G, k) *be a yes-instance of* $\mathcal{H}$-FREE EDGE DELETION *where* G *has maximum degree at most* $\Delta > 0$. *For any MHDS* M *of* G, $l_M \leq 1 + \log_{\frac{2\Delta}{2\Delta-1}} k$.

□

Lemma 4. *Let* $\mathcal{H}$ *be a set of connected graphs with diameter at most* D. *Let* $V' \supseteq V_{\mathcal{H}}(G)$ *and let* $c \geq 0$. *Let* G' *be obtained by removing vertices of* G *at a distance more than* $c + D$ *from* V'. *Furthermore, assume that if* G' *is a yes-instance then there exists an MHDS* M' *of* G' *such that every vertex in* $V_{M'}$ *is at a distance at most* c *from* V' *in* G'. *Then* (G, k) *is a yes-instance if and only if* (G', k) *is a yes-instance of* $\mathcal{H}$-FREE EDGE DELETION.

Proof. Let G be a yes-instance with an MHDS M. Then $M' = M \cap E(G')$ is an HDS of G' such that $|M'| \leq k$. Conversely, let G' be a yes-instance. By the assumption, there exists an MHDS M' of G' such that every vertex in $V_{M'}$ is at a distance at most c from V' in G'. We claim that M' is an MHDS of G. For contradiction, assume $G \setminus M'$ has an induced $H \in \mathcal{H}$ with a vertex set V''. As G and G' has same set of induced copies of graphs in $\mathcal{H}$, at least one edge in every induced copy of graphs in $\mathcal{H}$ in G is in M'. Then, by Proposition 2, at least one vertex in V'' is in $V_{M'}$. We observe that for every vertex in G' the distance from V' is same in G and G'. Hence every vertex in V'' is at a distance at most $c + D$ from V' in G. Then, V'' induces a copy of H in $G' \setminus M'$ which is a contradiction.

□

Lemma 5. *Let* G *be a graph and let* $d > 1$ *be a constant. Let* $V' \subseteq V(G)$ *such that all vertices in* G *with degree more than* d *is in* V'. *Partition* V' *into* V_1 *and* V_2 *such that* V_1 *contains all the vertices in* V' *with degree at most* d *and* V_2 *contains all the vertices with degree more than* d. *If* *every vertex in G is at a distance at most $c > 0$ from V', then $|V(G)| \leq |V_1| \cdot d^{c+1} + |N_G(V_2)| \cdot d^c$.*

Proof. To enumerate the number of vertices in G, consider the d-ary breadth first trees rooted at vertices in V_1 and in $N_G[V_2]$.

$$\begin{aligned} |V(G')| &\leq |V_1| \left(\frac{d^{c+1} - 1}{d - 1} \right) + |N_G[V_2]| \left(\frac{d^c - 1}{d - 1} \right) \\ &\leq |V_1| d^{c+1} + |N_G[V_2]| d^c \end{aligned}$$

□

3 Polynomial Kernels

In this section, we assume that $\mathcal{H}$ is a fixed finite set of connected graphs with diameter at most D. First we devise an algorithm to obtain polynomial kernel for $\mathcal{H}$-FREE EDGE DELETION for bounded degree input graphs. Then we prove a stronger result - a polynomial kernel for K_t-free input graphs (for some fixed $t > 2$) when $\mathcal{H}$ contains $K_{1,s}$ for some $s > 1$.

We assume that the input graph G has maximum degree at most $\Delta > 1$ and G has at least one induced copy of H. We observe that if these conditions are not met, obtaining polynomial kernel is trivial.

Now we state the kernelization rule which is the single rule in the kernelization.

Rule 0: Delete all vertices in G at a distance more than $(1+\log_{\frac{2\Delta}{2\Delta-1}} k)D$ from $V_{\mathcal{H}}(G)$.

We note that the rule can be applied efficiently with the help of breadth first search from vertices in $V_{\mathcal{H}}(G)$. Now we prove the safety of the rule.

