OPTIMAL COINCIDENCE BEST APPROXIMATION SOLUTION IN NON-ARCHIMEDEAN FUZZY METRIC SPACES

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Abstract. In this paper, we introduce the concept of best proximal contraction theorems in non-Archimedean fuzzy metric space for two mappings and prove some proximal theorems. As a consequence, it provides the existence of an optimal approximate solution to some equations which contains no solution. The obtained results extend further the recently development proximal contractions in non-Archimedean fuzzy metric spaces and famous Banach contraction principle.

1. **Introduction and Preliminaries**

Let K be any nonempty subset of a metric space X and $T : K \to X$ be a nonself mapping. A fixed point problem is to find a point x^* in K such that $d(x^*, Tx^*) = 0$. A point x^* in K where $\inf\{d(y, Tx^*) : y \in K\}$ is attained, that is, $d(x^*, Tx^*) = \inf\{d(y, Tx^*) : y \in K\}$ holds is called an approximate fixed point of T or an approximate solution of an equation $Tx = x$. In case, if it is not possible to solve fixed point problem, it could be interesting to study the conditions that assure existence and uniqueness of approximate fixed point of a mapping T.

A well-known best approximation theorem due to Ky Fan [2], states that if K is a nonempty compact convex subset of a Hausdorff locally convex topological vector space E and $T : K \to E$ is a continuous mapping, then there exists an element x^* in K such that $d(x^*, Tx^*) = \inf_{y \in K} d(Tx^*, y) = d(Tx^*, K)$.

The problem of finding an optimal best approximation solution is also of great interest in optimization theory.

Let A and B be two nonempty subsets of X and $T : A \to B$. Suppose that $\triangle_{AB} = d(A, B) = \inf(\{d(a, b) : a \in A, b \in B\})$ is the measure of a distance between two sets A and B. A point x^* is called a best proximity point of T if $d(x^*, Tx^*) = \triangle_{AB}$, which is a solution of the following optimization problem

$$\begin{aligned} f(x) &= d(x, Tx) \leftarrow \min \\ & \quad \text{subject to the constraint} \\ x &\in A. \end{aligned}$$

Indeed, if T is a multifunction from A to B, then the following

$$f(x) = d(x, Tx) \geq dist(A, B),$$

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always hold. So, the most optimal solution to the problem of minimizing the real valued function $x \to d(x, Tx)$ over the domain A of the multifunction T will be one for which the value

$$\triangle_{AB} = dist(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\}$$

is attained. If $A = B$, best proximity point problem reduces to a fixed point problem. In this way, best proximity point problem can be viewed as a natural generalization of a fixed point problem. Furthermore, results dealing with existence and uniqueness of best proximity point of certain mappings are more general than the ones dealing with fixed point problem of those mappings ([1, 9, 11, 12, 13]). A coincidence best proximity point problem is defined as follows: find a point x^* in A such that $d(gx^*, Tx^*) = \triangle_{AB}$ where g is a self mapping on A. This is an extension of a best proximity point problem.

On the other hand, Zadeh [16] introduced the concept of a fuzzy set. Kramosil and Michalek [8] defined fuzzy metric spaces. George and Veeramani [4, 5] modified and studied the notion of fuzzy metric spaces with the help of continuous $t-norm$ and generalized the concept of a probabilistic metric space to fuzzy situation.

In this paper, we prove the existence and uniqueness of an optimal coincidence best approximation of a solution of a function $M(gx, Tx, t)$ over a nonempty subset of non-Archimedean fuzzy metric space, where g is a self mapping on A and T is a nonself mapping. Our results extend and strengthen various known results in [10].

Consistent with [6], [14] and [15], the following definitions and results will be needed in the sequel.

Definition 1.1. [14] A binary operation $* : [0, 1]^2 \longrightarrow [0, 1]$ is said to be a *continuous* $t-norm$ if the following conditions hold:

(1): $*$ is associative and commutative;
(2): $*$ is continuous;
(3): $a * 1 = a$ for all $a \in [0, 1]$;
(4): $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

The most commonly used continuous $t-norms$ in fuzzy logic are minimum $t-norm\ \wedge$, product $t-norm \cdot$ and Lukasiewicz $t-norm *_L$, where $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$, and $a *_L b = \max\{a + b - 1, 0\}$. It is easy to check that $*_L \leq \cdot \leq \wedge$.

Definition 1.2. (compare [5]) Let X be a nonempty set, and $*$ a continuous $t-norm$. A fuzzy set M on $X \times X \times [0, +\infty)$ is said to be a fuzzy metric if for any $x, y, z \in X$, the following conditions hold:

(i): $M(x, y, t) > 0$,
(ii): $x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0$,
(iii): $M(x, y, t) = M(y, x, t)$,
(iv): $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s > 0$,
(v): $M(x, y, \cdot) : [0, +\infty) \to [0, 1]$ is left continuous.

