# Output consensus of multi-agent systems with delayed and sampled-data

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**Abstract:** This paper considers the output consensus problem of high-order leader-following multi-agent systems with unknown nonlinear dynamics, in which the delayed and sampled outputs of the system are the only available data. The unknown nonlinear dynamics are assumed to satisfy the Lipschitz condition and the interconnected topologies are assumed to be undirected and connected. A distributed observer-based output feedback controller is proposed for the system to reach output consensus. Both of the bounds of the allowable delay and sampling period are also obtained. Stability analysis shows that the considered systems are globally exponentially stable under the output feedback controller. Finally, a simulation example is given to validate our theoretical results.

## 1. Introduction

The autonomy, distribution and coordination render the multi-agent systems have strong robustness and reliability in solving practical problems. Its wide range of applications can be found in various areas, including flocking, swarming, distributed sensor fusion, distributed coordination of mobile robots, congestion control in communication networks, synchronization of dynamical networks, and so on. The distributed coordination problem also attracted the attention of scientists in control theory and many well known works have been done in the context of control theory, for example, [1, 2, 3, 4, 5], to name just a few.

In the past decades, the related topics on consensus problems have been comprehensively further studied in different situations, for example, consensus in networks with time-delays [6, 7, 8], finite time consensus [9, 10, 11, 12, 13], consensus in stochastic networks [14, 15], quantized consensus [16, 17, 18], and so on.

With the rapid development of intelligent instrument and digital measurement, the information of modern control systems tend to be sampled and sent periodically furthermore through the digital communication channels. Thus, the consensus with sampled-data and time-delay is meaningful research topic. The articles studying sampled-data systems mainly include: discrete-time models [19, 20, 21], impulsive models [22, 23, 24], quantized models [16, 25], and time-delay systems [26, 27, 28]. Most of the works mentioned above are concerned with state consensus. In practical cases, most real world systems are uncertain, and the full states of agents maybe unknown. Therefore, output consensus has attracted numerous papers' attention in recent years, such as, [26, 29, 30, 31, 32]. Some representative works are summarized as follows. In [19], the authors studied the multi-consensus problem and multi-tracking problem of second-order multi-agent systems by using only sampled current and past position data, a necessary and sufficient condition on gains and sampled period was given under directed topology. In [22], two kinds of impulsive distributed consensus algorithms which only utilize the sampled information were proposed to investigate the distributed consensus problem for second-order continuous-time multi-agent systems with sampled-data communication. In [27, 33], the authors studied the state consensus in multi-agent dynamical systems with sampled-data, and distributed linear consensus protocols were proposed. In [33], both the current and some past sampled position data were utilized to design consensus protocols. While, using less information and saving energy, a protocol with more sampled data and no current data was further designed in [27]. To the best of our knowledge, the studies on the output consensus problem of multi-agent systems with unknown nonlinear dynamics and both delayed and sampled data are rare.

In this paper, the output consensus problem of high-order leader-following multi-agent systems with unknown nonlinear dynamics is considered. The only available data of the system outputs is assumed to be sampled and delayed. The unknown nonlinear dynamics are assumed to satisfy the Lipschitz condition, and the interconnected graphs of the multi-agent systems are assumed to be undirected and connected. A distributed observer-based output feedback controller is proposed for the sampled-data multi-agent systems to reach output consensus. The stability analysis is conducted based on Lyapunov theory and algebraic graph theory, and shows that the error system is globally exponentially stable. Finally, the bounds of the allowable delay and sampling period are also given.

The contributions of this paper are mainly in three aspects. First, a novel distributed observer-based output feedback controller is presented for the considered sampled-data multi-agent systems to reach output consensus. Second, a sufficient condition is obtained to ensure the global exponential stability of the considered systems. Finally, we give the bounds of the allowable sampling period and time delays.

This paper is organized as follows. In Section 2, some preliminary results and model formulation are presented, the related observer-based output feedback controller is also proposed in this section. In Section 3, the main results are given and proved. In Section 4, an example is given to illustrate the validity of the proposed design method. Finally, the paper is concluded in Section 5.

## 2. Problem Statement

For the multi-agent system, the information exchange among $N$ agents can be conveniently described by a simple and undirected graph $\mathcal{G}(\mathcal{V},\mathcal{E})$. $\mathcal{V}=\{1,2,\cdots,N\}$ is the node set in which each node can be regarded as the $N$ agents, respectively. $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$ is the edge set, a pair $(i,j)\in\mathcal{E}$ if and only if $(j,i)\in\mathcal{E}$, and an edge $(i,j)$ in $\mathcal{G}$ means that agents $i$ and $j$ can obtain information from each other. The set of neighbors of node $i$ is denoted by $\mathcal{N}_i=\{j\in\mathcal{V}|(i,j)\in\mathcal{E},j\neq i\}$. A path is a sequence of connected edges in a graph. If there is a path between any two nodes of a graph $\mathcal{G}$, then graph $\mathcal{G}$ is said to be connected, otherwise disconnected. To model the interconnection relationship between $N$ agents and the leader, we introduce another graph $\bar{\mathcal{G}}$ on nodes $0,1,2,\cdots,N$, where $0$ represents the leader agent. Obviously, $\mathcal{G}$ is a subgraph of $\bar{\mathcal{G}}$. Let $\mathcal{A}=[a_{ij}]\in R^{N\times N}$ be the adjacency matrix of graph $\mathcal{G}$, where $a_{ij}=1$ if $(i,j)\in\mathcal{G}$, otherwise $a_{ij}=0$. The index number between the $i$th agent and the leader agent is denoted by $b_i$, where $b_i=1$ if the leader agent is the neighbor of the $i$th agent, otherwise $b_i=0$. The degree matrix $\mathcal{D}$ of graph $\mathcal{G}$ is a diagonal matrix with the $i$th diagonal element being $|\mathcal{N}_i|$. The Laplacian of graph $\mathcal{G}$ is defined as $\mathcal{L}=\mathcal{D}-\mathcal{A}$, which is symmetric. Let $H=\mathcal{L}+\mathcal{B}$, and $\mathcal{B}$ be a diagonal matrix with diagonal elements $b_1, b_2, \cdots, b_N$.

We consider a multi-agent system consisting of $N$ agents and a leader. The dynamics of the $i$th agent, $i = 1, 2, \cdots, N$, are described by

$$\begin{cases} \dot{x}^i(t) = A_0 x^i(t) + f^i(x^i(t)) + \bar{b}u^i(t), \\ y^i(t) = Cx^i(t), \end{cases} \tag{1}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \bar{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, C = (1, 0, \cdots, 0) \ ,$$

$x^i(t) = (x_1^i(t), x_2^i(t), \cdots, x_n^i(t))^T \in R^n$ is the position state of the $i$th agent, $u^i(t) \in R$ is the control input of the $i$th agent, which will be designed later, $f^i(x^i(t)) = (f_1^i(x^i(t)), f_2^i(x^i(t)), \cdots,$ $f_n^i(x^i(t)))^T$ is assumed to be unknown and nonlinear with $f_m^i(x^i(t)) = f_m^i(x_1^i(t), x_2^i(t), \cdots, x_m^i(t))$, $m = 1, 2, \cdots, n$, and $y^i(t)$ is the output of the system.

We assume that the leader-agent moves in $R^n$ and its underlying dynamics are described by

$$\begin{cases} \dot{x}^0(t) = A_0 x^0(t) + f^0(x^0(t)), \\ y^0(t) = Cx^0(t), \end{cases} \tag{2}$$

where $x^0(t) = (x_1^0(t), x_2^0(t), \cdots, x_n^0(t))^T \in R^n$ is the position state of the leader agent, and $f^0(x^0(t)) = (f_1^0(x^0(t)), f_2^0(x^0(t)), \cdots, f_n^0(x^0(t)))^T$ is a vector function with $f_m^0(x^0(t)) = f_m^0(x_1^0(t),$ $x_2^0(t), \cdots, x_m^0(t))$, which is also assumed to be unknown and nonlinear, and $y^0(t)$ is its output.

**Remark 1.** From the linear system theory, if the linear system $\dot{x} = Fx + gu$ is controllable, then it can be transformed into the Brunovsky controller form [34], that is the linear part of the system (1). The system (1) is widely applied to some real world systems, such as two-link planar robots, aircraft wing rock control systems and induction motor systems [35, 36, 37, 38].

In this paper, the output $y^i(t)$ of system (1) and $y^0(t)$ of system (2) are assumed to be sampled at time instants $t_k$ and are available at $t_k + \tau_k$, where $\{t_k\}, k = 1, 2, \cdots, \infty$, is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = \infty$ and $\tau_k \geqslant 0$, that is the sampled-data $y^i(t_k)$ and $y^0(t_k)$ are available with a time-delay $\tau_k$. The sampling interval $[t_{k-1}, t_k)$ satisfies $0 < T_{\text{min}} \leqslant t_k - t_{k-1} = T_k \leqslant T_{\text{max}}$ for all $k = 1, 2, \cdots, \infty$, where $T_k$ is the length of the $k$th sampling interval, $T_{\text{min}} = \min\{T_k\}$ and $T_{\text{max}} = \max\{T_k\}$. We assume that $\tau$ is one of the upper bound of $\tau_k$, that is $\tau_k \leqslant \tau$, and satisfy $\tau < T_{\text{min}}$, which means that the sampled-data at time $t_k$ can be used before next sampling time instant.

