Iterative approximation of fixed points of Prešić operators on partial metric spaces

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We study the convergence of the Prešić type k-step iterative process for a class of operators $f : X^k \to X$ satisfying Prešić type F-contractive condition in the setting of metric spaces. As an applications of the result presented herein, we derive global attractivity results for a class of matrix difference equations. Numerical experiments are also presented to illustrate the theoretical findings.

Keywords: Fixed point approximation, iterative method, F-contraction, equilibrium point, matrix difference equation, global attractivity

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1. Introduction and Preliminaries

Banach contraction mapping principle [?] is simple and powerful result with a wide range of applications, including iterative methods for solving linear, nonlinear, differential, integral, and difference equations. There are several generalizations and extensions of the Banach contraction principle in the existing literature. Recently, Wardowski [?] introduced a new class of contractions called $F-$contraction and proved a fixed point result as a generalization of the Banach contraction principle [?].

We begin with some basic known definitions and results which will be used in the sequel. Throughout this article, $\mathbb{N}, \mathbb{R}^+, \mathbb{R}$ denote the set of natural numbers, the set of positive real numbers and the set of real numbers, respectively.

Let $\mathcal{F}$ be the collection of all mappings $F : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the following conditions:

(F_1) F is strictly increasing, that is, for all $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$ implies that $F(\alpha) < F(\beta)$.

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(F_2) For every sequence $\{\alpha_n\}$ of positive real numbers, $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$ are equivalent.

(F_3) There exists $h \in (0,1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^h F(\alpha) = 0$.

Definition 1.1. [23] Let (X,d) be a metric space and $F \in \mathcal{F}$. A mapping $f : X \to X$ is said to be an $F-$contraction on X if there exists $\tau > 0$ such that

$$d(fx, fy) > 0 \text{ implies that } \tau + F\left(d(fx, fy)\right) \le F\left(d(x,y)\right)$$

for all $x, y \in X$.

Note that every $F-$contraction is continuous (see [23]).

Wardowski obtained the following result.

Theorem 1.1. Let (X,d) be a complete metric space and $f : X \to X$ an $F-$contraction. Then there exists a unique x in X such that $x = fx$. Moreover, for any $x_0 \in X$, the iterative sequence $\{x_n\}$ defined by $x_{n+1} = f(x_n)$ converges to x

Let $f : X^k \to X$, where $k \ge 1$ is a positive integer. A point $x^* \in X$ is called a fixed point of f if $x^* = f(x^*, \ldots, x^*)$.

Consider the k-th order nonlinear difference equation:

$$(1) \qquad\qquad x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}), \ n = 1, 2, \ldots$$

with the initial values $x_1, \ldots, x_k \in X$.

Equation (1) can be studied by means of fixed point theory in view of the fact that x in X is a solution of (1) if and only if x is a fixed point of mapping $T : X \to X$ given by

$$T(x) = f(x, x, \ldots, x), \text{ for all } x \in X.$$

One of the most important results in this direction is due to Prešić [16] given as follows:

Theorem 1.2. [16] Let (X,d) be a complete metric space, k a positive integer. If a mapping $f : X^k \to X$ satisfies the following contractive condition:

$$d(f(x_1, x_2, \ldots, x_k), f(x_2, \ldots, x_k, x_{k+1})) \le q_1 d(x_1, x_2) + q_2 d(x_2, x_3) + \ldots + q_k d(x_k, x_{k+1}),$$

for every $x_1, \ldots, x_{k+1} \in X$, where $q_1, q_2, \ldots, q_k$ are non-negative constants such that $q_1 + q_2 + \ldots + q_k < 1$. Then there exists a unique point $x^* \in X$ such that $f(x^*, \ldots, x^*) = x^*$. Moreover, for any arbitrary points $x_1, \ldots, x_k \in X$, the sequence (1) converges to x^*.

Note that, for $k = 1$, Theorem 1.2 reduces to the Banach contraction principle.

Ćirić and Prešić [7] generalized the above result as follows.

Theorem 1.3. [7] Let (X,d) be a complete metric space, and k a positive integer. If $f : X^k \to X$ satisfies the following contractive condition:

$$d(f(x_1, x_2, \ldots, x_k), f(x_2, \ldots, x_k, x_{k+1})) \le q \max\{d(x_1, x_2), d(x_2, x_3), \ldots, d(x_k, x_{k+1})\},$$

for any $x_1, \ldots, x_{k+1} \in X$, where $0 < q < 1$. Then there exists $x^* \in X$ such that $f(x^*, \ldots, x^*) = x^*$. Moreover, for any arbitrary points $x_1, \ldots, x_k \in X$, the sequence (1.1) is convergent and

$$\lim_{n\to\infty} x_n = f(\lim_{n\to\infty} x_n, \ldots, \lim_{n\to\infty} x_n).$$

If in addition,
$$d(T(u,\ldots,u),T(v,\ldots,v)) < d(u,v)$$
holds for all $u,v \in X$, with $u \neq v$, then x^* is the unique point in X with $f(x^*,\ldots,x^*) = x^*$.

