Clifford A-algebras of Quadratic A-Modules

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Abstract. A *Clifford $\mathcal{A}$-algebra* of a quadratic $\mathcal{A}$-module $(\mathcal{E}, q)$ i san as - sociative and unital $\mathcal{A}$-algebra (i.e. sheaf of $\mathcal{A}$-algebras) associated with the quadratic $\mathcal{S}h\mathcal{S}et_X$-morphism q, and satisfying a certain universal property. By introducing sheaves of sets of orthogonal bases (or simply sheaves of orthogonal bases), we show that *with every Riemannian quadratic free $\mathcal{A}$-module of finite rank, say, n, one can associate a Clifford free $\mathcal{A}$-algebra of rank* 2^n. This "*main*" result is stated in Theorem 3.2.

Keywords. Clifford $\mathcal{A}$-morphism, quadratic $\mathcal{A}$-module, Riemannian qua-dratic $\mathcal{A}$-module, Clifford $\mathcal{A}$-algebra, principal $\mathcal{A}$-automorphism, even sub-$\mathcal{A}$-algebra, $\mathcal{A}$-antiautomorphism, sub-$\mathcal{A}$-module of odd products.

Introduction

Abstract Differential Geometry (ADG, in short) offers ways to circumvent singularities, which arise in physics due to the inefficiency of the classical differential geometry (CDG, viz. differential geometry of smooth manifolds) when applied in the quantum domain (quantum field theory, for instance). Mallios in [7] expounds on the entanglement of the notion of smooth manifolds in the quantum domain. To remedy this undesirable shortfall of CDG, Mallios suggests a *sheaf-theoretic* démarche. In this *axiomatic* setting, smooth manifolds are replaced by *vector sheaves* (i.e. *locally free $\mathcal{A}$-modules*). See [4, 5, 6].

It is in this purely algebraic setting that we envisage to cast a look at *sheaves of Clifford algebras* over a given sheaf $\mathcal{A}$ of unital and commutative algebras. Sheaves of Clifford algebras over $\mathcal{A}$ are also called *Clifford $\mathcal{A}$-algebras.*

Clifford $\mathcal{A}$-algebras are defined in a way to be a generalization of Clifford algebras of quadratic vector spaces, but instead of considering arbitrary quadratic free $\mathcal{A}$-modules $(\mathcal{E}, q)$ of finite rank (in the pair $(\mathcal{E}, q)$, $\mathcal{E}$ is a free $\mathcal{A}$-module of finite rank and q is a quadratic $\mathcal{SETS}$-morphism sending the underlying sheaf of sets of $\mathcal{E}$ into the underlying sheaf of sets of $\mathcal{A}$) we consider *Riemannian quadratic free $\mathcal{A}$-modules* $(\mathcal{E}, q)$ of finite rank; these are quadratic free $\mathcal{A}$-modules such that the *q-induced* $\mathcal{A}$-bilinear morphism is a *Riemannian $\mathcal{A}$-metric,* viz. a symmetric positive definite $\mathcal{A}$-bilinear morphism, cf. [4, p. 318, Definition 8.2]. The motivation behind this restricting to Riemannian quadratic free $\mathcal{A}$-modules of finite rank lays in the fact that the Gram-Schmidt orthogonalization process affords one with orthogonal gauges of Riemannian free $\mathcal{A}$-modules of finite rank, where the ordered algebraized space $(X, \mathcal{A})$ is enriched with square root and satisfies the inverse-closed section condition. See [4, pp, 335-340] and [10] for details on the Gram-Schmidt orthogonalization process and its generalization to symplectic free $\mathcal{A}$-modules of finite rank, respectively.

Making use of techniques underlying the proof in [2, pp 294, 295, Theorem VIII.2.B] that every quadratic vector space of finite dimension admits a Clifford algebra, we show in Theorem 3.2, deemed to be the main result of the paper, that *with every Riemannian quadratic free $\mathcal{A}$-module of finite rank is associated up to $\mathcal{A}$-isomorphism a Clifford free $\mathcal{A}$-algebra of rank* 2^n *if the rank of the Riemannian quadratic free $\mathcal{A}$-module is n.*

Throughout the paper, the pair $(X, \mathcal{A})$, or just $\mathcal{A}$, will denote a fixed $\mathbb{C}$-algebraized space, with X a topological space and $\mathcal{A}$ a sheaf (over X) of unital and commutative algebras. We will assume that all sheaves encountered herein are defined over the topological space X. On the other hand, we will also mainly use the notation of [4]; thus, for instance, $\mathcal{A}$-$\mathcal{M}od_X$ will stand for the category of $\mathcal{A}$-modules with their respective $\mathcal{A}$-morphisms.

1. Clifford $\mathcal{A}$-Morphisms

In this section, we introduce Clifford $\mathcal{A}$-morphims, a notion derived from the classical one, viz. Clifford maps of quadratic vector spaces (cf. [2, p. 287-289]).

Definition 1.1. Let $\mathcal{E}$ be an $\mathcal{A}$-module and $F : \mathcal{A}\text{-}\mathcal{M}od_X \longrightarrow \mathcal{S}h\mathcal{S}et_X$ the forgetful functor of the category of $\mathcal{A}$-modules into the category of sheaves of sets. A morphism $q \in \mathrm{H}om_{\mathcal{S}h\mathcal{S}et_X}(F(\mathcal{E}), F(\mathcal{A}))$ is called **$\mathcal{A}$-quadratic** on $\mathcal{E}$ if the following are satisfied:

(1) Given any open subset U of X and scalar $\lambda \in \mathcal{A}(U)$, define $\lambda \in \mathrm{H}om_{\mathcal{A}(U)}(\mathcal{A}(U), \mathcal{A}(U)) \equiv \mathrm{E}nd_{\mathcal{A}(U)}\mathcal{A}(U) \simeq \mathcal{A}(U)$ by

$$\lambda(s) := \lambda s,$$

for every $s \in \mathcal{A}(U)$. Then,

$$q_U \circ \lambda \equiv q \circ \lambda := ev(\lambda^2, q(-)) \equiv ev_U(\lambda^2, q_U(-)),$$

where $ev \in \mathrm{H}om_{\mathcal{S}h\mathcal{S}et_X}(F(\mathcal{E}nd_{\mathcal{A}}\mathcal{A}) \oplus F(\mathcal{A}), F(\mathcal{A}))$ (ev is called the *evaluation morphism*) is given by

$$ev_U(\psi, \alpha) \equiv ev(\psi, \alpha) := \psi_U(\alpha) \equiv \psi_U \cdot \alpha$$

for any open $U \subseteq X$ and sections $\alpha \in \mathcal{A}(U)$ and $\psi \in (\mathcal{E}nd_{\mathcal{A}}\mathcal{A})(U)$.

