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Existence of Common Fixed Point for b-Metric Rational Type Contraction

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Abstract. The necessary conditions for existence of a common fixed point of two mappings satisfying generalized b-order contractive condition in the setting of a partially ordered b-complete b-metric space are presented. Also, we study well-posedness of common fixed point problem for generalized b-order contractive mappings. We employ our result to establish an existence of a solution of an integral equation.

1. Introduction

Alber and Guerre-Delabrere [6] proved that a weakly contractive mapping defined on a Hilbert space is a Picard operator. Rhoades [35] established the same result considering the domain of mapping a complete metric space instead of Hilbert space. The study of common fixed points of mappings satisfying certain contractive conditions can be employed to establish existence of solutions of certain operator equations such as differential and integral equations. Beg and Abbas [10] obtained common fixed points extending a weak contractive condition to two maps . In 2009, Dorić [17] proved common fixed point theorems for generalized (ψ, ϕ)-weakly contractive mappings. Abbas and Dorić [4] obtained a common fixed point theorem for four maps. For more work in this direction, we refer to $[1-4, 8, 10, 15, 16, 21-28, 33-35, 37-39]$ and references mentioned therein.

The study of fixed points of mappings on complete metric spaces equipped with a partial ordering $\preceq$ was first investigated in [34] Ran and Reurings, and then by Nieto and Lopez [31] subsequently extended the result of Ran and Reurings [34] for non-decreasing mappings and applied to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions (see also [31]).

The concept of b-metric space was introduced by Czerwik in [14]. Since then, several papers have been published on the fixed point theory of various classes of single-valued and multi-valued operators in b-metric spaces (see also [11, 12, 36]). Pacurar [32] proved some results on sequences of almost contractions and fixed points in b-metric spaces. Hussain and Shah [20] obtained some results on KKM mappings in cone b-metric spaces. Recently, Khamsi [29] and Khamsi and Hussain [30] have dealt with spaces of this

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kind, although under different names (in the spaces called " metric-type") and obtained (common) fixed point results. In particular, they showed that most of the new fixed point existence results of contractive mappings defined on such metric spaces are merely copies of the classical ones.

Consistent with [14] and [36], the following definitions and results will be needed in the sequel.

Definition 1.1. *([14]) Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a b-metric iff, for all $x, y, z \in X$, the following conditions are satisfied:*

(b_1) $d(x, y) = 0$ iff $x = y$,

(b_2) $d(x, y) = d(y, x)$,

(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric when $s = 1$.

Following example show that in general a b-metric need not necessarily be a metric. (see also [36, p. 264]):

Example 1.2. *Let (X, d) be a metric space, and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.*

Obviously conditions (b_1) and (b_2) of definition 1.1 are satisfied.

If $1 < p < \infty$, then the convexity of the function $f(x) = x^p$ $(x > 0)$ implies

$$\left(\frac{a+b}{2}\right)^p \leq \frac{1}{2}\left(a^p + b^p\right),$$

and hence, $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ holds.

Thus, for each $x, y, z \in X$ we obtain

$$\rho(x, y) = (d(x, y))^p \leq (d(x, z) + d(z, y))^p \leq 2^{p-1}\left((d(x, z))^p + (d(z, y))^p\right) = 2^{p-1}(\rho(x, z) + \rho(z, y)).$$

So condition (b_3) of definition 1.1 is satisfied and ρ is a b-metric.

However, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space.

For example, if $X = \mathbb{R}$ (set of real numbers) and $d(x, y) = \left|x - y\right|$ is the usual metric, then $\rho(x, y) = (x - y)^2$ is a b-metric on $\mathbb{R}$ with $s = 2$, but is not a metric on $\mathbb{R}$.

Also the following example of a b-metric space is given in ([30]**)**.

Example 1.3. *([30]) Let X be the set of Lebesgue measurable functions on $[0, 1]$ such that $\int_0^1 \left|f(x)\right|^2 dx < \infty$. Define $D : X \times X \to [0, \infty)$ by*

$$D(f, g) = \int_0^1 \left|f(x) - g(x)\right|^2 dx.$$

Then D satisfies the following properties

(b_4) $D(f, g) = 0$ if and only if $f = g$,
(b_5) $D(f, g) = D(g, f)$, for any $f, g \in X$,
(b_6) $D(f, g) \leq 2(D(f, h) + D(h, g))$, for any points $f, g, h \in X$.

Khamsi ([29]) also showed that each cone metric space has a b-metric structure. In fact he proved the following interesting result.

Theorem 1.4. *([29]) Let (X, d) be a metric cone over the Banach space E with the cone P which is normal with the normal constant K. The mapping $D : X \times X \to [0, \infty)$ defined by $D(x, y) = \left\|d(x, y)\right\|$ satisfies the following properties*

(b_7) $D(x, y) = 0$ *if and only if* $x = y$,

(b_8) $D(x, y) = D(y, x)$, *for any* $x, y \in X$,

(b_9) $D(x, y) \leq K(D(x, z_1) + D(z_1, z_2) + \cdots + D(z_n, y))$, *for any points* $x, y, z_i \in X, i = 1, 2, \ldots, n$.

Let S be the class of all mappings $\beta : [0, \infty) \to [0, 1)$ which satisfy the condition: $\beta t_n \to 1$ whenever $t_n \to 0$. Note that $S \neq \phi$. For instance, a mapping $f : [0, \infty) \to [0, 1)$ given by $fx = \frac{1}{1+x}$ qualifies for a membership of S. Let f and g be two self mappings on a nonempty set X. If $x = fx$ (respectively, $fx = gx$ and $x = fx = gx$) for some x in X, then x is called a fixed point of f (respectively, coincidence and common fixed point of f and g). We define the following sets $F(f) = \{x \in X : x = fx\}$ and $C(f, g) = \{x \in X := x = gx = fx\}$. For a complete metric space (X, d), we say that f is a Picard operator if the sequence $x_{n+1} = fx_n = f^n x_0$, $n = 0, 1, 2, ...,$ converges to x^* for each $x_0 \in X$, that is, the set $F(f) = \{x^*\}$. The set $\{x_0, fx_0, f^2x_0, f^3x_0, \cdots\}$ is called an orbit of f at the point x_0 and denoted by $O_f(x_0)$.

In 1973, Geraghty [19] gave following generalization of a Banach fixed point theorem.

