Invariant Approximation Results of Generalized Contractive Mappings

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Abstract. Abbas, Ali and Salvador [Fixed and periodic points of generalized contractions in metric spaces, Fixed Point Theory Appl. 2013, 2013:243] extended the concept of $F-$ contraction mapping introduced in [21], to two mappings. The aim of this paper is to introduce the notion of a generalized F_g- weak contraction mapping and to study sufficient conditions for the existence of common fixed points for such class of mappings. As applications, related invariant approximation results are derived. The results obtained herein unify, generalize and complement various known results in the literature.

1. Introduction and Preliminaries

The Banach contraction principle appeared in explicit form in Banach's thesis [3] in 1922 where it was used to establish the existence of a solution for an integral equation. Since then, it has become a very popular tool in solving existence problems in many branches of mathematics. Extensions of this principle were obtained either by generalizing the domain of mappings or by extending the contractive condition on the mappings (see, e.g., [1],[2],[4]-[22] and references mentioned therein).

Recently, Wardowski [21] introduced a new type of contractive mapping called F-contraction and proved a fixed point theorem as a generalization of the Banach contraction principle. Abbas et al. ([2]) introduced the concept of $F-$ contraction with respect to a self mapping and obtained common fixed point results in an ordered metric space. Meinardus [15] was the first who employed fixed point theorem to prove the existence of invariant approximation in Banach spaces. Subsequently, several interesting and valuable results have appeared about invariant approximations ([4], [5], [13]). Following this research direction, we introduce a notion of generalized F_g-contraction mappings and obtain common fixed point results employing such contractions. Some results on invariant approximation for such mappings are also obtained which in turn extend and strengthen various known results.

In the sequel the letters $\mathbb{N}, \mathbb{R}_+$ and and $\mathbb{R}$ will denote the set of all natural numbers, the set of all nonnegative real numbers and the set of all real numbers, respectively.

Next we give some definitions which will be used in the sequel.

Definition 1.1. *([21]) Let F be the collection of all mappings $F : \mathbb{R}_+ \to \mathbb{R}$ such that*
  (F1) F is strictly increasing, that is, for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta$ implies that $F(\alpha) < F(\beta)$;
  (F2) For any sequence $\{\alpha_n\}$ of positive numbers, $\lim\limits_{n \to \infty} \alpha_n = 0$ and $\lim\limits_{n \to \infty} F(\alpha_n) = -\infty$ are equivalent;
  (F3) There exists $k \in (0, 1)$ such that $\lim\limits_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

For examples of mappings satisfying conditions ($F1 - F3$), we refer to [21] and [22]. Let $\mathfrak{F}$ be the collection of all mappings $F : \mathbb{R}_+ \to \mathbb{R}$ which satisfy conditions (F1) and (F3).

Definition 1.2. *([21]) Let (X, d) be a metric space. A map $T : X \to X$ is said to be F-contraction on X if there exist $F \in F$ and $\tau > 0$ such that the following holds:*

$$\tau + F(d(Tx, Ty)) \le F(d(x, y))$$

for all $x, y \in X$ with $Tx \ne Ty$.

Abbas et al. ([2]) extended the above definition to two mappings.

Recently, Wardowski and Dung [22] used a contractive inequality introduced and investigated by Ćirić [10], [11] and extended the concept of $F-$ contraction mappings to $F-$ weak contraction mappings as follows:

Definition 1.3. *Let (X, d) be a metric space. A map $T : X \to X$ is said to be F-weak contraction on X if there exist $F \in F$ and $\tau > 0$ such that for any $x, y \in X$ with $d(Tx, Ty) > 0$, the following hold:*

$$\tau + F(d(Tx, Ty)) \le F(\max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}).$$

Every $F-$ contraction mapping is $F-$weak contraction but converse does not hold in general ([22]).
To begin with, we recall the following result of [22].

Theorem 1.4. *Let (X, d) be a complete metric space and $T : X \to X$ be F-weak contraction. If T or F is continuous, then we have*
  (1) T has a unique fixed point $x_1 \in X$.
  (2) For all $x \in X$, the sequence $\{T^n x\}$ is convergent to x_1.

Now, we give the following definition.