(2) The *morphism* $B_q \in \mathrm{H}om_{\mathcal{S}h\mathcal{S}et_X}(F(\mathcal{E}) \oplus F(\mathcal{E}), F(\mathcal{A}))$, given by

$$B_q := (q \circ +) - (q \circ pr_1) - (q \circ pr_2),$$

where $pr_i, + : F(\mathcal{E}) \oplus F(\mathcal{E}) \longrightarrow F(\mathcal{E})$ are the i-th projection and addition morphisms, respectively, *is* $\mathcal{A}$*-bilinear.*

The pair $(\mathcal{E}, q)$ is called a **quadratic $\mathcal{A}$-module.**

We shall denote by

$$\mathrm{Q}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U) \tag{1}$$

the set of $\mathcal{A}|_U$-quadratic morphisms on $\mathcal{E}|_U$. The set (1) is an $\mathcal{A}(U)$-module. In fact, for any $\alpha \in \mathcal{A}(U)$ and $q \equiv (q_V)_{U \supseteq V,\ open} \in \mathrm{Q}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U)$, one sets the following:

$$(\alpha \cdot q)_V := \alpha|_V \cdot q_V \equiv \alpha \cdot q_V,$$

which thus provides the $\mathcal{A}(U)$-module structure of (1). On the other hand, it is readily verified that the *collection* $(\mathrm{Q}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U), \sigma^U_V)$ *is a complete presheaf of modules* (the restriction maps are defined as follows: if $q \in \mathrm{Q}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U)$, then $\sigma^U_V(q) := (q_W)_{V \supseteq W,\ open}$). The sheaf generated by this complete presheaf is called the **sheaf of quadratic morphisms** of $\mathcal{E}$ and is denoted

$$\mathcal{Q}(\mathcal{E}, \mathcal{A}) \equiv \mathcal{Q}(\mathcal{E}).$$

Given an arbitrary $\mathcal{A}$-bilinear form b on $\mathcal{E}$, the *morphism*

$$q_b := b \circ \Delta, \tag{2}$$

where Δ is the diagonal $\mathcal{A}$-morphism of $\mathcal{E}$ (that is, for every open U in X and section s in $\mathcal{E}(U)$, $\Delta_U(s) \equiv \Delta(s) := (s, s)$), *is clearly a quadratic $\mathcal{A}$-morphism on $\mathcal{E}$.*

Let $\mathcal{B}(\mathcal{E}) \equiv \mathcal{L}^2_{\mathcal{A}}(\mathcal{E}, \mathcal{E}; \mathcal{A})$ be the $\mathcal{A}$-module of $\mathcal{A}$-bilinear forms (cf. [9]), the $\mathcal{S}h\mathcal{S}et_X$*- morphism* $\Xi : \mathcal{B}(\mathcal{E}) \longrightarrow \mathcal{Q}(\mathcal{E})$ such that

$$\Xi_U(b) := q_b, \tag{3}$$

for any open $U \subseteq X$ and section $b \in \mathcal{B}(\mathcal{E})(U) := \mathrm{L}^2_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U; \mathcal{A}|_U)$, where q_b is given as in (2), *is clearly an $\mathcal{A}$-morphism, with the sub-$\mathcal{A}$-module $\mathcal{A}(\mathcal{E})$ of skew-symmetric $\mathcal{A}$-bilinear forms being the kernel of* Ξ. On the other hand, it is immediate that $\Theta : \mathcal{Q}(\mathcal{E}) \longrightarrow \mathcal{B}(\mathcal{E})$, such that, for every open $U \subseteq X$ and section $q \in \mathcal{Q}(\mathcal{E})(U)$,

$$\Theta_U(q) := B_q, \tag{4}$$

is an $\mathcal{A}$-morphism of $\mathcal{Q}(\mathcal{E})$ into the sub-$\mathcal{A}$-module $\mathcal{S}(\mathcal{E})$ of $\mathcal{B}(\mathcal{E})$ of symmetric $\mathcal{A}$-bilinear forms.

Suppose now that the *characteristic* of $\mathcal{A}$ is not 2, that is the characteristic of every *individual* algebra $\mathcal{A}(U)$, where U is an open subset of X, is not 2. In the above $\mathcal{A}$-morphism Θ, let's replace B_q in (4) by the symmetric $\mathcal{A}|_U$-bilinear form

$$b_q := \frac{1}{2} B_q \tag{5}$$

for every quadratic $\mathcal{A}|_U$-form $q \in \mathcal{Q}(\mathcal{E})(U)$. So, one has

$$b_q = \frac{1}{2}\{(q \circ +) - (q \circ pr_1) - (q \circ pr_2)\}, \tag{6}$$

where $pr_i : \mathcal{E}|_U \oplus \mathcal{E}|_U \longrightarrow \mathcal{E}|_U$ $(i = 1, 2)$ is the i-th projection and, as expected, $+ : \mathcal{E}|_U \oplus \mathcal{E}|_U \longrightarrow \mathcal{E}|_U$ is the addition $\mathcal{A}|_U$-morphism. Clearly,

$$b_q \circ \Delta = q, \tag{7}$$

with Δ the diagonal $\mathcal{A}|_U$-morphism on $\mathcal{E}|_U$.

Setting

$$\widetilde{\Theta} = \frac{1}{2}\Theta,$$

one has

$$\Xi \circ \widetilde{\Theta} = \mathrm{Id}_{\mathcal{Q}(\mathcal{E})},$$

which implies that $\widetilde{\Theta}$ is injective and Ξ surjective. Clearly, $\mathrm{Im}\widetilde{\Theta} \subseteq \mathcal{S}(\mathcal{E})$. Conversely, for any symmetric $b \in \mathcal{B}(\mathcal{E})(U)$, one has that $\Xi_U(b) := q_b = b \circ \Delta$ and

$$b_q = b.$$

Thus, if we consider $\Xi|_{\mathcal{S}(\mathcal{E})}$, it is clear that

$$\widetilde{\Theta} \circ \Xi = \mathrm{Id}_{\mathcal{S}(\mathcal{E})}.$$

Hence, we have proved:

Proposition 1.1. *Let $(\mathcal{E}, q)$ be a quadratic $\mathcal{A}$-module, with $\mathcal{A}$ a sheaf of algebras of characteristic other than 2. Then,*

$$\mathcal{Q}(\mathcal{E}) = \mathcal{S}(\mathcal{E}), \tag{8}$$

within an $\mathcal{A}$-isomorphism.

So, we come now to the following crucial notion.

Definition 1.2. Let $(\mathcal{E}, q)$ be a quadratic $\mathcal{A}$-module, and $\mathcal{K}$ an *associative* and *unital $\mathcal{A}$-algebra.* A sheaf morphism $\varphi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{K})$ is called a **Clifford sheaf morphism** if

$$\varphi^2 = ev(q, -) \cdot 1, \tag{9}$$

where: (a) $ev : \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A}) \oplus \mathcal{E} \longrightarrow \mathcal{A}$ is the evaluation $\mathcal{A}$-morphism, namely

$$ev_U(\psi, s) := \psi_U(s),$$

for any open U in X and sections $s \in \mathcal{E}(U)$, $\psi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A})(U)$, and (b) $1 \in \mathrm{H}om_{\mathcal{S}h\mathcal{S}et_X}(\mathcal{K}, \mathcal{K})$ is the constant $\mathcal{S}h\mathcal{S}et_X$-morphism $1_W(t) = 1_{\mathcal{K}(W)}$ for every open $W \subseteq X$ and section $t \in \mathcal{K}(W)$.

For every open $U \subseteq X$ and section $s \in \mathcal{E}(U)$, (9) becomes

$$\varphi_U(s)^2 \equiv \varphi(s)^2 := q(s) \cdot 1 \equiv q_U(s) \cdot 1_{\mathcal{K}(U)}.$$

On the other hand, let us consider another section $t \in \mathcal{E}(U)$; then, we have

$$\varphi_U(s)\varphi_U(t) + \varphi_U(t)\varphi_U(s) = 2b_U(s, t) \cdot 1_{\mathcal{K}(U)}, \tag{10}$$

where

$$b := \widetilde{\Theta}_X(q).$$

We will call b the *$\mathcal{A}$-bilinear morphism induced by* the quadratic $\mathcal{A}$-morphism q.

