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Exact solution of the Mindlin-Herrmann model for longitudinal vibration of an isotropic rod

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Abstract This paper presents a new approach to the problem of coupled longitudinal and transversal propagation of stress waves in an isotropic thick and elastic rod, based on the Mindlin-Herrmann theory. The novelty is that, the Hamilton vriational principle is used not only for derivation of the governing equations and set of natural boundary conditions, but also for obtaining the exact solution in terms Green functions directly from the Langrangian. The success of this approach is based on the existence of multiple orthogonalities of the eigenfuctions. The proposed method is much easier than the standard approach of building Green functions. A numerical example illustrates the method of finding eigenfrequencies and eigenfunctions for isotropic Mindlin-Herrmann rod.

Keywords eigenfunction orthogonalities · Green function · longitudinal-transversal vibration · Mindlin-Herrmann model · self-adjoint operator

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1 Introduction

Modern theories of one-dimensional rod vibrations take into account the lateral effects, which are substantial in the case of relatively thick rods. For example, in the Rayleigh-Love [1–3] and Rayleigh-Bishop [4,5] models, the lateral displacements are supposed to be proportional to the product of longitudinal strain of the rod, its Poisson ratio and the distance from the neutral line of the cross-section. Lack of physical clarity in the interpretation of certain higher effects, such as independent shear displacement and radial motion which describe transverse deformation have also been associated with these approaches [2]. The theory of longitudinal stress wave propagation in an elastic rod which couples axial and independent lateral displacements was first established by Mindlin and Herrmann in 1950 [6] and later on, was developed in more details by Graff in his book [7]. Krishnaswamy and Batra have investigated the same model with various boundary conditions and shown that the analysis of the corresponding frequency spectrum can be considered in three situations, depending on the domain in which the eigenfrequency belongs to [8–10]. Recently Krawczuk, Grabowska and Palacz, Zak and Krawczuk, and Anderson [2,11, 12] in their work on different theories of longitudinal vibrations of rods have analysed the frequency equation of each theory including the one of Mindlin and Herrmann, also providing a comparison regarding their accuracy and applicability. These authors have based their approach to wave propagation in a elastic continuum medium only on the analysis of the frequency equation, which shows the relationship between the governing factors of the phenomena, these being: time frequency, spacial frequency or wave number and phase velocity.

In the Mindlin-Herrmann model, which is the main focus of this paper, the lateral displacements are independent of the longitudinal strain and the Poisson ratio. They are described by the product of an unknown function, called transverse deformation and the distance from the neutral line of the rod. Compared to single mode models, this representation increases the number of unknown functions and hence, the model is described by two partial differential equations. Consequently, the model is more accurate as it has wider frequency range in which the effect of longitudinal vibration of the rod can be analysed [2, 12, 13]. Nevertheless, on needs to be mindful of the fact that, similar to Classical, Rayleigh-Love, Rayleigh-Bishop models, the Mindlin-Herrman model is based on plane cross-section theory. Therefore, it is not suitable to high frequency vibrations where the effect of the cross-section deplanation is substantial. In our approach, the derivation of the system of equations of motion is based on an application of the energy and variational method in the process of which the associated natural and essential boundary conditions are automatically obtained. In the course of this work, the multiple orthogonalities method for vibration problems in [14] is applied to solve the Mindlin-Herrmann model analytically in terms of Green functions. The great advantage of using Green functions lies in the fact that, it allows for analysis of the influences of the initial conditions on the wave propagation. It is necessary to emphasise that our approach for deriving Green functions is different from the various methods available in the literature, which either use Driac delta function or a single orthogonality condition of the eigenfunctions [15–17]. The originality of the method is that, it combines the Ritz method and the variational principle. It consists of two main steps. Firstly we prove two orthogonality conditions of the system of eigenfunctions of the corresponding Sturm-Liouville problem. Secondly, assuming that the general solution of the problem can be decomposed in the form of a Fourier series with respect to the eigenfunction system, this can then be substituted into the Lagrangian, and after this by applying the orthogonality conditions of the eigenfunction and the corresponding norms, the simple form of the Lagrangian is obtained which holds the Euler-Lagrange equation. The solution of the resulting time-ordinary differential equation is substituted back into the assumed solution to construct the Green function that is equivalent to obtaining the analytical (exact) solution of the problem. It is important to stress that the two orthogonality conditions arise naturally from the Lagrangian. The physical meaning of the first orthogonality consists of the orthogonality of eigen-velocities of the deformations involved in the expression of kinetic energy. The physical meaning of the second orthogonality consists of the orthogonality of two stress-strain terms involved in the expression of strain energy. Thus the first and the second orthogonalities originate respectively from the inertial forces and the strain energy, which jointly describe the elastic forces in the system . Here it is also necessary to emphasise that exact solutions of models including those for vibrations of distributed structures are important for both theoretical and numerical analysis of these models. In the later one these solutions form the reference results for testing the accuracy of the numerical algorithms using for example finite difference and finite element method. The Mindlin-Herrmann model and proposed approach, based on two orthogonalities of the eigenfunctions could also be applied to composite rods, because it does not need the Poisson ratio as in the Rayleigh-Love and Rayleigh-Bishop models.