The triplet $(X, M, *)$ is called a fuzzy metric space. Since M is a fuzzy set on $X \times X \times [0, +\infty)$, the value $M(x, y, t)$ is regarded as the degree of closeness of x and y with respect to t.

It is well known [6], and easy to see, that for each $x, y \in X$, $M(x, y, .)$ is a non-decreasing function on $[0, +\infty)$.

If we replace (iv) by

(vi): $M(x, z, \max\{t, s\}) \geq M(x, y, t) * M(y, z, s)$ for all $t, s > 0$.

Then triplet $(X, M, *)$ is said to be a non-Archimedean fuzzy metric space.

As (vi) implies (iv), each non-Archimedean fuzzy metric space is a fuzzy metric space. Further, if we take $s = t$, then (vi) becomes $M(x, z, t) \geq M(x, y, t) * M(y, z, t)$ for all $t > 0$ and M is said to be strong fuzzy metric.

Each fuzzy metric $(M, *)$ on a set X induces a Hausdorff topology τ_M on X which has a base the family of open balls $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in (0, 1), t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$.

Observe that a sequence $\{x_n\}_{n \in \mathbb{N}}$ converges to $x \in X$ (with respect to τ_M) if and only if $\lim_{n \to \infty} M(x, x_n, t) = 1$ for all $t > 0$.

If (X, d) is a metric space and we define $M_d : X \times X \times [0, +\infty) \to [0, 1]$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)},$$

then $(X, M_d, \cdot)$ is a fuzzy metric space and $(M_d, \cdot)$ is called the standard fuzzy metric induced by d ([4]). The topologies τ_{M_d} and τ_d (the topology induced by the metric) on X are the same. Note that if d is a metric on a set X, then the fuzzy metric $(M_d, *)$ is strong for every continuous $t - norm$ "$*$" such that for all $* \leq \cdot$, where M_d is defined by $M_d(x, y, t) = \frac{t}{t + d(x, y)}$, for all $x, y \in X$ and $t > 0$ (for further details [3]).

A sequence $\{x_n\}_{n \in \mathbb{N}}$ in a fuzzy metric space $(X, M, *)$ is said to be a Cauchy sequence if for each $t > 0$ and $\varepsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$. A fuzzy metric space $(X, M, *)$ is said to be complete [5] if every Cauchy sequence converges in $(X, M, *)$. A subset $A \subseteq X$ is said to be closed if for each convergent sequence $\{x_n\}$ with $x_n \in A$ and $x_n \longrightarrow x$, implies $x \in A$. A subset $A \subseteq X$ is said to be compact if each sequence in A has a convergent subsequence.

Lemma 1.3. [6] *M is a continuous function on $X^2 \times (0, \infty)$.*

Definition 1.4. [15] Let A and B be two nonempty subsets of a fuzzy metric space $(X, M, *)$. Define;

$$\begin{aligned} A_0(t) &= \{x \in A : M(x, y, t) = M(A, B, t) \text{ for some } y \in B\}, \\ B_0(t) &= \{y \in B : M(x, y, t) = M(A, B, t) \text{ for some } x \in A\}, \end{aligned}$$

where $M(x, A, t) = \sup_{a \in A} M(x, a, t)$ is the distance of a point $x \in X$ from a nonempty set A for $t > 0$ and $M(A, B, t) = \sup\{M(a, b, t) \text{ for } a \in A \text{ and } b \in B\}$.

Let $\Omega = \{\eta : [0, 1] \to [0, 1]$ such that η is continuous and decreasing on $[0, 1]$ and $\eta(t) = 0$ if and only if $t = 1\}$.

2. Optimal Solutions Using Best Proximity Point Theorems in Non-archimedean Fuzzy Metric Space

Definition 2.1. Let A be a nonempty subset of a non-Archimedean fuzzy metric space $(X, M, *)$. A self mapping $f : A \to A$ is said to be fuzzy isometry if $M(fx, fy, t) = M(x, y, t)$ holds for all $x, y \in A$ and $t > 0$.

Example 2.2. Let $X = [0, 1] \times \mathbb{R}$, and $d : X \times X \to \mathbb{R}$ be a usual metric on X. Let $A = \{(0, x) : x \in \mathbb{R}\}$. Note that $(X, M_d, \cdot)$ is non-Archimedean fuzzy metric space, where M_d is standard fuzzy metric induced by d. Define the mapping $f : A \to A$ by $f(0, x) = (0, -x)$. Clearly, $M_d(w, u, t) = \frac{t}{t+|x-y|}$, where $w = (0, x)$, $u = (0, y) \in A$. Also, $M(fw, fu, t) = \frac{t}{t+|-x+y|}$, which shows that f is a fuzzy isometry.

