KURTOSIS OF THE LOGISTIC-EXPONENTIAL SURVIVAL DISTRIBUTION

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ABSTRACT

In this paper the kurtosis of the logistic-exponential distribution is analyzed. All the moments of this survival distribution are finite, but do not possess closed-form expressions. The standardized fourth central moment, known as Pearson's coefficient of kurtosis and often used to describe the kurtosis of a distribution, can thus also not be expressed in closed form for the logistic-exponential distribution. Alternative kurtosis measures are therefore considered, specifically quantile-based measures and the L-kurtosis ratio. It is shown that these kurtosis measures of the logistic-exponential distribution are invariant to the values of the distribution's **single** shape parameter and hence skewness-invariant.

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1. INTRODUCTION

The logistic-exponential distribution, introduced by Lan and Leemis (2008), is a useful model in survival analysis, since it encompasses failure rates that are increasing, decreasing, bathtub-shaped and upside-down bathtub-shaped, and because its cumulative distribution function, $F(x)$, probability density function, $f(x)$, and quantile function, $Q(u)$, all have closed-form expressions. This survival distribution has a two-parameter version with scale parameter $\lambda > 0$ and shape parameter $\kappa > 0$, and a three-parameter version which includes a location parameter, $\theta \geqslant 0$. In this paper the focus will be on the two-parameter version, included by Leemis and McQueston (2008) in their univariate distribution relationship chart (see `www.math.wm.edu/~leemis/chart/UDR/UDR.html`).

The cumulative distribution, probability density and quantile functions of the logistic-exponential distribution are

$$F(x) = 1 - \frac{1}{1 + (\exp[\lambda x] - 1)^{\kappa}}, \quad x > 0,$$

$$f(x) = \frac{\lambda\kappa \exp[\lambda x]\,(\exp[\lambda x] - 1)^{\kappa-1}}{\left(1 + (\exp[\lambda x] - 1)^{\kappa}\right)^2}, \quad x > 0,$$

and

$$Q(u) = \tfrac{1}{\lambda} \ln\left[1 + \left(\tfrac{u}{1-u}\right)^{\frac{1}{\kappa}}\right], \quad 0 < u < 1,$$

respectively. If $\kappa > 1$, the probability density function of the logistic-exponential distribution is unimodal and the distribution has upside-down bathtub-shaped failure rates. The logistic-exponential distribution's probability density function is J-shaped for $\kappa \leqslant 1$ and the distribution reduces to the exponential distribution with a constant failure rate when $\kappa = 1$. Bathtub-shaped failure rates are obtained for $\kappa < 1$. Density curves for the logistic-exponential distribution are illustrated in Figure 1 for $\lambda = 1$ and selected values of κ.

This paper investigates the kurtosis properties of the logistic-exponential distribution. The skewness and kurtosis of a distribution have historically been described with its standardized third and fourth central moments, α_3 and α_4, referred to as Pearson's coefficients of skewness and kurtosis in the literature (Pearson, 1905). But although all the moments of the logistic-exponential distribution exist, and hence also its α_3 and α_4, they do not have closed-form expressions. Of course, it is well known that $\alpha_3 = 2$ and $\alpha_4 = 9$ for $\kappa = 1$ (exponential distribution). It was furthermore proved by Lan and Leemis (2008) that $\alpha_3 \downarrow 2.1126$ and $\alpha_4 \downarrow 8.6876$ as $\kappa \downarrow 0$ and that $\alpha_3 \downarrow 0$ and $\alpha_4 \downarrow 4.2$ as $\kappa \to \infty$ (note that the logistic distribution has $\alpha_3 = 0$ and $\alpha_4 = 4.2$).

Figure 2 shows α_4 plotted against α_3 for various skewed two-parameter distributions with nonnegative support. The uniform, normal and logistic distributions at $(\alpha_3, \alpha_4) = (0, 1.8)$, $(\alpha_3, \alpha_4) = (0, 3)$ and $(\alpha_3, \alpha_4) = (0, 4.2)$ respectively are symmetric limiting or special cases of some of these skew distributions. Specifically, the generalized Pareto distribution reduces to the uniform distribution when its shape parameter equals one (Hosking and Wallis, 1987). As explained in Johnson et al. (1994), both the gamma distribution and the log-normal distribution tend to the normal distribution as the values of their shape parameters tend to infinity (or zero, depending on the parameterizations of the gamma distribution and the log-normal distribution considered). Likewise the logistic-exponential distribution and the log-logistic distribution tend to the logistic distribution as the values of their shape parameters tend to infinity - see Lan and Leemis (2008) and Tadikamalla and Johnson (1982) respectively for details.