The main theoretical results of the paper are as follows: the formulation and proof of two kinds of orthogonality conditions of the eigenfunctions; the method of obtaining the exact solution of the system of partial differential equations in terms of Green's functions using the Lagrangian functional of the system; transformation of the differential operator of the Sturm-Liouville problem to self-adjoint form; proof of the positivity of the operator and the corresponding eigenvalues. The self-adjoint properties of the operator are used as an alternative technique to prove the eigenfunction multiple orthogonality conditions.

The content of this paper is arranged in the following way: Section 2 presents the derivation of the system of equations of motion with the associated natural boundary conditions. Section 3 deals with free vibrations and the corresponding Sturm-Liouville problem is investigated. In Section 4 the two orthogonality conditions of the eigenfunctions are established. The Green's functions are derived in Section 5. Section 6 is devoted to the numerical simulation of the model and a critical comparison of Mindlin-Herrmann and Classical model. In order for the paper to be more self-contained, the derivation of the

Mindlin-Herrmann model is shown in Appendix A and the differential operator in the Sturm-Liouville problem and its properties are discussed in Appendix B.

2 Mindlin-Herrmann model

The aim of this section is to formulate the two mode Mindlin-Herrmann model and set notations that will be use throughout this paper. The model is presented in [2] and [7,11] in Cartesian and cylindrical coordinate respectively and used in [13,19] as improvement of the single mode classical, Rayleigh-Love and Rayleigh-Bishop models. According to Mindlin-Herrmann theory of longitudinal stress wave propagation, the axial displacements u and the transverse (lateral) displacements v and w are assumed to be function of the form:

$$u = u_1(x,t), \quad v = v(x,y,t) = yu_2(x,t), \quad w = w(x,z,t) = zu_2(x,t) \tag{1}$$

where $x \in D = (0,l)$ is the axial distance along the rod of length l, y and z are the lateral distance from the x-axis (neutral line), $t \geq 0$ is the time. Here u_2 is the transverse deformation. Using Hamilton variational principle the equation of motion or Mindlin-Herrmann model for vibrating isotropic rod is obtained [7]:

$$\begin{cases} \rho\ddot{u}_1 - (\lambda+2\mu)u_1'' - 2\lambda u_2' = f(x,t) \\ \rho I_p\ddot{u}_2 + 4(\lambda+\mu)Su_2 - \mu I_p u_2'' + 2S\lambda u_1' = 0 \end{cases} \tag{2}$$

with the following essential boundary conditions at fixed end

$$u_1|_{x=0,l} = 0, \quad u_2|_{x=0,l} = 0, \tag{3}$$

and natural boundary conditions at free ends

$$(\lambda+2\mu)u_1' + 2\lambda u_2|_{x=0,l} = 0, \quad \mu I_p u_2'|_{x=0,l} = 0, \tag{4}$$

where the upper dot and the prime denoted the derivative with respect to time and x respectively, λ and μ are Lame's constant defined by $\lambda = \frac{E\eta}{(1-2\eta)(1+\eta)}$ and $\mu = \frac{E}{2(1+\eta)}$, in which E is the Young modulus of elasticity, ρ is the mass density, $S = \int_{(s)} ds$ is the cross-sectional area, $I_p = \int_{(s)} \left(y^2+z^2\right) ds$ is the

Polar moment of inertial, η is the Poisson ratio and $f = f(x,t)$ is the external distributed force.