Lemma 6. *Rule 0 is safe.*

Proof. Let G' be obtained from G by applying Rule 0. Let M' be an MHDS of G'. If G' is a yes-instance, then by Lemma 1 and Corollary 1, every vertex in $V_{M'}$ is at a distance at most $D \log_{\frac{2\Delta}{2\Delta-1}} k$ from $V_{\mathcal{H}}(G')$. Hence, we can apply Lemma 4 with $V' = V_{\mathcal{H}}(G)$ and $c = D \log_{\frac{2\Delta}{2\Delta-1}} k$.

□

Lemma 7. *Let (G, k) be a yes-instance of $\mathcal{H}$-FREE EDGE DELETION. Let G' be obtained by one application of Rule 0 on G. Then, $|V(G')| \leq (2\Delta^{2D+1} \cdot k^{pD+1})$ where $p = \log_{\frac{2\Delta}{2\Delta-1}} \Delta$.*

Proof. Let M be an MHDS of G such that $|M| \leq k$. We observe that every vertex in $V_{\mathcal{H}}(G)$ is at a distance at most D from V_{M_1} in G. Hence, by construction, every vertex in G' is at a distance at most $(2+\log_{\frac{2\Delta}{2\Delta-1}} k)D$ from V_{M_1} in G and in G'. We note that $|V_{M_1}| \leq 2k$. To enumerate the number of vertices in G', we apply Lemma 5 with $V' = V_{M_1}$, $c = (2 + \log_{\frac{2\Delta}{2\Delta-1}} k)D$ and $d = \Delta$.

$$\begin{aligned} |V(G')| &\leq 2k\Delta^{(2+\log_{\frac{2\Delta}{2\Delta-1}} k)D+1} \\ &\leq 2\Delta^{2D+1} \cdot k^{pD+1} \end{aligned}$$

□

Now we present the algorithm to obtain a polynomial kernel. The algorithm applies Rule 0 on the input graph and according to the number of vertices in the resultant graph it returns the resultant graph or a trivial no-instance.

> Kernelization for $\mathcal{H}$-FREE EDGE DELETION
> ($\mathcal{H}$ is a finite set of connected graphs with maximum diameter D)
> Input:(G, k) where G has maximum degree at most Δ.
>
> **1.** Apply Rule 0 on G to obtain G'.
> **2.** If the number of vertices in G' is more than $2\Delta^{2D+1} \cdot k^{pD+1}$ where $p = \log_{\frac{2\Delta}{2\Delta-1}} \Delta$, then return a trivial no-instance $(H, 0)$ where H is the graph with minimum number of vertices in $\mathcal{H}$. Else return (G', k).

Theorem 1. *The kernelization for $\mathcal{H}$-FREE EDGE DELETION returns a kernel with the number of vertices at most $2\Delta^{2D+1} \cdot k^{pD+1}$ where $p = \log_{\frac{2\Delta}{2\Delta-1}} \Delta$.*

Proof. Follows from Lemma 6 and Lemma 7 and the observation that the number of vertices in the trivial no-instance is at most $2\Delta^{2D+1} \cdot k^{pD+1}$.

□

3.1 A stronger result for a restricted case

Here we give a polynomial kernel for $\mathcal{H}$-FREE EDGE DELETION when $\mathcal{H}$ is a fixed finite set of connected graphs and contains a $K_{1,s}$ for some $s > 1$ and when the input graphs are K_t-free, for any fixed $t > 2$.

It is proved in [17] that the maximum degree of a {claw, K_4}-free graph is at most 5. We give a straight forward generalization of this result for

$\{K_{1,s}, K_t\}$-free graphs. Let $R(s,t)$ denote the Ramsey number. Remember that the Ramsey number $R(s,t)$ is the least integer such that every graph on $R(s,t)$ vertices has either an independent set of order s or a complete subgraph of order t.