Theorem 1.5. *Let (X, d) be a complete metric space and f a self map on X. If there exists $\beta \in S$ such that*

$$d(fx, fy) \leq \beta(d(x, y))d(x, y)$$

for all $x, y \in X$. Then f is a Picard operator.

In the following Harandi and Emami [9] reconsidered Theorem 1.5 in the framework of a partially ordered metric spaces. Note that the condition that

$$\text{every pair of elements has a lower bound or an upper bound} \tag{1}$$

is equivalent to the condition that for every $x, y \in X$, there exists $z \in X$ which is comparable to x and y. For more details we refer to Nieto and Lopez [31]. This condition is crucial to prove the uniqueness of fixed point of mapping satisfying certain contractive condition in the framework of partially ordered metric spaces.

Theorem 1.6. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $f : X \to X$ be an increasing mapping such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. If*

$$d(fx, fy) \leq \alpha(d(x, y))d(x, y)$$

for each $x, y \in X$ with $x \succeq y$, where $\alpha \in S$. Then f has a fixed point provided that either f is continuous or X is such that if an increasing sequence $\{x_n\} \to x$ in X; then $x_n \preceq x$, for all n. Besides, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y. Then f has a unique fixed point.

We also need the following definitions:

Definition 1.7. *([25]) Let f and g be two self-maps defined on a set X. Then f and g are said to be weakly compatible if they commute at every coincidence point.*

Definition 1.8. *Let (X, d) be a metric space. Then the pair $\{f, g\}$ is said to be compatible if and only if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.*

Definition 1.9. *Let X be a nonempty set. Then $(X, d, \preceq)$ is called partially ordered b-metric space if and only if d is a b-metric on a partially ordered set $(X, \preceq)$.*

A subset $\mathcal{K}$ of a partially ordered set X is said to be totally ordered if every two elements of $\mathcal{K}$ are comparable.

Definition 1.10. *Let (X, d) be a metric space and $x, y, z \in X$. An element y is said to be closer to x than z if $d(x, y) \leq d(x, z)$.*

Definition 1.11. *Let $(X, \preceq)$ be a partially ordered set and f a self mapping on X. A mapping f is said to have a noncompetitive farthest property if $x, y \in X$ which are not comparable, there exists $z \in X$ which is (a) comparable to x and y (b) z is closer to x and y than fz.*

Example 1.12. *Let $X = \{1, 2, 3\}$. Define $a \preceq b$ if and only if $a \leq b$, where $\leq$ is usual order on $\mathbb{R}$. Thus*

$$\preceq := \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\} \subseteq X \times X.$$

Define self map f on X such that

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

Note that $(2,1), (3,1)$ and $(3,2) \notin \preceq$, there exits an element $3 \in X$, which is comparable to $(2,1), (3,1)$ and $(3,2)$. Also 3 is closer to $(2,1), (3,1)$ and $(3,2)$ than $f3$. Hence f is a noncompetitive farthest property.

Definition 1.13. *([7]) Let $(X, \preceq)$ be a partially ordered set. Two mappings $f, g : X \to X$ are said to be weakly increasing maps, if $fx \preceq gfx$ and $gx \preceq fgx$ hold for all $x \in X$.*

Example 1.14. *Let $X = \mathbb{R}^+$ be endowed with usual order and usual topology. Let $f, g : X \to X$ be defined by*

$$fx = \begin{cases} x^{\frac{1}{2}}, & \text{if } x \in [0,1] \\ x^2, & \text{if } x \in [1, \infty) \end{cases}, \quad gx = \begin{cases} x, & \text{if } x \in [0,1) \\ 2x, & \text{if } x \in [1, \infty) \end{cases}.$$

Then, the pair (f, g) is weakly increasing maps, where g is a discontinuous mapping on $\mathbb{R}^+$.

We also need the following definitions and propositions in the setup of b-metric spaces.

Definition 1.15. *([13]) Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:*

(a) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to +\infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$.
(b) Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to +\infty$.

Proposition 1.16. *(see remark 2.1 in [13]). In a b-metric space (X, d) the following assertions hold:*

(p_1) *a convergent sequence has a unique limit,*
(p_2) *each convergent sequence is Cauchy,*
(p_3) *in general, a b-metric is not continuous.*

Definition 1.17. *([13]) Let (X, d) be a b-metric space. If Y is a nonempty subset of X, then the closure $\overline{Y}$ of Y is the set of limits of all convergent sequences of points in Y, i.e.,*

$$\overline{Y} = \left\{ x \in X : \text{there exists a sequence } \{x_n\} \text{ in } Y \text{ such that } \lim_{n\to\infty} x_n = x \right\}.$$

Taking into account of the above definition, we have the following concepts.

Definition 1.18. *([13]) Let (X, d) be a b-metric space. Then a subset $Y \subset X$ is called closed if and only if for each sequence $\{x_n\}$ in Y which converges to an element x, we have $x \in Y$ (i.e. $\overline{Y} = Y$).*

Definition 1.19. *([13]) The b-metric space (X, d) is complete if every Cauchy sequence in X converges.*

Definition 1.20. *Let $(X, \preceq, d)$ be a partially ordered b-metric space. If there exist $\beta \in S$ such that*

$$sd(fx, gy) \leq \beta(d(x, y))M(x, y) + LN(x, y), \tag{2}$$

for all $x, y \in X$ with $x \preceq y$, where $L \geq 0$,

$$\begin{aligned} M(x, y) &= \max\{d(x, y), \frac{d(x, fx)d(y, gy)}{1 + d(fx, gy)}\}, \\ N(x, y) &= \min\{d(x, y), d(x, fx), d(x, gy), d(y, fx), d(y, gy)\}. \end{aligned}$$

Then f and g are said to satisfy generalized b-order rational contractive condition.

An ordered b-metric space $(X, d, \preceq)$ is said to have sequential limit comparison property if for every non-decreasing sequence (non-increasing sequence) $\{x_n\}$ in X such that $x_n \to x$ implies that $x_n \preceq x$ ($x \preceq x_n$).