Since no closed-form expressions exist for α_3 and α_4 of the logistic-exponential distribution, their values in Figure 2 were estimated as the averages of the sample coefficients of skewness and kurtosis of 20 simulated samples of size 50 000 each. The values for all the other distributions' coefficients of skewness and kurtosis were obtained using their corresponding theoretical expressions – see, for instance, Johnson et al. (1994, 1995).

It is evident from Figure 2 that, in terms of α_3 and α_4, the curve plotted for the logistic-exponential distribution differs from the curves plotted for the other distributions. As discussed by Lan and Leemis (2008), it is the only curve that is bounded and hence it is the only curve for which maximum values for α_3 and α_4 are achieved. In the next two sections the kurtosis properties of the logistic-exponential distribution are studied. Specifically Section 2 proves that the quantile-based kurtosis measures of the logistic-exponential distribution are

constant. Section 3 considers the L-moments of the logistic-exponential distribution. In particular, the L-kurtosis ratio is found to be constant**. Thus for this useful family of distributions, these two broad classes of kurtosis measures are constant, and, in particular, are skewness-invariant**.

2. QUANTILE-BASED KURTOSIS MEASURES

As indicated by Jones et al. (2011) and van Staden (2013), quantile-based kurtosis measures are typically of the general form

$$\frac{\sum_{j=1}^{n_1} a_j S(u_j)}{\sum_{k=1}^{n_2} b_k S(u_k)} = \frac{\sum_{j=1}^{n_1} a_j \left(Q(u_j) - Q(1-u_j)\right)}{\sum_{k=1}^{n_2} b_k \left(Q(u_k) - Q(1-u_k)\right)}, \tag{1}$$

where $a_j : j = 1, 2, ..., n_1$ and $b_k : k = 1, 2, ..., n_2$ are constants with n_1 and n_2 positive integers, and where

$$S(u) = Q(u) - Q(1-u), \quad \tfrac{1}{2} < u < 1,$$

is the spread function introduced by MacGillivray and Balanda (1988). Examples of kurtosis measures of the form in (1) include the measure by Kelley (1921),

$$K = \frac{S(\frac{3}{4})}{2S(\frac{9}{10})},$$

the octile-based measure of Moors (1988),

$$M = \frac{S(\frac{7}{8}) - S(\frac{5}{8})}{S(\frac{3}{4})},$$

the quintile-based measure given in Jones et al. (2011),

$$J = \frac{S(\frac{4}{5}) - 3S(\frac{3}{5})}{S(\frac{4}{5})},$$

and the ratio-of-spread functions,

$$R(u, v) = \frac{S(u)}{S(v)}, \quad \tfrac{1}{2} < v < u < 1,$$

proposed by MacGillivray and Balanda (1988).

Referred to as the spread-spread function by some researchers in the literature (see, for instance, Seier and Bonett (2003) and Kotz and Seier (2008)), the ratio-of-spread functions is a shape functional for kurtosis related to the plot of $S_G(S_F)^{-1}$ for distributions F and G, called the spread-spread plot by Balanda and MacGillivray (1990). Linking the spread-spread plot to kurtosis orderings, Balanda and MacGillivray (1990) extended van Zwet's ordering $\leqslant_S$, van Zwet (1964), to skewed distributions, defining $F \leqslant_S G \Leftrightarrow S_G\left((S_F(u))^{-1}\right)$ convex for $\frac{1}{2} < u < 1$. That is, if the spread-spread plot is convex (concave) for $\frac{1}{2} < u < 1$, then distribution G has greater (smaller) kurtosis than distribution F.