Lemma 8. *For integers $s > 1, t > 1$, any $\{K_{1,s}, K_t\}$-free graph has maximum degree at most $R(s, t-1) - 1$.*

Proof. Assume G is $\{K_{1,s}, K_t\}$-free. For contradiction, assume G has a vertex v of degree at least $R(s, t-1)$. By the definition of the Ramsey number there exist at least s mutually non-adjacent vertices or $t-1$ mutually adjacent vertices in the neighborhood of v. Hence there exist either an induced $K_{1,s}$ or an induced K_t in G.

□

We modify the proof technique used for devising polynomial kernelization for $\mathcal{H}$-FREE EDGE DELETION for bounded degree graphs to obtain polynomial kernelization for K_t-free input graphs for the case when $\mathcal{H}$ contains $K_{1,s}$ for some $s > 1$.

Let $s > 1$ be the least integer such that $\mathcal{H}$ contains $K_{1,s}$. Let $t > 2$, G be K_t-free and M be an MHDS of G. Let $d = R(s, t-1) - 1$. Let D be the maximum diameter of graphs in $\mathcal{H}$. We define the following.

$M_0 = \{e : e \in M$ and e is incident to a vertex with degree at least $d+1\}$.
$V_R(G) = \{v : v \in V(G)$ and v has degree at least $d+1$ in $G\}$.

Lemma 9. *$G \setminus M_0$ has degree at most d and every vertex in G with degree at least $d+1$ is incident to at least one edge in M_0.*

Proof. As $G \setminus M$ is $\{K_{1,s}, K_t\}$-free and every edge in M which is incident to at least one vertex of degree at least $d+1$ is in M_0, the result follows from Lemma 8.

□

Lemma 10. *Let M be an MHDS of G. Let $M' = M \setminus M_0$ and $G' = G \setminus M_0$. Then, M' is an MHDS of G' and every vertex in V_M is at a distance at most $Dl_{M'}$ from $V_{\mathcal{H}}(G) \cup V_R(G)$ in G .*

Proof. It is straight forward to verify that M' is an MHDS of G'. By Lemma 1, every vertex in $V_{M'}$ is at a distance at most $(l_{M'} - 1)D$ from $V_{\mathcal{H}}(G')$ in G'. Every induced $H \in \mathcal{H}$ in G' is either an induced H in G or formed by deleting M_0 from G. Therefore, every vertex in $V_{\mathcal{H}}(G')$ is at a distance at most D from $V_{\mathcal{H}}(G) \cup V_R(G)$ in G'. Hence, every vertex in $V_{M'}$ is at a distance at most $Dl_{M'}$ from $V_{\mathcal{H}}(G) \cup V_R(G)$ in G'. The result follows from the fact $M = M' \cup M_0$.

□

The single rule in the kernelization is:

Rule 1: Delete all vertices in G at a distance more than $(2+\log_{\frac{2d}{2d-1}} k)D$ from $V_{\mathcal{H}}(G) \cup V_R(G)$ where $d = R(s, t-1) - 1$.

Lemma 11. *Rule 1 is safe.*

Proof. Let G' be obtained from G by applying Rule 1. Let M' be an MHDS of G'. If G' is a yes-instance, then by Lemma 10 and Corollary 1, every vertex in $V_{M'}$ is at a distance at most $D(1+\log_{\frac{2d}{2d-1}} k)$ from $V_{\mathcal{H}}(G') \cup V_R(G')$ in G'. We note that $V_{\mathcal{H}}(G) = V_{\mathcal{H}}(G')$ and $V_R(G) = V_R(G')$. Hence, we can apply Lemma 4 with $V' = V_{\mathcal{H}}(G) \cup V_R(G)$, $c = D(1 + \log_{\frac{2d}{2d-1}} k)$ and $d = R(s, t-1) - 1$.