Remark 1.1. We shall assume throughout the paper the following: (a) the *characteristic* of the sheaf of algebras $\mathcal{A}$ is not 2, (b) $\mathcal{A}$-algebras $\mathcal{K}$, targets of Clifford $\mathcal{A}$-morphisms, must not have *zero divisors,* that is, for any open set U in X and nowhere-zero sections $s, t \in \mathcal{K}(U)$, the *product section* $s \cdot t \equiv st$ is nowhere zero, (c) if $s \in \mathcal{K}(U)$ is nowhere zero, then the *annihilator* of s is trivially zero, that is

$$\{\alpha \in \mathcal{A}(U) : \ \alpha s = 0\} = \{0\}.$$

It follows from (10) and Remark 1.1 that $\varphi_U(s) = 0$ implies that $b_U(s,t) = 0$ for any $t \in \mathcal{E}(U)$. If q is non-degenerate, the above condition $\varphi_U(s) = 0$ implies that $s = 0$. Thus, *any Clifford $\mathcal{A}$-morphism of a non-isotropic quadratic $\mathcal{A}$-module is injective.*

Assuming the notations of Definition 1.2, if $\mathcal{L}$ is another *associative* and *unital $\mathcal{A}$-algebra* and $\Phi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{K}, \mathcal{L})$ a unital $\mathcal{A}$-morphism, which means that, for every open $U \subseteq X$, $\Phi_U : \mathcal{K}(U) \longrightarrow \mathcal{L}(U)$ is an $\mathcal{A}(U)$-morphism of the associative and unital $\mathcal{A}(U)$-algebras $\mathcal{K}(U)$ and $\mathcal{L}(U)$, so that, for all sections $s, t \in \mathcal{K}(U)$,

$$\Phi_U(st) = \Phi_U(s)\Phi_U(t), \quad \Phi_U(1_{\mathcal{K}(U)}) = 1_{\mathcal{L}(U)},$$

then $\Phi \circ \varphi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{L})$ is a Clifford sheaf morphism. Indeed, for every open $U \subseteq X$ and section $s \in \mathcal{E}(U)$,

$$\begin{aligned}(\Phi_U \circ \varphi_U)(s)^2 &= \Phi_U(\varphi_U(s))\Phi_U(\varphi_U(s)) = \Phi_U(\varphi_U(s)^2) \\ &= \Phi_U(q_U(s) \cdot 1_{\mathcal{K}(U)}) = q_U(s)\Phi_U(1_{\mathcal{K}(U)}) = q_U(s) \cdot 1_{\mathcal{L}(U)}.\end{aligned}$$

Definition 1.3. A quadratic $\mathcal{A}$-module $(\mathcal{E}, q)$ is called **Riemannian** if the *q-induced $\mathcal{A}$-bilinear morphism b* is a *Riemannian $\mathcal{A}$-metric,* i.e., a *strongly non-degenerate $\mathcal{A}$-valued inner product,* which is *symmetric* and *positive definite.*

We recall (see [4, pp. 335-340]) that for any *ordered algebraized space* $(X, \mathcal{A})$ *satisfying the inverse-closed section condition* ([10]), i.e., every nowhere-zero section of $\mathcal{A}$ is invertible, and *enriched with square root,* i.e., every nonnegative section of $\mathcal{A}$ has a square root, *if $(\mathcal{E}, \rho)$ is a free Riemannian $\mathcal{A}$-module of finite rank $n \in \mathbb{N}$* and

$$(s_1, \dots, s_n) \subseteq \mathcal{E}(U)^n \simeq \mathcal{E}^n(U),$$

where U is open in X, is a (local) gauge of $\mathcal{E}$, *then there exists an orthonormal gauge of $\mathcal{E}$,* obtained from $(s_1, \dots, s_n)$, say,

$$(t_1, \dots, t_n) \subseteq \mathcal{E}(U)^n;$$

more accurately, $t_1, \dots, t_n$ are such that

$$\rho_U(t_i, t_j) = \delta_{ij},$$

for all $1 \leq i, j \leq n$, and

$$[t_1, \dots, t_m] = [s_1, \dots, s_m],$$

for every $1 \leq m \leq n$.

Hence, we have

Proposition 1.2. *Let $(X, \mathcal{A})$ be an ordered algebraized space, enriched with square root, and satisfying the inverse-closed section condition. Moreover, let $(\mathcal{E}, q)$ be a Riemannian quadratic free $\mathcal{A}$-module of rank n, $\mathcal{K}$ an associative and unital $\mathcal{A}$-algebra, and φ an $\mathcal{A}$-morphism of $\mathcal{E}$ into $\mathcal{K}$. Then, φ is Clifford if and only if*

$$\varphi_U(e_i)^2 = q_U(e_i) \cdot 1_{\mathcal{K}(U)}, \quad i = 1, \ldots, n \tag{11}$$

and

$$\varphi_U(e_i)\varphi_U(e_j) + \varphi_U(e_j)\varphi_U(e_i) = 0, \quad 1 \leq i \neq j \leq n, \tag{12}$$

for any open $U \subseteq X$ and orthogonal gauge $(e_1, \ldots, e_n)$ of $(\mathcal{E}(U), q_U) \equiv (\mathcal{E}(U), b_U)$, where $b \equiv (b_U)_{X \supseteq U,\ open}$ is the q-induced Riemannian $\mathcal{A}$-metric.

Proof. The condition is obviously necessary. Indeed, let us consider an open subset U of X, and an orthogonal basis $(e_1, \ldots, e_n)$ of $(\mathcal{E}(U), q_U)$. Clearly, for any $i = 1, \ldots, n$,

$$\varphi_U(e_i)^2 = q_U(e_i) \cdot 1_{\mathcal{K}(U)}.$$

As for (12), one easily applies (10) and the fact that $(e_1, \ldots, e_n)$ is orthogonal.

Conversely, for any open $U \subseteq X$ and section $s \in \mathcal{E}(U)$, with $s = \sum_{i=1}^n \alpha^i e_i$, we have

$$\begin{aligned} \varphi_U(s)^2 &= \left[\sum_{i=1}^n \alpha^i \varphi_U(e_i)\right]^2 = \sum_{i=1}^n (\alpha^i)^2 \varphi_U(e_i)^2 \\ &= \left[\sum_{i=1}^n (\alpha^i)^2 q_U(e_i)\right] 1_{\mathcal{K}(U)} = \left[\sum_{i=1}^n q_U(\alpha^i e_i)\right] 1_{\mathcal{K}(U)} = q_U(s) \cdot 1_{\mathcal{K}(U)}. \end{aligned}$$

□

Remark 1.2. For the remainder of the paper, unless otherwise mentioned, any pair $(\mathcal{E}, q)$ will denote a *Riemannian quadratic free $\mathcal{A}$-module of finite rank,* where the *sheaf $\mathcal{A}$ of algebras satisfies the inverse-closed section condition* and is *enriched with square root.* In this context, *if φ is a Clifford $\mathcal{A}$-morphism of $(\mathcal{E}, q)$ into $\mathcal{K}$, then, for any orthogonal gauge $(e_1, \ldots, e_n) \subseteq \mathcal{E}(U)^n$ of $\mathcal{E}$ on an open $U \subseteq X$, $\varphi_U(e_i)$, for any $i = 1, \ldots, n$, is nowhere zero.* Indeed, if b is the Riemannian $\mathcal{A}$-metric associated with q, then $q_U(e_i) = b_U(e_i, e_i)$; since b is Riemannian and e_i is nowhere zero, therefore $\varphi_U(e_i)$ is nowhere zero.

In the same vein, we observe the following. As in [2, p. 288], we reduce the number of terms in products over $\mathcal{K}(U)$ as follows: For a product

$$a \equiv \varphi_U(e_{i_1})\varphi_U(e_{i_2}) \cdots \varphi_U(e_{i_p}), \quad 1 \leq p \leq n,$$

i) if $i_k > i_{k+1}$, we interchange $\varphi_U(e_{i_k})$ and $\varphi_U(e_{i_{k+1}})$ and multiply by (-1): since

$$\varphi_U(e_{i_k})\varphi_U(e_{i_{k+1}}) + \varphi_U(e_{i_{k+1}})\varphi_U(e_{i_k}) = 0,$$

a does not change.

ii) if $i_k = i_{k+1}$, we replace $\varphi_U(e_{i_k})\varphi_U(e_{i_{k+1}}) = \varphi_U(e_{i_k})^2$ by $q_U(e_{i_k}) \cdot 1_{\mathcal{K}(U)}$. Here, as well, a does not change.