Definition 1.5. *Let (X, d) be a metric space and $T, g : X \to X$. A mapping T is called F_g-weak contraction on X if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for any $x, y \in X$ with $d(Tx, Ty) > 0$, the following holds:*

$$\tau + F(d(Tx, Ty)) \le F(\max\{d(gx, gy), d(gx, Tx), d(gy, Ty), \frac{d(gx, Ty) + d(gy, Tx)}{2}\}).$$

Definition 1.6. *Let (X, d) be a metric space and $T, g : X \to X$. A mapping T is called F_g-contraction on (X, d) if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that, for all $x, y \in X$ satisfying $d(Tx, Ty) > 0$, the following holds:*

$$\tau + F(d(Tx, Ty)) \le F(d(gx, gy)).$$

The following example shows that a mapping T is F_g-weak contraction but not F_g-contraction.

Example 1.7. *Let $X = [0, 1]$, and $d(x, y) = |x - y|$. Define $T, g : X \to X$ by*

$$T(x) = \begin{cases} \frac{1}{3}, x \in [0, \frac{1}{3}) \\ \frac{1}{2}, x \in [\frac{1}{3}, 1) \\ 0, x = 1 \end{cases}$$

and

$$g(x) = \begin{cases} \frac{8}{10}, x \in [0, \frac{1}{3}) \\ \frac{9}{10}, x \in [\frac{1}{3}, 1]. \end{cases}$$

Note that T is F_g-weak contraction when $F(t) = \ln(t)$, $t \in (0, +\infty)$ and $\tau = \ln 1.045$ but not F_g-contraction if $x = 1$ and $y = 0$.

A subset M of a normed space E is said to be (1) convex if M includes every line segment joining any two of its points (2) z–*starshaped* or starshaped with respect to $z \in M$ if for any $x \in M$ and $q \in [0, 1]$, $qz + (1 - q)x \in M$. In this case, z is called a *star-centre* of M.

A selfmapping f on M is said to be *demiclosed* at y if for any sequence $\{x_n\}$ in M with $\{x_n\}$ converges weakly to $x \in M$ and $\{fx_n\}$ converges to y strongly, then $fx = y$.

Definition 1.8. *Let (X, d) be a metric space and $f, g : X \to X$. The pair (f, g) is called (a) commuting if $fgx = gfx$ for all $x \in X$ (b) compatible if for any $\{x_n\}$ in X with $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t$ for some t in X, we have $\lim_{n \to \infty} d(fgx_n, gfx_n) = 0$ (c) weakly compatible if they commute at their coincidence points, that is, if $gfx = fgx$ whenever $gx = fx$.*

Let f and g be two self maps on X. A point $x \in X$ is called (1) a fixed point of f if $f(x) = x$; (2) a *common fixed point* of g and f if $x = gx = fx$; (3) coincidence point of g and f if $gx = fx$. The set of all common fixed points (coincidence points) of g and f is denoted by $F(g, f)$ ($C(g, T)$, respectively).

Let M be a subset of metric space (X, d). For any $x \in X$, define

$$P_M(x) = \{y \in M : d(x, y) = d(x, M)\},$$

where $d(x, M) = \inf\{d(x, y) : y \in M\}$. The set $P_M(x)$ is the set of best approximations of x from M. If for each $x \in X$, $P_M(x)$ is nonempty, then M is called proximinal. Observe that if M is closed, then $P_M(x)$ is also closed. The reader interested in the interplay of fixed points and approximation theory in normed spaces is referred to the pioneer work of Park [18] and Singh ([20]).

2. Main Results

Theorem 2.1. *Let D be a subset of a metric space (X, d) and g and T selfmappings of D with $\overline{T(D)} \subseteq g(D)$. If $\overline{T(D)}$ is complete, $F \in \mathfrak{F}$ is a continuous mapping, T is F_g-weak contraction mapping on D, then T and g have a unique coincidence point. Moreover if (g, T) is weakly compatible, then $F(T) \cap F(g)$ is a singleton.*