□

Lemma 12. *Let (G, k) be a yes-instance of $\mathcal{H}$-free Edge Deletion where G is K_t-free. Let G' be obtained by one application of Rule 1 on G. Then, $|V(G')| \leq 8d^{3D+1} \cdot k^{pD+1}$ where $p = \log_{\frac{2d}{2d-1}} d$.*

Proof. Let M be an MHDS of G such that $|M| \leq k$. We observe that every vertex in $V_{\mathcal{H}}(G)$ is at a distance at most D from V_{M_1} in G. Hence, by construction, every vertex in G' is at a distance at most $D(3+\log_{\frac{2d}{2d-1}} k)$ from $V_{M_1} \cup V_R(G)$. Clearly $|V_{M_1}| \leq 2k$. Using Lemma 9 we obtain $|N[V_R(G)]| \leq 2k(d+2)$. To enumerate the number of vertices in G', we apply Lemma 5 with $V' = V_{M_1} \cup V_R(G)$, $c = D(3 + \log_{\frac{2d}{2d-1}} k)$ and $d = R(s, t-1) - 1$.

$$\begin{aligned}|V(G')| &\leq 2kd^{D(3+\log_{\frac{2d}{2d-1}} k)+1} + 2k(d+2)d^{D(3+\log_{\frac{2d}{2d-1}} k)} \\ &\leq 8d^{3D+1} \cdot k^{pD+1}\end{aligned}$$

□

Now we present the algorithm.

Kernelization for $\mathcal{H}$-free Edge Deletion
($\mathcal{H}$ contains $K_{1,s}$ for some $s > 1$)
Input:(G, k) where G is K_t-free for some fixed $t > 2$.
Let $s > 1$ be the least integer such that $\mathcal{H}$ contains $K_{1,s}$.

1. Apply Rule 1 on G to obtain G'.
2. If the number of vertices in G' is more than $8d^{3D+1} \cdot k^{pD+1}$ where $d = R(s, t-1) - 1$ and $p = \log_{\frac{2d}{2d-1}} d$, then return a trivial no-instance $(K_{1,s}, 0)$. Else return (G', k).

For practical implementation, we can use any specific known upper bound for $R(s, t-1)$ or the general upper bound $\binom{s+t-3}{s-1}$.

Theorem 2. *The kernelization for* $\mathcal{H}$-FREE EDGE DELETION *when* $K_{1,s} \in \mathcal{H}$ *and the input graph is* K_t*-free returns a kernel with the number of vertices at most* $8d^{1+3D} \cdot k^{1+pD}$ *where* $d = R(s, t-1) - 1$ *and* $p = \log_{\frac{2d}{2d-1}} d$.

Proof. Follows from Lemma 11 and Lemma 12.

□

It is known that line graphs are characterized by a finite set of connected forbidden induced subgraphs including a claw ($K_{1,3}$). Both CLAW-FREE EDGE DELETION and LINE EDGE DELETION are NP-complete even for K_4-free graphs [23].

Corollary 2. CLAW-FREE EDGE DELETION *and* LINE EDGE DELETION *admit polynomial kernels for* K_t*-free input graphs for any fixed* $t > 3$.

□

We observe that the kernelization for $\mathcal{H}$-FREE EDGE DELETION when $K_{1,s} \in \mathcal{H}$ and the input graph is K_t-free works for the case when $K_t \in \mathcal{H}$ and the input graph is $K_{1,s}$-free.

Theorem 3. $\mathcal{H}$-FREE EDGE DELETION *admits polynomial kernelization when* $\mathcal{H}$ *is a finite set of connected graphs,* $K_t \in \mathcal{H}$ *for some* $t > 2$ *and the input graph is* $K_{1,s}$*-free for some fixed* $s > 1$.

4 Concluding Remarks

Our results may give some insight towards a dichotomy theorem on incompressibility of $\mathcal{H}$-FREE EDGE DELETION raised as an open problem in [4]. We conclude with an open problem: does $\mathcal{H}$-FREE EDGE DELETION admit polynomial kernel for planar input graphs?

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