Because the spread function of the logistic-exponential distribution is

$$\begin{aligned} S(u) &= Q(u) - Q(1-u) \\ &= \tfrac{1}{\lambda}\ln\left[1 + \left(\tfrac{u}{1-u}\right)^{\frac{1}{\kappa}}\right] - \tfrac{1}{\lambda}\ln\left[1 + \left(\tfrac{1-u}{u}\right)^{\frac{1}{\kappa}}\right] \\ &= \tfrac{1}{\lambda}\ln\left[\frac{1 + \left(\tfrac{u}{1-u}\right)^{\frac{1}{\kappa}}}{1 + \left(\tfrac{1-u}{u}\right)^{\frac{1}{\kappa}}}\right] \\ &= \tfrac{1}{\lambda}\ln\left[\left(\tfrac{u}{1-u}\right)^{\frac{1}{\kappa}}\right] \qquad \text{since } \frac{1+z}{1+z^{-1}} = z \text{ for } z \neq 0 \\ &= \tfrac{1}{\lambda\kappa}\left(\ln u - \ln[1-u]\right), \end{aligned}$$

the logistic-exponential distribution's quantile-based kurtosis measures of the form in (1) are given by

$$\frac{\sum_{j=1}^{n_1} a_j S(u_j)}{\sum_{k=1}^{n_2} b_k S(u_k)} = \frac{\sum_{j=1}^{n_1} a_j \left(\ln u_j - \ln[1-u_j]\right)}{\sum_{k=1}^{n_2} b_k \left(\ln u_k - \ln[1-u_k]\right)},$$

and are thus invariant to the value of the shape parameter κ. In particular, the logistic-exponential distribution's ratio-of-spread functions is simply

$$R(u,v) = \frac{\ln u - \ln[1-u]}{\ln v - \ln[1-v]},$$

and therefore, in terms of $\leqslant_S$, the kurtosis of the logistic-exponential distribution is the same for all values of κ.

None of the other two-parameter distributions considered in Figure 2 possess quantile-based kurtosis measures which are invariant to the values of their shape parameters. This is illustrated in Figure 3 depicting a shape functional diagram in which the ratio-of-spread functions is plotted against

$$\gamma(u) = \frac{Q(u) + Q(1-u) - 2Q(\frac{1}{2})}{S(u)}, \quad \tfrac{1}{2} < u < 1, \tag{2}$$

for the various survival distributions. The skewness functional in (2), originally suggested by David and Johnson (1956) and called the γ-functional, defines the **weak** skewness ordering $\underset{\gamma}{\overset{m}{\leqslant}}_2$, MacGillivray (1986), in that $F \underset{\gamma}{\overset{m}{\leqslant}}_2 G \Leftrightarrow \gamma_F(u) \leqslant \gamma_G(u)$. In effect, distribution G has greater skewness to the right (left) than distribution F if $\gamma_G(u) > (<)\, \gamma_F(u)$. Note that in Figure 3 the γ-functional and the ratio-of-spread functions are evaluated at $u = 0.9$ and $v = 0.75$.

The γ-functional of the logistic-exponential distribution is

$$\gamma(u) = \frac{\kappa\left(\ln\left[1+\left(\frac{u}{1-u}\right)^{\frac{1}{\kappa}}\right] + \ln\left[1+\left(\frac{1-u}{u}\right)^{\frac{1}{\kappa}}\right] - 2\ln 2\right)}{\ln u - \ln[1-u]},$$

and thus depends on the value of κ. **However, as shown above and illustrated in Figure 3, the ratio-of-spread functions, and all the quantile-based kurtosis measures of the form in (1), of the logistic-exponential distribution remain constant for different levels of skewness, including for different values of the γ-functional.**

3. L-MOMENTS

The theory of L-moments was compiled by Hosking (1990). L-moments are expectations of linear combinations of order statistics. Let $X_{1:n} \leqslant X_{2:n} \leqslant ... \leqslant X_{n:n}$ denote the order statistics for a random sample of size n from the distribution of X. The rth order L-moment and L-moment ratio of X are then defined by

$$L_r \equiv \tfrac{1}{r}\sum_{k=0}^{r-1}(-1)^k \tbinom{r-1}{k} E(X_{r-k:r}), \quad r = 1, 2, 3, ...,$$

and

$$\tau_r \equiv \frac{L_r}{L_2}, \quad r = 3, 4, 5, ...,$$

where L_1 and L_2 are the L-location and L-scale and where τ_3 and τ_4 are known as the L-skewness and L-kurtosis ratios. Note that in this paper the rth order L-moment is denoted by L_r instead of λ_r, as is usually done in the literature, to avoid confusion with the scale parameter of the logistic-exponential distribution. Hosking (1990) showed that the rth order

L-moment of X can be written as $L_r = \int_0^1 Q(u)P^*_{r-1}(u)du$, where

$$P^*_r(u) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} \binom{r+k}{k} u^k$$

is the rth order shifted Legendre polynomial, related to $P_r(u)$, the rth order Legendre polynomial, by $P^*_r(u) = P_r(2u-1)$.