This process will ultimately yield the following expression:

$$a = \lambda\varphi_U(e_{j_1})\varphi_U(e_{j_2})\cdots\varphi_U(e_{j_m}),$$

where $\lambda \in \mathcal{A}(U)$, and $J \equiv (1 \leq j_1 < j_2 < \cdots < j_m \leq n)$ an increasing sequence of indices.

As a convention, we let

$$\varphi_U(e_J) := \varphi_U(e_{j_1})\varphi_U(e_{j_2})\cdots\varphi_U(e_{j_m}),$$

where $J \equiv (1 \leq j_1 < j_2 < \cdots < j_m \leq n)$, and

$$\varphi_U(e_\emptyset) := 1_{\mathcal{K}(U)}$$

for the empty sequence $\emptyset$. Clearly, the 2^n elements $\varphi_U(e_J)$ of $\mathcal{K}(U)$ linearly span the sub-$\mathcal{A}(U)$-algebra $\mathcal{L}(U)$ of $\mathcal{K}(U)$, generated by $1_{\mathcal{K}(U)}$ and $\varphi_U(\mathcal{E}(U)) \equiv \varphi(\mathcal{E})(U)$. Thus, we have proved the following.

Theorem 1.1. *Let φ be a Clifford $\mathcal{A}$-morphism of a Riemannian quadratic free $\mathcal{A}$-module $(\mathcal{E}, q)$ of rank n into an associative and unital $\mathcal{A}$-algebra $\mathcal{K}$. Then, $\mathcal{K}$ contains a generalized locally free $\mathcal{A}$-module with maximum rank $\leq 2^n$, and containing the unital line sub-$\mathcal{A}$-module and the sub-$\mathcal{A}$-module $\varphi(\mathcal{E})$.*

2. Clifford $\mathcal{A}$-Algebras of Quadratic $\mathcal{A}$-Modules

Roughly speaking, a *sheaf of Clifford algebras* or a *Clifford $\mathcal{A}$-algebra* of a quadratic $\mathcal{A}$-module $(\mathcal{E}, q)$ is a *universal $\mathcal{A}$-algebra* in which we can embed $\mathcal{E}$, and such that the square $\mathcal{A}$-morphism in the sought $\mathcal{A}$-algebra corresponds to the quadratic $\mathcal{A}$-morphism on $\mathcal{E}$. This loose definition of a Clifford $\mathcal{A}$-algebra may be traced back to [3, p. 749].

Definition 2.1. By a **Clifford $\mathcal{A}$-algebra** of a quadratic $\mathcal{A}$-module $(\mathcal{E}, q)$, we mean any pair $(\mathcal{C}, \varphi_{\mathcal{C}})$, where $\mathcal{C}$ is an *associative* and *unital $\mathcal{A}$-algebra* and $\varphi_{\mathcal{C}} \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{C})$ is a *Clifford $\mathcal{A}$-morphism,* which satisfies the following conditions:

(1) $\mathcal{C}$ is generated by the sub-$\mathcal{A}$-algebra $\varphi_{\mathcal{C}}(\mathcal{E})$ and the unital line sub-$\mathcal{A}$-algebra $1_{\mathcal{C}}$ of $\mathcal{C}$.
(2) Every Clifford $\mathcal{A}$-morphism $\varphi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{K})$, where $\mathcal{K}$ is an associative and unital $\mathcal{A}$-algebra, factors through the Clifford $\mathcal{A}$-morphism $\varphi_{\mathcal{C}}$, i.e., there is a 1-respecting $\mathcal{A}$-morphism $\Phi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K})$ such that

$$\varphi = \Phi \circ \varphi_{\mathcal{C}}.$$

Since $\Phi(\varphi_{\mathcal{C}}(\mathcal{E})) = \varphi(\mathcal{E})$, $\mathcal{C}$ is generated by its *unital* line sub-$\mathcal{A}$-algebra and the sub-$\mathcal{A}$-algebra $\varphi_{\mathcal{C}}(\mathcal{E})$, and Φ is 1-respecting, it follows that Φ is *uniquely determined* by the Clifford $\mathcal{A}$-morphism φ. If we denote by

$$\mathcal{H}om^{Cl}_{\mathcal{A}}(\mathcal{E}, \mathcal{K})$$

the *sheaf of Clifford maps,* then $\mathcal{H}om^{Cl}_{\mathcal{A}}(\mathcal{E}, \mathcal{K})$ *is isomorphic to a subsheaf of* $\mathcal{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K})$. In fact, given any open subset U of X, let $\vartheta \in \mathcal{H}om^{Cl}_{\mathcal{A}}(\mathcal{E}, \mathcal{K})(U)$, that is, $\vartheta \in \mathrm{H}om^{Cl}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{K}|_U)$. Since $\mathcal{C} \equiv (\mathcal{C}, \varphi_{\mathcal{C}})$ is a Clifford $\mathcal{A}$-algebra of

$(\mathcal{E}, q)$, for any open $V \subseteq U$, there is a $\Theta_V \in \mathrm{H}om_{\mathcal{A}(V)}(\mathcal{C}(V), \mathcal{K}(V))$ such that $\Theta_V(1_{\mathcal{C}(V)}) = 1_{\mathcal{K}(V)}$ and $\vartheta_V = \Theta_V \circ (\varphi_{\mathcal{C}})_V$. We contend that the family $\Theta \equiv (\Theta_V)_{U \supseteq V,\ open}$ defines an $\mathcal{A}|_U$-morphism $\Theta \in \mathrm{H}om_{\mathcal{A}|_U}(\mathcal{C}|_U, \mathcal{K}|_U) \equiv \mathcal{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K})(U)$. Since, for any open $V \subseteq U$, $\Theta_V \in \mathrm{H}om_{\mathcal{A}(V)}(\mathcal{C}(V), \mathcal{K}(V))$ and $\Theta_V(1_{\mathcal{C}(V)}) = 1_{\mathcal{K}(V)}$, we need only to show that if (λ^U_V), (ρ^U_V) and (σ^U_V) are the families of restriction maps of the sheaves $\mathcal{K}$, $\mathcal{E}$ and $\mathcal{C}$, respectively, then

$$\lambda^U_V \circ \Theta_U = \Theta_V \circ \sigma^U_V,$$

for any open sets U, V in X with $V \subseteq U$. With no loss of generality, let $s \in \mathcal{C}(U)$, with $s = (\varphi_{\mathcal{C}})_U(e)$ for some $e \in \mathcal{E}(U)$. Then, based on the diagram below

$$\begin{tikzcd}
& \mathcal{E}(U) \arrow[r, "\vartheta_U"] \arrow[dl, "\rho^U_V"'] \arrow[d, "(\varphi_{\mathcal{C}})_U"] & \mathcal{K}(U) \arrow[dd, "\lambda^U_V"] \\
\mathcal{E}(V) \arrow[dr, "(\varphi_{\mathcal{C}})_V"'] \arrow[drr, "\vartheta_V"] & \mathcal{C}(U) \arrow[ur, "\Theta_U"'] \arrow[d, "\sigma^U_V"] & \\
& \mathcal{C}(V) \arrow[r, "\Theta_V"'] & \mathcal{K}(V)
\end{tikzcd},$$

clearly, one has

$$\begin{aligned}(\lambda^U_V \circ \Theta_U)((\varphi_{\mathcal{C}})_U(e)) &= (\lambda^U_V \circ \alpha_U)(e) = (\alpha_V \circ \rho^U_V)(e) \\ &= (\Theta_V \circ (\varphi_{\mathcal{C}})_V \circ \rho^U_V)(e) = (\Theta_V \circ \sigma^U_V)((\varphi_{\mathcal{C}})_U(e)).\end{aligned}$$

