A simple method to obtain the stochastic decomposition structure of the busy period in *Geo/Geo/1/N* vacation queue

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Abstract In this paper, some tips and tricks for algebraic manipulations are utilized to explicitly get the mean and variance of the duration of the busy period in a discrete-time finite-buffer vacation queue. Applying the law of total expectation, the closed-form expressions for the first two moments of the busy period initiated with an arbitrary number of customers are firstly derived. Then, by employing the queue length distri-bution at vacation termination and the quantities that mentioned above, we give the stochastic decomposition structure of the busy period. Finally, in order to ensure the reliability of the analytical approach, an effective way to validate the correctness of our results along with a numerical example is also provided. We may find that these simple tips and tricks can greatly reduce the difficulty of problem solving.

Keywords: Discrete-time queue, Finite-buffer, Vacation, Busy period, Stochastic decomposition structure

Mathematics Subject Classification: 60K25 · 68M20 · 90B22

1 Introduction

Busy period of a single server classical queueing system is defined easily as the period of time during which the server is continuously busy. More precisely, busy period is a time interval that starts when an arriving customer finds the server idle, and ends at the first subsequent time at which the system becomes empty. This performance measure plays a very significant role in optimal control of queueing systems. For the majority of queues, it is usually impossible to find the exact distribution of the busy period. In such situations, the typical approach to studying the busy periods is through their Laplace–Stieltjes or Z transforms, which tend to derive simple relations for infinite-buffer systems since the number of customers served in the busy period has the structure of a Galton–Watson branching process. However, this nice property does not hold anymore for finite-buffer systems. Thus, due to methodological limitation, the research progress on a busy period in finite-buffer queue is relatively slow over the past few decades. As far as we are aware only a handful of papers have been explicitly dealing with this important measure. One of the early papers to address related issues is that of Natvig (1975). He dealt with a general birth-and-death finite-buffer queue where the service and input rates are state-dependent. The first and second order moments of the length of a busy period are reported. But the demonstrated formulae in his paper are too complex to use easily. Recently, a similar model was studied by Al Hanbali and Boxma (2010). The transient behavior of a state-dependent M/M/1/K queue during the busy period was extensively investigated by using the theory of absorbing Markov chains. Later, Al Hanbali (2011) further extended the above results by considering the level dependent PH/PH/1/K queue. The closed-form expression for the probability density function (p.d.f.) of the length of the busy period in M/M/1/K queue was firstly studied by Sharma and Shobha (1986). Nonetheless, it is somewhat regrettable that the expression of p.d.f. given by them involves the eigenvalues of a transition matrix. To overcome this weakness, some further analysis was recently conducted by Takagi and Tarabia (2009). In addition, the busy period analysis of non-Markovian continuous-time finite-buffer queue has been addressed by Cooper and Tilt (1976), Harris (1971), Miller (1975), Rosenlund (1978), Shanthikumar and Sumita (1985), Pacheco and Ribeiro (2008), and references therein. Particularly, Lee (1984) investigated the M/G/1/N queue with vacation time and derived the joint transform of the length of the busy period and the number of customers served during the busy period. His work can be viewed as an extension of the work given by Miller (1975) and Cooper and Tilt (1976).

As one can see from above, there are very few results about the busy period in discrete-time finite-buffer queue. Except the work done by Chaudhry and Zhao (1994) and Alfa (2010), no work in this direction has come to our notice. Especially, the research results concerning the discrete-time finite-buffer vacation queue which can be used to get the moments of the busy period do not exist as far as we know. Actually, analyzing the finite-buffer vacation queue gives some obvious advantages. The first one is that the finite-buffer system is prevalent in many real world situations and taking the limit as capacity N approaches infinity allows us to get the result of the corresponding infinite-buffer system. Moreover, in the case of finite-buffer queue, we can analyze the unstable system, that is to say, the average arrival rate is higher than the average service rate. Thus, the shortcomings in existing research constitute the main motivation for writing this paper. Meanwhile, we will attempt to fill these gaps in this work. Since the number of customers present in the system at epochs of vacation termination is a random variable that can take only a finite number of positive integers, we will use an extended definition of the busy period that might be initiated by multiple customers in model analysis. This definition is richer and more natural than the usual definition of a busy period when discussing the systems with server vacations, and it is in line with that of the busy period initiated with i customers studied in Chae and Kim (2007) and Chae and Lim (2008). Additionally, it is worth noting that through some tips and tricks for algebraic manipulations we are very easy to get the stochastic decomposition structure of the busy period in a discrete-time finite-buffer vacation queue. To the best of our knowledge, this result has not been reported in the previous literature.

The remainder of this paper is organized as follows. Section 2 gives the description of the model. In Sect. 3, we discuss the first two moments of the busy period initiated by $i(1 \leq i \leq N)$ customers, and then give the stochastic decomposition structure of the busy period which indicates the relationship with that of the $Geo/Geo/1/N$ queue without server vacations. Moreover, we further provide an effective way to validate the correctness of the analytical results in Sect. 4, and some numerical examples for special cases are illustrated in Sect. 5. Finally, conclusions are given in Sect. 6.

2 Model description

In this section, we develop a discrete-time Geo/Geo/1/N model by incorporating the concept of multiple server vacations, where $N(N \geq 2)$ denotes the maximum number of customers allowed in the system including the one in service. Under discrete situation, the time axis is divided into fixed-length contiguous period, called slots, and all queueing activities occur around slot boundaries. To be more specific, we suppose that potential departures occur in the time interval (t^-, t), while potential arrivals and the beginning or ending of the vacations take place in the time interval (t, t^+) (see Fig. 1). This also means that the queue is analyzed for the early arrival system (EAS). We further assume that the inter-arrival times A of customers are independent and geometrically distributed with probability mass function (p.m.f.) $\Pr\{A = k\} = \lambda\bar{\lambda}^{k-1}$, $0 < \lambda < 1$, $k \geq 1$, where we use symbol $\bar{x} = 1 - x$, for any real number $x(0 < x < 1)$. The customers are served individually according

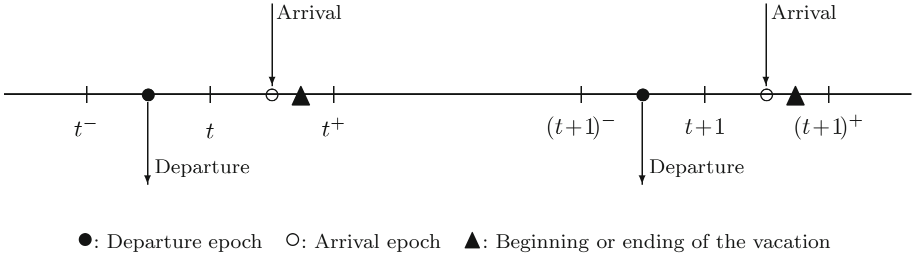

Fig. 1 Various time epochs in an early arrival system

to the first-come-first-served (FCFS) basis, and service time S has a geometric distribution with common p.m.f. $\Pr\{S = k\} = \mu\bar{\mu}^{k-1}$, $0 < \mu < 1$, $k \geq 1$. After all the customers are served in the queue exhaustively, the server immediately takes a vacation, where the vacation time is a discrete random variable, denoted by V, with p.m.f. $\Pr\{V = k\} = \theta\bar{\theta}^{k-1}$, $0 < \theta < 1$, $k \geq 1$. On returning from a vacation, if the server finds no customer waiting in the queue, he will take another vacation and so on until at least one customer presents in the system.

3 Busy period analysis: explicit expressions for the first two moments

The goal in this section is to find the explicit expressions for mean and variance of the busy period. As for multiple vacations queue, the busy period B is defined to be the time elapsed from the server's return to the system after taking a vacation until the queue becomes empty again and the next vacation begins. Because more than one customer can be accumulated in the system during the server's vacation period, deriving the mean and variance of the busy period is implemented by means of the analysis of the busy period that starts with $i\,(i \geq 1)$ customers, namely B_i. Since the service discipline is exhaustive in this model, i represents the number of arrivals during the preceding vacation time. Now, it is clear that to find $\mathbb{E}[B]$ and $\mathrm{Var}[B]$, the key step is to calculate the first two moments of B_i. In what follows, we will use the law of total expectation and the probability generating function (p.g.f.) technique to achieve our objective.

3.1 The first moment of $B_i\,(i = 1, 2, \ldots, N)$

Let $B_i(z)$, $i = 1, 2, \ldots, N$ denote the p.g.f. of the discrete random variable B_i. For determining this function, we first state the following lemma, which has been proved in Margolin and Winokur (1967).