Recently, Karapınar [28] studied well-posed problem for a cyclic weak ϕ contraction mapping on a complete metric space. For our purpose, we define the concept of well-posedness of common fixed point problems as follows:

Definition 1.21. *Let $f, g : X \to X$ be two self-mappings. Then a fixed point problem for f and g is well posed if for any sequence $\{x_n\}$ in X and x^* in $F(f) \cap F(g)$ with $x_n \preceq x^*$, $\lim_{n\to\infty} d(x_n, gx_n) = 0$ or $\lim_{n\to\infty} d(x_n, fx_n) = 0$ implies that $\lim_{n\to\infty} x_n = x^*$.*

The aim of this paper is to establish the necessary conditions for existence of a common fixed point of two mappings satisfying generalized b- order contractive condition in the setting of a partially ordered b-complete b-metric space .It is worth mentioning that we do not employ any form of commutativity condition on mappings involved herein. Presented results extend very recent results obtained by F. Zabihi and A. Razani [38]. We also study well-posedness of common fixed point problem for generalized b- order contractive mappings. We employ our result to establish an existence of a solution of an integral equation.

2. Common Fixed Point Results

Since in general a b-metric is not continuous, we need the following simple lemma about the b-convergent sequences in the proof of our main result.

Lemma 2.1. *([5]) Let (X, d) be a b-metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y respectively, then we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\longrightarrow\infty} d(x_n, y_n) \leq \limsup_{n\longrightarrow\infty} d(x_n, y_n) \leq s^2 d(x, y),$$

In particular, if $x = y$, then we have $\lim_{n\longrightarrow\infty} d(x_n, y_n) = 0$. Moreover for each $z \in X$ we have $\frac{1}{s} d(x, z) \leq \liminf_{n\longrightarrow\infty} d(x_n, z)$ $\leq \limsup_{n\longrightarrow\infty} d(x_n, z) \leq sd(x, z)$.

We start this section with the following result.

Theorem 2.2. *Let $(X, \preceq, d)$ be a partially ordered complete b-metric space. Suppose that $f, g : X \to X$ are two weakly increasing maps which satisfy a generalized b-order rational contractive condition. Also, there exists $x_0 \in X$ such that $fx_0 \preceq gfx_0$. Assume that either f, g are continuous or X has a sequential limit comparison property. Then f and g have a common fixed point.*

Proof. By given assumption there exists x_0 in X such that $fx_0 \preceq gfx_0$. Define a sequence $\{x_n\}$ in X in the following way:

$$x_{2n+1} = fx_{2n} \text{ and } x_{2n+2} = gx_{2n+1} \text{ for all } n \geq 0.$$

Note that

$$x_1 = fx_0 \preceq gfx_0 = gx_1 = x_2 = gx_1 \preceq fgx_1 = fx_2 = x_3.$$

Iteratively, we obtain that $x_1 \preceq x_2 \preceq x_3 \preceq x_4 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$. Note that

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\left\{d(x_{2n}, x_{2n+1}), \frac{d(x_{2n}, fx_{2n})d(x_{2n+1}, gx_{2n+1})}{1 + d(fx_{2n}, gx_{2n+1})}\right\} \\ &= \max\left\{d(x_{2n}, x_{2n+1}), \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+1}, x_{2n+2})}\right\} \\ &= d(x_{2n}, x_{2n+1}), \end{aligned} \tag{3}$$

and

$$\begin{aligned} N(x_{2n}, x_{2n+1}) &= \min\{d(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}), d(x_{2n}, gx_{2n+1}) \\ &\quad , d(x_{2n+1}, fx_{2n}), d(x_{2n+1}, gx_{2n+1})\} \\ &= \min\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \\ &= \min\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+2}), 0, d(x_{2n+1}, x_{2n+2})\} = 0. \end{aligned}$$

As x_{2n} and x_{2n+1} are comparable and f and g satisfy a generalized b-order rational contractive condition, so we have

$$\begin{aligned} sd(x_{2n+1}, x_{2n+2}) &= sd(fx_{2n}, gx_{2n+1}) \\ &\leq \beta(d(x_{2n}, x_{2n+1}))M(x_{2n}, x_{2n+1}) + LN(x_{2n}, x_{2n+1}) \\ &\leq \beta(d(x_{2n}, x_{2n+1}))d(x_{2n}, x_{2n+1}) \\ &< \frac{1}{s}d(x_{2n}, x_{2n+1}) < d(x_{2n}, x_{2n+1}). \end{aligned}$$

That is,

$$sd(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1}). \tag{4}$$

Similarly,

$$\begin{aligned} M(x_{2n+2}, x_{2n+1}) &= \max\left\{d(x_{2n+2}, x_{2n+1}), \frac{d(x_{2n+2}, fx_{2n+2})d(x_{2n+1}, gx_{2n+1})}{1 + d(fx_{2n+2}, gx_{2n+1})}\right\} \\ &= \max\left\{d(x_{2n+2}, x_{2n+1}), \frac{d(x_{2n+2}, x_{2n+3})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+3}, x_{2n+2})}\right\} \\ &= d(x_{2n+2}, x_{2n+1}), \end{aligned}$$

and

$$\begin{aligned} N(x_{2n+2}, x_{2n+1}) &= \min\{d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, fx_{2n+2}), d(x_{2n+2}, gx_{2n+1}) \\ &\quad , d(x_{2n+1}, fx_{2n+2}), d(x_{2n+1}, gx_{2n+1})\} \\ &= \min\{d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, x_{2n+3}), d(x_{2n+2}, x_{2n+2}) \\ &\quad , d(x_{2n+1}, x_{2n+3}), d(x_{2n+1}, x_{2n+2})\} \\ &= \min\{d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, x_{2n+3}), 0, d(x_{2n+1}, x_{2n+3}), d(x_{2n+1}, x_{2n+2})\} \\ &= 0 \end{aligned}$$

As x_{2n+3} and x_{2n+2} are comparable and f and g satisfy a generalized b-order rational contractive condition, so we have

$$\begin{aligned} sd(x_{2n+3}, x_{2n+2}) &= sd(x_{2(n+1)+1}, x_{2n+2}) \\ &= sd(fx_{2n+2}, gx_{2n+1}) \\ &\leq \beta(d(x_{2n+2}, x_{2n+1}))M(x_{2n+2}, x_{2n+1}) + LN(x_{2n+2}, x_{2n+1}) \\ &\leq \beta(d(x_{2n+2}, x_{2n+1}))d(x_{2n+2}, x_{2n+1}) \\ &< \frac{1}{s}d(x_{2n+2}, x_{2n+1}) \\ &< d(x_{2n+2}, x_{2n+1}). \end{aligned} \tag{5}$$