In this setting, the considered system (1) and (2) can be rewritten as:

$$\begin{cases} \dot{x}^i(t) = A_0 x^i(t) + f^i(x^i(t)) + \bar{b}u^i(t), \\ y^i(t) = Cx^i(t_k), i = 1, 2, \cdots, N, \\ t \in [t_k + \tau_k, t_{k+1} + \tau_{k+1}), k \geqslant 0, \\ x^i(t_{k+1} + \tau_{k+1}) = \lim_{t\to(t_{k+1}+\tau_{k+1})^-} x^i(t), \end{cases} \tag{3}$$

and

$$\begin{cases} \dot{x}^0(t) = A_0 x^0(t) + f^0(x^0(t)), \\ y^0(t) = Cx^0(t_k), \\ t \in [t_k + \tau_k, t_{k+1} + \tau_{k+1}), k \geqslant 0, \\ x^0(t_{k+1} + \tau_{k+1}) = \lim_{t\to(t_{k+1}+\tau_{k+1})^-} x^0(t). \end{cases} \tag{4}$$

Since the states of (3) and (4) are unmeasurable, an observer-based output feedback controller is proposed as follows:

$$\begin{cases} \dot{\hat{x}}^i(t) = A_0\hat{x}^i(t) + Mae_1^i(t_k) + f^i(\hat{x}^i(t)) + \bar{b}u^i(t), \\ \dot{\hat{x}}^0(t) = A_0\hat{x}^0(t) + Mae_1^0(t_k) + f^0(\hat{x}^0(t)), \\ \hat{x}^i(t_{k+1}+\tau_{k+1}) = \lim_{t\to(t_{k+1}+\tau_{k+1})^-}\hat{x}^i(t), \\ \hat{x}^0(t_{k+1}+\tau_{k+1}) = \lim_{t\to(t_{k+1}+\tau_{k+1})^-}\hat{x}^0(t), \\ u^i(t) = -KW[\sum_{j\in N^i} a_{ij}(\hat{x}^i(t)-\hat{x}^j(t)) + b_i(\hat{x}^i(t)-\hat{x}^0(t))], \\ t\in[t_k+\tau_k, t_{k+1}+\tau_{k+1}), k\geqslant 0, \end{cases} \tag{5}$$

where $\hat{x}^i(t) = (\hat{x}_1^i(t), \hat{x}_2^i(t), \cdots, \hat{x}_n^i(t))^T$, $e_1^i(t_k) = x_1^i(t_k) - \hat{x}_1^i(t_k), i = 1, 2, \cdots, N$, $\hat{x}^0(t) = (\hat{x}_1^0(t), \hat{x}_2^0(t), \cdots, \hat{x}_n^0(t))^T$, $e_1^0(t_k) = x_1^0(t_k) - \hat{x}_1^0(t_k)$, $W = \text{diag}(l^n, l^{n-1}, \cdots, l)^T$, with $l\geqslant 1$, $M = \text{diag}(l, l^2, \cdots, l^n)$, $f^i(\hat{x}^i(t)) = (f_1^i(\hat{x}^i(t)), f_2^i(\hat{x}^i(t)), \cdots, f_n^i(\hat{x}^i(t)))^T, i = 0, 1, \cdots, N$, and $f_m^i(\hat{x}^i(t)) = f_m^i(\hat{x}_1^i(t), \hat{x}_2^i(t), \cdots, \hat{x}_m^i(t)), m = 1, 2, \cdots, n$. $K\in R^{1\times n}$ and $a = (a_1, a_2, \cdots, a_n)^T$ will be determined later.

**Remark 2.** From the literature, the consensus problem of multi-agent systems with sampled-data, time-delay, or sampled-data and time-delay has been extensively studied in the past years. Most of the existing literature are on state consensus problem. The papers on output consensus problem are rare. We consider the output consensus problem of multi-agent systems with delayed sampled-data and unknown nonlinear dynamics in this paper. To the best of our knowledge, for the considered problem in this paper, no similar results appear in the existing literature. In this paper, a novel distributed controller is proposed for the considered system to reach output consensus globally exponentially. The essential difficulties to solve the problem are mainly in three aspects: firstly, the construction of the distributed controller. The distribution of the controller is very important in multi-agent systems, the design of distributed controller depends on the choice of the Lyapunov function; secondly, the construction of appropriate Lyapunov function and auxiliary integral function to deal with the delayed sampled-data and derive the global exponential stability; thirdly, the derivation of the sufficiency conditions guaranteing global exponential convergence. For this end, some inequality techniques are applied to solve the considered problem and give both the bounds of the allowable delay and sampling period.

Note that

$$\lim_{t\to(t_{k+1}+\tau_{k+1})^-} x^i(t) = \lim_{t\to(t_{k+1}+\tau_{k+1})^+} x^i(t),$$

and

$$\lim_{t\to(t_{k+1}+\tau_{k+1})^-} \hat{x}^i(t) = \lim_{t\to(t_{k+1}+\tau_{k+1})^+} \hat{x}^i(t).$$

Therefore, $x^i(t), \hat{x}^i(t)$ are continuous in all of the time intervals $[t_k+\tau_k, t_{k+1}+\tau_{k+1}), k = 0, 1, 2, \cdots, \infty$, that is to say $x^i(t), \hat{x}^i(t)$ are continuous in the time interval $[t_0, \infty)$. Since $e_1^i(t_k) = x_1^i(t_k) - \hat{x}_1^i(t_k)$, we have the sequence $\{e_1^i(t)\}, i = 0, 1, \cdots, N$ is also continuous in the time interval $[t_0, \infty)$.

**Definition 1.** [39] Let $x_1^i(t) = x_0^i$, $\hat{x}_1^i(t) = \hat{x}_0^i$ , $i = 0, 1, \cdots, N$ for $t \in [t_0 - T_{\max} - \tau, t_0]$. We call that the system (3-4) is globally exponentially stabilizable , if there exists a system (5) such that the system (3-4) and (5) satisfies $\|\hat{x}_1^i(t) - \hat{x}_1^0(t)\| \leqslant e^{-\lambda(t-t_0)}\varphi(\|\hat{x}_0^i\|, \|\hat{x}_0^0\|)$ and $\|\hat{x}_1^i(t) - x_1^i(t)\| \leqslant e^{-\lambda(t-t_0)}\varphi(\|\hat{x}_0^i\|, \|x_0^i\|)$ for any $x_0^i$, $\hat{x}_0^i$, where $\lambda > 0$ and $\varphi : R^+ \to R^+$ is a non-decreasing function. Or, we call that system (5) globally exponentially stabilizes the system (3-4).

The control objective of this paper is to design distributed controllers $u^i(t), i = 1, 2, \cdots, N$, such that the system (3) and (4) reach output consensus, that is, $\lim_{t\to\infty} |y^i(t) - y^0(t)| = 0, i = 1, 2, \cdots, N$.

To proceed, some related assumptions and lemmas are given:

**Assumption 1.** Graph $\mathcal{G}$ is connected.

**Lemma 1.** [40] Laplacian matrix $\mathcal{L}$ of graph $\mathcal{G}$ has at least one zero eigenvalue with $\mathbf{1}_N = (1, 1, \cdots, 1)^T \in \mathcal{R}^N$ as its eigenvector, and all the non-zero eigenvalues of $\mathcal{L}$ are positive. Laplacian $\mathcal{L}$ has a simple zero eigenvalue if and only if graph $\mathcal{G}$ is connected.

**Lemma 2.** [41] For any positive definite matrix $G \in R^{n\times n}$, scalar $\gamma > 0$, vector-function $\omega : [0, \gamma] \to R^n$ such that the integrations concerned are well defined, the following inequality holds:

$$[\int_0^\gamma \omega(s)ds]^T G[\int_0^\gamma \omega(s)ds] \leqslant \gamma[\int_0^\gamma \omega(s)^T G\omega(s)ds]$$

.

**Assumption 2.** The nonlinear functions $f^i(x_1(t), x_2(t), \cdots, x_m(t))$ and $f^i(y_1(t), y_2(t), \cdots, y_m(t))$, $m = 1, 2, \cdots, n$, are assumed to be continuous in the time interval $[t_0, \infty)$ and satisfying the following Lipschitz condition

$$\begin{aligned} & |f^i(x_1(t), x_2(t), \cdots, x_m(t)) - f^i(y_1(t), y_2(t), \cdots, y_m(t))| \\ \leqslant\ & l_m^i(|x_1(t) - y_1(t)| + |x_2(t) - y_2(t)| + \cdots + |x_m(t) - y_m(t)|), \end{aligned}$$

where $l_m^i > 0$ is the Lipschitz constants.

## 3. Main results

In this section, we firstly derive the state estimation error system, then, a sufficient condition is given to guarantee global exponential convergence of the consensus error. Since the systems are continuous in the time interval $[t_0, \infty)$, so we can only consider a sampling non-switching time interval $[t_k + \tau_k, t_{k+1} + \tau_{k+1}), k \geqslant 0$, in the following derivation.