Lemma 1 *Let X_1 and X_2 be independent random variables having geometric distributions with parameters η_1 and η_2, respectively. If $\mathbb{X}$ is the random variable $\min(X_1, X_2)$, then $\mathbb{X}$ has a geometric distribution with parameter $1 - \bar{\eta}_1\bar{\eta}_2$.*

Using memoryless property of geometric distribution and a first-step argument (i.e. by conditioning on the events that may occur in the next step), for $i = 1, 2, \ldots, N$, the p.g.f. of B_i can be decomposed as

$$\begin{aligned}
B_i(z) &= \mathbb{E}\left[z^{B_i}\right] = \mathbb{E}\left[z^{B_i}\mathbb{I}_{\{A<S\}}\right] + \mathbb{E}\left[z^{B_i}\mathbb{I}_{\{A=S\}}\right] + \mathbb{E}\left[z^{B_i}\mathbb{I}_{\{A>S\}}\right] \\
&= \mathbb{E}\left[z^{\min(A,S)}\right]\left(\mathbb{E}\left[z^{B_{i+1}}\right]\frac{\lambda\bar{\mu}}{1-\bar{\lambda}\bar{\mu}} + \mathbb{E}\left[z^{B_i}\right]\frac{\lambda\mu}{1-\bar{\lambda}\bar{\mu}} + \mathbb{E}\left[z^{B_{i-1}}\right]\frac{\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}}\right) \\
&= \frac{\left(1-\bar{\lambda}\bar{\mu}\right)z}{1-\bar{\lambda}\bar{\mu}z}\left(B_{i+1}(z)\frac{\lambda\bar{\mu}}{1-\bar{\lambda}\bar{\mu}} + B_i(z)\frac{\lambda\mu}{1-\bar{\lambda}\bar{\mu}} + B_{i-1}(z)\frac{\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}}\right) \\
&= \frac{z}{1-\bar{\lambda}\bar{\mu}z}\left(B_{i+1}(z)\lambda\bar{\mu} + B_i(z)\lambda\mu + B_{i-1}(z)\bar{\lambda}\mu\right), \quad i = 1, 2, \ldots, N-1, \quad (1)
\end{aligned}$$

$$
\begin{aligned}
B_N(z) &= \mathbb{E}\left[z^{B_N}\right] = \mathbb{E}\left[z^{B_N}\mathbb{I}_{\{A\le S\}}\right] + \mathbb{E}\left[z^{B_{N-1}}\mathbb{I}_{\{A>S\}}\right] \\
&= \mathbb{E}\left[z^{\min(A,S)}\right]\left(\mathbb{E}\left[z^{B_N}\right]\frac{\lambda}{1-\bar{\lambda}\bar{\mu}} + \mathbb{E}\left[z^{B_{N-1}}\right]\frac{\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}}\right) \\
&= \frac{\left(1-\bar{\lambda}\bar{\mu}\right)z}{1-\bar{\lambda}\bar{\mu}z}\left(B_N(z)\frac{\lambda}{1-\bar{\lambda}\bar{\mu}} + B_{N-1}(z)\frac{\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}}\right) \\
&= \frac{z}{1-\bar{\lambda}\bar{\mu}z}\left(B_N(z)\lambda + B_{N-1}(z)\bar{\lambda}\mu\right), \qquad (2)
\end{aligned}
$$

where $\mathbb{I}_C$ is an indicator function such that $\mathbb{I}_C = \begin{cases} 1, & \text{if event C occurs,} \\ 0, & \text{if event C does not occur.} \end{cases}$ Moreover, according to the model assumptions, if the server finds the queue is empty upon his return from vacation, he immediately takes another vacation, and the busy period in this case is defined to be of zero length for mathematical convenience. Thus, in Eq. (1), when $i = 1$, $B_0(z) = \mathbb{E}\left[z^{B_0}\right] = 1$.

Let $B_i'(1)$ represent the first derivative of $B_i(z)$ at $z = 1$. Taking the first derivative of Eqs. (1) and (2) with respect to z, setting $z = 1$ and noting that $B_0(z) = 1$, we have

$$
\begin{cases}
B_1'(1) = \dfrac{1}{1-\bar{\lambda}\bar{\mu}} + \dfrac{1}{1-\bar{\lambda}\bar{\mu}}\left(B_2'(1)\lambda\bar{\mu} + B_1'(1)\lambda\mu\right), \\
B_i'(1) = \dfrac{1}{1-\bar{\lambda}\bar{\mu}} + \dfrac{1}{1-\bar{\lambda}\bar{\mu}} \\
\qquad \times\left(B_{i+1}'(1)\lambda\bar{\mu} + B_i'(1)\lambda\mu + B_{i-1}'(1)\bar{\lambda}\mu\right), \quad i = 2, 3, \ldots, N-1, \\
B_N'(1) = \dfrac{1}{1-\bar{\lambda}\bar{\mu}} + \dfrac{1}{1-\bar{\lambda}\bar{\mu}}\left(B_N'(1)\lambda + B_{N-1}'(1)\bar{\lambda}\mu\right).
\end{cases} \qquad (3)
$$

According to the property of p.g.f., the expectation of B_i is given by $\mathbb{E}[B_i] = B_i'(1)$. Therefore, the above difference equations with constant coefficients may be written in matrix format as

$$
\boldsymbol{\Lambda}_N\left(\mathbb{E}[B_1], \mathbb{E}[B_2], \ldots, \mathbb{E}[B_N]\right)^\top = \left(\frac{1}{1-\bar{\lambda}\bar{\mu}}, \frac{1}{1-\bar{\lambda}\bar{\mu}}, \ldots, \frac{1}{1-\bar{\lambda}\bar{\mu}}\right)^\top, \qquad (4)
$$

where

$$
\boldsymbol{\Lambda}_N = \begin{pmatrix}
1-\dfrac{\lambda\mu}{1-\bar{\lambda}\bar{\mu}} & \dfrac{-\lambda\bar{\mu}}{1-\bar{\lambda}\bar{\mu}} & 0 & \cdots & 0 & 0 \\
\dfrac{-\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}} & 1-\dfrac{\lambda\mu}{1-\bar{\lambda}\bar{\mu}} & \dfrac{-\lambda\bar{\mu}}{1-\bar{\lambda}\bar{\mu}} & \cdots & 0 & 0 \\
0 & \dfrac{-\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}} & 1-\dfrac{\lambda\mu}{1-\bar{\lambda}\bar{\mu}} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1-\dfrac{\lambda\mu}{1-\bar{\lambda}\bar{\mu}} & \dfrac{-\lambda\bar{\mu}}{1-\bar{\lambda}\bar{\mu}} \\
0 & 0 & 0 & \cdots & \dfrac{-\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}} & 1-\dfrac{\lambda}{1-\bar{\lambda}\bar{\mu}}
\end{pmatrix}_{N\times N},
$$

and the superscript $\top$ represents the transpose of a vector or matrix.

Theorem 1 *Using Cramer's rule,* Eq. (4) *gives*

$$\mathbb{E}\left[B_1\right] = \frac{\det(\boldsymbol{\Lambda}_N^{(1)})}{\det(\boldsymbol{\Lambda}_N)} = \frac{1}{\bar{\lambda}\mu}\left(1+\rho+\rho^2+\cdots+\rho^{N-1}\right)$$
$$= \frac{1-\rho^N}{\mu-\lambda}, \quad 0<\lambda,\mu<1, \quad \lambda\neq\mu,$$

where $\boldsymbol{\Lambda}_N^{(1)}$ is derived from the matrix $\boldsymbol{\Lambda}_N$ by replacing the first column with the vector $\left(\frac{1}{1-\bar{\lambda}\bar{\mu}}, \frac{1}{1-\bar{\lambda}\bar{\mu}}, \ \ldots, \frac{1}{1-\bar{\lambda}\bar{\mu}}\right)^\top$ and $\rho = \dfrac{\lambda\bar{\mu}}{\bar{\lambda}\mu}$.