Next, for every open U in X, we denote by $\underline{\mathrm{H}}om_{\mathcal{A}|_U}(\mathcal{C}|_U, \mathcal{K}|_U)$ the $\mathcal{A}(U)$-module consisting with $\mathcal{A}|_U$-morphisms Θ, uniquely determined by Clifford $\mathcal{A}|_U$-morphisms $\vartheta \in \mathrm{H}om^{Cl}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{K}|_U)$. Furthermore, let (α^U_V) be the collection of restriction maps for the $\mathcal{A}$-module $\mathcal{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K})$. The collection

$$(\underline{\mathrm{H}}om_{\mathcal{A}|_U}(\mathcal{C}|_U, \mathcal{K}|_U), \alpha^U_V) \tag{13}$$

clearly determines a presheaf. Moreover, it is a complete presheaf. Indeed, if $U = \cup_{i \in I} U_i$ and $\Theta_1,\ \Theta_2 \in \underline{\mathrm{H}}om_{\mathcal{A}|_U}(\mathcal{C}|_U, \mathcal{K}|_U)$ with

$$\alpha^U_{U_i}(\Theta_1) \equiv \Theta_1|_{U_i} = \Theta_2|_{U_i} \equiv \alpha^U_{U_i}(\Theta_2)$$

for every $i \in I$, then, clearly, $\Theta_1 = \Theta_2$. Now, let $(\Theta_i) \in \prod_{i \in I} \underline{\mathrm{H}}om_{\mathcal{A}|_U}(\mathcal{C}|_U, \mathcal{K}|_U)$ such that, for any $U_{ij} \equiv U_i \cap U_j \neq \emptyset$ in $\mathcal{U} \equiv \{U_i,\ i \in I\}$, one has

$$\Theta_i|_{U_{ij}} = \Theta_j|_{U_{ij}}.$$

Then, since $\underline{\mathrm{H}}om_{\mathcal{A}|_U}(\mathcal{C}|_U, \mathcal{K}|_U) \subseteq \mathrm{H}om_{\mathcal{A}|_U}(\mathcal{C}|_U, \mathcal{K}|_U) = \mathcal{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K})(U)$, for any open U in X, there is $\Theta \in \mathcal{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K})(U)$ such that

$$\Theta|_{U_i} = \Theta_i,$$

for every $i \in I$. It follows that Θ is 1-respecting; in addition, since it is linear, $\Theta \in \underline{\mathrm{H}}om_{\mathcal{A}|_U}(\mathcal{C}|_U, \mathcal{K}|_U)$. Hence,

$$\mathcal{H}om^{Cl}_{\mathcal{A}}(\mathcal{E}, \mathcal{K}) \simeq \underline{\mathcal{H}}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K}) \subseteq \mathcal{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K}),$$

where $\underline{\mathcal{H}}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K})$ is the sheafification of the complete presheaf (13).

Condition (2) of Definition 2.1 could therefore be restated as follows:

(2′) For every associative and unital $\mathcal{A}$-algebra $\mathcal{K}$, $\mathcal{H}om^{Cl}_{\mathcal{A}}(\mathcal{E},\mathcal{K})$ is isomorphic to a subsheaf of $\mathcal{H}om_{\mathcal{A}}(\mathcal{C},\mathcal{K})$.

Lemma 2.1. *Let $(\mathcal{C},\varphi_{\mathcal{C}})$ be a Clifford $\mathcal{A}$-algebra of a quadratic $\mathcal{A}$-module $(\mathcal{E},q)$. Then, $(\mathcal{C}',\varphi_{\mathcal{C}'})$ is also a Clifford $\mathcal{A}$-algebra of $(\mathcal{E},q)$ if and only if there is an $\mathcal{A}$-isomorphism $\Phi:\mathcal{C}\longrightarrow\mathcal{C}'$ such that $\Phi\circ\varphi_{\mathcal{C}}=\varphi_{\mathcal{C}'}$.*

Proof. Suppose $(\mathcal{C}',\varphi_{\mathcal{C}'})$ is also a Clifford $\mathcal{A}$-algebra. Then, there exist unique $\mathcal{A}$-morphisms $\Phi:\mathcal{C}\longrightarrow\mathcal{C}'$ and $\Phi':\mathcal{C}'\longrightarrow\mathcal{C}$ such that $\Phi\circ\varphi_{\mathcal{C}}=\varphi_{\mathcal{C}'}$ and $\Phi\circ\varphi_{\mathcal{C}'}=\varphi_{\mathcal{C}}$. Since $\Phi'\circ\Phi\circ\varphi_{\mathcal{C}}=\Phi'\circ\varphi_{\mathcal{C}'}=\varphi_{\mathcal{C}}$, the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\varphi_{\mathcal{C}}"] \arrow[d, "\varphi_{\mathcal{C}}"'] & \mathcal{C} \\
\mathcal{C} \arrow[ur, "\Phi'\circ\Phi"'] &
\end{tikzcd}$$

is commutative. But, only one $\mathcal{A}$-morphism exists making the above diagram commutative; and clearly $\mathrm{Id}_{\mathcal{C}}$ does just that. As $\mathcal{C}$ is generated by $\varphi_{\mathcal{C}}(\mathcal{E})$ and its unital line sub-$\mathcal{A}$-algebra, $\Phi'\circ\Phi=\mathrm{Id}_{\mathcal{C}}$. In a similar way, one shows that $\Phi\circ\Phi'=\mathrm{Id}_{\mathcal{C}'}$, whence we see that Φ is an $\mathcal{A}$-isomorphism with $\Phi^{-1}=\Phi'$. □

Now, if $\varphi_{\mathcal{C}}$ is the Clifford $\mathcal{A}$-morphism of a Riemannian quadratic free $\mathcal{A}$-module $(\mathcal{E},q)$ into its Clifford $\mathcal{A}$-algebra $\mathcal{C}\equiv\mathcal{C}(\mathcal{E},q)$, the $\mathcal{A}$-morphism φ', such that $\varphi':=-\varphi_{\mathcal{C}}$, is another Clifford $\mathcal{A}$-morphism of $\mathcal{E}$ into $\mathcal{C}$; thus there exists an $\mathcal{A}$-endomorphism Π of (the unital $\mathcal{A}$-algebra) $\mathcal{C}$ such that

$$\Pi(1_{\mathcal{C}})=1_{\mathcal{C}},\quad \Pi\circ\varphi_{\mathcal{C}}=\varphi'=-\varphi_{\mathcal{C}}. \tag{14}$$

Clearly,

$$\Pi^2(1_{\mathcal{C}})=1_{\mathcal{C}},\quad \Pi^2\circ\varphi_{\mathcal{C}}=\varphi_{\mathcal{C}}; \tag{15}$$

it follows that Π is an $\mathcal{A}$-*involutive automorphism* of $\mathcal{C}$, called **the principal $\mathcal{A}$-automorphism of** the Clifford $\mathcal{A}$-algebra $\mathcal{C}(\mathcal{E},q)$.