The applicability of the above result to the study of global asymptotic stability of the equilibrium for the nonlinear difference equation (1) is well known ([5]).

Păcurar [15] derived a convergence result for Prešić-Kannan operators as follows:

Theorem 1.4. [15] Let (X,d) be a complete metric space, k a positive integer and $f : X^k \to X$ a given mapping. Suppose that there exists a constant $a \in \mathbb{R}$ with $0 < ak(k+1) < 1$ such that
$$d(f(x_1,\ldots,x_k),f(x_2,\ldots,x_{k+1})) \leq a \sum_{i=1}^{k+1} d(x_i, f(x_i,\ldots,x_i)),$$
holds for all $(x_1,\ldots,x_{k+1}) \in X^{k+1}$. Then,
(i) f has a unique fixed point $x^* \in X$;
(ii) for any arbitrary points $x_1,\ldots,x_k \in X$, the sequence $\{x_n\}$ defined by (1) converges to x^*.

For more results in this direction, we refer to [2, 3, 5, 7, 9, 10, 11, 14, 15, 17, 18, 19, 20, 21].

Motivated by the work of Wardowski [23], we give the following definition:

Definition 1.2. Let (X,d) be a metric space and $F \in \mathcal{F}$. A mapping $f : X^k \to X$ is said to be a Prešić type $F-$contraction if there exists $\tau > 0$ such that
$$d(f(x_1,\ldots,x_k),f(x_2,\ldots,x_{k+1})) > 0$$
implies that

(2) $\tau + F\left(d(f(x_1,\ldots,x_k),f(x_2,\ldots,x_{k+1}))\right) \leq F\left(\max\{d(x_i,x_{i+1}) : 1 \leq i \leq k\}\right)$

for all $(x_1,\ldots,x_{k+1}) \in X^{k+1}$.

Note that, for $F(\alpha) = \ln \alpha$, Prešić type $F-$contraction condition becomes

(3) $d(f(x_1,\ldots,x_k),f(x_2,\ldots,x_{k+1})) \leq e^{-\tau} \max\{d(x_i,x_{i+1}) : 1 \leq i \leq k\}$

for all $(x_1,\ldots,x_{k+1}) \in X^{k+1}$, $f(x_1,\ldots,x_k) \neq f(x_2,\ldots,x_{k+1})$.

Furthermore, for $(x_1,\ldots,x_{k+1}) \in X^{k+1}$ such that $f(x_1,\ldots,x_k) = f(x_2,\ldots,x_{k+1})$, the inequality (3) also holds, that is, f is a Ćirić-Prešić contraction (see [7]).

Remark 1.1. It follows from (F_1) and (2) that every Prešić type $F-$contraction mapping f is a Prešić contractive mapping, that is,
$$d(f(x_1,\ldots,x_k),f(x_2,\ldots,x_{k+1})) < \max\{d(x_i,x_{i+1}) : 1 \leq i \leq k\}$$
for all $(x_1,\ldots,x_{k+1}) \in X^{k+1}$, $f(x_1,\ldots,x_k) \neq f(x_2,\ldots,x_{k+1})$.

Thus every Prešić type $F-$contraction mapping f is a continuous mapping.

The aim of this paper is to study the convergence of the sequence $\{x_n\}$ defined by (1), where $f : X^k \to X$ is a Prešić type $F-$ contraction mapping. We also give examples to support the results presented herein. Applications to matrix difference equations are also presented.

2. **Main results**

In this section, we obtain some fixed point results for Prešić type $F-$contraction mappings. We start with the following result.

Theorem 2.1. *Let (X, d) be a complete metric space and $f : X^k \to X$ a Prešić type $F-$contraction, where k is a positive integer. Then, for any arbitrary points $x_1, \ldots, x_k \in X$, the sequence $\{x_n\}$ defined by (1) converges to u in X and u is a fixed point of f. In addition, if*

$$d(f(x, \ldots, x), f(y, \ldots, y)) > 0$$

implies that

$$\tau + F\left(d(f(x, \ldots, x), f(y, \ldots, y))\right) \leq F\left(d(x, y)\right)$$

for all $x, y \in X$ with $x \neq y$, then f has a unique fixed point.

Proof. First, let us observe that f has at most one fixed point. Indeed, if $u, v \in X$ such that $u = f(u, ..., u)$ and $v = f(v, ..., v)$ with $u \neq v$. Thus $d(f(u, ..., u), f(v, ..., v)) > 0$. Hence by given assumption, we have

$$
\begin{aligned}
\tau + F\left(d(u, v)\right) &= \tau + F\left(d(f(u, ..., u), f(v, \ldots, v))\right) \\
&\leq F(d(u, v)),
\end{aligned}
$$

a contradiction as $\tau > 0$. Therefore $u = v$.