L-moments possess several advantages compared to conventional moments. Firstly, all L-moment ratios are bounded, simplifying their interpretation. In particular, Hosking (1990) and Jones (2004) have shown that $-1 < \tau_3 < 1$ and $\frac{1}{4}(5\tau_3^2 - 1) \leqslant \tau_4 < 1$. Secondly, Hosking (1990) proved that if the mean of a distribution exists, then all its L-moments exist and the distribution is uniquely characterized by its L-moments. **All moments and all** L-moments of the logistic-exponential distribution exist.

Closed-form expressions for all the L-moments of the logistic-exponential distribution so far elude us. Table 1 reports values of $L_r : r = 1, 2, ..., 10$ for $\lambda = 1$ without loss of generality ($L_{r;\lambda,\kappa} = \lambda^{-1} L_{r;1,\kappa}$ because λ is a scale parameter) and for the values of κ represented in Figure 1. The values in Table 1 were obtained with *Mathematica* 8.0 notebooks (Wolfram, 2010). For example, Table 2 gives the *Mathematica* 8.0 source code for calculating $L_r : r = 1, 2, ..., 10$ for $\kappa = 0, 2$. Regarding the L-kurtosis ratio, it is known that $\tau_4 = \frac{1}{6}$ for the exponential distribution ($\kappa = 1$) as well as the logistic distribution ($\kappa \to \infty$). So immediately one wonders about the values of τ_4 for other values of κ. Using numerical methods in *Mathematica* 8.0, we obtained

$$L_r = \frac{1}{r(r-1)\lambda\kappa}, \quad r = 2, 4, 6, ...,$$

from which it follows that

$$\tau_r = \frac{2}{r(r-1)}, \quad r = 2, 4, 6, ...,$$

is independent of λ and κ, and, in particular, $\tau_4 = \frac{1}{6}$ and hence skewness-invariant. No expressions for L_r or τ_r are available yet for $r = 1, 3, 5, ...$, but numerical analysis using *Mathematica* 8.0 indicated that $\tau_3 \downarrow 0$ as $\kappa \to \infty$ and $\tau_3 \uparrow \frac{1}{2}$ as $\kappa \downarrow 0$.

Figure 4 presents an L-moment ratio diagram for the positively skewed two-parameter survival distributions considered in Figures 2 and 3. Values and expressions for the L-moment ratios of the uniform, normal, logistic and exponential distributions as well as for the generalized Pareto distribution, the Weibull distribution (which is a reflected generalized extreme-value distribution) and the log-logistic distribution (which can be reparameterized as a generalized logistic distribution, Hosking and Wallis (1997)) are available from Hosking (1990) and Hosking and Wallis (1997). Approximate values of τ_3 and τ_4 for the gamma and log-normal distributions in Figure 4 were calculated using rational-function approximations given by Hosking and Wallis (1997).

As was the case with Pearson's coefficients of skewness and kurtosis illustrated in Figure 2 and the shape functionals depicted in Figure 3, the L-moment ratios of the logistic-exponential distribution behave distinctly differently compared to the L-moment ratios of the other survival distributions considered in Figure 4. Firstly, and most apparent from Figure 4, among the distributions considered, the logistic-exponential distribution is the only survival distribution possesing a skewness-invariant L-kurtosis ratio. Secondly, although the focus of this paper is on kurtosis and not skewness, it is interesting to note that the logistic-exponential distribution has $\tau_3 \uparrow \frac{1}{2}$, whereas $\tau_3 \uparrow 1$ for all the other survival distributions in Figure 4.

4. CONCLUDING REMARKS

The concept of kurtosis is a complex topic in statistical research – the interested reader is referred to Balanda and MacGillivray (1988) and Seier (2003) for detailed reviews on kurtosis. Recently Jones et al. (2011) presented a seminal discussion on skewness-invariant kurtosis measures - also see van Staden (2013). Turning to survival distributions, this paper showed that the logistic-exponential distribution occupies a special place in the realm of two-parameter distributions with nonnegative support in that its quantile-based kurtosis measures and its L-kurtosis ratio are invariant to the values of its shape parameter and are consequently skewness-invariant. However, it is important to note that the shape parameter of the logistic-exponential distribution cannot be **universally** labeled a skewness parameter.