Proof The determinant of matrix $\boldsymbol{\Lambda}_k (k=2,3,\ldots,N)$ may be expanded along the first column as

$$\det(\boldsymbol{\Lambda}_k) = \begin{vmatrix} 1-\dfrac{\lambda\mu}{1-\bar{\lambda}\bar{\mu}} & \dfrac{-\lambda\bar{\mu}}{1-\bar{\lambda}\bar{\mu}} & 0 & \cdots & 0 & 0 \\ \dfrac{-\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}} & 1-\dfrac{\lambda\mu}{1-\bar{\lambda}\bar{\mu}} & \dfrac{-\lambda\bar{\mu}}{1-\bar{\lambda}\bar{\mu}} & \cdots & 0 & 0 \\ 0 & \dfrac{-\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}} & 1-\dfrac{\lambda\mu}{1-\bar{\lambda}\bar{\mu}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-\dfrac{\lambda\mu}{1-\bar{\lambda}\bar{\mu}} & \dfrac{-\lambda\bar{\mu}}{1-\bar{\lambda}\bar{\mu}} \\ 0 & 0 & 0 & \cdots & \dfrac{-\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}} & 1-\dfrac{\lambda}{1-\bar{\lambda}\bar{\mu}} \end{vmatrix}_{k\times k}$$
$$= \left(1-\frac{\lambda\mu}{1-\bar{\lambda}\bar{\mu}}\right)\det\left(\boldsymbol{\Lambda}_{k-1}\right) - \frac{\bar{\lambda}\mu\lambda\bar{\mu}}{\left(1-\bar{\lambda}\bar{\mu}\right)^2}\det\left(\boldsymbol{\Lambda}_{k-2}\right).$$

This implies that

$$\frac{\det\left(\boldsymbol{\Lambda}_k\right) - \dfrac{\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}}\det\left(\boldsymbol{\Lambda}_{k-1}\right)}{\det\left(\boldsymbol{\Lambda}_{k-1}\right) - \dfrac{\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}}\det\left(\boldsymbol{\Lambda}_{k-2}\right)} = \frac{\lambda\bar{\mu}}{1-\bar{\lambda}\bar{\mu}}.$$

Hence, we have

$$\det\left(\boldsymbol{\Lambda}_k\right) - \frac{\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}}\det\left(\boldsymbol{\Lambda}_{k-1}\right)$$
$$= \left[\det\left(\boldsymbol{\Lambda}_3\right) - \frac{\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}}\det\left(\boldsymbol{\Lambda}_2\right)\right]\left(\frac{\lambda\bar{\mu}}{1-\bar{\lambda}\bar{\mu}}\right)^{k-3}, \quad k=4,\ldots,N.$$

Notice that $\det(\boldsymbol{\Lambda}_3) - \frac{\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}} \det(\boldsymbol{\Lambda}_2) = 0$, thus, we further have

$$\det(\boldsymbol{\Lambda}_k) = \frac{\left(\bar{\lambda}\mu\right)^k}{\left(1-\bar{\lambda}\bar{\mu}\right)^k}, \quad k = 2, 3, \ldots, N.$$

On the other hand, for $k = 2, 3, \ldots, N$, let

$$\det(\boldsymbol{D}_k) = \begin{vmatrix} \frac{1}{1-\bar{\lambda}\bar{\mu}} & \frac{-\lambda\bar{\mu}}{1-\bar{\lambda}\bar{\mu}} & 0 & \cdots & 0 & 0 \\ \frac{1}{1-\bar{\lambda}\bar{\mu}} & 1-\frac{\lambda\mu}{1-\bar{\lambda}\bar{\mu}} & \frac{-\lambda\bar{\mu}}{1-\bar{\lambda}\bar{\mu}} & \cdots & 0 & 0 \\ \frac{1}{1-\bar{\lambda}\bar{\mu}} & \frac{-\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}} & 1-\frac{\lambda\mu}{1-\bar{\lambda}\bar{\mu}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \frac{1}{1-\bar{\lambda}\bar{\mu}} & 0 & 0 & \cdots & 1-\frac{\lambda\mu}{1-\bar{\lambda}\bar{\mu}} & \frac{-\lambda\bar{\mu}}{1-\bar{\lambda}\bar{\mu}} \\ \frac{1}{1-\bar{\lambda}\bar{\mu}} & 0 & 0 & \cdots & \frac{-\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}} & 1-\frac{\lambda}{1-\bar{\lambda}\bar{\mu}} \end{vmatrix}_{k\times k}$$

Employing $\det(\boldsymbol{D}_k)$, we can expand the determinant of matrix $\boldsymbol{\Lambda}_k^{(1)}$ along the last column, and get the following relationship between $\boldsymbol{\Lambda}_{k-1}^{(1)}$ and $\boldsymbol{\Lambda}_k^{(1)}$

$$\begin{aligned} \boldsymbol{\Lambda}_k^{(1)} &= \left(1 - \frac{\lambda}{1-\bar{\lambda}\bar{\mu}}\right)\left[\boldsymbol{\Lambda}_{k-1}^{(1)} + \frac{\lambda\bar{\mu}}{1-\bar{\lambda}\bar{\mu}}\boldsymbol{D}_{k-2}\right] \\ &\quad + \frac{\lambda\bar{\mu}}{1-\bar{\lambda}\bar{\mu}}\left[\frac{-\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}}\boldsymbol{D}_{k-2} + \frac{(\lambda\bar{\mu})^{k-2}}{\left(1-\bar{\lambda}\bar{\mu}\right)^{k-1}}\right] \\ &= \frac{\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}}\boldsymbol{\Lambda}_{k-1}^{(1)} + \frac{(\lambda\bar{\mu})^{k-1}}{\left(1-\bar{\lambda}\bar{\mu}\right)^k}, \quad k = 4, \ldots, N, \end{aligned}$$

Therefore, we also have

$$\boldsymbol{\Lambda}_k^{(1)} = \frac{\sum_{i=1}^{k}\left(\bar{\lambda}\mu\right)^{k-i}(\lambda\bar{\mu})^{i-1}}{\left(1-\bar{\lambda}\bar{\mu}\right)^k}.$$

This will then finally lead to the desired results presented in Theorem 1. □

Next, employing the above results, we can also give the explicit expression of the expected length of the busy period that starts with $i\,(i = 2, \ldots, N)$ customers.

Remark 1 In fact, $\mathbb{E}[B_1]$ is the average busy period in $Geo/Geo/1/N$ queue without server vacation. To obtain the stochastic decomposition structure of the busy period in the corresponding vacation queue, here, we denote $\mathbb{E}[B_1] = \mathbb{E}\left[B_{\mathrm{Geo/Geo/1/N}}\right]$.

Notice that $\mathbb{E}[B_0] = 0$, from Eq. (3), we may obtain the following equation for the first moment of $B_i (i = 1, \ldots, N-1)$.

$$\mathbb{E}[B_i] = \frac{1}{1-\bar{\lambda}\bar{\mu}} + \frac{1}{1-\bar{\lambda}\bar{\mu}} \left\{ \mathbb{E}\left[B_{i+1}\right] \lambda\bar{\mu} + \mathbb{E}[B_i]\, \lambda\mu + \mathbb{E}\left[B_{i-1}\right] \bar{\lambda}\mu \right\}. \quad (5)$$

Rearranging Eq. (5), we can arrive at

$$\begin{aligned} &\frac{\lambda\bar{\mu}}{1-\bar{\lambda}\bar{\mu}} \left(\mathbb{E}\left[B_{i+1}\right] - \mathbb{E}[B_i]\right) \\ &= -\frac{1}{1-\bar{\lambda}\bar{\mu}} + \frac{\bar{\lambda}\mu}{1-\bar{\lambda}\bar{\mu}} \left(\mathbb{E}[B_i] - \mathbb{E}\left[B_{i-1}\right]\right), \quad i = 1, \ldots, N-1. \end{aligned} \quad (6)$$

Writing Eq. (6) in the following form:

$$\psi_i = -\frac{1}{\lambda\bar{\mu}} + \frac{\bar{\lambda}\mu}{\lambda\bar{\mu}} \psi_{i-1}, \quad i = 1, \ldots, N-1, \quad (7)$$

with $\psi_i = \mathbb{E}\left[B_{i+1}\right] - \mathbb{E}[B_i]$, $i = 0, 1, \ldots, N-1$. Since $\psi_0 = \mathbb{E}[B_1]$, from Eq. (7), we can iteratively obtain the following expression for ψ_i,

$$\psi_i = \frac{-1}{\lambda\bar{\mu}} \sum_{k=1}^{i} \frac{1}{\rho^{k-1}} + \frac{\mathbb{E}[B_1]}{\rho^i}, \quad i = 1, 2, \ldots, N-1. \quad (8)$$