Definition 2.3. Let A be a nonempty subset of non-Archimedean fuzzy metric space $(X, M, *)$. A self mapping $f : A \to A$ is said to be fuzzy expansive if for any $x, y \in A$, and $t > 0$, we have $M(fx, fy, t) \leq M(x, y, t)$.

Note that every fuzzy isometry is fuzzy expansive but converse does not hold in general.

Example 2.4. Let $X = \mathbb{R}$, and $d : X \times X \to \mathbb{R}$ be a usual metric on X. Let $A = [0, 4]$. Note that $(X, M_d, \cdot)$ is a non-Archimedean fuzzy metric space, where M_d is standard fuzzy metric induced by d. Define the mapping $f : A \to A$ by $fx = 100x$. If $x = 0$ and $y = 4$, then $M(x, y, t) = \frac{t}{t+4}$, and $M(fx, fy, t) = \frac{t}{t+400}$, which shows that f is fuzzy expansive but not a fuzzy isometry.

Definition 2.5. Let A, B be nonempty subsets of a non-Archimedean fuzzy metric space $(X, M, *)$. A set B is said to be fuzzy approximately compact with respect to A if for every sequence $\{y_n\}$ in B and some $x \in A$, $M(x, y_n, t) \longrightarrow M(x, B, t)$ imply that $x \in A_0(t)$.

Definition 2.6. [7] A sequence $\{t_n\}$ of positive real numbers is an $s-$increasing sequence if there exists $n_0 \in \mathbb{N}$ such that $t_{n+1} \geq t_n + 1$ for all $n \geq n_0$.

Definition 2.7. A fuzzy metric space $(X, M, *)$ is said to satisfy the *property*$-T$ if for an $s-$ increasing sequence, there exists some n_0 in $\mathbb{N}$ such that $\prod_{n \geq n_0}^{\infty} M(x, y, t_n) \geq 1 - \varepsilon$, for all $n \geq n_0$.

Theorem 2.8. *Let $(X, M, *)$ be non-Archimedean complete fuzzy metric space, A and B are two nonempty closed subsets of X such that B is approximately compact with respect to A and $T : A \to B$. Suppose that there exists $\eta \in \Omega$ such that*

$$\left.\begin{array}{l} M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array}\right\} \Rightarrow \eta[M(u, v, t)] \leq \omega(t)\eta[(M(x, y, t)],$$

for all $x, y, u, v \in A$ and $t > 0$, where $\omega : (0, +\infty) \to (0, 1)$ is any function. If $g : A \to A$ is a fuzzy expansive mapping, $T(A_0(t)) \subseteq B_0(t) \neq \phi$ and $\phi \neq$

$A_0(t) \subseteq g(A_0(t))$ *for each* $t > 0$. *Then, there exists an element* $x \in A$ *such that* $M(gx, Tx, t) = M(A, B, t)$. *Further, for any fixed element* $x_0 \in A_0$, *the sequence* $\{x_n\}$ *defined by* $M(gx_{n+1}, Tx_n, t) = M(A, B, t)$, *converges to the unique element* x.

Proof. Let x_0 be a given point in $A_0(t)$. Since $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, we can choose an element $x_1 \in A_0(t)$ such that

$$M(gx_1, Tx_0, t) = M(A, B, t).$$

Also since $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$, and $A_0(t) \subseteq g(A_0(t))$, it follows that there exists an element $x_2 \in A_0(t)$ such that $M(gx_2, Tx_1, t) = M(A, B, t)$. Continuing this way, we can obtain a sequence $\{x_n\}$ in $A_0(t)$ such that it satisfies :

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t) \text{ and } M(gx_n, Tx_{n-1}, t) = M(A, B, t), \quad (1)$$

for each positive integer n. Now by given assumption, we have

$$\eta[M(gx_{n+1}, gx_n, t)] \leq \omega(t)\eta[M(x_n, x_{n-1}, t)]$$

for all $n \geq 0$. As g is fuzzy expansive and η is a decreasing mapping on $[0, 1]$, we have

$$\eta[M(x_{n+1}, x_n, t)] \leq \eta[M(gx_{n+1}, gx_n, t)] \leq \omega(t)\eta[M(x_n, x_{n-1}, t)].$$