Hence we conclude that

$$sd(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{6}$$

for each n. That is, the sequence $\{d(x_n, x_{n+1})\}$ is a nonincreasing. Thus $\lim\limits_{n\to\infty} d(x_n, x_{n+1})$ exists. Let $r \geq 0$ be such that $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = r$. We claim that $r = 0$. If not, that is, $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = r > 0$. Then

$$\frac{1}{s}r \leq sr \leq \lim_{n\to\infty} \beta(d(x_{n-1}, x_n))r \leq r. \tag{7}$$

Thus we have $\lim\limits_{n\to\infty} \beta(d(x_{n-1}, x_n)) \geq \frac{1}{s}$. As $\beta \in S$, so we have $\lim\limits_{n\to\infty} d(x_{n-1}, x_n) = 0$, a contradiction. Hence $r = 0$. Now, we prove that the sequence $\{x_n\}$ is Cauchy in X. Note that

$$\begin{aligned} d(x_n, x_m) &\leq M(x_n, x_m) \\ &= \max\{d(x_n, x_m), \frac{d(x_n, fx_n)d(x_m, gx_m)}{1 + d(fx_n, gx_m)}\} \\ &= \max\{d(x_n, x_m), \frac{d(x_n, x_{n+1})d(x_m, x_{m+1})}{1 + d(x_{n+1}, x_{m+1})}\}. \end{aligned} \tag{8}$$

Taking limit $m, n \to \infty$ in the above inequality, we have

$$\lim_{n,m\to\infty} M(x_n, x_m) = \lim_{n,m\to\infty} d(x_n, x_m). \tag{9}$$

Also,

$$\begin{aligned} \lim_{n,m\to\infty} N(x_n, x_m) &= \lim_{n,m\to\infty} \min\{d(x_n, x_m), d(x_n, f(x_n)), \\ &\quad d(x_n, g(x_m)), d(x_m, f(x_n)), d(x_m, g(x_m))\} \\ &= \lim_{n,m\to\infty} \min\{d(x_n, x_m), d(x_n, x_{n+1}), \\ &\quad d(x_n, x_{m+1}), d(x_m, x_{n+1}), d(x_m, x_{m+1})\} \\ &= 0. \end{aligned}$$

Now

$$\begin{aligned} d(x_n, x_m) &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{m+1}) + s^2 d(x_{m+1}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2 d(x_{m+1}, x_m) \\ &\quad + s\{\beta(d(x_n, x_m))M(x_n, x_m) + LN(x_n, x_m)\}. \end{aligned}$$

Taking limit as $m, n \to \infty$ in the above inequality, we obtain that

$$\begin{aligned} \lim_{n,m\to\infty} d(x_n, x_m) &\leq s \lim_{n,m\to\infty} \beta(d(x_n, x_m)) \lim_{n,m\to\infty} d(x_n, x_m) \\ \frac{1}{s} &\leq \lim_{n,m\to\infty} \beta(d(x_n, x_m)). \end{aligned} \tag{10}$$

As $\beta \in S$, so $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Hence a sequence $\{x_n\}$ is a b-Cauchy sequence in X. Since (X, d) is a complete b-metric space, there exist $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

Now, we consider two alternative cases. First, suppose that f is continuous, then it is clear that x^* is a fixed point of f.

As $x^* \preceq x^*$ and f and g satisfy a generalized b-order rational contractive condition so we have

$$\begin{aligned} sd(x^*, gx^*) &= sd(fx^*, gx^*) \\ &\leq \beta(d(x^*, x^*))M(x^*, x^*) + LN(x^*, x^*) = 0. \end{aligned}$$

Hence $x^* = gx^*$ and x^* is the common fixed point of f and g.

Now assume that X has a sequential limit comparison property. Hence $x^* \preceq x_n$. As f and g satisfy a generalized b-order rational contractive condition, so we have

$$\begin{aligned} sd(fx^*, x_{n+1}) &= sd(fx^*, gx_n) \\ &\leq \beta(d(x^*, x_n))M(x^*, x_n) + LN(x^*, x_n) \\ &= \beta(d(x^*, x_n))d(x^*, x_n). \end{aligned}$$

Taking limit as $n \to \infty$, we obtain that $sd(fx^*, x^*)) \leq 0$, which shows that $fx^* = x^*$. Similarly $gx^* = x^*$. □

Remark 2.3. *Above theorem gives that $C(f, g) \neq \phi$. Note that if $\tilde{x}, x^* \in C(f, g)$ are comparable then $\tilde{x} = x^*$, where f and g satisfy generalized b-order rational contractive condition. Indeed, if $\tilde{x}$ and x^* are two comarable elements in $C(f, g)$, then from inequality (2) we have*

$$\begin{aligned} sd(x^*, \tilde{x}) &= sd(fx^*, g\tilde{x}) \\ &\leq \beta(d(x^*, \tilde{x}))M(x^*, \tilde{x}) + LN(x^*, \tilde{x}) \\ &= \beta(d(x^*, \tilde{x}))d(x^*, \tilde{x}) < d(x^*, \tilde{x}), \end{aligned}$$

a contradiction. Hence $\tilde{x} = x^*$.

Remark 2.4. *If $\tilde{x}, x^* \in C(f, g)$ are incomparable and there exists $z \in X$ such that every element in the orbit $O_g(z) = \{z, gz, g^2z, ...\}$ is comparable to $\tilde{x}$ and x^*. Then $\tilde{x} = x^*$ provided that f and g satisfy generalized b-order rational contractive condition.*

Proof. Suppose that $\tilde{x}$ and x^* are two incomarable elements in $C(f, g)$ and there exists an element $z \in X$ such that every element of $O_g(z) = \{z, gz, g^2z, ...\}$ is comparable to $\tilde{x}$ and x^* (and hence to $f^n(\tilde{x})$ and $f^n(x^*)$ for each n in $\mathbb{N}$). Then

$$\begin{aligned} sd(x^*, \tilde{x}) &\leq sd(x^*, g^nz) + sd(\tilde{x}, g^nz) \\ &= sd(f^nx^*, g^nz) + sd(f^n\tilde{x}, g^nz) \\ &\leq \beta(d(f^{n-1}x^*, g^{n-1}z))M(f^{n-1}x^*, g^{n-1}z) + LN(f^{n-1}x^*, g^{n-1}z) \\ &\quad +\beta(d(f^{n-1}\tilde{x}, g^{n-1}z))M(f^{n-1}\tilde{x}, g^{n-1}z) + LN(f^{n-1}\tilde{x}, g^{n-1}z) \\ &< \frac{1}{s}M(f^{n-1}x^*, g^{n-1}z) + LN(f^{n-1}x^*, g^{n-1}z) \\ &\quad +\frac{1}{s}M(f^{n-1}\tilde{x}, g^{n-1}z) + LN(f^{n-1}\tilde{x}, g^{n-1}z), \end{aligned} \tag{11}$$