Now we show that f has a fixed point. Let $x_1, \cdots, x_k$ be arbitrary k elements in X. Define the sequence $\{x_n\}$ in X by

$$x_{n+k} = f(x_n, x_{n+1}, ..., x_{n+k-1}), \ n = 1, 2, \ldots.$$

If for some $n_0 \in \{1, 2, 3, ..., k\}$, we have $x_{n_0} = x_{n_0+1}$, then we have

$$x_{n_0+k} = f(x_{n_0}, x_{n_0+1}, ..., x_{n_0+k-1}) = f(x_{n_0+k}, x_{n_0+k}, ..., x_{n_0+k}),$$

that is, x_{n_0+k} is a fixed point of f and the proof is finished.

We assume that $x_{n+k} \neq x_{n+k+1}$ for all $n \in \mathbb{N}$. Denote $\gamma_{n+k} = d(x_{n+k}, x_{n+k+1})$ for $n = 1, 2, \ldots$ and $\theta = \max\{d(x_1, x_2), d(x_2, x_3), ..., d(x_k, x_{k+1})\}$, then we have $\gamma_{n+k} > 0$ for all $n \in \mathbb{N}$ and $\theta > 0$.

Now for $n \leq k$, we have the following inequalities:

$$
\begin{aligned}
F\left(\gamma_{k+1}\right) &= F\left(d\left(x_{k+1}, x_{k+2}\right)\right) \\
&= F\left(d\left(f(x_1, x_2, ..., x_k), f(x_2, x_3, ..., x_{k+1})\right)\right) \\
&\leq F\left(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}\right) - \tau \\
&= F\left(\theta\right) - \tau,
\end{aligned}
$$

$$\begin{aligned}
F\left(\gamma_{k+2}\right) &= F\left(d\left(x_{k+2}, x_{k+3}\right)\right) \\
&= F\left(d\left(f(x_2, x_3, ..., x_{k+1}), f(x_3, x_4, ..., x_{k+2})\right)\right) \\
&\leq F\left(\max\{d(x_i, x_{i+1}) : 2 \leq i \leq k+1\}\right) - \tau \\
&\leq F(\theta) - 2\tau, \\
&\vdots
\end{aligned}$$

and so on. Hence

$$\begin{aligned}
F\left(\gamma_{k+n}\right) &= F\left(d\left(x_{n+k}, x_{n+k+1}\right)\right) \\
&= F\left(d\left(f(x_n, x_{n+1}, ..., x_{n+k-1}), f(x_{n+1}, x_{n+2}, ..., x_{n+k})\right)\right) \\
&\leq F\left(\theta\right) - n\tau
\end{aligned}$$

for $n \geq 1$. On taking limit as $n \to \infty$, we obtain that $\lim\limits_{n \to \infty} F\left(\gamma_{k+n}\right) = -\infty$ and therefore $\lim\limits_{n \to \infty} \gamma_{k+n} = 0$ by (F_2).

Now by (F_3), there exists $h \in (0,1)$ such that

$$\lim_{n \to \infty} \gamma_{k+n}^h F\left(\gamma_{k+n}\right) = 0.$$

By (4), we have

$$\gamma_{k+n}^h F\left(\gamma_{k+n}\right) - \gamma_{k+n}^h F\left(\theta\right) \leq \gamma_{k+n}^h(F\left(\theta\right) - n\tau) - \gamma_{k+n}^h F\left(\theta\right) = -\gamma_{k+n}^h n\tau \leq 0.$$

On taking limit as $n \to \infty$, we obtain

(4)
$$\lim_{n \to \infty} n\gamma_{k+n}^h = 0.$$

Thus from (4), there exists $n_0 \in \mathbb{N}$ such that $n\gamma_{k+n}^h \leq 1$ for all $n \geq n_0$. Consequently we have

$$\gamma_{k+n} \leq \frac{1}{n^{1/h}} \text{ for all } n \geq n_0.$$

For any $n, m \in \mathbb{N}$ with $m \geq n \geq n_0$, we have

$$\begin{aligned}
d(x_{k+n}, x_{k+m}) &= d(f(x_n, \ldots, x_{k+n-1}), f(x_m, \ldots, x_{k+m-1})) \\
&\leq d(f(x_n, \ldots, x_{k+n-1}), f(x_{n+1}, \ldots, x_{k+n})) + d(f(x_{n+1}, \ldots, x_{k+n}), f(x_{n+2}, \ldots, x_{k+n+1})) \\
&\quad + ... + d(f(x_{m-1}, \ldots, x_{k+m-2}), f(x_m, \ldots, x_{k+m-1})) \\
&= d(x_{n+k}, x_{n+k+1}) + d(x_{n+k+1}, x_{n+k+2}) + ... + d(x_{m+k-1}, x_{m+k}) \\
&= \gamma_{n+k} + \gamma_{n+k+1} + ... + \gamma_{m+k-1} \\
&< \sum_{i=n}^{\infty} \gamma_{i+k} \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/h}} \to 0.
\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists u in X such that