For every open $U\subseteq X$, let $C_+(U)$ denote the sub-$\mathcal{A}(U)$-module of the $\mathcal{A}(U)$-algebra $\Gamma(U,\mathcal{C})\equiv\mathcal{C}(U)$ consisting of the *eigenvector sections* of Π_U for the *eigenvalue section* $+1$ (cf. [9]). It is evident that $C_+(U)$ *is a sub-$\mathcal{A}(U)$-algebra of $\mathcal{C}(U)$, containing any product of any even number of nowhere-zero sections in $(\varphi_{\mathcal{C}})_U(\mathcal{E}(U))$*:

$$(\varphi_{\mathcal{C}})_U(s_1)(\varphi_{\mathcal{C}})_U(s_2)\cdots(\varphi_{\mathcal{C}})_U(s_{2p}).$$

Conversely, if $(e_1,e_2,\ldots,e_n)$ is an orthogonal basis of $\mathcal{E}(U)$, reducing the number of terms in any product

$$(\varphi_{\mathcal{C}})_U(e_{i_1})(\varphi_{\mathcal{C}})_U(e_{i_2})\cdots(\varphi_{\mathcal{C}})_U(e_{i_p}),$$

as described in Section 1, does not change the parity of the number of terms involved. Thus, $C_+(U)$ is linearly generated by the elements $(\varphi_{\mathcal{C}})_U(e_J)$, with $J=(1\leq j_1<\cdots<j_m\leq n)$ for an even m.

By letting U vary over the open subsets of X, the family $(C_+(U),{}_+\lambda^U_V)$, where ${}_+\lambda^U_V:=\sigma^U_V|_{C_+(U)}$, with the (σ^U_V) being the restriction maps for the (complete) presheaf of sections $\Gamma\mathcal{C}$ of the $\mathcal{A}$-algebra $\mathcal{C}$, forms a *complete* *presheaf of algebras on X.* Indeed, let $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ be an open covering, and let $s, t \in C_+(U)$ such that

$$_+\lambda^U_{U_\alpha}(s) \equiv s_\alpha = t_\alpha \equiv {}_+\lambda^U_{U_\alpha}(t)$$

for every $\alpha \in I$. Since $C_+(U) \subseteq \mathcal{C}(U)$, and $\mathcal{C}$ is an $\mathcal{A}$-algebra, it follows that $s = t$. Thus, axiom $(S1)$ (cf. [4, p. 46, Definition 11.1]) is fulfilled. For axiom $(S2)$ (ibid.), let $s_\alpha \in C_+(U_\alpha)$, $\alpha \in I$, such that for any $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta \neq \emptyset$ in $\mathcal{U}$, one has

$$_+\lambda^{U_\alpha}_{U_{\alpha\beta}}(s_\alpha) \equiv s_\alpha|_{U_{\alpha\beta}} = s_\beta|_{U_{\alpha\beta}} \equiv {}_+\lambda^{U_\beta}_{U_{\alpha\beta}}(s_\beta).$$

Without loss of generality, suppose that

$$s_\alpha = (\varphi_{\mathcal{C}})_{U_\alpha}(s_{\alpha,1}) \cdots (\varphi_{\mathcal{C}})_{U_\alpha}(s_{\alpha,2p})$$

and

$$s_\beta = (\varphi_{\mathcal{C}})_{U_\beta}(s_{\beta,1}) \cdots (\varphi_{\mathcal{C}})_{U_\beta}(s_{\beta,2q})$$

with $s_{\alpha,1}, \ldots, s_{\alpha,2p} \in \mathcal{E}(U_\alpha)$ and $s_{\beta,1}, \ldots, s_{\beta,2q} \in \mathcal{E}(U_\beta)$. It is evident that there exists an $s \in \mathcal{C}(U)$ such that

$$\sigma^U_{U_\alpha}(s) \equiv s|_{U_\alpha} = s_\alpha,$$

for every $\alpha \in I$. Clearly, s is of the form

$$s = (\varphi_{\mathcal{C}})_U(s^1_1) \cdots (\varphi_{\mathcal{C}})_U(s^1_{2p_1}) + \cdots + (\varphi_{\mathcal{C}})_U(s^1_k) \cdots (\varphi_{\mathcal{C}})_U(s^k_{2p_k}),$$

where for any $i = 1, \ldots, k$, p_i is an integer $\leq n$, and every $s^i_1, \ldots, s^i_{2p_i} \in \mathcal{E}(U)$. Indeed, if s contains a product of an odd number of terms, then, for any $U_\alpha \in \mathcal{U}$, $s|_{U_\alpha} := s_\alpha \notin C_+(U)$. Thus, $C_+ \equiv (C_+(U), {}_+\lambda^U_V)$ is a complete presheaf of algebras. The sheafification of the presheaf C_+, denoted $\mathcal{C}_+ \equiv \mathbf{S}C_+$, is called the **even sub-$\mathcal{A}$-algebra of** the Clifford $\mathcal{A}$-algebra $\mathcal{C}$.

Now, let $C_-(U)$ be the eigen sub-$\mathcal{A}(U)$-module of $\mathcal{C}(U)$ for the eigenvalue section -1. Clearly, *elements of $C_-(U) \subseteq \mathcal{C}(U)$ are products of an odd number of terms of $(\varphi_{\mathcal{C}})_U(\mathcal{E}(U))$.* One proceeds as above to show that pairs $(C_-(U), {}_-\lambda^U_V)$, where ${}_-\lambda^U_V = \sigma^U_V|_{C_-(U)}$, *yield a complete presheaf.* However, we notice that every $C_-(U)$ *is not an algebra;* so the presheaf $(C_-(U), {}_-\lambda^U_V)$ is not a presheaf of algebras, but a presheaf of modules instead. Its sheafification, denoted $\mathcal{C}_-$, is called the **sub-$\mathcal{A}$-module of odd products of $\mathcal{C}$**.

Definition 2.2. Let $\mathcal{C}$ be an $\mathcal{A}$-algebra. The $\mathcal{A}$-algebra $\mathcal{C}^*$, in which products are defined to be products in $\mathcal{C}$ but in the reverse order, is called the **opposite $\mathcal{A}$-algebra of** $\mathcal{C}$.

Specifically, let U be open in X and $s, t \in \mathcal{C}(U)$; then, if $*$ denotes product in $\mathcal{C}^*(U)$, one has

$$s * t := ts.$$

Now, considering still $\varphi_{\mathcal{C}}$ as a Clifford $\mathcal{A}$-morphism of the Riemannian quadratic free $\mathcal{A}$-module $(\mathcal{E}, q)$ into its Clifford $\mathcal{A}$-algebra $\mathcal{C}$, $\varphi_{\mathcal{C}}$, which we denote by $\varphi_{\mathcal{C}}^*$, as an $\mathcal{A}$-morphism from $\mathcal{E}$ into $\mathcal{C}^*$, is again a Clifford $\mathcal{A}$-morphism. Thus, there exists a 1-respecting $\mathcal{A}$-morphism $\tau : \mathcal{C} \longrightarrow \mathcal{C}^*$ such that

$$\tau \circ \varphi_{\mathcal{C}} = \varphi_{\mathcal{C}}^*.$$

But, $\tau_U((\varphi_{\mathcal{C}})_U(s)) = (\varphi_{\mathcal{C}}^*)_U(s) = (\varphi_{\mathcal{C}})_U(s)$ for any open set U in X and section $s \in \mathcal{E}(U)$, and since $1_{\mathcal{C}}$ and $\varphi_{\mathcal{C}}(\mathcal{E}) = \varphi_{\mathcal{C}}^*(\mathcal{E})$ generate both $\mathcal{C}$ and $\mathcal{C}^*$, it follows that τ is bijective, hence, a 1-respecting $\mathcal{A}$-isomorphism of $\mathcal{C}$ into $\mathcal{C}^*$. We conclude that τ is the only **$\mathcal{A}$-antiautomorphism**, fixing the sections of $\varphi_{\mathcal{C}}(\mathcal{E})$. As for any open U in X and sections $s_1, \ldots, s_k \in \mathcal{E}(U)$, $\tau_U((\varphi_{\mathcal{C}})_U(s_1) \cdots (\varphi_{\mathcal{C}})_U(s_k)) = (\varphi_{\mathcal{C}})_U(s_k) \cdots (\varphi_{\mathcal{C}})_U(s_1)$, it follows that $\tau^2 = 1$, i.e., τ is an $\mathcal{A}$-*involution.*

Using sections, one easily sees that $\Pi \circ \tau = \tau \circ \Pi$, which is the only $\mathcal{A}$-antiautomorphism of $\mathcal{C}$ sending sections of $\varphi_{\mathcal{C}}(\mathcal{E})$ into their opposites. On the other hand, $\Pi \circ \tau$ is an $\mathcal{A}$-involution and is called the *conjugate* of $\mathcal{C}$.