where

$$\begin{aligned}
M(f^{n-1}x^*, g^{n-1}z) &= \max\{d(f^{n-1}x^*, g^{n-1}z), \frac{d(x^*, f^n x^*)d(z, g^n z)}{1 + d(f^n x^*, g^n z)}\} \\
&= d(f^{n-1}x^*, g^{n-1}z), \\
M(f^{n-1}\tilde{x}, g^{n-1}z) &= \max\{d(f^{n-1}\tilde{x}, g^{n-1}z), \frac{d(\tilde{x}, f^n \tilde{x})d(z, g^n z)}{1 + d(f^n \tilde{x}, g^n z)}\} \\
&= d(f^{n-1}\tilde{x}, g^{n-1}z), \\
N(f^{n-1}x^*, g^{n-1}z) &= \min\{d(f^{n-1}x^*, g^{n-1}z), d(f^{n-1}x^*, f^n x^*), \\
&\quad d(f^{n-1}x^*, g^n z), d(g^{n-1}z, f^n x^*), d(g^{n-1}z, g^n z)\} \\
&= 0 \text{ and} \\
N(f^{n-1}\tilde{x}, g^{n-1}z) &= \min\{d(f^{n-1}\tilde{x}, g^{n-1}z), d(f^{n-1}\tilde{x}, f^n \tilde{x}) \\
&\quad , d(f^{n-1}\tilde{x}, g^n z), d(g^{n-1}z, f^n \tilde{x}), d(g^{n-1}z, g^n z)\} \\
&= 0.
\end{aligned}$$

Thus, we have

$$sd(x^*, \tilde{x}) < \frac{1}{s}[d(f^{n-1}x^*, g^{n-1}z) + d(f^{n-1}\tilde{x}, g^{n-1}z)].$$

Continuing this way, we obtain that

$$\begin{aligned}
sd(x^*, \tilde{x}) &< \frac{1}{s}[d(f^{n-1}x^*, g^{n-1}z) + d(f^{n-1}\tilde{x}, g^{n-1}z)] \\
&= \frac{1}{s^2}[sd(f^{n-1}x^*, g^{n-1}z) + sd(f^{n-1}\tilde{x}, g^{n-1}z)] \\
&< \frac{1}{s^2}[\frac{1}{s}d(f^{n-2}x^*, g^{n-2}z) + \frac{1}{s}d(f^{n-2}\tilde{x}, g^{n-2}z)] \\
&\vdots \\
&< \frac{1}{s^{2n-1}}[d(f^0 x^*, g^0 z) + d(f^0 \tilde{x}, g^0 z)].
\end{aligned}$$

On taking limit as $n \to \infty$, we have $x^* = \tilde{x}$. □

Remark 2.5. *If $\tilde{x}, x^* \in C(f, g)$ are incomparable and the mapping g has noncompetitive farthest property, then $\tilde{x} = x^*$ provided that f and g satisfy generalized b-order rational contractive condition.*

Proof. Suppose that $\tilde{x}$ and x^* are two incomarable elements in $C(f, g)$. By given assumption there exists an element $z \in X$ comparable to $\tilde{x}$ and x^* and closer to $\tilde{x}$ and x^* than gz. Thus

$$\begin{aligned}
sd(x^*, \tilde{x}) &\leq sd(x^*, z) + sd(z, \tilde{x}) \\
&\leq sd(x^*, gz) + sd(\tilde{x}, gz) \\
&= sd(fx^*, gz) + sd(f\tilde{x}, gz) \\
&\leq \beta(d(x^*, z))M(x^*, z) + LN(x^*, z) \\
&\quad +\beta(d(\tilde{x}, z))M(\tilde{x}, z) + LN(\tilde{x}, z) \\
&= \beta(d(x^*, z))d(x^*, z) + \beta(d(\tilde{x}, z))d(\tilde{x}, z) \\
&< d(x^*, z) + d(z, \tilde{x}),
\end{aligned}$$

gives a contradiction. Hence $\tilde{x} = x^*$. □

Let Ψ be the class of all nondecreasing mappings $\psi : [0,\infty) \to [0,\infty)$ which satisfy the condition: $\lim_{n\to\infty} \psi^n(t) = 0$ whenever $t > 0$.

Lemma 2.6. *([38]) If $\psi \in \Psi$, then the following are satisfied,*
(a) $\psi(t) < t$ for all $t > 0$;
(b) $\psi(0) = 0$.

Theorem 2.7. *Let $(X, \preceq, d)$ be a partially ordered b-complete b-metric space. Suppose that $f, g : X \to X$ are two weakly increasing maps and there exists $x_0 \in X$ such that $fx_0 \preceq gfx_0$. Suppose that if there exist $\psi \in \Psi$ such that*

$$sd(fx, gy) \leq \psi(M(x, y)), \tag{12}$$

for all $x, y \in X$ with $x \preceq y$, where

$$M(x, y) = \max\{d(x, y), \frac{d(x, fx)d(y, gy)}{1 + d(fx, gy)}\}.$$

Assume that either f, g are continuous or X has a sequential limit comparison property. Then f and g have a common fixed point.

Proof. By given assumption there exists x_0 in X such that $fx_0 \preceq gfx_0$. Define a sequence $\{x_n\}$ in X in the following way:

$$x_{2n+1} = fx_{2n} \text{ and } x_{2n+2} = gx_{2n+1} \text{ for all } n \geq 0.$$

Note that

$$x_1 = fx_0 \preceq gfx_0 = gx_1 = x_2 = gx_1 \preceq fgx_1 = fx_2 = x_3.$$

Iteratively, we obtain that $x_1 \preceq x_2 \preceq x_3 \preceq x_4 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$. Note that

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\left\{d(x_{2n}, x_{2n+1}), \frac{d(x_{2n}, fx_{2n})d(x_{2n+1}, gx_{2n+1})}{1 + d(fx_{2n}, gx_{2n+1})}\right\} \\ &= \max\left\{d(x_{2n}, x_{2n+1}), \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+1}, x_{2n+2})}\right\} \\ &= d(x_{2n}, x_{2n+1}). \end{aligned}$$