$$\lim_{n,m \to \infty} d(x_n, x_m) = \lim_{n \to \infty} d(x_n, u) = 0.$$

Now by the continuity of f, we have

$$\begin{aligned} u &= \lim_{n\to\infty} x_{n+k} = \lim_{n\to\infty} f(x_n, x_{n+1}, \ldots, x_{n+k-1}) \\ &= f(\lim_{n\to\infty} x_n, \lim_{n\to\infty} x_{n+1}, \ldots, \lim_{n\to\infty} x_{n+k-1}) \\ &= f(u, u, \ldots, u). \end{aligned}$$

$\square$

Example 2.1. *Let* $X = \{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\}$ *and* $d(x, y) = |x - y|$. *Then* (X, d) *is a complete metric space. Define the mapping* $f : X^2 \to X$ *by*

$$f(x, y) = \begin{cases} \dfrac{x_{n-1} + y_{m-1}}{2}, & \text{if } x = x_n, \ y = y_m \text{ for } n, m > 1, \\[3mm] \dfrac{x_1 + y_1}{2}, & \text{otherwise.} \end{cases}$$

Note that for $n > 3$, *we have*

$$\begin{aligned} &d(f(x_{n-2}, x_{n-1}), f(x_{n-1}, x_n)) \\ &= \frac{1}{4}\left|\left(\frac{(n-3)(n-2)}{2} + \frac{(n-2)(n-1)}{2}\right) - \left(\frac{(n-2)(n-1)}{2} + \frac{(n-1)n}{2}\right)\right| \\ &= \frac{1}{4}(4n - 6) = n - \frac{3}{2} \end{aligned}$$

and

$$\begin{aligned} &\max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\} \\ &= \max\left\{\left|\frac{(n-2)(n-1)}{2} - \frac{(n-1)n}{2}\right|, \left|\frac{n(n-1)}{2} - \frac{n(n+1)}{2}\right|\right\} \\ &= \max\{n - 1, n\} = n. \end{aligned}$$

Now

$$\lim_{n\to\infty} \frac{d(f(x_{n-2}, x_{n-1}), f(x_{n-1}, x_n))}{\max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}} = \lim_{n\to\infty} \frac{n - 3/2}{n} = 1.$$

Thus

$$d(f(x_{n-2}, x_{n-1}), f(x_{n-1}, x_n)) \le q \max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}$$

does not hold for $q \in (0, 1)$. *Hence the condition of Theorem 2 in* [7] *is not satisfied.*

On the other hand by taking $F(\alpha) = \ln \alpha + \alpha$, $\alpha > 0$ *and* $\tau = 1$, *for* $f(x_i, x_{i+1}) \ne f(x_{i+1}, x_{i+2})$ $i = 1, 2, \ldots,$. *We consider the following cases:*
(1) If $x = x_1$, $y = x_2$, *then*

$$\begin{aligned} &d(f(x_1, x_2), f(x_2, x_3))e^{d(f(x_1,x_2),f(x_2,x_3)) - \max\{d(x_1,x_2),d(x_2,x_3)\}} \\ &= |1 - 2| e^{|1-2|-3} \\ &= e^{-2} < 3e^{-1} = e^{-1} \max\{d(x_1, x_2), d(x_2, x_3)\}. \end{aligned}$$

(2) For $x = x_n$, $y = x_{n+1}$ with $n > 1$, we have

$$d(f(x_n, x_{n+1}), f(x_{n+1}, x_{n+2}))e^{d(f(x_n, x_{n+1}), f(x_{n+1}, x_{n+2})) - \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}}$$

$$= \left(n + \frac{1}{2}\right)e^{(n+\frac{1}{2}) - (n+2)} = (n + \frac{1}{2})e^{-\frac{3}{2}}$$

$$< e^{-1}(n + 2) = e^{-1}\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}.$$

Thus f is the Prešić type $F-$contraction on X and $(1,1)$ is a unique fixed point of f, that is, $f(1,1) = 1$. □

Example 2.2. *Let $X = [0,2]$ and d be a usual metric of X. Let k be a positive integer and $f : X^k \to X$ be the mapping defined by*

$$f(x_1, \ldots, x_k) = \frac{x_1 + x_k}{4k} \ \text{for all } x_1, \ldots, x_k \in X.$$