Proof. Let x_0 in D be arbitrary. Choose a point $x_1 \in D$ such that $gx_1 = Tx_0$. This can be done because $T(D) \subseteq g(D)$. As $gx_1 \in g(D)$, there exists $x_2 \in D$ such that $gx_2 = Tx_1$. Continuing this process, having chosen x_n in D with $g(x_n) \in D$, we obtain x_{n+1} in D such that $gx_{n+1} = Tx_n$. Thus we obtain a sequence $\{x_n\}$ in D such that $gx_{n+1} = Tx_n$ for $n \geq 0$. If there exists some n_0 in $\mathbb{N} \cup \{0\}$ such that $Tx_{n_0} = Tx_{n_0+1}$, then $Tx_{n_0} = gx_{n_0+1}$ implies that $gx_{n_0+1} = Tx_{n_0+1}$ and x_{n_0+1} is coincidence point of g and T. We assume that $Tx_{n+1} \neq Tx_n$ for all n in $\mathbb{N} \cup \{0\}$. Consider

$$F(d(Tx_{n+1}, Tx_n))$$

$$\leq F(\max\{d(gx_{n+1}, gx_n), d(gx_{n+1}, Tx_{n+1}), d(gx_n, Tx_n)\frac{d(gx_{n+1}, Tx_n) + d(gx_n, Tx_{n+1})}{2}\}) - \tau$$

$$= F(\max\{d(Tx_n, Tx_{n-1}), d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n)\frac{d(Tx_n, Tx_n) + d(Tx_{n-1}, Tx_{n+1})}{2}\}) - \tau$$

$$\leq F(\max\{d(Tx_n, Tx_{n-1}), d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n), \frac{d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})}{2}\}) - \tau$$

$$= F(\max\{d(Tx_n, Tx_{n-1}), d(Tx_n, Tx_{n+1})\}) - \tau.$$

If for some $n \geq 1$,

$$\max\{d(Tx_n, Tx_{n-1}), d(Tx_n, Tx_{n+1})\} = d(Tx_n, Tx_{n+1})$$

then we have

$$F(d(Tx_{n+1}, Tx_n)) \leq F(d(Tx_{n+1}, Tx_n)) - \tau < F(d(Tx_{n+1}, Tx_n))$$

a contradiction. Hence

$$\max\{d(Tx_n, Tx_{n-1}), d(Tx_n, Tx_{n+1})\} = d(Tx_n, Tx_{n-1})$$

for all $n \geq 1$. Thus, we have

$$F(d(Tx_{n+1}, Tx_n)) \leq F(d(Tx_n, Tx_{n-1})) - \tau$$

for all $n \geq 1$. This further implies that

$$F(d(Tx_{n+1}, Tx_n)) \leq F(d(Tx_1, Tx_0)) - n\tau$$

for all $n \geq 1$. On taking limit as $n \to \infty$, we have $\lim_{n\to\infty} F(d(Tx_{n+1}, Tx_n)) = -\infty$. We now show that $\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = 0$. Assume on contrary that $\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = \alpha \neq 0$. Using Archmidean property of real numbers, there exists some $\alpha_1 \in \mathbb{R}_+$ such that $0 < \alpha_1 < \alpha$. By nondecreasing property of F, we obtain that $F(\alpha_1) < -\infty$, a contradiction. Hence $\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = 0$. By (F3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (d(Tx_{n+1}, Tx_n))^k F(d(Tx_{n+1}, Tx_n)) = 0.$$

As $F(d(Tx_{n+1}, Tx_n)) \leq F(d(Tx_1, Tx_0)) - n\tau$, so

$$(d(Tx_{n+1}, Tx_n))^k [F(d(Tx_{n+1}, Tx_n)) - F(d(Tx_1, Tx_0))] \leq -(d(Tx_{n+1}, Tx_n))^k n\tau \leq 0,$$

and hence

$$\lim_{n\to\infty} (d(Tx_{n+1}, Tx_n))^k n = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that $(d(Tx_{n+1}, Tx_n))^k n \leq 1$ for all $n \geq n_1$. That is, $d(Tx_{n+1}, Tx_n) \leq \frac{1}{n^{\frac{1}{k}}}$ for all $n \geq n_1$. Therefore for all $m > n \geq 1$, we have

$$
\begin{aligned}
d(Tx_m, Tx_n) &\leq d(Tx_m, Tx_{m-1}) + \ldots + d(Tx_{n+1}, Tx_n) \\
&< \sum_{i=n}^{\infty} d(Tx_{i+1}, Tx_i) \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}} < \infty
\end{aligned}
$$