From(3)-(5) we get the state estimation error system:

$$\dot{e}^i(t) = A_0 e^i(t) - Mae_1^i(t_k) + \tilde{f}^i, i = 1, 2, \cdots, N, \tag{6}$$

and

$$\dot{e}^0(t) = A_0 e^0(t) - Mae_1^0(t_k) + \tilde{f}^0, \tag{7}$$

where $e^i(t) = x^i(t) - \hat{x}^i(t)$, $\tilde{f}^i = f^i(x^i(t)) - f^i(\hat{x}^i(t)), i = 0, 1, \cdots, N$.

Let $\varepsilon^i(t) = (\varepsilon_1^i(t), \varepsilon_2^i(t), \cdots, \varepsilon_n^i(t))^T, \varepsilon_m^i(t) = \frac{e_m^i(t)}{l^{m-1}}, i = 0, 1, \cdots, N, m = 1, 2, \cdots, n$, we have

$$\dot{\varepsilon}^i(t) = lA_0\varepsilon^i(t) - la\varepsilon_1^i(t_k) + \tilde{g}^i, i = 1, 2, \cdots, N, \tag{8}$$

and

$$\dot{\varepsilon}^0(t) = lA_0\varepsilon^0(t) - la\varepsilon_1^0(t_k) + \tilde{g}^0, \tag{9}$$

where $\varepsilon_1^i(t_k) = x_1^i(t_k) - \hat{x}_1^i(t_k) = e_1^i(t_k), \tilde{g}^i = (\tilde{g}_1^i, \tilde{g}_2^i, \cdots, \tilde{g}_n^i)^T, \tilde{g}_m^i = \frac{\tilde{f}_m^i}{l^{m-1}}, i = 0, 1, \cdots, N$.

Let $z^i(t) = (z_1^i(t), z_2^i(t), \cdots, z_n^i(t))^T, z_m^i(t) = \frac{\hat{x}_m^i(t) - \hat{x}_m^0(t)}{l^{m-1}}, i = 1, 2, \cdots, N$, we get

$$\dot{z}^i(t) = lA_0z^i(t) + la(\varepsilon_1^i(t_k) - \varepsilon_1^0(t_k)) + \bar{g}^i + \frac{\bar{b}}{l^{n-1}}u^i(t), \tag{10}$$

where $\bar{g}^i = (\bar{g}_1^i, \bar{g}_2^i, \cdots, \bar{g}_n^i)^T, \bar{g}_m^i = \frac{\bar{f}_m^i}{l^{m-1}}, \bar{f}_m^i = f_m^i(\hat{x}^i(t)) - f_m^0(\hat{x}^0(t))$.

Let $\varepsilon(t) = ((\varepsilon^0(t))^T, (\varepsilon^1(t))^T, \cdots, (\varepsilon^N(t))^T)^T$, $\varepsilon_1(t_k) = (\varepsilon_1^0(t_k), \varepsilon_1^1(t_k), \cdots, \varepsilon_1^N(t_k))^T, i = 0, 1, \cdots, N$, we get the matrix form of equation (8) and (9):

$$\dot{\varepsilon}(t) = l(I_{N+1} \otimes A_0)\varepsilon(t) - l\varepsilon_1(t_k) \otimes a + \tilde{g}, \tag{11}$$

where $\tilde{g} = ((\tilde{g}^0)^T, (\tilde{g}^1)^T, \cdots, (\tilde{g}^N)^T)^T$, $I_{N+1}$ is the $(N+1) \times (N+1)$ identity matrix.

Let $z(t) = ((z^1(t))^T, (z^2(t))^T, \cdots, (z^N(t))^T)^T$, we get the matrix form of equation (10) as follows:

$$\dot{z}(t) = l(I_N \otimes A_0)z(t) + l(\eta_1(t_k) - 1_N \otimes \varepsilon_1^0(t_k)) \otimes a + \bar{g} + \frac{u \otimes \bar{b}}{l^{n-1}}, \tag{12}$$

where $\eta_1(t_k) = (\varepsilon_1^1(t_k), \varepsilon_1^2(t_k), \cdots, \varepsilon_1^N(t_k))^T$, $\bar{g} = ((\bar{g}^1)^T, (\bar{g}^2)^T, \cdots, (\bar{g}^N)^T)^T, u = (u^1(t), u^2(t), \cdots, u^N(t))^T$, $I_N$ is the $N \times N$ identity matrix.

**Theorem 1.** Under Assumption 1 and 2, there exists an output feedback controller (5) such that the multi-agent system (3)-(4) is globally exponentially stable, if there exist symmetric positive definite matrices $P$ and $Q$ satisfying

$$D_1^TP + PD_1 \leqslant -r_1I, \tag{13}$$

$$A_0^TQ + QA_0 - 2\underline{\lambda}_H K^TK \leqslant -r_2I, K = \bar{b}^TQ \tag{14}$$

and $l$ satisfying

$$l > \max\{1, l_1, \frac{4n\sqrt{N}l_1\bar{\lambda}_P}{r_1}, \frac{4n\sqrt{N}l_1\bar{\lambda}_Q}{r_2}\}, \tag{15}$$

and

$$T_{\max} + \tau \leqslant \frac{1}{c_3l}, T_{\min} - \tau > (\frac{c_2}{c_1 - c_2})\frac{1}{c_1l}, \tag{16}$$

where $r_1, r_2$ are two positive constants, $c_1 = \min\{\frac{r_1}{8\bar{\lambda}_P}, \frac{r_2}{8\bar{\lambda}_Q}\}, c_2 = \frac{\kappa_1}{c_3^2}$ $,\kappa_1 = \frac{\bar{\lambda}_P^2}{r_1\underline{\lambda}_P}(12n\bar{a}_1 + 8n\bar{a}_1c_3^2)$,$c_3 > \max\{4, \frac{12n\bar{\lambda}_P^2\bar{a}_1}{r_1} + c_1, \sqrt{\frac{\kappa_1+1}{c_1}}, \frac{\kappa_1+\sqrt{\kappa_1^2+4c_1^3\kappa_1}}{2c_1^2}\}$, $\bar{a}_1 = \max\{a_i^2\}$, $l_1 = \max\{l_m^i\}$, $D_1 = \begin{pmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{pmatrix}$, $\bar{\lambda}_P = \lambda_{\max}(P), \underline{\lambda}_P = \lambda_{\min}(P), \bar{\lambda}_Q = \lambda_{\max}(Q), \underline{\lambda}_Q = \lambda_{\min}(Q)$, in which $\lambda_{\max}(\cdot)$, $\lambda_{\min}(\cdot)$ denote the maximum, minimum eigenvalues of a matrix, respectively.

*Proof.* For system (11) and (12), we consider two Lyapunov function candidates as follows

$$V_P = \varepsilon(t)^T(I_{N+1} \otimes P)\varepsilon(t),$$

and

$$V_Q = z(t)^T(I_N \otimes Q)z(t).$$

Obviously, $V_P$ and $V_Q$ are continuously differentiable at any time except for the switching time instants. At a non-switching time $t$, we obtain the time derivatives of the two Lyapunov functions, respectively.

$$\begin{array}{rcl} \dot{V}_P &=& l\varepsilon(t)^T[I_{N+1} \otimes (D_1^TP + PD_1)]\varepsilon(t) + 2\varepsilon(t)^T(I_{N+1} \otimes P)\tilde{g} \\ && +2l\varepsilon(t)^T(I_{N+1} \otimes P)[(\varepsilon_1(t) - \varepsilon_1(t_k)) \otimes a], \end{array} \tag{17}$$

and

$$\begin{array}{rcl} \dot{V}_Q &=& -lz(t)^T[(H \otimes \bar{b}K)^T(I_N \otimes Q) + (I_N \otimes Q)(H \otimes \bar{b}K)]z(t) \\ && +lz(t)^T[(I_N \otimes A_0)^T(I_N \otimes Q) + (I_N \otimes Q)(I_N \otimes A_0)]z(t) \\ && +2lz(t)^T(I_N \otimes Q)[\eta_1(t_k) - 1_N \otimes \varepsilon_1^0(t_k)] \otimes a \\ && +2z(t)^T(I_N \otimes Q)\bar{g}, \end{array} \tag{18}$$

where $\varepsilon_1(t) = (\varepsilon_1^0(t), \varepsilon_1^1(t), \cdots, \varepsilon_1^N(t))^T$.