Define $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(\alpha) = \alpha + \ln(\alpha)$. Note that $F \in \mathcal{F}$ ([23]).
Note that, for $\tau = \ln(2k) > 0$ and $x_1, x_2, \ldots, x_{k+1} \in X$, with

$$d(f(x_1, \ldots, x_k), f(x_2, \ldots, x_{k+1})) > 0,$$

we have

$$\tau + F\left(d(f(x_1, \ldots, x_k), f(x_2, \ldots, x_{k+1}))\right)$$

$$= \ln(2k) + F\left(\left|\frac{x_1 + x_k}{4k} - \frac{x_2 + x_{k+1}}{4k}\right|\right)$$

$$= \ln(2k) + F\left(\frac{1}{4k}|(x_1 - x_2) + (x_k - x_{k+1})|\right)$$

$$= \ln(2k) + \frac{1}{4k}|(x_1 - x_2) + (x_k - x_{k+1})| + \ln\left(\frac{1}{4k}|(x_1 - x_2) + (x_k - x_{k+1})|\right)$$

$$\leq \ln(2k) + \frac{1}{2k}\max\{d(x_1, x_2), d(x_k, x_{k+1})\} + \ln\left(\frac{1}{2k}\max\{d(x_1, x_2), d(x_k, x_{k+1})\}\right)$$

$$= \frac{1}{2k}\max\{d(x_1, x_2), d(x_k, x_{k+1})\} + \ln(\max\{d(x_1, x_2), d(x_k, x_{k+1})\})$$

$$\leq \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\} + \ln(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\})$$

$$= F(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}).$$

Moreover, for all $x, y \in X$ with $x \neq y$, we have

$$d(f(x, \ldots, x), f(y, \ldots, y)) = \frac{|x - y|}{2k} > 0$$

and

$$F\left(d(f(x,\ldots,x),f(y,\ldots,y)))\right) = F\left(\frac{|x-y|}{2k}\right)$$
$$= \frac{|x-y|}{2k} + \ln\left(\frac{|x-y|}{2k}\right)$$
$$= \frac{1}{2k}d(x,y) + \ln\left(d\left(x,y\right)\right) - \ln\left(2k\right)$$
$$\leq d(x,y) + \ln\left(d\left(x,y\right)\right) - \ln\left(2k\right)$$
$$= F\left(d\left(x,y\right)\right) - \tau.$$

Thus, all the required hypotheses of Theorem 2.1 are satisfied. Moreover, for any arbitrary points $x_1, ..., x_k \in X$, the sequence $\{x_n\}$ defined by (1) converges to $u = 0$, the unique fixed point of f. $\square$

The following result is an immediate consequence of Theorem 2.1 by taking $F\left(\alpha\right) = \ln\alpha$.

Corollary 2.1. *Let (X,d) be a complete metric space, k a positive integer and $f : X^k \to X$ a given mapping. Suppose that there exists $\tau > 0$ such that*

$$(5) \qquad d(f(x_1,\ldots,x_k),f(x_2,\ldots,x_{k+1})) \leq e^{-\tau}\max\{d(x_i,x_{i+1}) : 1 \leq i \leq k\},$$

for all $(x_1,\ldots,x_{k+1}) \in X^{k+1}$ with $f(x_1,\ldots,x_k) \neq f(x_2,\ldots,x_{k+1})$. Then, for any arbitrary points $x_1,\ldots,x_k \in X$, the sequence $\{x_n\}$ defined by (1) converges to u, and u is a fixed point of f, that is, $u = f(u,\ldots,u)$. Moreover, if

$$d(f(x,\ldots,x),f(y,\ldots,y)) \leq e^{-\tau}d(x,y),$$

holds for all $x,y \in X$ with $x \neq y$, then u is the unique fixed point of f.

Corollary 2.2. *Let (X,d) be a complete metric space, k a positive integer and $f : X^k \to X$ a given mapping. Suppose that there exist $\lambda_1,\ldots,\lambda_k$ non-negative constants with $\lambda_1 + \lambda_2 + \ldots + \lambda_k < 1$ such that*

$$(6) \quad d(f(x_1,\ldots,x_k),f(x_2,\ldots,x_{k+1})) \leq \lambda_1 d(x_1,x_2) + \lambda_2 d(x_2,x_3) + \ldots + \lambda_k d(x_k,x_{k+1}),$$

for all $(x_1,\ldots,x_{k+1}) \in X^{k+1}$ with $f(x_1,\ldots,x_k) \neq f(x_2,\ldots,x_{k+1})$. Then, for any arbitrary points $x_1, ..., x_k \in X$, the sequence $\{x_n\}$ defined by (1) converges to u, where u is the unique fixed point of f.