Substituting $\mathbb{E}[B_1] = \dfrac{1-\rho^N}{\mu-\lambda}$ into Eq. (8), it follows that

$$\psi_i = \frac{1}{\bar{\lambda}\mu} \frac{1-\rho^{N-i}}{1-\rho}, \quad i = 1, 2, \ldots, N-1.$$

Thus, a simple iteration formula based on initial value $\mathbb{E}[B_0]$ can be expressed as

$$\mathbb{E}\left[B_{i+1}\right] = \mathbb{E}[B_i] + \frac{1}{\bar{\lambda}\mu} \frac{1-\rho^{N-i}}{1-\rho}, \quad i = 0, 1, \ldots, N-1. \quad (9)$$

Also, from Eq. (9), the explicit expression of $\mathbb{E}[B_i]$ $(i = 1, 2, \ldots, N)$ is given by

$$\mathbb{E}[B_i] = \frac{1}{\mu-\lambda} \left[i - \frac{\rho^N \left(1-\rho^{i-1}\right)}{\rho^{i-1}(1-\rho)} - \rho^N \right]. \quad (10)$$

So far, we have given the explicit expression for the first moment of $B_i (i = 1, 2, \ldots, N)$ in a very concise form. In the next subsection, a similar technique will be used to derive the second moment of $B_i (i = 1, 2, \ldots, N)$ and thus leads to the variance of $B_i (i = 1, 2, \ldots, N)$.

3.2 The second moment of $B_i\,(i = 1, 2, \ldots, N)$

By differentiating Eqs. (1) and (2) twice with respect to z and evaluating at $z = 1$, we get

$$\begin{cases} B_1''(1) = \dfrac{2\bar{\lambda}\bar{\mu}}{\left(1-\bar{\lambda}\bar{\mu}\right)^2} + \dfrac{2}{\left(1-\bar{\lambda}\bar{\mu}\right)^2}\left(B_2'(1)\,\lambda\bar{\mu} + B_1'(1)\,\lambda\mu\right) \\ \quad + \dfrac{1}{1-\bar{\lambda}\bar{\mu}}\left(B_2''(1)\,\lambda\bar{\mu} + B_1''(1)\,\lambda\mu\right), \\ B_i''(1) = \dfrac{2\bar{\lambda}\bar{\mu}}{\left(1-\bar{\lambda}\bar{\mu}\right)^2} + \dfrac{2}{\left(1-\bar{\lambda}\bar{\mu}\right)^2}\left(B_{i+1}'(1)\,\lambda\bar{\mu} + B_i'(1)\,\lambda\mu + B_{i-1}'(1)\,\bar{\lambda}\mu\right) \\ \quad + \dfrac{1}{1-\bar{\lambda}\bar{\mu}}\left(B_{i+1}''(1)\,\lambda\bar{\mu} + B_i''(1)\,\lambda\mu + B_{i-1}''(1)\,\bar{\lambda}\mu\right), \quad i = 2, 3, \ldots, N-1, \\ B_N''(1) = \dfrac{2\bar{\lambda}\bar{\mu}}{\left(1-\bar{\lambda}\bar{\mu}\right)^2} + \dfrac{2}{\left(1-\bar{\lambda}\bar{\mu}\right)^2}\left(B_N'(1)\,\lambda + B_{N-1}'(1)\,\bar{\lambda}\mu\right) \\ \quad + \dfrac{1}{1-\bar{\lambda}\bar{\mu}}\left(B_N''(1)\,\lambda + B_{N-1}''(1)\,\bar{\lambda}\mu\right). \end{cases} \tag{11}$$

Since $\mathbb{E}\left[B_i^2\right] = B_i'(1) + B_i''(1)$, adding Eqs. (3) and (11) together, $\mathbb{E}\left[B_i^2\right]$ is obtained as

$$\begin{cases} \mathbb{E}\left[B_1^2\right] = \dfrac{1}{1-\bar{\lambda}\bar{\mu}}\left(\mathbb{E}\left[B_2^2\right]\lambda\bar{\mu} + \mathbb{E}\left[B_1^2\right]\lambda\mu\right) + \dfrac{2\mathbb{E}\left[B_1\right]-1}{1-\bar{\lambda}\bar{\mu}}, \\ \mathbb{E}\left[B_i^2\right] = \dfrac{1}{1-\bar{\lambda}\bar{\mu}}\left(\mathbb{E}\left[B_{i+1}^2\right]\lambda\bar{\mu} + \mathbb{E}\left[B_i^2\right]\lambda\mu + \mathbb{E}\left[B_{i-1}^2\right]\bar{\lambda}\mu\right) + \dfrac{2\mathbb{E}\left[B_i\right]-1}{1-\bar{\lambda}\bar{\mu}}, \\ \quad i = 2, 3, \ldots, N-1, \\ \mathbb{E}\left[B_N^2\right] = \dfrac{1}{1-\bar{\lambda}\bar{\mu}}\left(\mathbb{E}\left[B_N^2\right]\lambda + \mathbb{E}\left[B_{N-1}^2\right]\lambda\mu\right) + \dfrac{2\mathbb{E}\left[B_N\right]-1}{1-\bar{\lambda}\bar{\mu}}. \end{cases} \tag{12}$$

Equation (12) may be written compactly in a matrix form

$$\begin{aligned} &\boldsymbol{\Lambda}_N\left(\mathbb{E}\left[B_1^2\right], \mathbb{E}\left[B_2^2\right], \ldots, \mathbb{E}\left[B_N^2\right]\right)^\top \\ &\quad = \left(\frac{2\mathbb{E}\left[B_1\right]-1}{1-\bar{\lambda}\bar{\mu}}, \frac{2\mathbb{E}\left[B_2\right]-1}{1-\bar{\lambda}\bar{\mu}}, \ldots, \frac{2\mathbb{E}\left[B_N\right]-1}{1-\bar{\lambda}\bar{\mu}}\right)^\top. \end{aligned} \tag{13}$$

Use Cramer's rule again to find the value of $\mathbb{E}\left[B_1^2\right]$ in the solution to the Eq. (13), we have

$$\mathbb{E}\left[B_1^2\right] = \frac{\det(\bar{\boldsymbol{\Lambda}}_N^{(1)})}{\det(\boldsymbol{\Lambda}_N)},$$

where the matrix $\bar{\boldsymbol{\Lambda}}_N^{(1)}$ in the numerator is obtained by replacing the first column of $\boldsymbol{\Lambda}_N$ with the right hand side of Eq. (13). It is similar to the proof of Theorem 1, $\mathbb{E}\left[B_1^2\right]$ can be expressed in terms of $\mathbb{E}\left[B_i\right]$ as

$$\begin{aligned}\mathbb{E}\left[B_1^2\right] &= \sum_{i=1}^{N} \frac{1}{\bar{\lambda}\mu}\rho^{i-1}\left(2\mathbb{E}\left[B_i\right]-1\right) = \sum_{i=1}^{N} \frac{2}{\bar{\lambda}\mu}\rho^{i-1}\mathbb{E}\left[B_i\right] - \mathbb{E}\left[B_1\right] \\ &= \frac{2\left[1-(2N+1)\rho^N(1-\rho)-\rho^{2N+1}\right]}{\left(\bar{\lambda}\mu\right)^2(1-\rho)^3} - \frac{1-\rho^N}{\mu-\lambda}. \end{aligned} \tag{14}$$

Remark 2 Let $\mathrm{Var}\left[B_{\mathrm{Geo/Geo/1/N}}\right]$ represent the variance of the busy period of the standard $Geo/Geo/1/N$ queue without vacations. So, we have

$$\begin{aligned}\mathrm{Var}\left[B_{\mathrm{Geo/Geo/1/N}}\right] &= \mathbb{E}\left[B_1^2\right] - \mathbb{E}\left[B_1\right]^2 \\ &= \frac{2\left[1-(2N+1)\rho^N(1-\rho)-\rho^{2N+1}\right]}{\left(\bar{\lambda}\mu\right)^2(1-\rho)^3} - \frac{1-\rho^N}{\mu-\lambda}\left[1+\frac{1-\rho^N}{\mu-\lambda}\right].\end{aligned}$$

Because $\mathbb{E}\left[B_0^2\right]=0$, from Eq. (12), we can also derive the following relationship on the second moment of B_i.