3. Construction of Clifford $\mathcal{A}$-Algebras

We construct Clifford $\mathcal{A}$-algebras mimicking the classical case, as presented by [2, p. 294, Théorème VIII. 2. B]. Chevalley [1] and Lang [3] construct (classical) Clifford algebras of modules (or vector spaces) by considering the tensor algebra of the module (vector space) concerned. Since the problem is universal in its nature, the two approaches result into isomorphic algebras. For our approach, we first need the following.

Proposition 3.1. *Let $(\mathcal{E}, q)$ be a Riemannian free $\mathcal{A}$-module of rank n. For every open U in X, let $B(U)$ be the set consisting of all the orthogonal bases of $\mathcal{E}(U)$. If, for every $U, V \in \tau_X$ with $V \subseteq U$,*

$$\rho_V^U : B(U) \longrightarrow B(V)$$

denotes the natural restriction, the collection $B := (B(U), \rho_V^U)$ determines a complete presheaf of sets (of orthogonal bases).

Proof. That B is a presheaf is immediate. Now, let U be an open subset of X and $\mathcal{U} \equiv (U_i)_{i \in I}$ a covering of U. Next, let $s \equiv (s^1, \ldots, s^n)$ and $t \equiv (t^1, \ldots, t^n)$ be bases of $\mathcal{E}(U)$, i.e. $s, t \in B(U)$, such that $s|_{U_i} = t|_{U_i}$ for every $i \in I$. More explicitly, $s^j|_{U_i} = t^j|_{U_i}$ for every $i \in I$ and $j = 1, \ldots, n$. Since $s^j, t^j \in \mathcal{E}(U)$ $(j = 1, \ldots, n)$, it follows that $s^j = t^j$; thus, $s = t$. Hence, axiom $(S1)$ (cf. [4, p. 46, Definition 11. 1]) is fulfilled.

For axiom $(S2)$ (ibid.), let $s_i \in B(U_i)$ such that, for every $U_i \cap U_j \equiv U_{ij} \neq \emptyset$ in $\mathcal{U}$,

$$s_i|_{U_{ij}} = s_j|_{U_{ij}}.$$

Again, using the fact that $\Gamma(\mathcal{E})$ is complete, one has that there exists $t^k \in \mathcal{E}(U)$ such that $t^k|_{U_i} = s_i^k$, $k = 1, \ldots, n$. Therefore, $t \equiv (t^1, \ldots, t^n)$ is such that $t|_{U_i} = s_i$, $i \in I$. Clearly, t is orthogonal. $\square$

Keeping with the notations of Proposition 3.1, we will call the sheaf generated by B the *sheaf of orthogonal bases of* $\mathcal{E}$, and will denote it by $\mathcal{B}$, i.e. $\mathcal{B} = \mathbf{S}B$.

Theorem 3.1. *Let $(\mathcal{E}, q)$ be a Riemannian quadratic free $\mathcal{A}$-module of rank n, $\mathcal{C}$ an associative and unital $\mathcal{A}$-algebra, and $\varphi_{\mathcal{C}} \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{C})$ a Clifford $\mathcal{A}$-morphism such that, given the sheaf of orthogonal bases $e_U := (e_{U,1}, \dots, e_{U,n})$ of $\mathcal{E}$, the sheaf of sets, consisting of elements of the form*

$$(\varphi_{\mathcal{C}})_U(e_{U,J}) := (\varphi_{\mathcal{C}})_U(e_{U,j_1})(\varphi_{\mathcal{C}})_U(e_{U,j_2}) \cdots (\varphi_{\mathcal{C}})_U(e_{U,j_m}),$$

where $J = (1 \leq j_1 < j_2 < \cdots < j_m \leq n)$, assuming that $(\varphi_{\mathcal{C}})_U(e_{U,\emptyset}) = 1_{\mathcal{C}(U)}$, is a sheaf of bases for the underlying free $\mathcal{A}$-module of $\mathcal{C}$. Then, the pair $(\mathcal{C}, \varphi_{\mathcal{C}})$ is a Clifford $\mathcal{A}$-algebra of $(\mathcal{E}, q)$.

Proof. In fact, let φ be a Clifford $\mathcal{A}$-morphism of $(\mathcal{E}, q)$ into some associative and unital $\mathcal{A}$-algebra $\mathcal{K}$. Moreover, let Φ be an $\mathcal{A}$-morphism of $\mathcal{C}$ into $\mathcal{K}$, given by:

$$\Phi_U((\varphi_{\mathcal{C}})_U(e_{U,J})) := \varphi_U(e_{U,J}),$$

where

$$\varphi_U(e_{U,J}) := \varphi_U(e_{U,j_1})\varphi_U(e_{U,j_2}) \cdots \varphi_U(e_{U,j_m})$$

with $J := (1 \leq j_1 < j_2 < \cdots < j_m \leq n)$.

We claim that Φ is multiplicative and 1-respecting, hence an $\mathcal{A}$-morphism of the $\mathcal{A}$-algebras $\mathcal{C}$ and $\mathcal{K}$. To this end, it suffices to show that every Φ_U is multiplicative on $((\varphi_{\mathcal{C}})_U(e_{U,J}))_{J \in \mathcal{P}(I_n)}$, where $I_n = \{1, \dots, n\}$.

Let us consider the product in $\mathcal{C}(U)$:

$$\begin{aligned}(\varphi_{\mathcal{C}})_U(e_{U,J}) &\cdot (\varphi_{\mathcal{C}})_U(e_{U,J'}) \\ &= (\varphi_{\mathcal{C}})_U(e_{U,j_1}) \cdots (\varphi_{\mathcal{C}})_U(e_{U,j_m})(\varphi_{\mathcal{C}})_U(e_{U,j'_1}) \cdots (\varphi_{\mathcal{C}})_U(e_{U,j'_p}),\end{aligned} \tag{16}$$

and the following product in $\mathcal{K}(U)$:

$$\begin{aligned}(\varphi_U)(e_{U,J}) &\cdot (\varphi_U)(e_{U,J'}) \\ &= (\varphi_U)(e_{U,j_1}) \cdots (\varphi_U)(e_{U,j_m})(\varphi_U)(e_{U,j'_1}) \cdots (\varphi_U)(e_{U,j'_p}).\end{aligned} \tag{17}$$

The right-hand sides of (16) and (17) reduce to

$$\begin{aligned}(\varphi_{\mathcal{C}})_U(e_{U,J}) \cdot (\varphi_{\mathcal{C}})_U(e_{U,J'}) &= \lambda(\varphi_{\mathcal{C}})_U(e_{U,L}) \\ \varphi_U(e_{U,J}) \cdot \varphi_U(e_{U,J'}) &= \lambda\varphi_U(e_{U,L}),\end{aligned}$$

where $L = (1 \leq l_1 < l_2 < \cdots < l_r \leq n)$. Therefore, given J and J':

$$\begin{aligned}\Phi_U((\varphi_{\mathcal{C}})_U(e_{U,J})(\varphi_{\mathcal{C}})_U(e_{U,J'})) &= \Phi_U(\lambda(\varphi_{\mathcal{C}})_U(e_{U,L})) = \lambda\Phi_U((\varphi_{\mathcal{C}})_U(e_{U,L})) \\ = \lambda\varphi_U(e_{U,L}) &= \varphi_U(e_{U,J})\varphi_U(e_{U,J'}) = \Phi_U((\varphi_{\mathcal{C}})_U(e_{U,J}))\Phi_U((\varphi_{\mathcal{C}})_U(e_{U,J'}));\end{aligned}$$

furthermore, since $\Phi_U(1_{\mathcal{C}(U)}) = \Phi_U((\varphi_{\mathcal{C}})_U(e_{U,\emptyset})) = \varphi_U(e_{U,\emptyset}) = 1_{\mathcal{K}(U)}$, Φ_U is an $\mathcal{A}(U)$-morphism, taking $\mathcal{C}(U)$ into $\mathcal{K}(U)$. Hence, $\Phi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K})$ and is 1-respecting.