This proves that $\{Tx_n\}$ is a Cauchy sequence in g(X). As $gx_n = Tx_{n-1}$, $\{gx_n\}$ is also a Cauchy sequence in $\overline{T(D)}$. By the completeness of $\overline{T(D)}$, there is some $w \in \overline{T(D)} \subseteq g(D)$ such that $\lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} Tx_n = w$. Let $z \in D$ be such that $g(z) = w$, that is,

$$\lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} Tx_n = gz. \tag{1}$$

Let us assume that $d(Tz, Tx_n) > 0$ and $d(gz, Tz) > 0$. As T is a weak F-contraction, we have

$$F(d(Tz, Tx_n)) \leq F(max\{d(gz, gx_n), d(gz, Tz), d(gx_n, Tx_n), \frac{d(gz, Tx_n) + d(gx_n, Tz)}{2}\}) - \tau. \tag{2}$$

Then from (1) and (2) there is a sufficiently large n_0, such that

$$F(d(Tz, Tx_n)) \leq F(d(Tz, gz)) - \tau, \tag{3}$$

for all $n \geq n_0$.

Note that if $\lim_{n \to \infty} d(Tz, Tx_n) = d(Tz, gz) > 0$, then by (F2), $\lim_{n \to \infty} F(d(Tz, Tx_n)) \neq -\infty$.

Now, taking the limit as $n \to \infty$ in the above inequality (3), we get, as

$$\lim_{n \to \infty} F(d(Tz, Tx_n)) = F(d(Tz, gz)),$$

$$F(d(Tz, gz)) \leq F(d(Tz, gz)) - \tau.$$

Hence we have

$$\tau \leq 0,$$

a contradiction with the assumption $\tau > 0$. Therefore, our supposition $d(gz, Tz) > 0$ was wrong. Thus $d(gz, Tz) = 0$. Hence $Tz = gz$. So we proved that z is a coincidence point of T and g.

Now suppose that T and g are weakly compatible. Since $w = T(z) = g(z)$, then $T(w) = T(g(z)) = g(T(z)) = g(w)$. Now we show that $g(w) = w$. If $T(z) \neq T(w)$, then we have

$$\begin{aligned}
F(d(T(z), T(w))) &\leq F(\max\{d(gz, gw), d(gz, Tz), d(gw, Tw), \frac{d(gz, Tw) + d(gw, Tz)}{2}\}) - \tau \\
&= F(\max\{d(Tz, Tw), \frac{d(Tz, Tw) + d(Tw, Tz)}{2}\}) - \tau \\
&= F(d(Tz, Tw)) - \tau,
\end{aligned}$$

a contradiction. Hence $T(z) = T(w) = g(w) = w$.

If z_1, z_2 are two common fixed points of T and g such that $z_1 \neq z_2$, then we have

$$\begin{aligned}
F(d(z_1, z_2)) &= F(d(T(z_1), T(z_2))) \\
&\leq F(\max\{d(gz_1, gz_2), d(gz_1, Tz_1), d(gz_2, Tz_2), \frac{d(gz_1, Tz_2) + d(gz_2, Tz_1)}{2}\}) - \tau \\
&= F(\max\{d(z_1, z_2), d(z_1, z_1), d(z_2, z_2), \frac{d(z_1, z_2) + d(z_2, z_1)}{2}\}) - \tau \\
&= F(d(z_1, z_2)) - \tau.
\end{aligned}$$

a contradiction. Hence $z_1 = z_2$. $\square$

Corollary 2.2. *Let D be a subset of a metric space (X, d) and g and T self-mappings on D with $\overline{T(D)} \subseteq g(D)$. If $\overline{T(D)}$ is complete, $F \in \mathfrak{F}$ is a continuous mapping, T is F_g-contraction mapping on D, then T and g have a unique coincidence point in D. Moreover if (g, T) is weakly compatible, then $F(T) \cap F(g)$ is a singleton.*

If $g =$identity mapping, then we have the following results.

Corollary 2.3. *Let D be a nonempty subset of a metric space (X, d) and $T : D \to D$. If $\overline{T(D)} \subseteq D$, $\overline{T(D)}$ is complete, $F \in \mathfrak{F}$ is a continuous mapping, and T is an F-weak contraction, then $F(T)$ is singleton.*

Corollary 2.4. *Let D be a nonempty subset of a metric space (X, d) and $T : D \to D$. If $\overline{T(D)} \subseteq D$, $\overline{T(D)}$ is complete, $F \in \mathfrak{F}$ is a continuous mapping, and T is an F-contraction mapping, then $F(T)$ is a singleton.*

As an application of Corollary 2.3, we obtain the following general result.