$$\begin{aligned}\mathbb{E}\left[B_{i+1}^2\right]-\mathbb{E}\left[B_i^2\right] &= \frac{1-2\mathbb{E}\left[B_i\right]}{\lambda\bar{\mu}} + \frac{\bar{\lambda}\mu}{\lambda\bar{\mu}}\left(\mathbb{E}\left[B_i^2\right]-\mathbb{E}\left[B_{i-1}^2\right]\right) \\ &= \frac{1}{\rho^i}\sum_{k=1}^{i}\frac{1}{\bar{\lambda}\mu}\rho^{k-1}\left(1-2\mathbb{E}\left[B_k\right]\right)+\frac{\mathbb{E}\left[B_1^2\right]}{\rho^i}, \quad i=1,2,\ldots,N-1.\end{aligned} \tag{15}$$

Then, from Eqs. (14) and (15), we get

$$\mathbb{E}\left[B_{i+1}^2\right] = \frac{1}{\rho^i}\sum_{k=i+1}^{N}\frac{1}{\bar{\lambda}\mu}\rho^{k-1}\left(2\mathbb{E}\left[B_k\right]-1\right)+\mathbb{E}\left[B_i^2\right], \quad i=1,2,\ldots,N-1. \tag{16}$$

Hence, for $i=2,3,\ldots,N$, the expression of $\mathbb{E}\left[B_i^2\right]$ can be obtained recursively using Eq. (16) as follows

$$
\begin{aligned}
\mathbb{E}\left[B_i^2\right] &= \mathbb{E}\left[B_1^2\right] \frac{\rho^i - 1}{\rho^{i-1}(\rho-1)} + \sum_{r=1}^{i-1} \frac{1}{\rho^r} \sum_{k=1}^{r} \frac{1}{\bar{\lambda}\mu} \rho^{k-1} \left(1 - 2\mathbb{E}\left[B_k\right]\right) \\
&= \mathbb{E}\left[B_1^2\right] \frac{\rho^i - 1}{\rho^{i-1}(\rho-1)} + \sum_{r=1}^{i-1} \frac{1}{\bar{\lambda}\mu} \left(1 - 2\mathbb{E}\left[B_r\right]\right) \frac{\rho^{i-r} - 1}{\rho^{i-r}(\rho-1)}. \quad (17)
\end{aligned}
$$

3.3 Stochastic decomposition structure of the busy period

Let a_j be the probability that exactly j customers arrive during a vacation V. Thus

$$
\begin{cases}
a_0 = \sum_{k=1}^{\infty} \theta \bar{\theta}^{k-1} \bar{\lambda}^k = \dfrac{\theta \bar{\lambda}}{1 - \bar{\lambda}\bar{\theta}}, \\
a_j = \sum_{k=j}^{\infty} \theta \bar{\theta}^{k-1} \binom{k}{j} \lambda^j \bar{\lambda}^{k-j} = \dfrac{\theta \lambda \left(\bar{\theta}\lambda\right)^{j-1}}{\left(1 - \bar{\lambda}\bar{\theta}\right)^{j+1}}, \quad j = 1, 2, \ldots, N-1, \\
a_N = 1 - \sum_{j=0}^{N-1} a_j = \dfrac{\lambda^N \bar{\theta}^{N-1}}{\left(1 - \bar{\lambda}\bar{\theta}\right)^N}.
\end{cases}
$$

For $j \geq 1$, $\{V_c = j\}$ represents the case in which the busy period starts with the ending of a vacation during which j customers have arrived. It follows that

$$
\begin{cases}
\Pr\{V_c = j\} = \dfrac{a_j}{1 - a_0} = \dfrac{\theta \left(\lambda\bar{\theta}\right)^{j-1}}{\left(1 - \bar{\lambda}\bar{\theta}\right)^j} = \dfrac{\theta \xi^{j-1}}{\left(1 - \bar{\lambda}\bar{\theta}\right)}, \quad j = 1, 2, \ldots, N-1, \\
\Pr\{V_c = N\} = \dfrac{a_N}{1 - a_0} = \dfrac{\left(\lambda\bar{\theta}\right)^{N-1}}{\left(1 - \bar{\lambda}\bar{\theta}\right)^{N-1}} = \xi^{N-1},
\end{cases} \quad (18)
$$

where $\xi = \frac{\lambda\bar{\theta}}{1-\bar{\lambda}\bar{\theta}}$. By conditioning on the number of customers who have arrived during a vacation period, we have

$$
\begin{cases}
\mathbb{E}[B] = \sum_{l=1}^{N} \mathbb{E}[B_l] \Pr\{V_c = l\}, \\
\mathbb{E}\left[B^2\right] = \sum_{l=1}^{N} \mathbb{E}\left[B_l^2\right] \Pr\{V_c = l\}.
\end{cases} \quad (19)
$$

Also, substituting Eqs. (10), (17), (18) into Eq. (19) and then after some algebraic manipulations, we can get

$$\begin{aligned}
\mathbb{E}[B] &= \sum_{j=1}^{N-1} \frac{\theta\xi^{j-1}}{1-\bar{\lambda}\bar{\theta}} \frac{1}{\mu-\lambda}\left[j - \frac{\rho^N\left(1-\rho^{j-1}\right)}{\rho^{j-1}(1-\rho)} - \rho^N\right] \\
&\quad + \frac{\xi^{N-1}}{\mu-\lambda}\left[N - \frac{\rho^N\left(1-\rho^{N-1}\right)}{\rho^{N-1}(1-\rho)} - \rho^N\right] \\
&= \frac{-\rho^N}{\mu-\lambda}\left(\sum_{j=1}^{N-1}\frac{\theta\xi^{j-1}}{1-\bar{\lambda}\bar{\theta}} + \xi^{N-1}\right) + \frac{\theta}{1-\bar{\lambda}\bar{\theta}}\frac{1}{\mu-\lambda}\sum_{j=1}^{N} j\xi^{j-1} + \frac{N\xi^N}{\mu-\lambda} \\
&\quad - \frac{\theta\xi\rho^{N-1}}{(1-\rho)\left(1-\bar{\lambda}\bar{\theta}\right)(\mu-\lambda)}\sum_{j=1}^{N-1}\left(1-\rho^j\right)\left(\frac{\xi}{\rho}\right)^{j-1} - \frac{\xi^N}{\mu-\lambda}\frac{\rho\left(1-\rho^{N-1}\right)}{1-\rho} \\
&= \frac{-\rho^N}{\mu-\lambda} + \frac{1-\bar{\lambda}\bar{\theta}}{\theta(\mu-\lambda)}\left(1-\xi^N\right) - \frac{1-\bar{\lambda}\bar{\theta}}{\theta(\mu-\lambda)}N\xi^N \\
&\quad + \frac{1-\bar{\lambda}\bar{\theta}}{\theta(\mu-\lambda)}N\xi^{N+1} - \frac{N\xi^N}{\mu-\lambda} - \frac{\theta\xi\rho^N\left[1-\left(\frac{\xi}{\rho}\right)^{N-1}\right]}{\left(1-\bar{\lambda}\bar{\theta}\right)(\mu-\lambda)(1-\rho)(\rho-\xi)} \\
&\quad + \frac{\theta\xi\rho^N\left(1-\xi^{N-1}\right)}{\left(1-\bar{\lambda}\bar{\theta}\right)(\mu-\lambda)(1-\rho)(1-\xi)} - \frac{\xi^N\rho\left(1-\rho^{N-1}\right)}{(\mu-\lambda)(1-\rho)} \\
&= \frac{-\rho^N}{\mu-\lambda} + \frac{1-\bar{\lambda}\bar{\theta}}{\theta(\mu-\lambda)}\left(1-\xi^N\right) - \frac{\theta\xi\rho^N}{\left(1-\bar{\lambda}\bar{\theta}\right)(\mu-\lambda)(1-\rho)(\rho-\xi)} \\
&\quad + \frac{\xi\rho}{(1-\rho)(\mu-\lambda)}\left[\frac{(1-\xi)\rho^{N-1}\left(\frac{\xi}{\rho}\right)^{N-1}}{\rho-\xi} + \rho^{N-1}\left(1-\xi^{N-1}\right) - \xi\left(1-\rho^{N-1}\right)\right] \\
&= \frac{1-\rho^N}{\mu-\lambda} + \frac{1-\bar{\lambda}\bar{\theta}}{\theta(\mu-\lambda)}\left(1-\xi^N\right) \\
&\quad - \frac{1}{\mu-\lambda}\left\{1 + \frac{\rho^N}{1-\rho}\left[\frac{\theta\xi}{\left(1-\bar{\lambda}\bar{\theta}\right)(\rho-\xi)} - \left(\frac{\xi}{\rho}\right)^{N-1}\frac{\xi(1-\rho)}{\rho-\xi} - \xi\right]\right\} \\
&= \mathbb{E}\left[B_{\mathrm{Geo/Geo/1/N}}\right] + \frac{1-\bar{\lambda}\bar{\theta}}{\theta(\mu-\lambda)}\left(1-\xi^N\right) \\
&\quad - \frac{1}{\mu-\lambda}\left\{1 + \frac{\rho^N}{1-\rho}\left[\frac{\theta\xi}{\left(1-\bar{\lambda}\bar{\theta}\right)(\rho-\xi)} - \left(\frac{\xi}{\rho}\right)^{N-1}\frac{\xi(1-\rho)}{\rho-\xi} - \xi\right]\right\}. \qquad (20)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[B^2\right] &= \mathbb{E}\left[B_1^2\right] + \mathbb{E}\left[B_1^2\right]\frac{\xi\left[\rho^{N-1}-\xi^{N-1}\right]}{\rho^{N-1}(\rho-\xi)} \\
&\quad + \frac{1}{\bar{\lambda}\mu}\sum_{k=1}^{N-1}\left(1-2\mathbb{E}[B_k]\right)\frac{\xi^k\left(\rho^{N-k}-\xi^{N-k}\right)}{\rho^{N-k}(\rho-\xi)}
\end{aligned}$$