As $\mathcal{C}$ is generated by the sub-$\mathcal{A}$-algebra $\varphi_{\mathcal{C}}(\mathcal{E})$ and the unital line sub-$\mathcal{A}$-algebra $1_{\mathcal{C}}$ of $\mathcal{C}$, then $\mathcal{C}$ is a Clifford $\mathcal{A}$-algebra of $(\mathcal{E}, q)$. $\square$

Theorem 3.2. *With every Riemannian quadratic free $\mathcal{A}$-module $(\mathcal{E}, q)$, there is an associated Clifford free $\mathcal{A}$-algebra $\mathcal{C} \equiv \mathcal{C}(\mathcal{E}, q)$; moreover, rank $\mathcal{C} = 2^n$ if $n =$ rank $\mathcal{E}$.*

Proof. Let $\mathcal{B}$ be a sheaf of orthogonal bases of $\mathcal{E} \equiv (\mathcal{E}, q)$, and $\mathcal{P}$ the sheaf of algebras of anticommutative polynomials over $\mathcal{A}$, such that if $p \in \mathcal{P}(U)$, for some open U in X, then p is an anticommutative polynomial in $e_1, e_2, \ldots, e_n$, where $(e_1, e_2, \ldots, e_n)$ is a fixed orthogonal basis in $\mathcal{B}(U)$. If U and V are open subsets of X with V a subopen of U, we fix orthogonal bases $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ in $\mathcal{E}(U)$ and $\mathcal{E}(V)$, respectively, in such a way that $\rho^U_V(e_i) \equiv e_i|_V = f_i$, for every $i = 1, \ldots, n$, where the $\{\rho^U_V\}$ are restriction maps for the (complete) presheaf of sections of $\mathcal{E}$. Furthermore, we denote by $1_{\mathcal{P}(U)} \equiv 1$ the polynomial $e_1^{m_1} e_2^{m_2} \ldots e_n^{m_n}$, where $m_i = 0$, $i = 1, \ldots, n$. On every open $U \subseteq X$, define the product in $\mathcal{P}(U)$ as follows:

$$(e_1^{p_1} e_2^{p_2} \ldots e_n^{p_n}) \cdot (e_1^{q_1} e_2^{q_2} \ldots e_n^{q_n}) = (-1)^{\sum_{i<j} q_i p_j} e_1^{p_1+q_1} \ldots e_n^{p_n+q_n}. \tag{18}$$

Moreover, still under the assumption that $(e_1 \ldots e_n)$ is the fixed orthogonal basis of $\mathcal{E}(U)$, the section $e_1^{m_1} e_2^{m_2} \ldots e_n^{m_n}$ of $\mathcal{P}$ over U such that

$$m_i = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

is denoted e_j. This notation ensures an identification of $\mathcal{E}$ with a sub-$\mathcal{A}$-module of $\mathcal{P}$. On the other hand, in every $\mathcal{P}(U)$, one has

$$e_i e_j = -e_j e_i \quad i \neq j.$$

The product thus defined on every $\mathcal{P}(U)$ is associative, for one easily shows that, by multiplying both members of (18) on the right by a polynomial $e_1^{r_1} e_2^{r_2} \ldots e_n^{r_n}$, one obtains the following equality:

$$\sum_{i<j} q_i p_j + \sum_{i<j} r_i(p_j + q_j) = \sum_{i<j} (q_i + r_i) p_j + \sum_{i<j} r_i q_i.$$

For every i, $1 \leq i \leq n$, let $q_U(e_i) := a_i \in \mathcal{A}(U)$. Next, consider the correspondence

$$U \longmapsto \mathcal{C}(U) \subseteq \mathcal{P}(U), \tag{19}$$

where $\mathcal{C}(U)$ is the free $\mathcal{A}(U)$-module, with a basis consisting of the 2^n sections

$$e_1^{m_1} e_2^{m_2} \ldots e_n^{m_n},$$

where $0 \leq m_i \leq 1$ for every i. It is clear that the correspondence (19) together with the restriction maps restricted to the $\mathcal{C}(U)$, where U runs over the open subsets of X, yield a free $\mathcal{A}$-module of rank 2^n. We will denote the free $\mathcal{A}$-module thus obtained by $\mathcal{C} \equiv (\mathcal{C}(U), \lambda^U_V)$. Let's also consider the *projection $\mathcal{A}$-morphism* $\pi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}, \mathcal{C})$, defined by:

$$\pi_U(e_1^{p_1} e_2^{p_2} \cdots e_n^{p_n}) := a_1^{\lfloor \frac{p_1}{2} \rfloor} \cdots a_n^{\lfloor \frac{p_n}{2} \rfloor} e_1^{\overline{p_1}} e_2^{\overline{p_2}} \cdots e_n^{\overline{p_n}},$$

where $(e_1, e_2, \cdots, e_n)$ is the fixed orthogonal basis of $\mathcal{E}(U)$,

$$\lfloor \frac{p_i}{2} \rfloor = \max\{x \in \mathbb{Z} : \ x \leq \frac{p_i}{2}\}$$

for every $1 \leq i \leq n$, and

$$p_i = 2l_i + \overline{p_i}, \quad l_i \in \mathbb{Z}, \overline{p_i} \in \mathbb{Z},$$

viz. $\overline{p_i}$ is the remainder of p_i modulo 2.

Given sections f and g of the $\mathcal{A}$-algebra $\mathcal{P}$ over an open subset U of X, one has

$$\pi_U(f \cdot g) = \pi_U(\pi_U(f) \cdot \pi_U(g)),$$

which is easily verified by taking $f = e_1^{p_1} \cdots e_n^{p_n}$ and $g = e_1^{q_1} \cdots e_n^{q_n}$.

Finally, we define on the free $\mathcal{A}$-module $\mathcal{C}$ the following multiplication: if $s, t \in \mathcal{C}(U)$, where U is open in X, then

$$s * t := \pi_U(s \cdot t).$$

$*$ is associative; the proof of this fact may be found in [2, p. 295]. Hence, $\mathcal{C}$ is an associative and unital free $\mathcal{A}$-algebra, which contains $\mathcal{E}$. Let's denote by $\iota_{\mathcal{C}}$ the inclusion $\mathcal{E} \subseteq \mathcal{C}$. Since, for every open $U \subseteq X$ and the corresponding orthogonal basis $(e_1, \cdots, e_n)$ of $\mathcal{E}(U)$,

$$(\iota_{\mathcal{C}})_U(e_i)^2 = a_i \cdot 1_{\mathcal{C}(U)}$$

and

$$(\iota_{\mathcal{C}})_U(e_i)(\iota_{\mathcal{C}})_U(e_j) + (\iota_{\mathcal{C}})_U(e_j)(\iota_{\mathcal{C}})_U(e_i) = 0, \quad 1 \leq i \neq j \leq n,$$

the pair $\mathcal{C} \equiv (\mathcal{C}, \iota_{\mathcal{C}})$ is a Clifford $\mathcal{A}$-algebra of $(\mathcal{E}, q)$, by virtue of Theorem 3.1. $\square$

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