Proof. Clearly, condition (6) implies condition (5) with $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_k$. Now, let $x,y \in X$ with $x \neq y$. From (6), we have

$$d(f(x,x,\ldots,x),f(y,y,\ldots,y)) \leq d(f(x,\ldots,x),f(x,\ldots,x,y)) + d(f(x,\ldots,x,y),f(x,\ldots,x,y,y))$$
$$+ \ldots + d(f(x,y,\ldots,y),f(y,y,\ldots,y))$$
$$\leq (\lambda_k + \lambda_{k-1} + \ldots + \lambda_1)d(x,y) = \lambda d(x,y),$$

where $\lambda = \lambda_k + \lambda_{k-1} + \ldots + \lambda_1 \in (0,1)$. Hence all the hypotheses of Corollary **??** are satisfied and the result follows. $\square$

If we take $F(\alpha) = -\dfrac{1}{\sqrt{\alpha}}$ for $\alpha \in \mathbb{R}^+$, then $F \in \mathcal{F}$ and we obtain the following non-linear contraction that generalizes the contraction of type $d(fx, fy) \leq \alpha\,(d\,(x,y))\,d\,(x,y)$ (see Boyd and Wong [6]).

Corollary 2.3. *Let* (X, d) *be a complete metric space,* k *a positive integer and* $f : X^k \to X$ *a given mapping. Suppose that there exists* $\tau > 0$ *such that*

$$d(f(x_1, \ldots, x_k), f(x_2, \ldots, x_{k+1})) \leq \frac{1}{(1 + \tau\sqrt{\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}})^2} \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\},$$

for all $(x_1, \ldots, x_{k+1}) \in X^{k+1}$ *with* $f(x_1, \ldots, x_k) \neq f(x_2, \ldots, x_{k+1})$. *Then, for any arbitrary points* $x_1, \ldots, x_k \in X$, *the sequence* $\{x_n\}$ *defined by (1.1) converges to* u, *that is,* $u = f(u, \ldots, u)$. *Moreover, if*

$$d(f(x, \ldots, x), f(y, \ldots, y)) \leq \lambda d(x, y),$$

holds for all $x, y \in X$ *with* $x \neq y$ *where* $\lambda \in [0, 1)$, *then* u *is the unique fixed point of* f.

Remark 2.1.   (1) *Theorem 2.1 extends and generalizes Theorem 1.2 in* [7], *and Theorem 1.1 in* [16].
  (2) *If* $k = 1$, *Theorem 2.1 reduces to the Theorem 2.1 of Wardowski in* [23].
  (3) *If* $k = 1$, *Corollaries 2.2-?? reduces to Theorem 1 of Banach in* [1].
  (4) *If* $k = 1$, *Corollary ?? reduces to the Theorem of Boyd and Wong in* [6].

3. Global Attractivity Results for a Class of Matrix Difference Equations

In this section, we investigate the global attractivity of the recursive sequence $\{X_n\} \subset P(N)$ defined by

(7) $$X_{n+k} = Q + \frac{1}{k} \sum_{i=0}^{k-1} A^* \varphi(X_{n+i}) A, \quad n = 1, 2, \ldots,$$

where $P(N)$ is the set of $N \times N$ Hermitian positive definite matrices, k is a positive integer, Q is an $N \times N$ Hermitian positive semidefinite matrix, A is an $N \times N$ nonsingular matrix, A^* is the conjugate transpose of A and $\varphi : P(N) \to P(N)$.

At first, we recall some definitions and preliminary results.

Definition 3.1. *Let* k *be a positive integer,* M *a nonempty set and* $f : M^k \to M$. *For given* $x_1, x_2, \ldots, x_k \in M$, *consider the recursive sequence* $\{x_n\} \subset M$ *defined by*

(8) $$x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}), \quad n = 1, 2, \ldots,$$

The equilibrium point $\overline{x}$ *of the equation (3.2) is the point that satisfies the condition:*

$$\overline{x} = f(\overline{x}, \ldots, \overline{x}).$$

Definition 3.2. *Let* (M, d) *be a metric space and* $\overline{x}$ *an equilibrium point of Equation (8). The equilibrium point* $\overline{x}$ *is called a global attractor if for all* $x_1, x_2, \ldots, x_k \in M$, *we have* $d(x_n, \overline{x}) \to 0$ *as* $n \to \infty$.

We denote by $P(N)$ (for $N \geq 2$), the open convex cone of all $N \times N$ Hermitian positive definite matrices. We endow $P(N)$ with the Thompson metric defined by

$$A, B \in P(N), \quad d(A, B) = \max\{\ln M(A/B), \ln M(B/A)\},$$

where $M(A/B) = \inf\{\theta > 0 : A \leq \theta B\} = \theta^+(B^{-1/2}AB^{-1/2})$, the maximal eigenvalue of $B^{-1/2}AB^{-1/2}$. Here, $X \leq Y$ means that $Y - X$ is positive semidefinite and $X < Y$ means that $Y - X$ is positive definite. From Nussbaum [13], $P(N)$ is a complete metric space with respect to the Thompson metric d and $d(A, B) = \|\ln(A^{-1/2}BA^{-1/2})\|$, where $\|\cdot\|$ stands for the spectral norm. The Thompson metric exists on any open normal convex cones of real Banach spaces [13, 22]; in particular, the open convex cone of positive definite operators of a Hilbert space. Now we shortly introduce the elegant properties of the Thompson metric. It is invariant under the matrix inversion and congruence transformations, that is,