$$= \left\{ \frac{2\left[1-(2N+1)\rho^N(1-\rho)-\rho^{2N+1}\right]}{\left(\bar{\lambda}\mu\right)^2(1-\rho)^3} - \frac{1-\rho^N}{\mu-\lambda} \right\} \frac{\rho^N-\xi^N}{\rho^{N-1}(\rho-\xi)}$$
$$+\frac{1}{\bar{\lambda}\mu}\sum_{k=1}^{N-1}\left\{1-2\left[k-\frac{\rho^N\left(1-\rho^{k-1}\right)}{\rho^{k-1}(1-\rho)}-\rho^N\right]\right\}\frac{\xi^k\left(\rho^{N-k}-\xi^{N-k}\right)}{\rho^{N-k}(\rho-\xi)}. \tag{21}$$

From the first and second moments of the busy period we can compute the variance as $\mathrm{Var}[B] = \mathbb{E}\left[B^2\right] - \mathbb{E}[B]^2$. It is clear that the variance of the busy period also have the stochastic decomposition structure. Just because the expressions of $\mathbb{E}\left[B^2\right]$ and $\mathbb{E}[B]^2$ are slightly cumbersome to write, we do not indent to substitute Eqs. (20) and (21) into above formula.

3.4 Expectation of the busy cycle

The time interval from the instant of commencement of a service at the end of a vacation to the instant of starting the next service after availing at least one vacation is a busy cycle, denoted by B_{c}. The duration between two consecutive busy periods is called a whole vacation period, denoted by V_{w}, which is composed of some vacations V. According to the above definitions, the expectation of the busy cycle can be expressed as $\mathbb{E}[B_{\mathrm{c}}] = \mathbb{E}[B] + \mathbb{E}[V_{\mathrm{w}}]$. That is to say, as long as the value of $\mathbb{E}[V_{\mathrm{w}}]$ is determined, we can obtain the expectation of the busy cycle.

Let H be the number of consecutive vacations taken by the server. Based on the multiple vacation policy, it is easy to verify that $\Pr\{H=h\} = (1-a_0)\,a_0^{h-1}$. Then, using Wald's equation, we have

$$\mathbb{E}[V_{\mathrm{w}}] = \mathbb{E}\left[\sum_{k=1}^{H} V_k\right] = \mathbb{E}[H]\,\mathbb{E}[V] = \frac{1-\bar{\lambda}\bar{\theta}}{\theta\lambda}. \tag{22}$$

From Eq. (20), it follows that

$$\mathbb{E}[B_{\mathrm{c}}] = \mathbb{E}\left[B_{\mathrm{Geo/Geo/1/N}}\right] + \frac{1-\bar{\lambda}\bar{\theta}}{\theta(\mu-\lambda)}\left(1-\xi^N\right)$$
$$-\frac{1}{\mu-\lambda}\left\{1+\frac{\rho^N}{1-\rho}\left[\frac{\theta\xi}{\left(1-\bar{\lambda}\bar{\theta}\right)(\rho-\xi)} - \left(\frac{\xi}{\rho}\right)^{N-1}\frac{\xi(1-\rho)}{\rho-\xi} - \xi\right]\right\}$$
$$+\frac{1-\bar{\lambda}\bar{\theta}}{\theta\lambda}. \tag{23}$$

Further, let P_{busy} and P_{vacation} be the probabilities of the server's being busy and on vacation, respectively. We then have $P_{\text{busy}} = \frac{\mathbb{E}[B]}{\mathbb{E}[B_{\mathrm{c}}]}$ and $P_{\text{vacation}} = \frac{\mathbb{E}[V_{\mathrm{w}}]}{\mathbb{E}[B_{\mathrm{c}}]}$.

4 Validate the correctness of the analytical results

In the previous section, we have presented a concise approach for analysis of the first two moments of the busy period in the Geo/Geo/1/N vacation queue. Is there any effective way to validate the correctness of the analytical results? This problem merits our attention. In what follows, by the calculation of the steady-state probabilities of the system, we may numerically verify the correctness of our analysis.

At an arbitrary time t^-, the steady state of the system can be characterized by the following random variables:

$N(t)$: the number of customers in the system at time t^-;
$Y(t)$: the state of the server at time t^-, i.e., $Y(t) = 1$ or 0 corresponding to whether the server is busy, or on vacation, respectively.

Then $\{(N(t), Y(t)) : t = 0, 1, \ldots\}$ forms a two-dimensional Markov chain with a state space

$$\Omega = \{(0,0)\} \cup \left\{ \bigcup_{i=1}^{N} \{(i,0), (i,1)\} \right\}.$$

In limiting case, let us define the joint probability $\pi_{i,j} = \lim_{t\to\infty} \Pr\left\{N(t) = i, Y(t) = j\right\}$, $i = 0, 1, \ldots, N$, $j = 0, 1$. Here, we consider $N(t)$ as the level variable and $Y(t)$ as auxiliary variable. For $i = 1, 2, \ldots, N$, level $l(i)$ denotes the union of two states given by $l(i) = \{(i,0), (i,1)\}$. If the states in Ω are listed in lexicographical order then the transition probability matrix of the vector-valued discrete-time Markov chain governing the system has the following block tridiagonal matrix form

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{P}_{0,0} & \boldsymbol{P}_{0,1} & & & & & \\ \boldsymbol{P}_{1,0} & \boldsymbol{P}_{1,1} & \boldsymbol{P}_{1,2} & & & & \\ & \boldsymbol{P}_{2,1} & \boldsymbol{P}_{2,2} & \boldsymbol{P}_{2,3} & & & \\ & & \boldsymbol{P}_{3,2} & \boldsymbol{P}_{3,3} & \boldsymbol{P}_{3,4} & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \boldsymbol{P}_{N-1,N-2} & \boldsymbol{P}_{N-1,N-1} & \boldsymbol{P}_{N-1,N} \\ & & & & & \boldsymbol{P}_{N,N-1} & \boldsymbol{P}_{N,N} \end{pmatrix}, \qquad (24)$$

where the coefficient matrices appearing in (24) are given by

$$\boldsymbol{P}_{0,0} = \left(\bar{\lambda}\right), \quad \boldsymbol{P}_{0,1} = \begin{pmatrix}\lambda\bar{\theta} & \lambda\theta\end{pmatrix}, \quad \boldsymbol{P}_{1,0} = \begin{pmatrix} 0 \\ \mu\bar{\lambda} \end{pmatrix},$$

$$\boldsymbol{P}_{i,i-1} = \begin{pmatrix} 0 & 0 \\ 0 & \mu\bar{\lambda} \end{pmatrix}, \quad i = 2, \ldots, N, \quad \boldsymbol{P}_{i,i} = \begin{pmatrix} \bar{\lambda}\bar{\theta} & \bar{\lambda}\theta \\ 0 & \mu\lambda + \bar{\mu}\bar{\lambda} \end{pmatrix}, \quad i = 1, \ldots, N-1,$$

$$\boldsymbol{P}_{i,i+1} = \begin{pmatrix} \lambda\bar{\theta} & \lambda\theta \\ 0 & \bar{\mu}\lambda \end{pmatrix}, \quad i = 1, \ldots, N-1, \quad \boldsymbol{P}_{N,N} = \begin{pmatrix} \theta & \bar{\theta} \\ 0 & \mu\lambda + \bar{\mu} \end{pmatrix}.$$

Obviously, $\boldsymbol{P}$ is what is known as a finite quasi-birth-and-death (QBD) process. This is reflected by having the block tridiagonal structure. Also, in matrix $\boldsymbol{P}$, the blocks for which the entries are zero are not written. Since $\{(N(t), Y(t)), t \geq 0\}$ is an irreducible Markov chain on a finite state space, it must be positive recurrent. Next, we shall provide an effective and numerically stable algorithm to compute the stationary probability vector of the Markov chain.