Due to the symmetricity of $H$, there exists an orthogonal matrix $S$ such that

$$SHS^T = \Lambda = \text{diag}\{\lambda_H^1, \lambda_H^2, \cdots, \lambda_H^N\},$$

where $\lambda_H^1, \lambda_H^2, \cdots, \lambda_H^N$ are the $N$ eigenvalues of $H$, let $\underline{\lambda}_H$ denotes the minimum nonzero eigenvalues of the matrix $H$, and $K = \bar{b}^TQ$. Then we have

$$\begin{array}{rl} & -lz(t)^T[(H \otimes \bar{b}K)^T(I_N \otimes Q) + (I_N \otimes Q)(H \otimes \bar{b}K)]z(t) \\ = & -lz(t)^T[(S^T\Lambda S \otimes \bar{b}K)^T(I_N \otimes Q) + (I_N \otimes Q)(S^T\Lambda S \otimes \bar{b}K)]z(t) \\ = & -lz(t)^T(S^T \otimes I_n)[(\Lambda \otimes \bar{b}K)^T(I_N \otimes Q) + (I_N \otimes Q)(\Lambda \otimes \bar{b}K)](S \otimes I_n)z(t) \\ = & -lz(t)^T(S^T \otimes I_n)[(\Lambda \otimes \bar{b}\bar{b}^TQ)^T(I_N \otimes Q) + (I_N \otimes Q)(\Lambda \otimes \bar{b}\bar{b}^TQ)](S \otimes I_n)z(t) \\ = & -2lz(t)^T(S^T \otimes I_n)(\Lambda \otimes Q^T\bar{b}\bar{b}^TQ)(S \otimes I_n)z(t) \\ \leqslant & -2lz(t)^T(I_N \otimes \underline{\lambda}_HQ^T\bar{b}\bar{b}^TQ)z(t). \end{array} \tag{19}$$

From (18) and (19) we get:

$$\begin{array}{rcl} \dot{V}_Q &\leqslant& lz(t)^T\{I_N \otimes [A_0^TQ + QA_0 - 2\underline{\lambda}_HQ^T\bar{b}\bar{b}^TQ]\}z(t) \\ && +2lz(t)^T(I_N \otimes Q)[\eta_1(t_k) - 1_N \otimes \varepsilon_1^0(t_k)] \otimes a \\ && +2z(t)^T(I_N \otimes Q)\bar{g}. \end{array} \tag{20}$$

From (13) and (14), we have

$$l\varepsilon(t)^T[I_{N+1}\otimes(D_1^TP+PD_1)]\varepsilon(t)\leqslant -r_1l\varepsilon(t)^T\varepsilon(t), \tag{21}$$

and

$$lz(t)^T\{I_N\otimes[A_0^TQ+QA_0-2\underline{\lambda}_HQ^T\bar{b}\bar{b}^TQ]\}z(t)\leqslant -r_2lz(t)^Tz(t). \tag{22}$$

According to the inequality $2a^Tb\leqslant a^TXa+b^TX^{-1}b$, we obtain

$$\begin{array}{ll} & 2l\varepsilon(t)^T(I_{N+1}\otimes P)[(\varepsilon_1(t)-\varepsilon_1(t_k))\otimes a] \\ \leqslant & \frac{1}{4}r_1l\varepsilon(t)^T\varepsilon(t)+4nl\bar{a}_1\bar{\lambda}_P^2(\varepsilon_1(t)-\varepsilon_1(t_k))^T(\varepsilon_1(t)-\varepsilon_1(t_k))/r_1, \end{array} \tag{23}$$

and

$$\begin{array}{ll} & 2lz(t)^T(I_N\otimes Q)(\bar{\varepsilon}_1(t_k)-1_N\otimes\varepsilon_1^0(t_k))\otimes a \\ \leqslant & \frac{1}{4}r_2lz(t)^Tz(t)+4nl\bar{a}_1\bar{\lambda}_Q^2(\bar{\varepsilon}_1(t_k)-1_N\otimes\varepsilon_1^0(t_k))^T(\bar{\varepsilon}_1(t_k)-1_N\otimes\varepsilon_1^0(t_k))/r_2 \\ \leqslant & \frac{1}{4}r_2lz(t)^Tz(t)+8nl\bar{a}_1\bar{\lambda}_Q^2\varepsilon(t_k)^T\varepsilon(t_k)/r_2. \end{array} \tag{24}$$

From the Lipschitz condition, we get

$$2\varepsilon(t)^T(I_{N+1}\otimes P)\tilde{g}\leqslant 2n\sqrt{N}l_1\bar{\lambda}_P\varepsilon(t)^T\varepsilon(t), \tag{25}$$

and

$$2z(t)^T(I_N\otimes Q)\bar{g}\leqslant 2n\sqrt{N}l_1\bar{\lambda}_Qz(t)^Tz(t). \tag{26}$$

Based on the Cauchy inequality and the **Lemma 2**, it follows that

$$\begin{array}{lll} |\varepsilon_1^i(t)-\varepsilon_1^i(t_k)|^2 & \leqslant & (t-t_k)\int_{t_k}^t|\dot{\varepsilon}_1^i(s)|^2ds \\ & = & (t-t_k)\int_{t_k}^t|l\varepsilon_2^i(s)-la_1\varepsilon_1^i(t_k)+\frac{\tilde{f}_1^{\,i}}{l^0}|^2ds \\ & \leqslant & 3l^2(t-t_k)\int_{t_k}^t[\varepsilon_2^i(s)^2+a_1^2\varepsilon_1^i(t_k)^2+\frac{l^2}{l^2}\varepsilon_1^i(s)^2]ds \\ & \leqslant & 3l^2(t-t_k)^2\bar{a}_1\varepsilon_1^i(t_k)^2 \\ & & +3l^2(t-t_k)\int_{t_k}^t[\varepsilon_1^i(s)^2+\varepsilon_2^i(s)^2]ds, i=0,1,\cdots,N, \end{array}$$

that is

$$\begin{array}{ll} & (\varepsilon_1(t)-\varepsilon_1(t_k))^T(\varepsilon_1(t)-\varepsilon_1(t_k)) \\ \leqslant & 3l^2(t-t_k)^2\bar{a}_1\varepsilon_1(t_k)^T\varepsilon_1(t_k)+3l^2(t-t_k)\int_{t_k}^t[\varepsilon_1(s)^T\varepsilon_1(s) \\ & +\varepsilon_2(s)^T\varepsilon_2(s)]ds. \end{array} \tag{27}$$

From (17)-(27), we have

$$\begin{array}{lll} \dot{V}_P & \leqslant & (-\frac{3}{4}r_1l+2n\sqrt{N}l_1\bar{\lambda}_P)\varepsilon(t)^T\varepsilon(t)+\frac{12n\bar{a}_1^2l^3\bar{\lambda}_P^2(t-t_k)^2\varepsilon_1(t_k)^T\varepsilon_1(t_k)}{r_1} \\ & & +\frac{12n\bar{a}_1l^3\bar{\lambda}_P^2(t-t_k)}{r_1}\int_{t_k}^t[\varepsilon_1(s)^T\varepsilon_1(s)+\varepsilon_2(s)^T\varepsilon_2(s)]ds, \\ \dot{V}_Q & \leqslant & (-\frac{3}{4}r_2l+2n\sqrt{N}l_1\bar{\lambda}_Q)z(t)^Tz(t)+8nl\bar{a}_1\bar{\lambda}_Q^2\varepsilon(t_k)^T\varepsilon(t_k)/r_2. \end{array}$$

Construct the following auxiliary integral function

$$V_R = \int_{t-T_{\max}-\tau}^{t} \int_{\rho}^{t} [\varepsilon(s)^T \varepsilon(s) + z(s)^T z(s)] ds d\rho.$$

We have

$$V_R \leqslant (T_{\max} + \tau) \int_{t-T_{\max}-\tau}^{t} [\varepsilon(s)^T \varepsilon(s) + z(s)^T z(s)] ds.$$

The derivative of $V_R$ is

$$\dot{V}_R = (T_{\max} + \tau)[\varepsilon(t)^T \varepsilon(t) + z(t)^T z(t)] - \int_{t-T_{\max}-\tau}^{t} [\varepsilon(s)^T \varepsilon(s) + z(s)^T z(s)] ds.$$

Let $V(t) = V_P + V_Q + l^2 V_R$, we have

$$\begin{aligned}
\dot{V}(t) \;\leqslant\; & (-\tfrac{3}{4} r_1 l + 2n\sqrt{N} l_1 \bar{\lambda}_P + (T_{\max} + \tau) l^2) \varepsilon(t)^T \varepsilon(t) \\
& + (-\tfrac{3}{4} r_2 l + 2n\sqrt{N} l_1 \bar{\lambda}_Q + (T_{\max} + \tau) l^2) z(t)^T z(t) \\
& + (\tfrac{12n\bar{a}_1^2 l^3 \bar{\lambda}_P^2 (t-t_k)^2}{r_1} + \tfrac{8n\bar{\lambda}_Q^2 \bar{a}_1 l}{r_2}) \varepsilon(t_k)^T \varepsilon(t_k) \\
& + [\tfrac{12n\bar{a}_1 l^3 \bar{\lambda}_P^2 (t-t_k)}{r_1} - l^2] \textstyle\int_{t-T_{\max}-\tau}^{t} [\varepsilon(s)^T \varepsilon(s) + z(s)^T z(s)] ds, \\
\leqslant\; & -(\tfrac{1}{4} r_1 - (T_{\max} + \tau) l) l \varepsilon(t)^T \varepsilon(t) - (\tfrac{1}{4} r_2 - (T_{\max} + \tau) l) l z^T(t) z(t) \\
& + (\tfrac{12n\bar{a}_1^2 l^3 \bar{\lambda}_P^2 (t-t_k)^2}{r_1} + \tfrac{8n\bar{\lambda}_Q^2 \bar{a}_1 l}{r_2}) \varepsilon(t_k)^T \varepsilon(t_k) \\
& + [\tfrac{12n\bar{a}_1 l^3 \bar{\lambda}_P^2 (t-t_k)}{r_1} - l^2] \textstyle\int_{t-T_{\max}-\tau}^{t} [\varepsilon(s)^T \varepsilon(s) + z(s)^T z(s)] ds, \\
\leqslant\; & -\tfrac{1}{\bar{\lambda}_P} (\tfrac{1}{4} r_1 - (T_{\max} + \tau) l) l V_P(t) - \tfrac{1}{\bar{\lambda}_Q} (\tfrac{1}{4} r_2 - (T_{\max} + \tau) l) l V_Q(t) \\
& + \tfrac{1}{\underline{\lambda}_P} (\tfrac{12n\bar{\lambda}_P^2 l^3 \bar{a}_1 (t-t_k)^2}{r_1} + \tfrac{8n\bar{\lambda}_Q^2 \bar{a}_1 l}{r_2}) V_P(t_k) \\
& + \tfrac{(12n\bar{\lambda}_P^2 l^3 \bar{a}_1 (t-t_k)/r_1) - l^2}{T_{\max} + \tau} V_R(t).
\end{aligned}$$