Theorem 2.5. *Let D be a nonempty subset of a metric space (X, d) and $g, T : D \to D$. If $F(g)$ is nonempty, $\overline{T(F(g))} \subseteq F(g)$, $\overline{T(D)}$ is complete, $F \in \mathfrak{F}$ is a continuous mapping, and T is F_g-weak contraction, then $F(g, T)$ is a singleton.*

Proof. Let $x, y \in F(g)$. Note that

$$
\begin{aligned}
F(d(Tx, Ty)) &\leq F(\max\{d(gx, gy), d(gx, Tx), d(gy, Ty), \frac{d(gx, Ty) + d(gy, Tx)}{2}\}) - \tau \\
&= F(\max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}) - \tau.
\end{aligned}
$$

Hence T is an $F-$ weak contraction on $F(g)$. The completeness of $\overline{T(D)}$ implies the completeness of $\overline{T(F(g))}$. It follows from Corollary 2.3 that $F(g, T) = F(g) \cap F(T)$ is singleton. $\square$

In the sequel, we assume that D is a q-starshaped subset of a normed space E and $g, T : D \to D$. Define $\delta(gx, Ty) := \inf\{\|gx - T_k y\| : 0 \leq k \leq 1\}$ for all $x, y \in D$ where $T_k x := kTx + (1 - k)q$.

For a subset D of a normed space E, $\overline{D}^w$ denotes weak closure of D in E

Definition 2.6. *Let X be a normed linear space and $T, g : X \to X$. A mapping T is called generalized F_g-nonexpansive on X if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for any $x, y \in X$ with $d(Tx, Ty) > 0$, we have*

$$
F(\|Tx - Ty\|) \leq F(\max\{\|gx - gy\|, \delta(gx, Tx), \delta(gy, Ty), \delta(gx, Ty), \delta(gy, Tx)\}) - \tau.
$$

Theorem 2.7. *Let D be a subset of a normed space E and $g, T : D \to D$. If $F(g)$ is q-starshaped, $\overline{T(F(g))} \subseteq F(g)$, $\overline{T(D)}$ is compact, T is continuous and generalized F_g-nonexpansive on D, $F \in \mathfrak{F}$ is a continuous mapping, then $F(g, T)$ is nonempty. Moreover, if we replace $\overline{T(F(g))} \subseteq F(g)$ with $\overline{T(F(g))}^w \subseteq F(g)$, compactness of $\overline{T(D)}$ with weak compactness of $\overline{T(D)}$ and continuity of T with demiclosedness of $g - T$ at zero, then the same conclusion holds provided that E is a Banach space.*

Proof. Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ such that $\lambda_n \to 1$ as $n \to \infty$. For each $n \geq 1$, define

$$
T_n x = \lambda_n Tx + (1 - \lambda_n)q
$$

for all $x \in F(g)$. As $F(g)$ is q-starshaped and $\overline{T(F(g))} \subseteq F(g)$, so we have $\overline{T_n(F(g))} \subseteq F(g)$ for each $n \geq 1$. Note that

$$
\begin{aligned}
F(\|T_n x - T_n y\|) &= F(\|\lambda_n Tx + (1 - \lambda_n)q - \lambda_n Ty - (1 - \lambda_n)q\|) \\
&= F(\|\lambda_n(Tx - Ty)\|).
\end{aligned}
$$

Since F is strictly increasing and $\lambda_n < 1$ for each $n \geq 1$, therefore

$$
\begin{aligned}
F(\|T_n x - T_n y\|) &< F(\|(Tx - Ty)\|) \\
&< F(\max\{\|gx - gy\|, \delta(gx, Tx), \delta(gy, Ty), \delta(gx, Ty), \delta(gy, Tx)\}) - \tau \\
&< F(\max\{\|gx - gy\|, \|gx - T_n x\|, \|gy - T_n y\|, \|gx - T_n y\|, \|gy - T_n x\|\}) - \tau
\end{aligned}
$$

holds for all $x, y \in F(g)$. As $\overline{T(D)}$ is compact, so $\overline{T_n(D)}$ is compact for each $n \geq 1$. Thus $\overline{T_n(D)}$ is complete for each $n \geq 1$. It follows from Theorem 2.5 that for each $n \geq 1$, $F(g, T_n) = F(g) \cap F(T_n) = \{x_n\}$ for some $x_n \in F(g)$.