The steady-state probability vector $\boldsymbol{\pi}$ for $\boldsymbol{P}$ is generally partitioned as $\boldsymbol{\pi} = \{\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ \ldots, \boldsymbol{\pi}_N\}$, where the subvectors $\boldsymbol{\pi}_0$ and $\boldsymbol{\pi}_i\,(i = 1, \ldots, N)$, are of dimension 1 and 2, respectively. For $i = 0$ and $i = 1, \ldots, N$, we also write $\boldsymbol{\pi}_0 = (\pi_{0,0})$ and $\boldsymbol{\pi}_i = \left(\pi_{i,0}, \pi_{i,1}\right)$, respectively. It is well known that the vector $\boldsymbol{\pi}$ is the unique solution to the following system of linear algebraic equations

$$\begin{cases} (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_N)\,(\boldsymbol{P} - \boldsymbol{I}_{2N+1}) = \mathbf{0}, \\ (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_N)\,\boldsymbol{e} = 1, \end{cases} \tag{25}$$

where $\boldsymbol{I}_{2N+1}$ represents the identity matrix of dimension $(2N+1) \times (2N+1)$, $\mathbf{0}$ denotes a zero matrix of appropriate dimension and $\boldsymbol{e}$ is a column vector of ones of suitable dimension. Employing LU-type RG-factorization (see Li 2010), we will give an RG-factorization solution to Eq. (25). The major advantage of RG-factorization is that it can avoid the calculation of high dimensional matrices by decomposing them into small ones. Essentially speaking, RG-factorization is iteratively constructed in terms of three probabilistic measures, namely R-, U- and G-measures. Now, we first define U-measures as

$$\begin{cases} \boldsymbol{U}_0 = \boldsymbol{P}_{0,0} - \boldsymbol{I}_1, \\ \boldsymbol{U}_k = \left(\boldsymbol{P}_{k,k} - \boldsymbol{I}_2\right) + \boldsymbol{P}_{k,k-1}\left(-\boldsymbol{U}_{k-1}^{-1}\right)\boldsymbol{P}_{k-1,k},\ k = 1, 2, \ldots, N. \end{cases}$$

Based on the matrices $\boldsymbol{U}_k (k = 0, 1, \ldots, N-1)$, the R-measures amd G-measures can be expressed as

$$\boldsymbol{R}_k = \boldsymbol{P}_{k,k-1}\left(-\boldsymbol{U}_{k-1}^{-1}\right), \quad k = 1, 2, \ldots, N,$$
$$\boldsymbol{G}_k = \left(-\boldsymbol{U}_k^{-1}\right)\boldsymbol{P}_{k,k+1}, \quad k = 0, 1, \ldots, N-1.$$

Using the matrix sequences $\{\boldsymbol{U}_k, k = 0, 1, \ldots, N\}$, $\{\boldsymbol{R}_k, k = 1, 2, \ldots, N\}$ and $\{\boldsymbol{G}_k, k = 0, 1, \ \ldots, N-1\}$, the LU-type RG-factorization for matrix $\boldsymbol{P} - \boldsymbol{I}_{2N+1}$ is given by

$$\boldsymbol{P} - \boldsymbol{I}_{2N+1} = (\boldsymbol{I}_{2N+1} - \boldsymbol{R}_L)\,\mathrm{diag}\,(\boldsymbol{U}_0, \boldsymbol{U}_1, \ldots, \boldsymbol{U}_N)\,(\boldsymbol{I}_{2N+1} - \boldsymbol{G}_U)\,,$$

where

$$\boldsymbol{R}_L = \begin{pmatrix} \mathbf{0} & & & & & \\ \boldsymbol{R}_1 & \mathbf{0} & & & & \\ & \boldsymbol{R}_2 & \mathbf{0} & & & \\ & & \ddots & \ddots & & \\ & & & \boldsymbol{R}_{N-1} & \mathbf{0} & \\ & & & & \boldsymbol{R}_N & \mathbf{0} \end{pmatrix} \quad \text{and} \quad \boldsymbol{G}_U = \begin{pmatrix} \mathbf{0} & \boldsymbol{G}_0 & & & & \\ & \mathbf{0} & \boldsymbol{G}_1 & & & \\ & & \mathbf{0} & \boldsymbol{G}_2 & & \\ & & & \ddots & \ddots & \\ & & & & \mathbf{0} & \boldsymbol{G}_{N-1} \\ & & & & & \mathbf{0} \end{pmatrix}.$$

Since the Markov chain is a QBD with finitely-many levels, using the LU-type RG-factorization, the steady-state probability vector is given as the matrix-product solution (see Li and Cao 2004)

$$\begin{cases} \boldsymbol{\pi}_N = \tau \boldsymbol{\omega}_N, \\ \boldsymbol{\pi}_k = \tau \boldsymbol{\omega}_N \displaystyle\prod_{i=0}^{N-k-1} \boldsymbol{R}_{N-i}, \; k = 0, 1, \ldots, N-1, \end{cases}$$

where $\boldsymbol{\omega}_N$ is the stationary probability vector of the generator $\boldsymbol{U}_N$ and the scalar τ is determined by $\pi_{0,0} + \sum_{k=1}^{N} \boldsymbol{\pi}_k \boldsymbol{e} = 1$, that is to say, τ is a normalization constant.

As soon as the vectors $\boldsymbol{\pi}_i$, $i = 0, 1, \ldots, N$, have been calculated, we are able to find various performance measures of the system under consideration. Particularly, if the previous analysis about the busy period is correct, then the following equalities must hold:

$$P_{\text{busy}} = \frac{\mathbb{E}[B]}{\mathbb{E}[B_{\text{c}}]} = \sum_{i=1}^{N} \pi_{i,1} \quad \text{and} \quad P_{\text{vacation}} = \frac{\mathbb{E}[V_{\text{w}}]}{\mathbb{E}[B_{\text{c}}]} = \sum_{i=0}^{N} \pi_{i,0}.$$

Actually, through a number of numerical experiments we found that the above equalities are indeed always true. For instance, taking $N = 10$, $\lambda = 0.21$, $\mu = 0.24$ and $\theta = 0.018$, we have

$$P_{\text{busy}} = \frac{\mathbb{E}[B]}{\mathbb{E}[B_{\text{c}}]} = \sum_{i=1}^{N} \pi_{i,1} = 0.707974 \quad \text{and}$$

$$P_{\text{vacation}} = \frac{\mathbb{E}[V_{\text{w}}]}{\mathbb{E}[B_{\text{c}}]} = \sum_{i=0}^{N} \pi_{i,0} = 0.292026.$$

This indicates that the analytical results presented in the previous section are accurate and reliable.