Although no closed-form expressions are available for the logistic-exponential distribution's coefficients of skewness and kurtosis, it follows from Figure 2 that neither of these classical measures of shape for the logistic-exponential distribution are invariant to the values of the shape parameter, since the corresponding curve is not a straight line.

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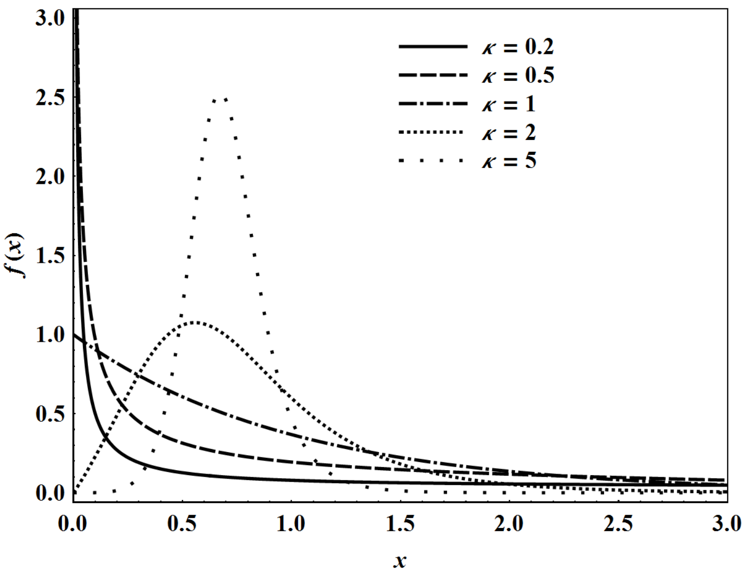

Figure 1. Density curves for the logistic-exponential distribution for $\lambda = 1$ and selected values of κ.

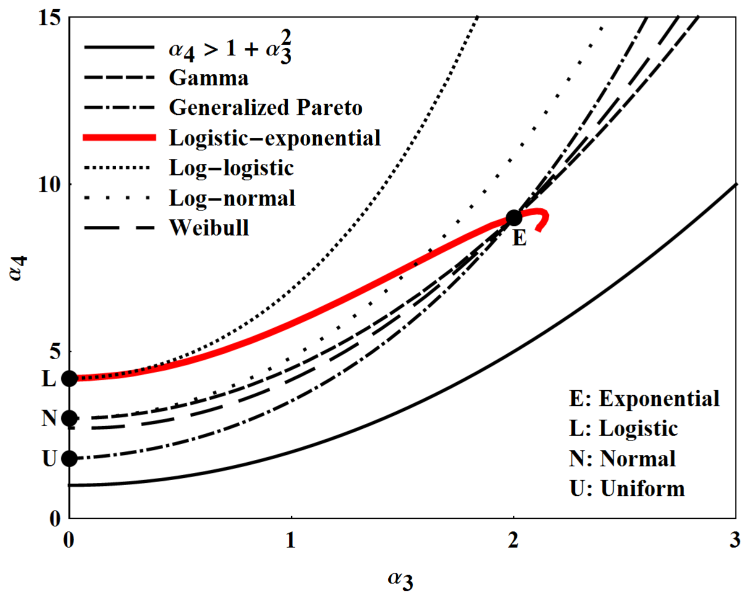

Figure 2. Moment ratio diagram in which Pearson's coefficient of kurtosis, α_4, is plotted against Pearson's coefficient of skewness, α_3, for the logistic-exponential and other skewed two-parameter survival distributions with nonnegative support. The lower boundary for all distributions is given by $\alpha_4 > 1 + \alpha_3^2$.