Since $\overline{T(D)}$ is compact, there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \to z \in \overline{T(D)}$. Since $\{Tx_n\}$ is a sequence in $T(F(g))$, then $z \in \overline{T(F(g))} \subseteq F(g)$. Moreover, $x_{n_k} = T_{n_k} x_{n_k} = \lambda_{n_k} Tx_{n_k} + (1 - \lambda_{n_k})q \to z$. As T is continuous on D, we have $Tz = z$.

Next, assume that $\overline{T(D)}^w$ is weakly compact and $g - T$ is demiclosed at zero. Thus there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \to^w z \in \overline{T(F(g))}^w$. Since $\{Tx_n\}$ is a sequence in $T(F(g))$, then

$z \in \overline{T(F(g))}^w \subseteq F(g)$. From the boundedness of $\{Tx_{n_k}\}$, we conclude that $\{x_{x_{n_k}}\}$ is bounded too. As $\lambda_{n_k} \to 1$ and

$$
\begin{aligned}
\|x_{n_k} - Tx_{n_k}\| &= \|gx_{n_k} - Tx_{n_k}\| \\
&= \|(1 - \lambda_{n_k})q + \lambda_{n_k}Tx_{n_k} - Tx_{n_k}\| \\
&\leq (1 - \lambda_{n_k}) \|q - Tx_{n_k}\| \\
&\leq (1 - \lambda_{n_k}) (\|q\| + \|Tx_{n_k}\|),
\end{aligned}
$$

for each $k \geq 1$. We have $x_{n_k} - Tx_{n_k} \to^w 0$. It follows from the demiclosedness of $g - T$ at zero that $z - Tz = (g - T)z = 0$ and hence $Tz = z$. The result follows. $\square$

Let M be a nonempty subset of a metric space (X, d). Suppose that $C = P_M(u) \cap C_M^f(u)$, where $C_M^g(u) = \{x \in M : gx \in P_M(u)\}$.

Corollary 2.8. *Let E be a normed linear space and T, g self mappings of E. If $u \in E$, $D \subseteq G$, $G = D \cap F(g)$ is q-starshaped, $\overline{T(G)} \subseteq G$, $\overline{T(D)}$ is compact, and T is continuous generalized F_g-nonexpansive on D, $F \in \mathfrak{F}$ is a continuous mapping, then $P_M(u) \cap F(g, T)$ is nonempty. Moreover, if we replace $\overline{T(G)} \subseteq G$ with $\overline{T(G)}^w \subseteq G$, compactness of $\overline{T(D)}$ with weak compactness of $\overline{T(D)}$ and continuity of T with demiclosedness of $g - T$ at zero, then the same conclusion holds provided that E is a Banach space.*

Corollary 2.9. *Let E be a normed linear space and T, f self mappings of X. If $u \in E$, $D \subseteq P_M(u)$, $G = D \cap F(f)$ is q-starshaped, $\overline{T(G)} \subseteq G$, $\overline{T(D)}$ is compact and T is continuous generalized F_g-nonexpansive on D, $F \in \mathfrak{F}$ is a continuous mapping, then $P_M(u) \cap F(f, T)$ is nonempty. Moreover, if we replace $\overline{T(G)} \subseteq G$ with $\overline{T(G)}^w \subseteq G$, compactness of $\overline{T(D)}$ with weak compactness of $\overline{T(D)}$ and continuity of T with demiclosedness of $f - T$ at zero, then the same conclusion holds provided that E is a Banach space.*

Let G_0 denotes the class of closed convex subsets of a normed space X containing 0. For $M \in G_0$ and $p \in X$, let $M_p = \{x \in M : \|x\| \leq 2\|p\|\}$. Then $P_M(p) \subset M_p \in G_0$ (see [4], [17]).