Put $M(x_{n+1}, x_n, t) = \tau_{n+1}(t)$ for all $t > 0, n \in \mathbb{N} \cup \{0\}$. So

$$\eta(\tau_{n+1}(t)) \leq \omega(t)\eta(\tau_n(t)) < \eta(\tau_n(t)). \quad (2)$$

Consequently, $\tau_{n+1}(t) > \tau_n(t)$ and hence $\{\tau_n(t)\}$ is an increasing sequence for all $t > 0$. Thus $\lim_{n\to+\infty} \tau_n(t) = \tau(t)$. Now we show that $\tau(t) = 1$ for all $t > 0$. If not, then there exists $t_0 > 0$ such that $0 < \tau(t_0) < 1$. Taking limit as $n \to \infty$ in (2), we have

$$\eta(\tau(t_0)) \leq \omega(t_0)\eta(\tau(t_0)) < \eta(\tau(t_0)),$$

a contradiction. Now we show that $\{x_n\}$ is a Cauchy sequence. If not, then there exist $\varepsilon \in (0, 1)$ and $t_0 > 0$ such that for all $k \in \mathbb{N}$, there are $m_k, n_k \in \mathbb{N}$, with $m_k \geq n_k \geq k$ such that

$$M(x_{m_k}, x_{n_k}, t_0) \leq 1 - \varepsilon.$$

Assume that m_k is the least such integer exceeding n_k, then we have

$$M(x_{m_k-1}, x_{n_k}, t_0) > 1 - \varepsilon.$$

So for all k,

$$\begin{aligned} 1 - \varepsilon &\geq M(x_{m_k}, x_{n_k}, t_0) \\ &\geq M(x_{m_k-1}, x_{m_k}, t_0) * M(x_{m_k-1}, x_{n_k}, t_0) \\ &> \tau_{m_k}(t_0) * (1 - \varepsilon). \end{aligned} \quad (3)$$

On taking limit as $k \to \infty$ on both sides of (3), we have

$$\lim_{k\to+\infty} M(x_{m_k}, x_{n_k}, t_0) = 1 - \varepsilon.$$

Also

$$M(x_{m_k+1}, x_{n_k+1}, t_0) \geq M(x_{m_k+1}, x_{m_k}, t_0) * M(x_{m_k}, x_{n_k}, t_0) * M(x_{n_k}, x_{n_k+1}, t_0),$$

and

$$M(x_{m_k}, x_{n_k}, t_0) \geq M(x_{m_k}, x_{m_k+1}, t_0) * M(x_{m_k+1}, x_{n_k+1}, t_0) * M(x_{n_k+1}, x_{n_k}, t_0),$$

we get $\lim_{k\to+\infty} M(x_{m_k+1}, x_{n_k+1}, t_0) = 1 - \varepsilon$. From (1), we have

$$M(gx_{m_k+1}, Tx_{m_k}, t_0) = M(A, B, t_0) \text{ and } M(gx_{n_k+1}, Tx_{n_k}, t_0) = M(A, B, t_0).$$

Hence

$$\eta[M(x_{m_k+1}, x_{n_{k+1}}, t_0)] \leq \eta[M(gx_{m_k+1}, gx_{n_{k+1}}, t_0)] \leq \omega(t_0)\eta[M(x_{m_k}, x_{n_k}, t_0)].$$

On taking limit as $k \to \infty$ on both sides of the above inequality, we get $\eta(1-\varepsilon) \leq \omega(t_0)\eta(1-\varepsilon)$. If $\eta(1-\varepsilon) = 0$, then by the definition of η, we have $\varepsilon = 0$, which gives a contradiction. If $\eta(1-\varepsilon) > 0$, then $\omega(t_0) \geq 1$, which is again a contradiction, since $0 < \omega(t_0) < 1$. Thus $\{x_n\}$ is a Cauchy sequence in a closed subset A of a complete fuzzy metric space $(X, M, *)$. Hence there exists some $x \in A$ such that $\lim_{n\to+\infty} M(x_n, x, t) = 1$, for all $t > 0$. Now

$$\begin{aligned} M(gx, B, t) &\geq M(gx, Tx_n, t) \\ &\geq M(gx, gx_{n+1}, t) * M(gx_{n+1}, Tx_n, t) \\ &= M(gx, gx_{n+1}, t) * M(A, B, t) \\ &\geq M(gx, gx_{n+1}, t) * M(gx, B, t). \end{aligned}$$