As x_{2n} and x_{2n+1} are comparable with f and g satisfy (12), so we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\leq sd(x_{2n+1}, x_{2n+2}) \\ &= sd(fx_{2n}, gx_{2n+1}) \\ &\leq \psi(M(x_{2n}, x_{2n+1})) \\ &\leq \psi(d(x_{2n}, x_{2n+1})). \end{aligned} \tag{13}$$

It is clear that the expression (13) turns into

$$d(x_{2n+1}, x_{2n+2}) \leq \psi(d(x_{2n}, x_{2n+1})). \tag{14}$$

Similarly,

$$\begin{aligned} M(x_{2n+2}, x_{2n+1}) &= \max\left\{d(x_{2n+2}, x_{2n+1}), \frac{d(x_{2n+2}, fx_{2n+2})d(x_{2n+1}, gx_{2n+1})}{1 + d(fx_{2n+2}, gx_{2n+1})}\right\} \\ &= \max\left\{d(x_{2n+2}, x_{2n+1}), \frac{d(x_{2n+2}, x_{2n+3})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+3}, x_{2n+2})}\right\} \\ &= d(x_{2n+2}, x_{2n+1}), \end{aligned}$$

As x_{2n+3} and x_{2n+2} are comparable with f and g satisfy (12), so we have

$$\begin{aligned} d(x_{2n+3}, x_{2n+2}) &\leq sd(x_{2(n+1)+1}, x_{2n+2}) \\ &= sd(fx_{2n+2}, gx_{2n+1}) \\ &\leq \psi(M(x_{2n+2}, x_{2n+1})) \\ &\leq \psi(d(x_{2n+2}, x_{2n+1})). \end{aligned} \tag{15}$$

Hence we conclude that

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \leq d(x_{n-1}, x_n)$$

for each n. That is, the sequence $\{d(x_n, x_{n+1})\}$ is a nonincreasing. Thus $\lim\limits_{n\to\infty} d(x_n, x_{n+1})$ exists. Now by induction

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \leq \psi^2(d(x_{n-2}, x_{n-1})) \leq ... \leq \psi^n(d(x_1, x_0)). \tag{16}$$

As $\psi \in \Psi$, we conclude that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{17}$$

In what follows, we shall prove that the sequence $\{x_n\}$ is Cauchy in X. So, it is sufficient to prove that $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$. Suppose, on the contrary that, there exists $\epsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \text{ and } d(x_{m_i}, x_{n_i}) \geq \epsilon. \tag{18}$$

Implies, we have

$$d(x_{m_i}, x_{n_i-1}) < \epsilon. \tag{19}$$

From (18) and using the triangular inequality, we get

$$\epsilon \leq d(x_{n_i}, x_{m_i}) \leq sd(x_{m_i}, x_{m_i+1}) + sd(x_{m_i+1}, x_{n_i}).$$

Taking $i \to \infty$ in the above inequality, we get

$$\frac{\epsilon}{s} \leq \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}). \tag{20}$$

Using the triangular inequality, we get

$$d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{n_i-1}) + sd(x_{n_i-1}, x_{n_i}).$$

Taking $i \to \infty$ in the above inequality, we get

$$\limsup_{i\to\infty} d(x_{m_i}, x_{n_i}) \leq \epsilon s.$$

From the definition of $M(x, y)$ and the above limits,

$$\begin{aligned} M(x_{m_i}, x_{n_i-1}) &= \max\{d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, fx_{m_i})d(x_{n_i-1}, gx_{n_i-1})}{1 + d(fx_{m_i}, gx_{n_i-1})}\} \\ &= \max\{d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, x_{m_i+1})d(x_{n_i-1}, x_{n_i})}{1 + d(x_{m_i+1}, x_{n_i})}\} \\ &= d(x_{m_i}, x_{n_i-1}) \end{aligned}$$

and if $i \to \infty$ by (19), we get

$$\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}) \leq \epsilon.$$

Now

$$\begin{aligned} sd(x_{m_i+1}, x_{n_i}) &= sd(fx_{m_i}, gx_{n_i-1}) \\ &\leq \psi(M(x_{m_i}, x_{n_i-1})) \\ &= \psi(d(x_{m_i}, x_{n_i-1})). \end{aligned}$$

Again, if $i \to \infty$ by (19) and (20), we obtain

$$s = s(\frac{\epsilon}{s}) \leq (s \limsup_{i\to\infty} d(x_{m_{i+1}}, x_{n_i})) \leq \psi(\epsilon) < \epsilon,$$

which is a contradiction. Thus, $\{x_n\}$ is a b-Cauchy sequence. Completeness of X yields that $\{x_n\}$ converges to a point $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

Now, we shall consider two alternative cases. First, suppose that f is continuous, then it is clear that x^* is a fixed point of f. Now we show that $x^* = gx^*$. Suppose, on the contrary, that $d(x^*, gx^*) > 0$. Regarding $x^* \leq x^*$ together with the inequality (12), we conclude that

$$\begin{aligned} sd(x^*, gx^*) &= sd(fx^*, gx^*) \\ &\leq \psi(M(x^*, x^*)) \\ &= \psi(0) = 0, \end{aligned}$$

a contradiction. Hence $x^* = gx^*$ and x^* is the common fixed point of f and g.

For the second case, we assume that X has a sequential limit comparison property. Thus, we have $x^* \leq x_n$. Consequently, we find that

$$\begin{aligned} sd(fx^*, x_{n+1}) &= sd(fx^*, gx_n) \\ &\leq \psi(M(x^*, x_n)) \\ &= \psi(d(x^*, x_n)). \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$sd(fx^*, x^*)) \leq 0,$$

which shows that $fx^* = x^*$. Similarly $gx^* = x^*$. □

Remark 2.8. *Above theorem gives that $C(f, g) \neq \phi$. Note that if $\tilde{x}, x^* \in C(f, g)$ are comparable then $\tilde{x} = x^*$, where f and g satisfy (12). Indeed, if $\tilde{x}$ and x^* are two comarable elements in $C(f, g)$, then from inequality (2) we have*

$$\begin{aligned} sd(x^*, \tilde{x}) &= sd(fx^*, g\tilde{x}) \\ &\leq \psi(M(x^*, \tilde{x})) \\ &\leq \psi(d(x^*, \tilde{x})) < d(x^*, \tilde{x}), \end{aligned}$$

a contradiction. Hence $\tilde{x} = x^*$.