$$(9) \qquad d(A, B) = d(A^{-1}, B^{-1}) = d(U^*AU, U^*BU),$$

for any nonsingular matrix U. The other useful result is the nonpositive curvature property of the Thompson metric

$$(10) \qquad d(X^r, Y^r) \leq r d(X, Y), \quad r \in [0, 1].$$

According to (9) and (10), we have

$$(11) \qquad d(U^*X^rU, U^*Y^rU) \leq |r| d(X, Y), \quad r \in [-1, 1].$$

Lemma 3.1. *Lemma 3.3. For any $A, B, C, D \in P(N)$,*

$$d(A + B, C + D) \leq \max\{d(A, C), d(B, D)\}.$$

Furthermore, for all positive semidefinite A and $B, C \in P(N)$,

$$d(A + B, A + C) \leq d(B, C).$$

Let $\varphi : P(N) \to P(N)$ be an F-contraction mapping with respect to the Thompson metric d. Let Q be an $N \times N$ Hermitian positive semidefinite matrix ($Q \geq 0$) and A an $N \times N$ nonsingular matrix (A^{-1} exists). For a positive integer k, for given $X_1, X_2, \ldots, X_k \in P(N)$, consider the sequence $\{X_n\} \subset P(N)$ defined by (7). Our main result in this section is the following.

Theorem 3.1. *Equation (7) has a unique equilibrium point $\overline{X} \in P(N)$. Moreover, $\overline{X}$ is global attractor.*

Proof. Define the mapping $f : P(N)^k \to P(N)$ by

$$f(U_1, U_2, \ldots, U_k) = Q + \frac{1}{k}[A^*\varphi(U_1)A + A^*\varphi(U_2)A + \ldots + A^*\varphi(U_k)A],$$

for all $U_1, U_2, \ldots, U_k \in P(N)$.

Let $U_1, U_2, \ldots, U_{k+1} \in P(N)$. By using Lemma 3.3, we have

$$d(f(U_1, U_2, \ldots, U_k), f(U_2, U_3, \ldots, U_{k+1}))$$

$$= d\left(Q + \frac{1}{k}\sum_{i=1}^{k} A^*\varphi(U_i)A, Q + \frac{1}{k}\sum_{j=2}^{k+1} A^*\varphi(U_j)A\right)$$

$$\leq d\left(\frac{1}{k}\sum_{i=1}^{k} A^*\varphi(U_i)A, \frac{1}{k}\sum_{j=2}^{k+1} A^*\varphi(U_j)A\right)$$

$$= d\left(\sum_{i=1}^{k} \left(\frac{1}{\sqrt{k}}A\right)^* \varphi(U_i)\left(\frac{1}{\sqrt{k}}A\right), \sum_{j=2}^{k+1} \left(\frac{1}{\sqrt{k}}A\right)^* \varphi(U_j)\left(\frac{1}{\sqrt{k}}A\right)\right).$$

Denote $V = \dfrac{1}{\sqrt{k}}A$. Then, using again Lemma 3.3, we have

$$d(f(U_1, U_2, \ldots, U_k), f(U_2, U_3, \ldots, U_{k+1}))$$

$$\leq d\left(\sum_{i=1}^{k} V^*\varphi(U_i)V, \sum_{j=2}^{k+1} V^*\varphi(U_j)V\right)$$

$$= d\left(V^*\varphi(U_1)V + V^*\varphi(U_2)V + \ldots + V^*\varphi(U_k)V, V^*\varphi(U_2)V + V^*\varphi(U_3)V + \ldots + V^*\varphi(U_{k+1})V\right)$$

$$\leq \max\left\{d(V^*\varphi(U_1)V, V^*\varphi(U_2)V), d(V^*\varphi(U_2)V, V^*\varphi(U_3)V), \ldots, d(V^*\varphi(U_k)V, V^*\varphi(U_{k+1})V)\right\}$$

$$= \max\left\{d(V^*\varphi(U_i)V, V^*\varphi(U_{i+1})V): \ i = 1, 2, \ldots, k\right\}.$$

Since A is nonsingular, the matrix V is also nonsingular. Using property (9), for all $i = 1, 2, \ldots, k$, we have

$$d(V^*\varphi(U_i)V, V^*\varphi(U_{i+1})V) = d(\varphi(U_i), \varphi(U_{i+1})).$$

But φ is an F-contraction. Then, for all $i = 1, 2, \ldots, k$, we have

$$\tau + F\left(d(V^*\varphi(U_i)V, V^*\varphi(U_{i+1})V)\right) \leq F\left(d(U_i, U_{i+1})\right).$$

Thus, we have

$$\tau + F\left(d(f(U_1, U_2, \ldots, U_k), f(U_2, U_3, \ldots, U_{k+1}))\right) \leq \max\left\{d(U_i, U_{i+1}): \ i = 1, 2, \ldots, k\right\}$$

for all $U_1, U_2, \ldots, U_{k+1} \in P(N)$.