Because $l > \max\{1, l_1, \frac{4n\sqrt{N} l_1 \bar{\lambda}_P}{r_1}, \frac{4n\sqrt{N} l_1 \bar{\lambda}_Q}{r_2}\}, T_{\max} + \tau < \frac{1}{c_3 l}$, we have

$$\begin{aligned}
\dot{V}(t) \;\leqslant\; & -\tfrac{1}{\bar{\lambda}_P} (\tfrac{r_1}{4} - \tfrac{1}{c_3}) l V_P(t) - \tfrac{1}{\bar{\lambda}_Q} (\tfrac{r_2}{4} - \tfrac{1}{c_3}) l V_Q(t) \\
& + \tfrac{1}{\underline{\lambda}_P c_3^2} (\tfrac{12n\bar{\lambda}_P^2 \bar{a}_1}{r_1} + \tfrac{8n\bar{\lambda}_Q^2 \bar{a}_1 c_3^2}{r_2}) l V_P(t_k) \\
& + (\tfrac{12n\bar{\lambda}_P^2 \bar{a}_1}{r_1} - c_3) l^3 V_R(t).
\end{aligned}$$

Since $c_3 > \max\{\frac{8}{r_1}, \frac{8}{r_2}, \frac{12n\bar{\lambda}_P^2 \bar{a}_1}{r_1} + c_1, \sqrt{\frac{\kappa_1+1}{c_1}}, \frac{\kappa_1 + \sqrt{\kappa_1^2 + 4c_1^3 \kappa_1}}{2c_1^2}\}, c_1 \leqslant \min\{\frac{r_1}{8\bar{\lambda}_P}, \frac{r_2}{8\bar{\lambda}_Q}\}, c_2 = \frac{\kappa_1}{c_3^2}$, we have

$$\begin{aligned}
\dot{V}(t) \;\leqslant\; & -c_1 l V_P(t) - c_1 l V_Q(t) - c_1 l^3 V_R(t) + c_2 l V_P(t_k) \\
\leqslant\; & -c_1 l V(t) + c_2 l V(t_k).
\end{aligned} \tag{28}$$

Multiplying $e^{c_1 l t}$ on both sides of (28), we have

$$e^{c_1 l t} \frac{d}{dt} V(t) + e^{c_1 l t} c_1 l t V(t) \leqslant e^{c_1 l t} c_2 l V(t_k). \tag{29}$$

Integrating both sides of (29) from $t_k + \tau_k$ to $t$, we have

$$V(t) \leqslant e^{-c_1 l(t-t_k-\tau_k)} V(t_k+\tau_k) + \frac{c_2}{c_1} V(t_k) - \frac{c_2}{c_1} e^{-c_1 l(t-t_k-\tau_k)} V(t_k).$$

Note that $\varepsilon(t)$ and $z(t)$ are continuous in the interval $[t_0, \infty)$, we have

$$V(t_{k+1}+\tau_{k+1}) \leqslant e^{-c_1 l(T_{k+1}+\tau_{k+1}-\tau_k)} V(t_k+\tau_k) + \frac{c_2}{c_1} V(t_k) - \frac{c_2}{c_1} e^{-c_1 l(T_{k+1}+\tau_{k+1}-\tau_k)} V(t_k), \quad (30)$$

and

$$V(t_{k+1}) \leqslant e^{-c_1 l(T_{k+1}-\tau_k)} V(t_k+\tau_k) + \frac{c_2}{c_1} V(t_k) - \frac{c_2}{c_1} e^{-c_1 l(T_{k+1}-\tau_k)} V(t_k). \quad (31)$$

From (30) and (31), we have

$$\begin{array}{rl} & V(t_{k+1}+\tau_{k+1}) + \rho_1 V(t_{k+1}) \\ \leqslant & (e^{-c_1 l(T_{k+1}+\tau_{k+1}-\tau_k)} + \rho_1 e^{-c_1 l(T_{k+1}-\tau_k)}) V(t_k+\tau_k) \\ & +\frac{c_2}{c_1}[1+\rho_1 - e^{-c_1 l(T_{k+1}-\tau_k)}(e^{-c_1 l \tau_{k+1}} + \rho_1)] V(t_k), \end{array} \quad (32)$$

where $\rho_1$ is a positive constant number.

Since $c_2 = \frac{\kappa_1}{c_3^2}$. Then $c_3 > \frac{\kappa_1}{c_1}$ implies that $c_1 > \frac{\kappa_1}{c_3} = c_2 c_3 > 8c_2$, i.e., $\frac{c_2}{c_1} < \frac{1}{8}$. From $T_{\min} - \tau > \frac{c_2}{c_1(c_1-c_2)l} > 0$, we can choose some $\rho_1 > 0$ such that

$$\eta_1 = \max_{k \geqslant 0} \{ \frac{c_2}{c_1 \rho_1} [1+\rho_1 - e^{-c_1 l(T_{k+1}-\tau_k)}(e^{-c_1 l \tau_{k+1}} + \rho_1)] \} < 1, \quad (33)$$

and

$$\eta_2 = \max_{k \geqslant 0} \{ \rho_1 e^{-c_1 l(T_{k+1}-\tau_k)} + e^{-c_1 l(T_{k+1}+\tau_{k+1}-\tau_k)} \} < 1. \quad (34)$$

Let $\eta = \max\{\eta_1, \eta_2\}$. From (33) and (34), we have $0 < \eta < 1$. Then, it follows that

$$V(t_{k+1}+\tau_{k+1}) + \rho_1 V(t_{k+1}) \leqslant \eta [V(t_k+\tau_k) + \rho_1 V(t_k)]. \quad (35)$$

Through the iteration for (35), we have

$$V(t_k+\tau_k) + \rho_1 V(t_k) \leqslant \eta^k [V(t_0+\tau_0) + \rho_1 V(t_0)].$$

Since $0 < \eta < 1$, it is clear that the sequence $\{V(t_k+\tau_k) + \rho_1 V(t_k)\}$ is convergent to zero, and

$$V(t) \leqslant V(t_k+\tau_k) + \rho_1 V(t_k) \leqslant \eta^k [V(t_0+\tau_0) + \rho_1 V(t_0)], t \in [t_k+\tau_k, t_{k+1}+\tau_{k+1}).$$

Since for any $t > t_0 + \tau_0$, there exists $k \geqslant 0$ such that $t \in [t_k+\tau_k, t_{k+1}+\tau_{k+1})$. It follows that $\frac{t-t_0-\tau_0}{T_{\max}+\tau} - 1 \leqslant k$. Then

$$\begin{array}{rcl} V(t) & \leqslant & \eta^{\frac{t-t_0-\tau_0}{T_{\max}+\tau}-1} [V(t_0+\tau_0) + \rho_1 V(t_0)] \\ & = & e^{\frac{t}{T_{\max}+\tau} \ln \eta} \eta^{\frac{-t_0-\tau_0}{T_{\max}+\tau}-1} [V(t_0+\tau_0) + \rho_1 V(t_0)]. \end{array}$$

From the **Definition 1**, we get the system is globally exponentially stable. This completes the proof of the theorem. □

From **Theorem 1**, we have the following results which give the estimates of the allowable sampling interval and time delays.

**Corollary 1.** There exists an output feedback control law in the form of (5), which globally exponentially stabilizes the system (3-4), if there exist symmetric positive definite matrices $P, Q$ and a constant $l$ such that (13), (14), (15) and

$$\tau < \frac{1}{\rho_2 l}(\frac{1}{c_3} - \frac{1}{c_4}), T_{\max} < \frac{1}{c_3 l} - \frac{1}{\rho_2 l}(\frac{1}{c_3} - \frac{1}{c_4}), T_{\min} > \frac{1}{c_4 l} + \frac{1}{\rho_2 l}(\frac{1}{c_3} - \frac{1}{c_4}), \tag{36}$$

are satisfied, where $\rho_2 > 2, c_4 = \frac{(c_1 - c_2)c_1}{c_2}, c_1, c_2, c_3$ and $\kappa_1$ are given in Theorem 1.