Thus

$$M(gx, B, t) \geq M(gx, Tx_n, t) \geq M(gx, gx_{n+1}, t) * M(gx, B, t).$$

As g is continuous and the sequence $\{x_n\}$ converges to x, the sequence $\{gx_n\}$ converges to $g(x)$, $M(gx, Tx_n, t) \to M(gx, B, t)$. Since $\{Tx_n\} \subseteq B$, and B is a fuzzy approximately compact with respect to A, $\{Tx_n\}$ has a subsequence which converges to some y in B, therefore $M(gx, y, t) = M(A, B, t)$, and hence $gx \in A_0(t)$. Now $A_0 \subseteq g(A_0)$ implies that $gx = gu$ for some $u \in A_0(t)$. From

$$\eta[M(x, u, t)] \leq \eta[M(gx, gu, t)] = \eta(1) = 0,$$

we have $M(x, u, t) = 1$ which implies that $u = x$. Thus $x \in A_0(t)$. As $T(A_0) \subseteq B_0$, so $M(z, Tx, t) = M(A, B, t)$ for some z in A. By given assumption, we have $\eta[M(gx_{n+1}, z, t)] \leq \omega(t)\eta[M(x_n, x, t)]$ which on taking limit as $n \to \infty$ gives

$$\lim_{n\to\infty} M(gx_n, z, t) = 1.$$

This further implies that $gx = z$. So $M(gx, Tx, t) = M(z, Tx, t) = M(A, B, t)$. Now we show that x is the unique fuzzy best proximity point of T. If not, then there is another point w such that $0 < M(x, w, t) < 1$ for all $t > 0$ and $M(gw, Tw, t) = M(A, B, t)$. Then it follows that

$$\eta[M(x, w, t)] \leq \eta[M(gx, gw, t)] \leq \omega(t)\eta[M(x, w, t)] < \eta[M(x, w, t)].$$

Thus $\eta[M(x, w, t)] < \eta[M(x, w, t)]$, a contradiction. Hence $M(x, w, t) = 1$ for all $t > 0$, that is $w = x$. □

Example 2.9. Let $X = [0,1] \times \mathbb{R}$, and $d : X \times X \to \mathbb{R}$ an Euclidean metric on X. Let $A = \{(x,1) : x \in \mathbb{R}\}$ and $B = \{(x,-1) : x \in \mathbb{R}\}$. Note that $(X, M_d, \cdot)$ is a non-Archimedean complete fuzzy metric space, where M_d an is standard fuzzy metric induced by d. Note that $M_d(A,B,t) = \dfrac{t}{t+2}$, $A_0(t) = A$ and $B_0(t) = B$.

Define a mapping $T : A \to B$ by $T(x,1) = (\frac{x}{2}, -1)$ and $g : A \to A$ by $g(x,1) = (2x,1)$. Clearly g is fuzzy expansive mapping, $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) = g(A_0(t))$.

Let us consider $u = (x_1,1),\ v = (x_2,1) \in A$, then there exists $x = (x_3,1)$ and $y = (x_4,1) \in A$ such that

$$\begin{aligned} M(u,Tx,t) &= M(A,B,t), \\ M(v,Ty,t) &= M(A,B,t), \end{aligned}$$

are satisfied. Solving the above two equations we have $x_1 = \frac{x_3}{2}$ and $x_2 = \frac{x_4}{2}$. Assume that

$$\eta(t) = 1 - t \text{ and } \omega(t) = \frac{t}{t+1}.$$

By condition (ii) of the above theorem,

$$\eta(M(u,v,t)) \leq \omega(t)\eta(M(x,y,t)),$$

holds true, So all conditions of the Theorem (2.8) are satisfied. Moreover $(0,1)$ is the unique element satisfying the conclusion of the theorem.

Following the similar arguments to those given in Theorem (2.8), we can prove the following result.

Corollary 2.10. *Let $(X, M, *)$ be a non-Archimedean complete fuzzy metric space, A and B two nonempty closed subsets of X such that B is approximately compact with respect to A and $T : A \to B$. Suppose that there exists $\eta \in \Omega$ such that*

$$\left.\begin{array}{l} M(u,Tx,t) = M(A,B,t) \\ M(v,Ty,t) = M(A,B,t) \end{array}\right\} \Rightarrow \eta[M(u,v,t)] \leq \omega(t)\eta[(M(x,y,t)],$$

for all $x,y,u,v \in A$ and $t > 0$, where $\omega : (0,+\infty) \to (0,1)$ is any function. If $g : A \to A$ is a fuzzy isometry, $T(A_0(t)) \subseteq B_0(t) \neq \phi$ and $\phi \neq A_0(t) \subseteq g(A_0(t))$ for each $t > 0$. Then, there exists an element $x \in A$ such that $M(gx,Tx,t) = M(A,B,t)$. Further, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ defined by $M(gx_{n+1}, Tx_n, t) = M(A,B,t)$, converges to the unique element x.