Remark 2.9. *If $\tilde{x}, x^* \in C(f, g)$ are incomparable and there exists $z \in X$ such that every element in the orbit $O_g(z) = \{z, gz, g^2z, ...\}$ is comparable to $\tilde{x}$ and x^*. Then $\tilde{x} = x^*$ provided that f and g satisfy (12).*

Proof. Suppose that $\tilde{x}$ and x^* are two incomarable elements in $C(f, g)$ and there exists an element $z \in X$ such that every element of $O_g(z) = \{z, gz, g^2z, ...\}$ is comparable to $\tilde{x}$ and x^* (and hence to $f^n(\tilde{x})$ and $f^n(x^*)$ for each n in $\mathbb{N}$). Then we have

$$\begin{aligned} sd(x^*, \tilde{x}) &\leq sd(x^*, g^n z) + sd(\tilde{x}, g^n z) \\ &= sd(f^n x^*, g^n z) + sd(f^n \tilde{x}, g^n z) \\ &\leq \psi(M(f^{n-1}x^*, g^{n-1}z)) + \psi(M(f^{n-1}\tilde{x}, g^{n-1}z)) \\ &= \psi(d(f^{n-1}x^*, g^{n-1}z)) + \psi(d(f^{n-1}\tilde{x}, g^{n-1}z)), \end{aligned}$$

where

$$\begin{aligned} M(f^{n-1}x^*, g^{n-1}z) &= \max\{d(f^{n-1}x^*, g^{n-1}z), \frac{d(x^*, f^n x^*)d(z, g^n z)}{1 + d(f^n x^*, g^n z)}\} \\ &= d(f^{n-1}x^*, g^{n-1}z), \text{ and} \\ M(f^{n-1}\tilde{x}, g^{n-1}z) &= \max\{d(f^{n-1}\tilde{x}, g^{n-1}z), \frac{d(\tilde{x}, f^n \tilde{x})d(z, g^n z)}{1 + d(f^n \tilde{x}, g^n z)}\} \\ &= d(f^{n-1}\tilde{x}, g^{n-1}z). \end{aligned}$$

Thus, we have

$$sd(x^*, \tilde{x}) < d(f^{n-1}x^*, g^{n-1}z) + d(f^{n-1}\tilde{x}, g^{n-1}z).$$

Continuing this way, we obtain that

$$\begin{aligned} sd(x^*, \tilde{x}) &< \frac{1}{s}[sd(f^{n-1}x^*, g^{n-1}z) + sd(f^{n-1}\tilde{x}, g^{n-1}z)] \\ &\vdots \\ &< \frac{1}{s^{n-1}}[d(f^0 x^*, g^0 z) + d(f^0 \tilde{x}, g^0 z)]. \end{aligned}$$

On taking limit as $n \to \infty$, we have $x^* = \tilde{x}$. □

Remark 2.10. *If $\tilde{x}, x^* \in C(f, g)$ are incomparable and the mapping g has noncompetitive farthest property, then $\tilde{x} = x^*$ provided that f and g satisfy (12).*

Proof. Suppose that $\tilde{x}$ and x^* are two incomarable elements in $C(f, g)$. By given assumption there exists an element $z \in X$ comparable to $\tilde{x}$ and x^* and z is closer to $\tilde{x}$ and x^* than gz. Thus

$$\begin{aligned} sd(x^*, \tilde{x}) &\leq sd(x^*, z) + sd(z, \tilde{x}) \\ &\leq sd(x^*, gz) + sd(\tilde{x}, gz) \\ &= sd(fx^*, gz) + sd(f\tilde{x}, gz) \\ &\leq \psi(M(x^*, z)) + \psi(M(\tilde{x}, z)) \\ &= \psi(d(x^*, z)) + \psi(d(\tilde{x}, z)) \\ &< d(x^*, z) + d(z, \tilde{x}), \end{aligned}$$

gives a contradiction. Hence $\tilde{x} = x^*$. □

Example 2.11. *Let $X = \{1, 2, 3, 4\}$. Define $a \preceq b$ if and only if $b \leq a$, where $\leq$ is usual order on $\mathbb{R}$. Thus we have*

$$\preceq := \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3), (4, 4)\} \subseteq X \times X.$$

Define two self maps f and g such that

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 1 \end{pmatrix}, \ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

It is straight forward to check that f and g are weakly increasing maps on X. Define the b-metric d on X by $d(x,x) = 0$, $d(x,y) = d(y,x)$, $d(1,2) = \frac{1}{5}$, $d(3,1) = d(4,1) = d(3,2) = d(4,2) = 10$, and $d(4,3) = 5$. Note that (X,d) is a b-metric space with $s = \frac{11}{10}$. Suppose that $L \geq 0$ is arbitrary real number and the function β is given by $\beta x = \dfrac{1}{1+x}$. Obviously, $sd(fx,gy) = 0$ when $(x,y) \in \{(1,1),(2,1),(2,2),(4,1),(4,2),(4,3)\}$. Also,

(x,y)	$sd(fx,gy)$	$\beta(d(x,y))M(x,y) + LN(x,y)$
$(3,1)$	11/50	10/11
$(3,2)$	11/50	10/11
$(3,3)$	11/50	$250/3 + 10L$
$(4,4)$	11/50	$250/3 + 10L$

Thus all the conditions of Theorem 2.1 are satisfied. Moreover 1 is the common fixed point of f and g. Note that, for $(x,y) = (3,3)$ or $(4,4)$, $d(fx,gy) \leq kd(x,y)$ is not satisfied for any value of k.

Example 2.12. *Let $X = \{1,2,3,4\}$. Define $a \preceq b$ if and only if $a \leq b$, where $\leq$ is usual order on $\mathbb{R}$. Thus we have*

$$\preceq := \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\} \subseteq X \times X.$$

Define two self maps f and g such that

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \end{pmatrix}, \ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{pmatrix}.$$

It is straight forward to check that f and g are weakly increasing maps on X. Define the b-metric d on X by $d(x,x) = 0$, $d(x,y) = d(y,x)$, $d(1,2) = \frac{1}{5}$, $d(1,3) = d(1,4) = d(2,3) = d(2,4) = 10$, and $d(3,4) = 5$. Then (X,d) is a b-metric space with $s = \frac{11}{10}$. Suppose that the function ψ is given by $\psi x = \ln(\frac{3}{2}x + 1)$. Obviously, $sd(fx,gy) = 0$ when $(x,y) \in \{(1,1),(1,3),(2,2),(2,4),(3,4)\}$. Note that

(x,y)	$sd(fx,gy)$	$\psi(M(x,y))$
$(1,2)$	0.22	0.2624
$(1,4)$	0.22	2.7726
$(2,3)$	0.22	2.7726
$(3,3)$	0.22	4.8363
$(4,4)$	0.22	4.8363

Thus all the conditions of Theorem 2.2 are satisfied. Moreover 1 is the common fixed point of f and g. Note that Theorem 2.1 is not satisfied when $(x,y) = (1,2)$ for any $\beta \in S$. Furthermore, for $(x,y) = (3,3)$ or $(4,4)$, $d(fx,gy) \leq kd(x,y)$ is not satisfied for any value of k.