5 Numerical results

To end this paper, we provide tables and figures of the moments of the busy period in Geo/Geo/1/N multiple vacation queue that obtained through a MATLAB code based

Table 1 Expected value and standard deviation of the busy period initiated with i customers

i	$Geo/Geo/1/7$		$Geo/Geo/1/8$		$Geo/Geo/1/9$	
	$\mathbb{E}[B_i]$	$\mathrm{SD}[B_i]$	$\mathbb{E}[B_i]$	$\mathrm{SD}[B_i]$	$\mathbb{E}[B_i]$	$\mathrm{SD}[B_i]$
1	15.9532	33.0717	16.7791	36.7498	17.4364	39.9869
2	30.8688	43.9410	32.7323	49.4397	34.2155	54.3093
3	44.4807	50.2279	47.6479	57.2831	50.1687	63.5865
4	56.4546	53.8111	61.2598	62.2366	65.0843	69.8592
5	66.3705	55.5940	73.2337	65.1628	78.6962	73.9753
6	73.7008	56.2460	83.1496	66.6426	90.6701	76.4535
7	77.7824	56.3577	90.4799	67.1875	100.5860	77.7187
8	–	–	94.5615	67.2810	107.9163	78.1864
9	–	–	–	–	111.9980	78.2668

on the results of the previous section. We use Eqs. (10) and (17) to compute the first two moments and the standard deviation (SD) of B_i. The SD is always equal to the square root of the variance, and it measures the amount of variation or dispersion from the average. Unlike the variance, SD is expressed in the same units as the data. Here, a particular case we consider is that $\lambda = 0.3$, $\mu = 0.35$, $\theta = 0.1$. The computational results for different system capacities N are listed in Table 1. We observe from Table 1 that the increasing rates of $\mathbb{E}[B_i]$ and $\mathrm{SD}[B_i]$ becomes slower as the values of i get larger. In addition, by fixing $N = 17$, $\mu = 0.32$ and $\theta = 0.05$, we look at the effect of varying λ on $\mathbb{E}[B]$ and $\mathrm{SD}[B]$. The corresponding results of the Geo/Geo/1/17 queue with/without server vacations are displayed in Figs. 2 and 3. From Fig. 2, we

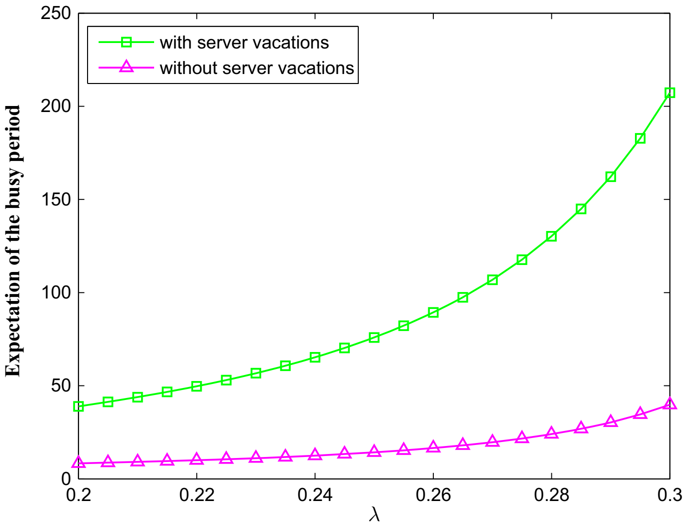

Fig. 2 Effect of λ on $\mathbb{E}[B]$ for Geo/Geo/1/N queue with/without server vacations $\lambda < \mu$

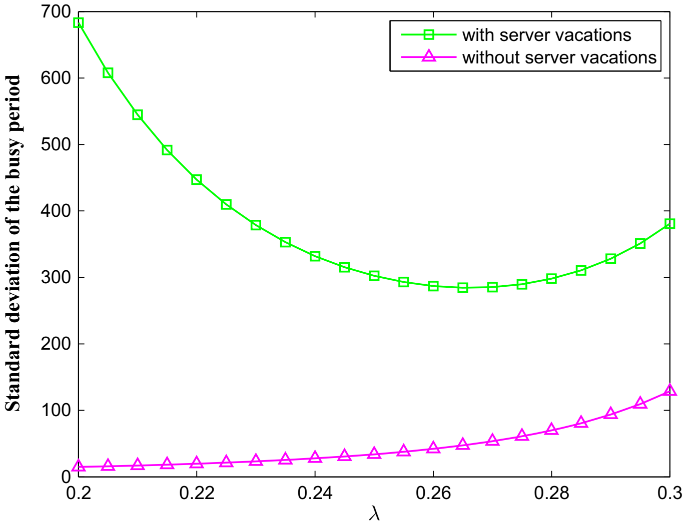

Fig. 3 Effect of λ on $\mathrm{SD}[B]$ for Geo/Geo/1/N queue with/without server vacations $\lambda < \mu$

observe that the measure $\mathbb{E}[B]$ increases as the arrival rate λ increases, and it also shows a very big variation when the server is allowed to take vacations. From Fig. 3, with respect to the measure $\mathrm{SD}[B]$, we may see some interesting trends. In vacation queue, $\mathrm{SD}[B]$ decreases initially and then increases with the increasing value of λ. This implies that the number of customers that arrive during the vacation time can cause larger fluctuations in length of the busy period. On the other hand, as is to be expected, in the ordinary Geo/Geo/1/N queue, $\mathrm{SD}[B]$ is a monotonically increasing function of λ. As we have already mentioned, the case in which the average arrival rate is higher than the average service rate can be considered in the finite-buffer queue. In Figs. 4 and 5, holding all other parameter values unchanged, the numerical example was performed by varying λ from 0.33 to 0.38. As we had expected, Figs. 4 and 5 indicate that $\mathbb{E}[B]$ and $\mathrm{SD}[B]$ are both strictly monotonically increasing functions of λ. Furthermore, a much greater difference of mean busy period will occur in vacation and non-vacation models.

6 Conclusions

The busy period analysis is just as important as the analysis of the queue length and the waiting time of customers because it helps us to study the traffic intensity and the cycle time of the queueing system. In vacation queueing model, since the busy period does not start at an arrival instant, we define a random variable B_i as the time length that starts from the instant when the number of customers is i to when the number of customers becomes zero for the first time. By calculating the first two moments of B_i, we easily get the stochastic decomposition structure of the busy period in discrete-time finite-buffer vacation queue. Based on the results of this research, further studies can be conducted in the following directions. First, research topics such as Geo/Geo/1/N

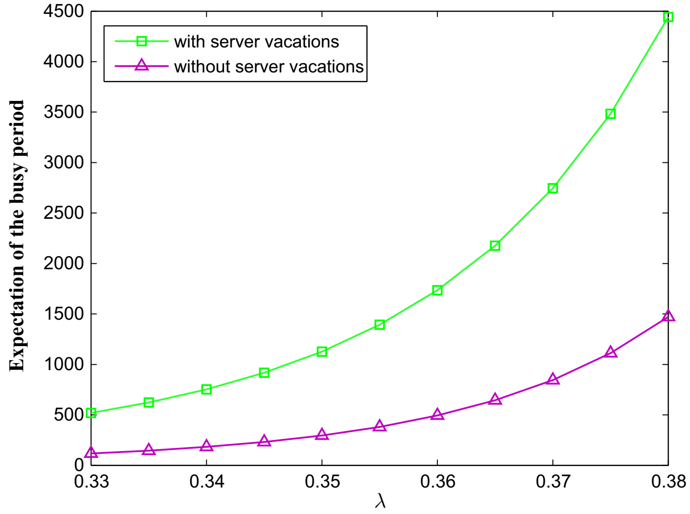

Fig. 4 Effect of λ on $\mathbb{E}[B]$ for Geo/Geo/1/N queue with/without server vacations $\lambda > \mu$

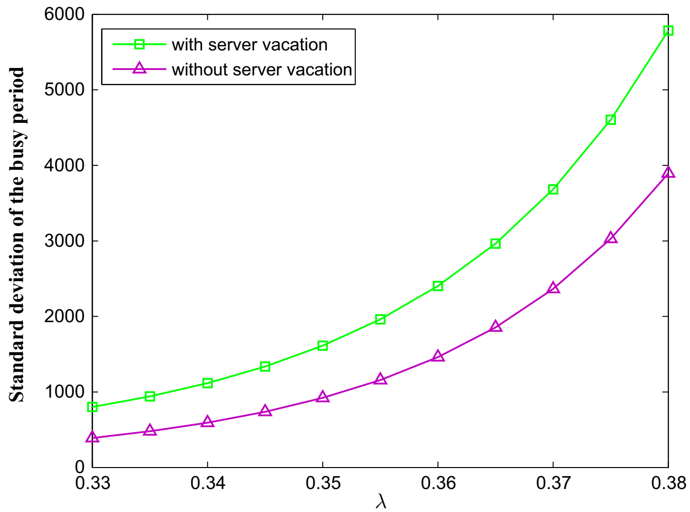

Fig. 5 Effect of λ on SD$[B]$ for Geo/Geo/1/N queue with/without server vacations $\lambda > \mu$

queue with single vacation policy and $\min(N, T)$ vacation policy (see Alfa and Frigui 1996) can be studied with the same analysis techniques. On the other hand, it is not impossible to get the higher moments of the busy period by extending our method. But owing to the growth of complexity, it is not a simple job. With the help of the computer algorithms, we think this difficulty might be overcome.

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