Example 2.11. Let $X = [0,1] \times \mathbb{R}$, and $d : X \times X \to \mathbb{R}$ an Euclidean metric on X. Let $A = \{(x,1) : x \in \mathbb{R}\}$ and $B = \{(x,-1) : x \in \mathbb{R}\}$. Note that $M_d(A,B,t) = \dfrac{t}{t+2}$, $A_0(t) = A$ and $B_0(t) = B$.

Define the mapping $T : A \to B$ by $T(x,1) = (-\frac{x}{2}, -1)$ and $g : A \to A$ by $g(x,1) = (-x,1)$. Clearly g is a fuzzy isometric mapping, $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) = g(A_0(t))$.

Let us consider $u = (x_1, 1),\ v = (x_2, 1) \in A$, then there exists $x = (x_3, 1)$ and $y = (x_4, 1) \in A$ such that

$$\begin{aligned} M(u, Tx, t) &= M(A, B, t), \\ M(v, Ty, t) &= M(A, B, t), \end{aligned}$$

are satisfied. Using above two equations we get $x_1 = \frac{-x_3}{2}$ and $x_2 = \frac{-x_4}{2}$. If

$$\eta(t) = 1 - t \text{ and } \omega(t) = \frac{t}{t+1},$$

then by the condition (ii) of the above theorem, we obtain that

$$\eta(M(u, v, t)) \leq \omega(t)\eta(M(x, y, t)).$$

All conditions of the Corollary (2.10) are satisfied. Moreover $(0, 1)$ is the unique element satisfying the conclusion of the corollary.

Corollary 2.12. *Let $(X, M, *)$ be non-Archimedean complete fuzzy metric space, A and B are two nonempty closed subsets of X such that B is approximately compact with respect to A and $T : A \to B$. Suppose that there exists $\eta \in \Omega$ such that*

$$\left.\begin{array}{l} M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array}\right\} \Rightarrow \eta[M(u, v, t)] \leq \omega(t)\eta[(M(x, y, t)],$$

for all $x, y, u, v \in A$ and $t > 0$, where $\omega : (0, +\infty) \to (0, 1)$ is any function. If $A_0(t) \neq \phi$ and $T(A_0(t)) \subseteq B_0(t) \neq \phi$ for each $t > 0$. Then, there exists an element $x \in A$ such that $M(x, Tx, t) = M(A, B, t)$. Further, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ defined by $M(x_{n+1}, Tx_n, t) = M(A, B, t)$, converges to the unique element x.

Proof. The result follows from Theorem (2.8), if $g = I_A$ (an identity mapping on A). □

Theorem 2.13. *Let $(X, M, *)$ be a non-Archimedean complete fuzzy metric space, such that for each $\varepsilon > 0$ and an $s-$ increasing sequence $\{t_n\}$ satisfying property$-T$, A and B are two nonempty closed subsets of X such that B is approximately compact with respect to A and $T : A \to B$. Suppose that there exists $\alpha \in (0, 1)$ such that*

$$\left.\begin{array}{l} M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array}\right\} \Rightarrow M(u, v, t) \geq M(x, y, \frac{t}{\alpha}),$$

for all $u, v, x, y \in A$, and $t > 0$. If $g : A \to A$ is a fuzzy expansive mapping, $T(A_0(t)) \subseteq B_0(t) \neq \phi$ and $\phi \neq A_0(t) \subseteq g(A_0(t))$ for each $t > 0$. Then, there exists an element $x \in A$ such that $M(gx, Tx, t) = M(A, B, t)$. Further, for any fixed element $x_0 \in A_0(t)$, the sequence $\{x_n\}$ defined by $M(gx_{n+1}, Tx_n, t) = M(A, B, t)$, converges to the unique element x.

Proof. Let x_0 be a given point in $A_0(t)$. As in the proof of Theorem (2.8), we can obtain a sequence $\{x_n\}$ in $A_0(t)$, such that

$$M(gx_n, Tx_{n-1}, t) = M(A, B, t), \text{ and } M(gx_{n+1}, Tx_n, t) = M(A, B, t) \tag{4}$$

hold for each positive integer n. By given assumption, it follows that

$$M(gx_n, gx_{n+1}, t) \geq M(x_{n-1}, x_n, \frac{t}{\alpha}).$$

Thus, we have

$$\begin{aligned} M(x_n, x_{n+1}, t) &\geq M(x_{n-1}, x_n, \frac{t}{\alpha}) \geq M(x_{n-2}, x_{n-1}, \frac{t}{\alpha^2}) \geq \\ \cdots &\geq M(x_0, x_1, \frac{t}{\alpha^n}). \end{aligned}$$