*Proof.* From $c_3 > \frac{\kappa_1 + \sqrt{\kappa_1^2 + 4c_1^3\kappa_1}}{2c_1^2}$, we have $c_1^2 c_3^2 - c_3\kappa_1 - c_1\kappa_1 > 0$. Then $c_3 < \frac{c_1(c_1 c_3^2 - \kappa_1)}{\kappa_1} = c_1(\frac{c_1}{c_2} - 1) = c_4$. Therefore $\frac{1}{c_3} - \frac{1}{c_4} > 0$. Since $\frac{c_2}{c_1} < \frac{1}{8}$ and $c_4 > 0$, it is easy to check that the conditions (36) imply that the conditions (16) are satisfied. The proof is completed. □

**Remark 3.** This corollary does not give the maximum sampling interval and time-delay. The maximum sampling interval and time-delay depend on the choices of $a_i, P, Q$.

Based on **Theorem 1** and **Corollary 1**, an algorithm to set the design parameters is presented as follows:

Step 1: Based on the interconnected topology, we construct the matrix $H$ and calculate the smallest eigenvalue of $H$ which is denoted by $\underline{\lambda}_H$;

Step 2: We choose the appropriate values of $a_i$ such that the inequality (13) hold and get the matrix $P$, $\bar{\lambda}_P, \underline{\lambda}_P$, then by solving the inequality (14) get the matrix $Q$, $\bar{\lambda}_Q, \underline{\lambda}_Q$, and let $K = \bar{b}^T Q$;

Step 3: We select $l$ such that the condition (15) is satisfied;

Step 4: Calculate $c_1$, select $\rho_2 > 2$ and $c_3$ such that $c_3 > \max\{4, \frac{12n\bar{\lambda}_P^2\bar{a}_1}{r_1} + c_1, \sqrt{\frac{\kappa_1+1}{c_1}}, \frac{\kappa_1 + \sqrt{\kappa_1^2 + 4c_1^3\kappa_1}}{2c_1^2}\}$, then calculate $c_2$ and $c_4$.

## 4. Simulations

In this section, we give an example to validate our theoretical results. In the example, we consider a multi-agent system consisting of four agents and a leader. The dynamics of the leader and the followers are described by

$$\begin{cases} \dot{x}_1^0(t) = x_2^0(t) + l_0\cos(t)x_1^0(t), \\ \dot{x}_2^0(t) = x_3^0(t) + l_0\cos(t)x_2^0(t), \\ \dot{x}_3^0(t) = l_0(\sin(t)x_2^0(t) + 2\cos(t)x_3^0(t)), \end{cases}$$

and

$$\begin{cases} \dot{x}_1^i(t) = x_2^i(t) + l_0\cos(t)x_1^i(t), \\ \dot{x}_2^i(t) = x_3^i(t) + l_0\cos(t)x_2^i(t), \\ \dot{x}_3^i(t) = u^i + l_0(\sin(t)x_2^i(t) + 2\cos(t)x_3^i(t)), \end{cases}$$

respectively, where $l$ is a constant.

An observer-based output feedback controller is proposed as follows:

$$\begin{cases} \dot{\hat{x}}_1^0(t) = \hat{x}_2^0(t) + la_1e_1^0(t_k) + l_0\cos(t)\hat{x}_1^0(t), \\ \dot{\hat{x}}_2^0(t) = \hat{x}_3^0(t) + l^2a_2e_1^0(t_k) + l_0\cos(t)\hat{x}_2^0(t), \\ \dot{\hat{x}}_3^0(t) = l^3a_3e_1^0(t_k) + l_0(\sin(t)\hat{x}_2^0(t) + 2\cos(t)\hat{x}_3^0(t)), \end{cases}$$

$$\begin{cases} \dot{\hat{x}}_1^i(t) = \hat{x}_2^i(t) + la_1 e_1^i(t_k) + l_0\cos(t)\hat{x}_1^i(t), \\ \dot{\hat{x}}_2^i(t) = \hat{x}_3^i(t) + l^2 a_2 e_1^i(t_k) + l_0\cos(t)\hat{x}_2^i(t), \\ \dot{\hat{x}}_3^i(t) = u^i + l^3 a_3 e_1^i(t_k) + l_0(\sin(t)\hat{x}_2^i(t) + 2\cos(t)\hat{x}_3^i(t)), \end{cases}$$

where

$$u^i(t) = -KW[\textstyle\sum_{j\in N^i} a_{ij}(\hat{x}^i(t) - \hat{x}^j(t)) + b_i(\hat{x}^i(t) - \hat{x}^0(t))],$$

and $W = \text{diag}(l^3, l^2, l)^T$.

Supposes that the interconnected topology is described as in Fig.1. We obtain that the largest and smallest nonzero eigenvalue of $H = \mathcal{L} + \mathcal{B}$ are $\bar{\lambda}_H = 2.618$ and $\underline{\lambda}_H = 0.382$. By choosing appropriate parameters of $a = (a_1, a_2, a_3) = (4, 4, 4)$, and solving the inequality (13), we get the solution $P > 0$, and $\bar{\lambda}_P = 7.0042, \underline{\lambda}_P = 0.2291$. By solving the inequality (14), we get the positive definite matrix $Q = \begin{pmatrix} 2.4841 & 2.5853 & 1.1440 \\ 2.5853 & 5.2780 & 2.8419 \\ 1.1440 & 2.8419 & 2.9577 \end{pmatrix}$, then $K = \bar{b}^T Q = (1.1440, 2.8419, 2.9577)$, $\bar{\lambda}_Q = 8.4858, \underline{\lambda}_Q = 0.6646$. Simulation is conducted in 30s time, the initial states of the whole system (3-4) are $x_0 = [1.5, 2.6, 0.82, 3, 1.31, 3.27, 3.44, 3.74, 2.25, 0.42, 1.14, 4.57, 0.76, 4.12, 2.69]^T$, and $\hat{x}_0 = [3.13, 1.46, 2.16, 0.07, 4.92, 0.84, 0.53, 1.86, 0.99, 2.45, 1.69, 4.76]^T$. Under the observer-based output feedback control law (5), selecting parameter $l_0 = 0.1, l = 1.5$. The sampling interval $T_k$ and the time-delay $\tau_k$ are given as $T_k = 0.01s$ and $\tau_k = 0.001s$, respectively. The simulation results are shown in Fig.2 and Fig.3. Fig.2 shows that the twelve components of the state estimation errors $e_m^i(t) = x_m^i(t) - \hat{x}_m^i(t), m = 1, 2, 3, i = 1, 2, 3, 4$, converge to zero. Fig.3 shows that the four components of the output error between the follower agents and the leader agents $\bar{x}_1^i(t) = x_1^i(t) - x_1^0(t), i = 1, 2, 3, 4$, converge to zero.

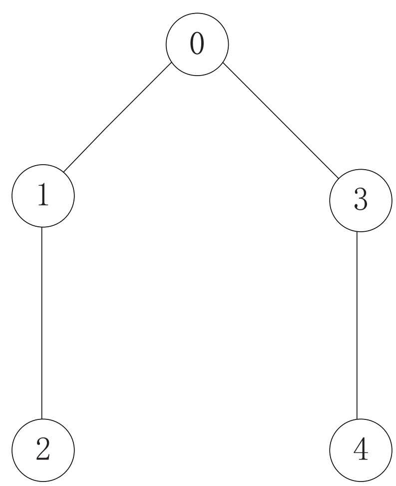


Fig. 1: Connected graph

**Remark 4.** So far, the consensus problem of multi-agent systems with the sampled-data [22, 33], sampled-data and time-delay [27, 28], nonlinear dynamics and time-delay [42, 43, 44] have been reported in the literatures. All of the above mentioned literatures are on the state consensus. The literatures on output consensus problem are summarized as follows:

cases 1: the leader-following systems with unknown nonlinear dynamics without considering sampled-data and time-delay [30, 45, 46],

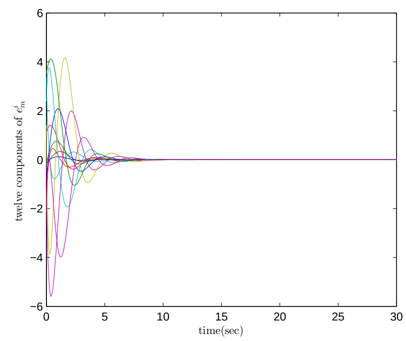


Fig. 2: The state estimation error

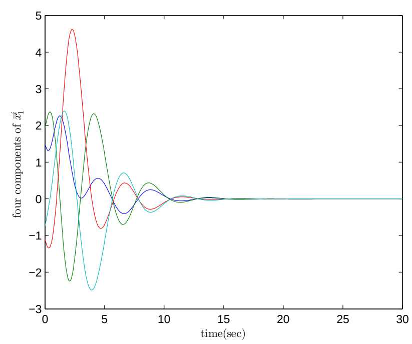


Fig. 3: The output error

cases 2: the leader-following systems with unknown nonlinear dynamics and time-delay without considering sampled-data [47],

cases 3: the leaderless systems with unknown nonlinear dynamics, time-delay and sampled-data [48].

In this paper, the output consensus problem for leader-following multi-agent systems with unknown nonlinear dynamics and delayed sampled-data is studied. The algorithm proposed in this paper is more general.