Theorem 2.10. *Let E be a normed linear space and T, g self mappings on X. If $p \in E$ and $M \in G_0$ such that $T(M_p) \subseteq M$, $\overline{T(M_p)}$ is compact, and $\|Tx - p\| \leq \|x - p\|$ for all $x \in M_p$, then $P_M(p)$ is nonempty, closed and convex with $T(P_M(p)) \subseteq P_M(p)$. If, in addition, D is a subset of $P_M(p)$, $G = D \cap F(g)$ is q-starshaped, $\overline{T(G)} \subseteq G$ and T is continuous generalized F_g-nonexpansive on D, $F \in \mathfrak{F}$ is a continuous mapping, then $P_M(p) \cap F(g, T)$ is nonempty. Moreover if we replace compactness of $\overline{T(M_p)}$ with weak compactness of $\overline{T(M_p)}^w$, $\overline{T(G)} \subseteq G$ with $\overline{T(G)}^w \subseteq G$ and continuity of T with demiclosedness of $g - T$ at 0, then the same conclusion holds provided that E is a Banach space.*

Proof. If $p \in M$ then the result follows. Assume that $p \notin M$. If $x \in M \backslash M_p$, then $\|x\| > 2\|p\|$ and hence

$$\|p - x\| \geq \|x\| - \|p\| > \|p\| \geq d(p, M).$$

Thus $\alpha = d(p, M) \leq \|p\|$. As $\overline{T(M_p)}$ is compact so by the continuity of norm, there exists $z \in \overline{T(M_p)}$ such that

$$\beta = d(p, cl \, T(M_p)) = \|z - p\|.$$

On the other hand, if $\overline{T(M_p)}^w$ is weakly compact, then by Lemma 5.5 in ([20]), there exists $z \in \overline{T(M_p)}^w$ such that $\beta = d(p, \overline{T(M_p)}^w) = \|z - p\|$.

Thus in both cases, we have

$$
\begin{aligned}
\alpha = d(p, M) &\leq d(p, \overline{T(M_p)}) = \beta \\
&= d(p, T(M_p)) \leq \|Tx - p\| \\
&\leq \|x - p\|
\end{aligned}
$$

for all $x \in M_p$. Therefore $\alpha = \beta = d(p, M)$, that is,

$$d(p, M) = d(p, \overline{T(M_p)}) = \|p - z\|$$

and $z \in P_M(P)$. Hence $P_M(p)$ is nonempty. The closedness and convexity of $P_M(p)$ follows from that of M. Now we prove that $T(P_M(p)) \subseteq P_M(p)$. If $y \in T(P_M(p))$. Then $y = Tx$ for some $x \in P_M(p)$. Since

$$\|p - y\| = \|p - Tx\| \leq \|p - x\| = d(p, M).$$

Therefore $y \in P_M(p) \subset M_p$. Now $T(P_M(p)) \subset M$ gives that $y \in M$.

The compactness of $\overline{T(M_p)}$ (weak compactness of $\overline{T(M_p)}^w$, respectively) implies that $\overline{T(D)}$ compact (respectively, $\overline{T(D)}^w$ is weakly compact). Hence the result follows from Corollary 2.9. $\square$

Example 2.11. *Let $M = [0, 1]$. Define $T, g : M \to M$ by*

$$T(x) = \begin{cases} 0, x \in [0, \frac{1}{2}) \\ \frac{1}{8}, x \in [\frac{1}{2}, 1] \end{cases}$$

and $g(x) = x^2$.
Note that

$$\tau + F(d(Tx, Ty)) \leq F(\max\{d(gx, gy), d(gx, Tx), d(gy, Ty), \frac{d(gx, Ty) + d(gy, Tx)}{2}\})$$

holds with $F(t) = \ln(t)$, $t \in (0, +\infty)$ and $\tau = \ln 1.5$. Moreover 0 is common fixed point of T and g.

Remarks 2.11.1. *1) It must be noted that the assumption of linearity or affinity for I is necessary in almost all known results dealing with common fixed points of maps T, and I when T is I-nonexpansive under the conditions of commuting, weakly commuting, R-subweakly commuting, compatibility or Banach operator pairs (see [4]-[9], [14] and the literature cited therein), but our results in this paper are independent of the linearity or affinity condition on I.*

2) In [22], it was shown that from different types of F-weak contractions, one can obtain the variety of well-known contractions. Using the concept the of Wardowski [21], we introduce the F_g-weak contraction mappings by omitting condition (F2) which was assumed in [21, 22]. Also, in Example 1.7, we showed that F_g-contraction is F_g-weak contraction but not conversely. Therefore, the obtained results generalize and extend many of the well-known results existing in the literature to F_g-weak contraction mappings.

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