So for each $t > 0$ and for all $m, n \in \mathbb{N}$ with $m \geq n$, we have

$$\begin{aligned} M(x_n, x_m, t) &\geq M(x_n, x_{n+1}, t) * M(x_{n+1}, x_{n+2}, t) * \cdots * M(x_{m-1}, x_m, t) \\ &\geq M(x_0, x_1, \frac{t}{\alpha^n}) * M(x_0, x_1, \frac{t}{\alpha^{n+1}}) * \cdots * M(x_0, x_1, \frac{t}{\alpha^{m-1}}) \\ &\geq \prod_{i=n}^{\infty} M(x_0, x_1, \frac{t}{\alpha^i}) = \prod_{i=n}^{\infty} M(x_0, x_1, t_i), \end{aligned}$$

where $t_i = \frac{t}{\alpha^i}$. Since $\lim_{n\to\infty}(t_{n+1} - t_n) = \infty$, therefore $\{t_n\}$ is an $s-$increasing sequence and satisfying property$-T$, there exist some n_0 in $\mathbb{N}$ and $\varepsilon > 0$ such that $\prod_{n=1}^{\infty} M(x_0, x_1, t_n) \geq 1 - \varepsilon$ for all $n \geq n_0$, and hence $M(x_n, x_m, t) \geq 1 - \varepsilon$ for all $n, m \geq n_0$. Thus $\{x_n\}$ is a Cauchy sequence in X. Thus $\{x_n\}$ is a Cauchy sequence in a closed subset A of a complete fuzzy metric space $(X, M, *)$. Hence there exists some $x \in A$ such that $\lim_{n\to+\infty} M(x_n, x, t) = 1$, for all $t > 0$. Now

$$\begin{aligned} M(gx, B, t) &\geq M(gx, Tx_n, t) \\ &\geq M(gx, gx_{n+1}, t) * M(gx_{n+1}, Tx_n, t) \\ &= M(gx, gx_{n+1}, t) * M(A, B, t) \\ &\geq M(gx, gx_{n+1}, t) * M(gx, B, t). \end{aligned}$$

On taking limit as $n \to \infty$, we have $M(gx, Tx_n, t) \to M(gx, B, t)$. Since B is a fuzzy approximately compact with respect to A, $\{Tx_n\}$ has a subsequence which converges to some y in B, so $M(gx, y, t) = M(A, B, t)$, and hence $gx \in A_0$. Using $A_0 \subseteq g(A_0)$, we have $gx = gu$ for some $u \in A_0(t)$. Now $M(x, u, t) \geq M(gx, gu, t) = 1$ implies that $x = u$ and $x \in A_0(t)$. Now $T(A_0(t)) \subseteq B_0(t)$ gives that

$$M(z, Tx, t) = M(A, B, t), \tag{5}$$

for some z in A. From (4) and (5), we have

$$M(gx_{n+1}, z, t) \geq M(x_n, x, \frac{t}{\alpha}).$$

On taking limit $n \to \infty$, we get $\lim_{n\to\infty} M(gx_{n+1}, z, t) \geq \lim_{n\to\infty} M(x_n, x, \frac{t}{\alpha}) = 1$, and hence $\lim_{n\to\infty} M(gx_{n+1}, z, t) = 1$. Also, $\lim_{n\to\infty} M(gx_{n+1}, gx, t) = 1$. Hence $gx = z$ and

$$M(gx, Tx, t) = M(A, B, t) = M(z, Tx, t). \tag{6}$$

If there is another element x^* such that

$$M(gx^*, Tx^*, t) = M(A, B, t). \tag{7}$$

Then from (6) and (7), we obtain

$$M(x, x^*, t) \geq M(gx, gx^*, t) \geq M(x, x^*, \frac{t}{\alpha}),$$

which further implies that

$$M(x, x^*, t) \geq M(x, x^*, \frac{t}{\alpha}),$$

a contradiction. Hence the result follows. □

Example 2.14. Let $X = [0, 1] \times \mathbb{R}$, and $d : X \times X \to \mathbb{R}$ is a metric on X defined below. Let $A = \{(0, x) : x \in \mathbb{R}\}$ and $B = \{(1, x) : x \in \mathbb{R}\}$. Let $(X, M, *)$ is a complete non-Archimedean fuzzy metric space, where $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $t > 0$, under product $t - norm$. Where