## 5. Conclusion

This paper considers the output consensus problem of high-order leader-following multi-agent systems with unknown nonlinear dynamics, in which the output data of the system are delayed and sampled. A novel continuous observer-based output feedback controller is presented such that the outputs of the system reach consensus exponentially. The connectedness of interconnected graphs is required to guarantee the exponential convergence of the considered systems. Moreover, the estimates of the allowable sampling period and time-delay were also obtained.

## 6. Acknowledgment

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## 7. References

[1] Fax, J.A., Murray, R.M.: 'Information flow and cooperative control of vehicle formations', IEEE Transactions on Automatic Control, 2004, **49**, (9), pp. 1465–1476

[2] Olfati-Saber, R., Murray, R.M.: 'Consensus problems in networks of agents with switching topology and time-delays', IEEE Transactions on Automatic Control, 2004, **49**, (9), pp. 1520–1533

[3] Olfati-Saber, R., Fax, J.R.M.: 'Murray, Consensus and cooperation in multi-agent networked systems', Proceedings of the IEEE, 2007, **95**, (1), pp. 215–233

[4] Ren, W., Beard, R.W.: 'Consensus seeking in multi-agent systems under dynamically changing interaction topologies', Automatica, 2005, **50**, (5), pp. 655–661

[5] Ren, W.: 'Multi-vehicle consensus with a time-varying reference state', Systems & Control Letters, 2007, **56**, (7-8), pp. 474–483

[6] Xiao, F., Wang, L.: 'Consensus protocols for discrete-time multi-agent systems with time-varying delays', Automatica, 2008, **44**, (10), pp. 2577–2582

[7] Lin, P., Jia, Y.M.: 'Consensus of second-order discrete-time multi-agent systems with nonuniform time-delays and dynamically changing topologies', Automatica, 2009, **45**, (9), pp. 2154–2158

[8] Zhu, W., Cheng, D.Z.: 'Leader-following consensus of second-order agents with multiple time-varying delays', Automatica, 2010, **46**, (12), pp. 1994–1999

[9] Li, S.H., Du, H.B., Lin, X.Z.: 'Finite-time consensus algorithm for multi-agent systems with double-integrator dynamics', Automatica, 2011, **47**, (8), pp. 1706–1712

[10] Cao, Y.C., Ren, W.: 'Finite-time consensus for multi-agent networks with unknown inherent nonlinear dynamics', Automatica, 2014, **50**, (10), pp. 2648–2656

[11] Yu, H., Shen, Y.J., Xia, X.H.: 'Adaptive finite-time consensus in multi-agent networks', Systems & Control Letters, 2013, **62**, (10), pp. 880–889

[12] Wang, L., Xiao, F.: 'Finite-time consensus problems for networks of dynamic agents', IEEE Transactions on Automatic Control, 2010, **55**, (4), pp. 950–955

[13] Khoo, S., Xie, L. , Man, Z.: 'Robust finite-time consensus tracking algorithm for multirobot systems', IEEE/ASME Transactions on Mechatronics, 2009, **14**, (2), PP. 219–228

[14] Tahbaz-Salehi, A., Jadbabaie, A.: 'A necessary and sufficient condition for consensus over random networks', IEEE Transactions on Automatic Control, 2008, **53**, (3), pp. 791–795

[15] Tang, Y., Gao, H.J., Zhang, W.B., Kurths, J.: 'Leader-following consensus of a class of stochastic delayed multi-agent systems with partial mixed impulses', Automatica, 2015, **53**, pp. 346–354

[16] Chen, W.S., Li, X.B., Jiao, L.C.: 'Quantized consensus of second-order continuous-time multi-agent systems with a directed topology via sampled data', Automatica, 2013, **49**, (7), pp. 2236–2242

[17] Wang, Y., Wu, Q.H., Wang, Y.Q., Yu, D.: 'Quantized consensus on first-order integrator networks', Systems & Control Letters, 2012, **61**, (12), pp. 1145–1150

[18] Li, D.Q., Liu, Q.P., Wang, X.F., Yin, Z.X.: 'Quantized consensus over directed networks with switching topologies', Systems & Control Letters, 2014, **65**, (65), pp. 13–22

[19] Chen, J., Guan, Z.H., He, D.X., Chi, M., Cheng, X.M.: 'Multi-consensus for second-order multi-agent systems based on sampled position information', IET Control Theory and Applications, 2015, **9**, (3), pp. 358–366

[20] Yan, H.C., Qian, F.F., Zhang, H., Yang, F.W., Guo, G.: '$H_\infty$ Fault Detection for Networked Mechanical Spring-Mass Systems with Incomplete Information', IEEE Transactions on Industrial Electronics, 2016, DOI 10.1109/TIE.2016.2559454

[21] Yan, H.C., Qian, F.F., Yang, F.W., Shi, H.B.: '$H_\infty$ filtering for nonlinear networked systems with randomly occurring distributed delays, missing measurements and sensor saturation', InformationSciences, 2015, pp. 1-11

[22] Guan, Z.H., Liu, Z.W., Feng, G., Jian, M.: 'Impulsive consensus algorithms for second-order multi-agent networks with sampled information', Automatica, 2012, **48**, (7), pp. 1397–1404

[23] Wang, Y.W., Yang, W., Xiao, J.W., Zeng, Z.G.: 'Impulsive Multisynchronization of Coupled Multistable Neural Networks With Time-Varying Delay', IEEE Transactions on Neural Networks and Learning Systems, 2016, pp. 1-12

[24] Wang, Y.W., Yi, J.W.: 'Consensus in second-order multi-agent systems via impulsive control using position-only information with heterogeneous delays', IET Control Theory and Applications, 2015, **9**, (3), pp. 336-345

[25] Wang, Y.W., Bian, T., Xiao, J.W., Wen, C.Y.: 'Global Synchronization of Complex Dynamical Networks Through Digital Communication With Limited Data Rate', IEEE Transactions on Neural Network and Learning Systems, 2015, **26**, (10), pp. 2487–2499

[26] Yu, H., J.Antsaklis, P.: 'Output Synchronization of Networked Passive Systems With Event-Driven Communication', IEEE Transactions on Automatic Control, 2014,**59**, (3), pp. 750-756

[27] Yu, W.W., Zhou, L., Yu, X.H.: 'Consensus in multi-agent systems with second-order dynamics and sampled data', IEEE Transactions on Industrial Informatics, 2013, **9**, (4), pp. 2137–2146

[28] Huang, N., Duan, Z.S., Chen, G.R.: 'Some necessary and sufficient conditions for consensus of second-order multi-agent systems with sampled position data', Automatica, 2016, **63**, pp. 148–155

[29] Liu, T., Cao, M., J.Hill, D.: 'Distributed Event-Triggered Control for Output Synchronization of Dynamical Networks with Non-identical Nodes', 53rd IEEE Conference on Decision and Control, 2014, (15-17), pp. 3554-3559

[30] Zhang, X., Chen, M.Y., Wang, L., Pang, Z.H., Zhou, D.H.: 'Connection-graph-based event-triggered output consensus in multi-agent systems with time-varying couplings', IET Control Theory and Applications, 2015,**9**, (1), pp. 1-9

[31] Hu, W.F., Liu, L.: 'Cooperative Output Regulation of Heterogeneous Linear Multi-Agent Systems by Event-Triggered Control', IEEE Transactions on Cybernetics, 2016, pp. 1-12

[32] Li, Z.K, Duan, Z.S., Chen, G.R., Huang, L.: 'Consensus of Multiagent Systems and Synchronization of Complex Networks: A Unified Viewpoint', IEEE Transactions on Circuits and Systems, 2010, **57**, (1), pp. 213-224

[33] Yu, W.W, Zheng, W.X., Chen, G.R., Ren, W., Cao, J.D.: 'Second-order consensus in multi-agent dynamical systems with sampled position data', Automatica, 2011, **47**, (7), pp. 1496–1503

[34] Riccardo, M., Patrizio, T.: 'Nonlinear Control Design: Geometric, Adaptive and Robust', Prentice Hall, 1995

[35] Wang, X.H., : 'Leader-Follower Consensus for a Class of Nonlinear Multi-Agent Systems', International Journal of Control, Automation, and Systems, 2012,**10**, (1), pp. 27-35

[36] Su, Y., Huang, J.: 'Global robust output regulation for nonlinear multiagent systems in strict feedback form', In 12th international conference on control automation robotics and vision , 2012, pp. 436-441

[37] Yoo, S.J.: 'Distributed adaptive containment control of uncertain nonlinear multi-agent systems in strict-feedback form', Automatica, 2013, **49**, (7), pp. 2145-2153

[38] Zhang, X.F., Liu, L., Feng, G.: 'Leader-follower consensus of time-varying nonlinear multi-agent systems', Automatica, 2015,**52**, pp. 8-14

[39] Shen, Y.J, Zhang, D.Y, Xia, X.H.: 'Continuous output feedback stabilization for nonlinear systems based on sampled and delayed output measurements', Int. J. Robust Nonlinear Control, 2015, DOI: 10.1002/rnc.3491

[40] Godsil, C.D., Royle, G.: 'Algebraic Graph Theory', New York Springer Press, 2001

[41] Liu, Y., Wang, Z., Liu, X.: 'On global exponential stability of generalized stochastic neural networks with mixed time-delays', Neurocomputing, 2006, **70**, (1-3), pp. 314–326