Theorem 2.13. *Let $(X,\preceq,d)$ be a partially ordered complete metric space. Let $f,g : X \to X$ be two self-mappings as in Theorem 2.1. Then the fixed point problem for f and g is well posed.*

Proof. Let $\{x_n\}$ be a sequence in X, and $x^* \in F(f) \cap F(g)$ such that $x_n \preceq x^*$. Suppose that $\lim\limits_{n\to\infty} d(x_n, fx_n) = 0$. If $\lim\limits_{n\to\infty} d(x_n, x^*) = 0$, then we are done. Suppose that $\lim\limits_{n\to\infty} d(x_n, x^*) = r > 0$. Consider

$$\begin{aligned} sd(x_n,x^*) &\leq s\{d(x_n,fx_n) + d(fx_n,x^*)\} \\ &= s\{d(x_n,fx_n) + d(fx_n,gx^*)\} \\ &\leq sd(x_n,fx_n) + \beta(d(x_n,x^*))M(x_n,x^*) + LN(x_n,x^*) \\ &\leq sd(x_n,fx_n) + \beta(d(x_n,x^*))d(x_n,x^*) \\ &< sd(x_n,fx_n) + \frac{1}{s}d(x_n,x^*) \end{aligned}$$

On taking limit as $n \to \infty$, we have a contradiction. Similarly, we obtain $\lim_{n\to\infty} x_n = x^*$ if we assume $\lim_{n\to\infty} d(x_n, gx_n) = 0$. □

3. Main Result

In this section, inspired by the work in [18], we prove the existence of a solution for the integral equation

$$p(t, u(x,t)) = \int_0^1 \int_0^1 q(t, \zeta, s, u(\zeta, s)) d\zeta ds. \tag{21}$$

where $u \in E = L^p(C[0,1] \times [0,1]), 1 < p < \infty$ and $t, d, s \in [0,1]$. Let E be ordered by the cone given by $K = \{u \in L^p(C[0,1] \times [0,1]) : u(x,t) \leq 0$, almost every where$\}$. The space with the b-metric given by

$$d(x, y) = \max_{t \in I} \left| x(t) - y(t) \right|^p$$

for all $x, y \in X$ is a complete b-metric space with $s = 2^{p-1}$ and $p \geq 1$. We assume the following:

(i) For all $u, v \in E$ with $u - v \in K$,

$$p(t, u(x,t)) - p(t, v(x,t)) \leq q(u(x,t) - v(x,t)),$$

for each $t \in [0,1]$.

(ii) $q(t, \zeta, s, u(\zeta, s)) \leq u(x,t)$ for all $t \in [0,1]$.

(iii) $p(t, \int_0^1 \int_0^1 q(t, \zeta, s, u(\zeta, s)) d\zeta ds) \leq \int_0^1 \int_0^1 q(t, \zeta, s, u(\zeta, s)) d\zeta ds$ for all $\zeta, t, s \in [0,1]$.

(iv) $q(t, \zeta, s, u(\zeta, s)) \geq p(t, u(x,t))$ for all $\zeta, t, s \in [0,1]$.

Then, the implicit integral equation (21) has a solution in $L^p(C[0,1] \times [0,1])$.

Proof. Define $f(u(x,t)) = p(t, u(x,t))$ and $g(u(x,t)) = \int_0^1 \int_0^1 q(t, \zeta, s, u(\zeta, s)) d\zeta ds$. Using (ii), (iii) and (iv), we obtain that

$$\begin{aligned} f(g(u(x,t))) &= p(t, \int_0^1 \int_0^1 q(t, \zeta, s, u(\zeta, s)) d\zeta ds) \\ &\leq \int_0^1 \int_0^1 q(t, \zeta, s, u(\zeta, s)) d\zeta ds = g(u(x,t)), \end{aligned}$$

and

$$\begin{aligned} g(f(u(x,t))) &= \int_0^1 \int_0^1 q(t, \zeta, s, f(u(\zeta, s))) d\zeta ds \\ &= \int_0^1 \int_0^1 q(t, \zeta, s, p(\zeta, u(\zeta, s))) d\zeta ds \\ &\leq \int_0^1 \int_0^1 p(t, u(x,t)) d\zeta ds = p(t, u(x,t)) = f(u(x,t)). \end{aligned}$$

So, f and g are weakly increasing maps. Finally, we have

$$\begin{aligned}
2^{p-1}\left|f(u(x,t)) - g(v(x,t))\right|^p &= 2^{p-1}\left|p(t,u(x,t)) - \int_0^1\int_0^1 q(t,\zeta,s,v(\zeta,s))d\zeta ds\right|^p \\
&\leq 2^{p-1}\left|p(t,u(x,t)) - \int_0^1\int_0^1 p(t,v(x,t))d\zeta ds\right|^p \\
&= 2^{p-1}\left|p(t,u(x,t)) - p(t,v(x,t))\right|^p \\
&= 2^{p-1}q^p(\max_{t\in I}|u(x,t)-v(x,t)|)^p \\
&= 2^{p-1}q^p d(u,v) \\
&\leq 2^{p-1}q^p d(u,v) + \frac{d(u,v)}{1+d(u,v)} \\
&\leq \beta(d(u,v))M(u,v) + LN(u,v),
\end{aligned}$$

where $L = 2^{p-1}q^p$. Take $\beta x = \frac{1}{1+x}$. Thus $\beta \in S$. Therefore f and g are generalized β-order contractive mappings. Thus all the conditions of Theorem 3 are satisfied. Hence given problem has a solution which in turn to solves the integral equation (21).

Remark 3.1. *Since a b-metric is a metric when $s = 1$, so our results can be viewed as the generalization and extension of corresponding results in [17, 38, 39] and several other comparable results.*

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