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

Note that $M(A, B, t) = \frac{t}{t+1}$, $A_0(t) = A$ and $B_0(t) = B$. Define $T : A \to B$ by $T(0, x) = (1, \frac{x}{5})$. Obviously $T(A_0(t)) \subseteq B_0(t)$. Define $g : A \to A$ by $g(0, x) = 10(0, x)$. Then g is a fuzzy expansive mapping and $A_0(t) = g(A_0(t))$. If $u = (0, x_1)$, $v = (0, x_2)$, $x = (0, x_3)$ and $y = (0, x_4)$ are such that the following equations are satisfied:

$$\begin{aligned} M(u, Tx, t) &= M(A, B, t), \\ M(v, Ty, t) &= M(A, B, t), \end{aligned}$$

then we get $x_1 = \frac{x_3}{5}$ and $x_2 = \frac{x_4}{5}$. Note that the condition

$$M(u, v, t) \geq M(x, y, \frac{t}{\alpha})$$

in Theorem (2.13) holds true for all $\alpha \geqslant \frac{1}{5}$. In particular choose $\alpha = \frac{2}{5}$. Moreover $(0, 0)$ is the unique element satisfying the conclusion of the theorem.

Corollary 2.15. *Let $(X, M, *)$ be a non-Archimedean complete fuzzy metric space, such that for each $\varepsilon > 0$ and an $s-$ increasing sequence $\{t_n\}$ satisfying property$-T$, A and B are two nonempty closed subsets of X such that B is approximately compact with respect to A and $T : A \to B$. Suppose that there exist $\alpha \in (0, 1)$ such that*

$$\left.\begin{array}{l} M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array}\right\} \Rightarrow M(u, v, t) \geq M(x, y, \frac{t}{\alpha}),$$

for all $u, v, x, y \in A$, and $t > 0$. If $g : A \to A$ is a fuzzy isometry, $T(A_0(t)) \subseteq$

$B_0(t) \neq \phi$ and $\phi \neq A_0(t) \subseteq g(A_0(t))$ for each $t > 0$. Then, there exists an element $x \in A$ such that $M(gx, Tx, t) = M(A, B, t)$. Further, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ defined by $M(gx_{n+1}, Tx_n, t) = M(A, B, t)$ converges to the unique element x.

Example 2.16. Let $X = [0, 1] \times \mathbb{R}$, and $d : X \times X \to \mathbb{R}$ is usual metric on X. Let $A = \{(0, x) : x \in \mathbb{R}\}$ and $B = \{(1, x) : x \in \mathbb{R}\}$.

Note that $M(A, B, t) = \frac{t}{t+1}$, $A_0(t) = A$ and $B_0(t) = B$. Define $T : A \to B$ by $T(0, x) = (1, \frac{x}{5})$. Obviously, $T(A_0(t)) \subseteq B_0(t)$. Define $g : A \to A$ by $g(0, x) = (0, -x)$. Then g is a fuzzy isometry and $A_0(t) = g(A_0(t))$. If $u = (0, x_1)$, $v = (0, x_2)$, $x = (0, x_3)$ and $y = (0, x_4)$ satisfy

$$\begin{aligned} M(u, Tx, t) &= M(A, B, t), \\ M(v, Ty, t) &= M(A, B, t). \end{aligned}$$

Then $x_1 = \frac{x_3}{5}$ and $x_2 = \frac{x_4}{5}$. Also

$$M(u, v, t) \geq M(x, y, \frac{t}{\alpha})$$

holds true for all $\alpha \geqslant \frac{1}{5}$. In particular choose $\alpha = \frac{2}{5}$. Moreover $(0, 0)$ is the unique element satisfying the conclusion of the theorem.

Corollary 2.17. *Let $(X, M, *)$ be non-Archimedean complete fuzzy metric space, such that for each $\varepsilon > 0$ and an $s-$ increasing sequence $\{t_n\}$ satisfying property$-T$, A and B are two nonempty closed subsets of X such that B is approximately compact with respect to A and $T : A \to B$. Suppose that there exist $\alpha \in (0, 1)$ such that*

$$\left.\begin{aligned} M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{aligned}\right\} \Rightarrow M(u, v, t) \geq M(x, y, \frac{t}{\alpha}),$$

for all $u, v, x, y \in A$, and $t > 0$. If $T(A_0(t)) \subseteq B_0(t) \neq \phi$ and $A_0(t) \neq \phi$ for each $t > 0$. Then, there exists an element $x \in A$ such that $M(x, Tx, t) = M(A, B, t)$. Further, for any fixed element $x_0 \in A_0(t)$, the sequence $\{x_n\}$ defined by $M(x_{n+1}, Tx_n, t) = M(A, B, t)$, converges to the unique element x.

Proof. The result follows from Theorem (2.13), if $g = I_A$ (an identity mapping on A). □

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