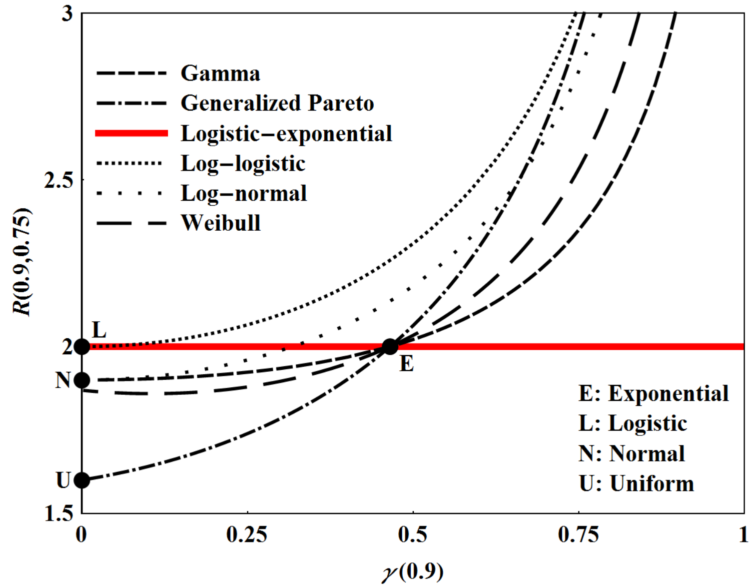

Figure 3. Shape functional diagram for the two-parameter survival distributions, including the logistic-exponential distribution, considered in Figure 2.

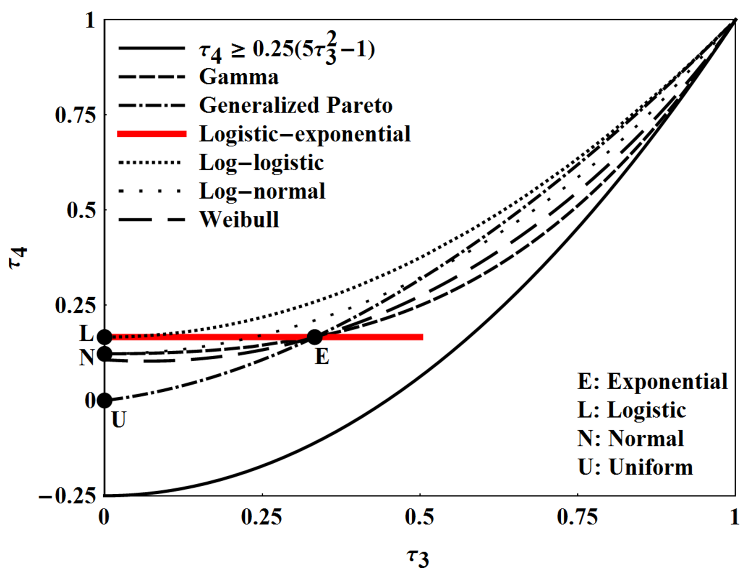

Figure 4. L-moment ratio diagram for the two-parameter survival distributions, including the logistic-exponential distribution, considered in Figure 2. The lower boundary for all distributions is given by $\tau_4 \geqslant \frac{1}{4}(5\tau_3^2 - 1)$.

Table 1. Values for the first ten L-moments of the logistic-exponential distribution for $\lambda = 1$ and the values of κ represented in Figure 1.

κ	0.2	0.5	1	2	5
L_1	3.5462	1.5708	1.0000	0.7854	0.7092
L_2	2.5000	1.0000	0.5000	0.2500	0.1000
L_3	1.2121	0.4292	0.1667	0.0537	0.0097
L_4	0.4167	0.1667	0.0833	0.0417	0.0167
L_5	0.1811	0.0937	0.0500	0.0197	0.0039
L_6	0.1667	0.0667	0.0333	0.0167	0.0067
L_7	0.1391	0.0482	0.0238	0.0101	0.0022
L_8	0.0893	0.0357	0.0179	0.0089	0.0036
L_9	0.0624	0.0277	0.0139	0.0061	0.0014
L_{10}	0.0556	0.0222	0.0111	0.0056	0.0022

Table 2. *Mathematica* 8.0 source code for calculating the first ten L-moments of the logistic-exponential distribution for $\kappa = 0.2$.

Source code:

$\boldsymbol{\kappa = 0.2;}$

$\boldsymbol{L_r = N\left[\text{Table}\left[\int_0^1 \left(\text{Log}\left[1 + \left(\frac{u}{1-u}\right)^{\frac{1}{\kappa}}\right] \text{LegendreP}[r-1, 2u-1]\right) du, \{r, 1, 10, 1\}\right]\right]}$

Solution: $\{3.54622, 2.5, 1.21214, 0.416667, 0.181089, 0.166667, 0.139118, 0.0892857, 0.0624484, 0.0555556\}$