[42] Hu, Y.X., Yu, W.W., Xuan, Q., Yu, L., Xie, G.M.: 'Consensus of multi-agent systems in the cooperationCcompetition network with inherent nonlinear dynamics: A time-delayed control approach', Neurocomputing, 2015, **158**, pp. 134-143

[43] Miao, G.Y., Wang, Z., Ma, Q., Lu, J.W.: 'Consensus of second-order multi-agent systems with nonlinear dynamics and time delays', Neural Comput & Applic, 2013, **23**, pp. 761-767

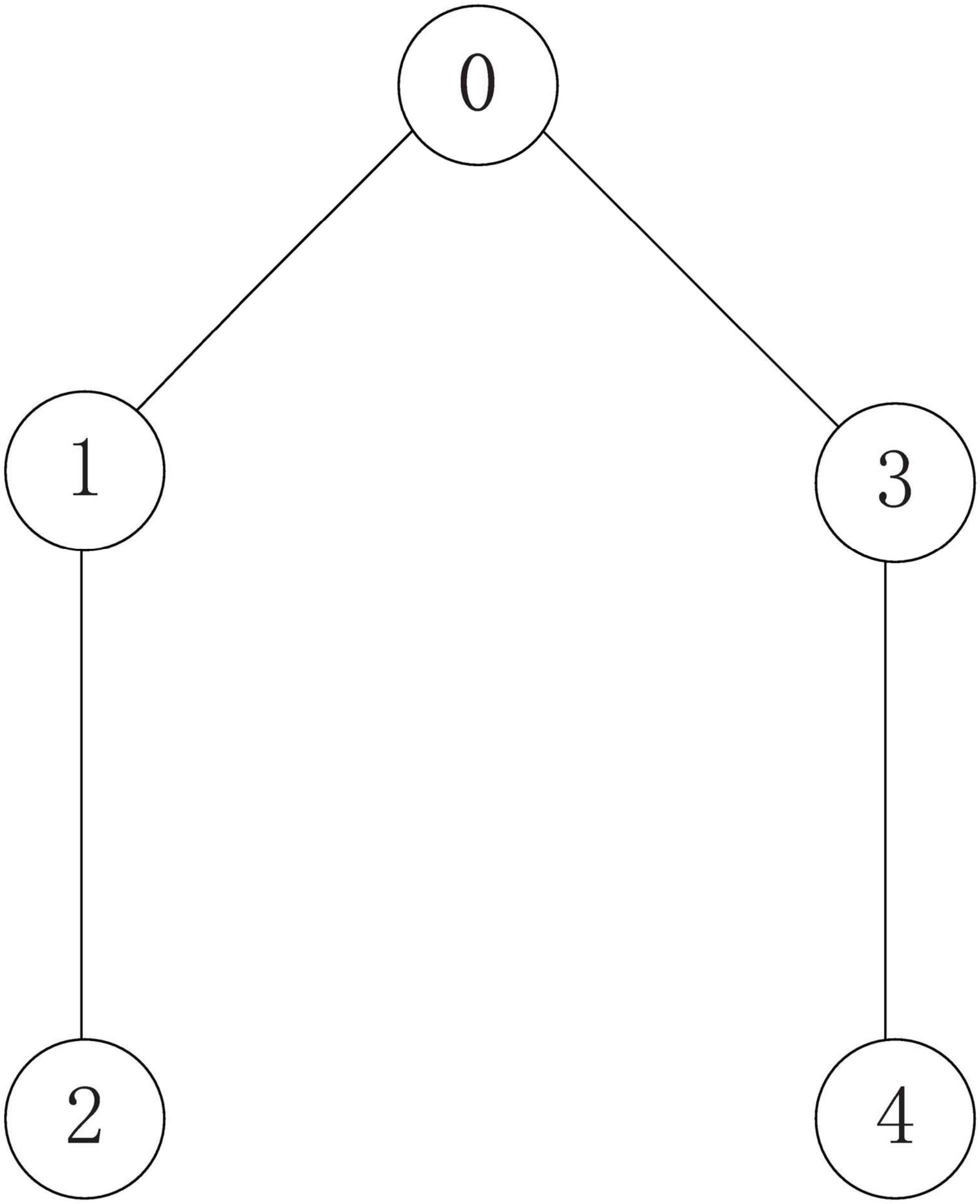
[44] Qian, Y.F., Wu, X.Q., Lu, J.H., Lu, J.A.: 'Consensus of second-order multi-agent systems with nonlinear dynamics and time delay', Nonlinear Dyn, 2014, **78**, pp. 495-503

[45] Zhang W.T., Kou, K.L., Lou, J.G., Liu, Y.: 'Observer based consensus for nonlinear multi-agent systems with communication failures', Neurocomputing, 2016, **173**, pp. 1034-1043

[46] Xu, X.L., Chen, S.Y., Gao, L.X.: 'Observer-based consensus tracking for second-order leader-following nonlinear multi-agent systems with adaptive coupling parameter design', Neurocomputing, 2015, **156**, pp. 297-305

[47] Yang, Y., Yue, D., Dou, C.X.: 'Distributed adaptive output consensus control of a class of heterogeneous multi-agent systems under switching directed topologies', Information Sciences, 2016, **345**, pp. 294-312

[48] Wen, G.H., Duan, Z.S., Yu, W.W., Chen, G.R.: 'Consensus of multi-agent systems with nonlinear dynamics and sampled-data information: a delayed-input approach', International Journal of Robust and Nonlinear Control, 2013, **23**, pp. 602-619




Connected graph

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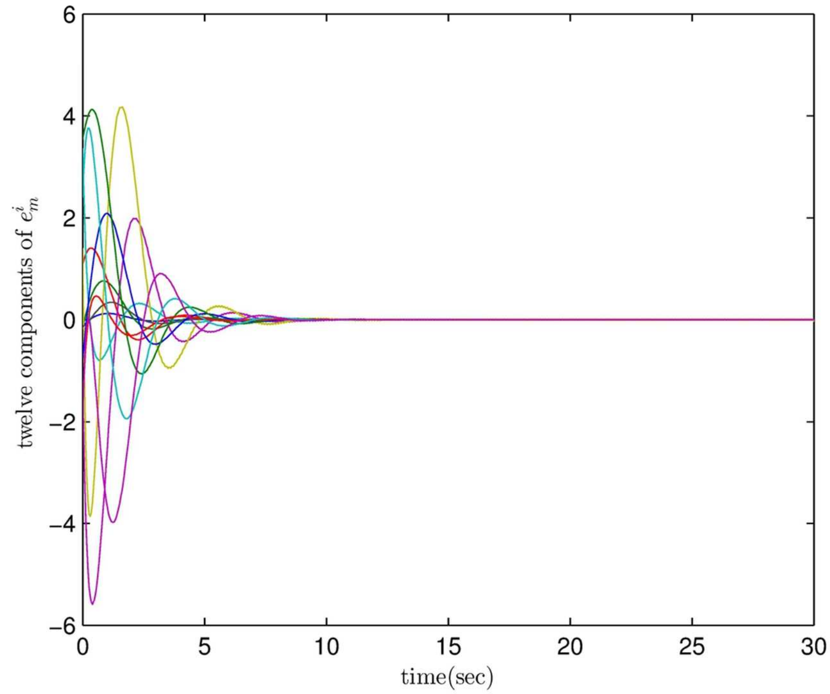
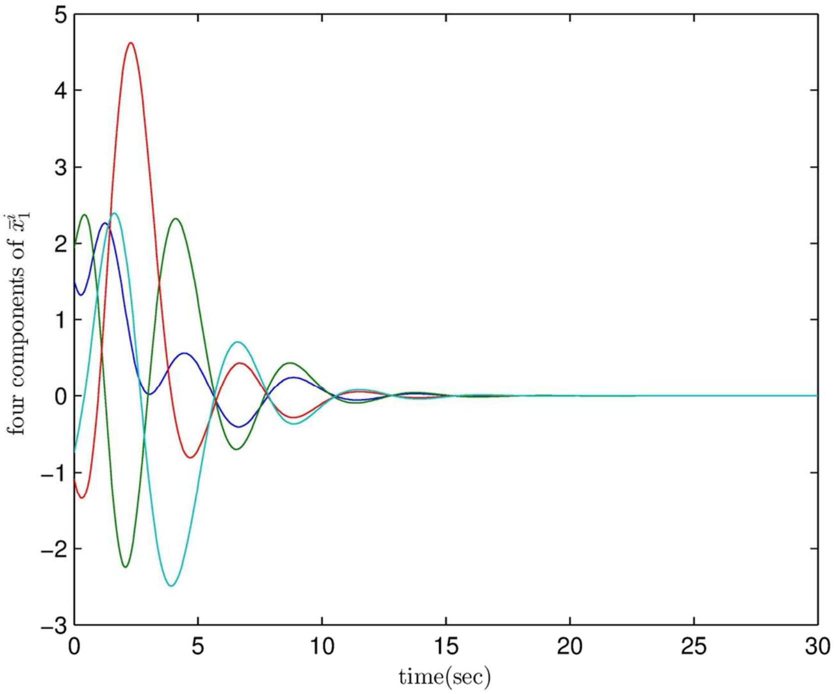



The state estimation error

111x83mm (300 x 300 DPI)




The output error

111x83mm (300 x 300 DPI)