Now, Applying Theorem 2.1, we obtain the existence of a global attractor equilibrium point $\overline{X} \in P(N)$.

On the other hand, for $U, W \in P(N)$ such that $d(f(U, U, \ldots, U) \neq f(W, W, \ldots, W))$, we have

$$
\begin{aligned}
F\left(d(f(U, U, \ldots, U), f(W, W, \ldots, W))\right) &= F\left(d(Q + A^*\varphi(U)A, Q + A^*\varphi(W)A)\right) \\
&\leq F\left(d(A^*\varphi(U)A, A^*\varphi(W)A)\right) \\
&= F\left(d(\varphi(U), \varphi(W))\right) \\
&\leq F\left(d(U, W)\right) - \tau,
\end{aligned}
$$

implies that

$$
\begin{aligned}
\tau + F\left(d(f(U, U, \ldots, U), f(W, W, \ldots, W))\right) &\leq \tau + F\left(d(\varphi(U), \varphi(W))\right) \\
&\leq F\left(d(U, W)\right).
\end{aligned}
$$

Again, applying Theorem 2.1, we obtain the uniqueness of the equilibrium point. $\qquad\square$

Now, we present some examples and numerical experiments.

For a positive integer k, consider the sequence $\{X_n\} \subset P(N)$ defined by

$$
\tag{12} X_{n+k} = Q + \frac{1}{k} \sum_{i=0}^{k-1} A^* X_{n+i}^\delta A, \quad n = 1, 2, \ldots
$$

for given $X_1, X_2, \ldots, X_k \in P(N)$, where $|\delta| \in [0, 1)$.

Corollary 3.1. Corollary 3.5. *Equation (12) has a unique equilibrium point $\overline{X} \in P(N)$. Moreover, $\overline{X}$ is global attractor.*

Proof. By using Properties (9) and (11), we show easily that $\varphi : P(N) \to P(N)$ defined by

$$
\varphi(X) = X^\delta, \quad \text{for all } X \in P(N)
$$

is $|\delta|$-contraction. Then, the result follows immediately from Theorem **??**. $\qquad\square$

Remark 3.6. The equilibrium point $\overline{X} \in P(N)$ of Equation (12) is the unique positive definite solution to the nonlinear matrix equation

$$
\tag{13} \overline{X} = Q + A^* \overline{X}^\delta A.
$$

In the last few years there has been a constantly increasing interest in developing the theory and numerical approaches for positive definite solutions to the nonlinear matrix equation of the form (13) (see, for example, [4, 8, 12]).

As an example, we consider for given $X_1, X_2 \in P(N)$, the recursive sequence $\{X_n\} \subset P(N)$ given by

$$
\tag{14} X_{n+2} = Q + \frac{1}{2}\left(A^* X_n^{1/2} A + A^* X_{n+1}^{1/2} A\right), \quad n = 1, 2, \ldots
$$

From Corollary 3.1, Equation (14) has a unique equilibrium point $\overline{X} \in P(N)$, that is, the unique positive definite solution to

$$
\overline{X} = Q + A^* \overline{X}^{1/2} A.
$$

To check our global attractivity result, we give the following numerical experiments.

We take $N = 3$, Q and A are given by

$$Q = \begin{pmatrix} 0.2 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}.$$

For each iteration i, we consider the residual error $E(i)$ given by

$$E(i) = \left\| X_i - (Q + A^* X_i^{1/2} A) \right\|,$$

where $\| \cdot \|$ is the spectral norm. All programs are written in MATLAB version 7.1.

Let us take

$$X_1 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 4 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} 5 & 5 & 1 \\ 5 & 11 & 7 \\ 1 & 7 & 17 \end{pmatrix},$$

then after 90 iterations of iterative method (14), we get the unique equilibrium point

$$(15) \qquad \overline{X} \approx X_{90} = \begin{pmatrix} 438.4 & 429.2 & 429.2 \\ 429.2 & 438.4 & 429.2 \\ 429.2 & 429.2 & 438.4 \end{pmatrix},$$

and its residual error $E(90) = 1.0503e - 013$.

For other initial points

$$X_1 = \begin{pmatrix} 120 & 7 & 7 \\ 7 & 120 & 7 \\ 7 & 7 & 120 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1003 & 3 & 3 \\ 3 & 2003 & 3 \\ 3 & 3 & 3003 \end{pmatrix},$$

after 90 iterations, we get the unique equilibrium point $\overline{X}$ given by (15), and its residual error $E(90) = 2.0196e - 013$. $\square$

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