Remark 1 The system of equation (2) can be solved mathematically with any combination of four of the above eight boundary conditions but not all of them have a physical meaning.

We consider the following mix boundary conditions (no longitudinal displacement and no transversal force are applied at the ends of the rod) in the rest of the paper:

$$u_1|_{x=0,l} = 0, \quad u_2'|_{x=0,l} = 0, \tag{5}$$

In addition to Equations (2) and (5) it is necessary to state the initial conditions to obtain a unique solution:

$$u_1|_{t=0} = g(x), \quad \frac{\partial u_1}{\partial t}|_{t=0} = h(x), \quad u_2|_{t=0} = \phi(x), \quad \frac{\partial u_2}{\partial t}|_{t=0} = \varphi(x). \tag{6}$$

More details on the derivation of Eqs. (2)-(4) are given in Appendix A.
In what follows, system of Eq. (2) and associated boundary conditions (5) will be transformed into matrix form, which will simplify the process of finding of the solution. Multiplying the first and second equations of system (2) by Sa and b (a and b are non zero arbitrary constants), respectively and dividing both equation by ρ leads to:

$$\begin{cases} Sa\ddot{u}_1 - \dfrac{(\lambda+2\mu)Sa}{\rho}u_1'' - \dfrac{2\lambda Sa}{\rho}u_2' = \dfrac{Sa}{\rho}f(x,t) \\ bI_p\ddot{u}_2 + \dfrac{2\lambda Sb}{\rho}u_1' - \dfrac{\mu I_p b}{\rho}u_2'' + \dfrac{4(\lambda+\mu)Sb}{\rho}u_2 = 0 \end{cases} \tag{7}$$

Letting

$$u_1 = \frac{1}{aS}v_1, \quad u_2 = \frac{1}{bI_p}v_2, \quad f_1(x,t) = \frac{Sa}{\rho}f(x,t), \; q^2 = \frac{b^2}{a^2} = \frac{S}{I_p}, \tag{8}$$

the system of equations (7) becomes:

$$\begin{cases} \ddot{v}_1 - \dfrac{(\lambda+2\mu)}{\rho}v_1'' - \dfrac{2\lambda q}{\rho}v_2' = f_1(x,t) \\ \ddot{v}_2 + \dfrac{2\lambda q}{\rho}v_1' - \dfrac{\mu}{\rho}v_2'' + \dfrac{4(\lambda+\mu)q^2}{\rho}v_2 = 0 \end{cases} \tag{9}$$

The system of equations of motion (9) can be written in the following matrix form:

$$\ddot{\mathbf{v}} - A\mathbf{v} = \mathbf{f}, \tag{10}$$

where

$$\mathbf{v}(x,t) = \begin{pmatrix} v_1(x,t) \\ v_2(x,t) \end{pmatrix},\ A = \begin{pmatrix} a_{11}\dfrac{\partial^2}{\partial x^2} & a_{12}\dfrac{\partial}{\partial x} \\ -a_{12}\dfrac{\partial}{\partial x} & a_{22}\dfrac{\partial^2}{\partial x^2} - c_{22} \end{pmatrix},\ \mathbf{f}(x,t) = \begin{pmatrix} f_1(x,t) \\ 0 \end{pmatrix} \tag{11}$$

and

$$a_{11} = \frac{(\lambda + 2\mu)}{\rho},\ \ a_{12} = \frac{2\lambda q}{\rho},\ \ a_{22} = \frac{\mu}{\rho},\ c_{22} = \frac{4(\lambda+\mu)q^2}{\rho}. \tag{12}$$

Using notations (8) and (12), the Lagrangian (A.8) can be rewritten in the following form

$$L = \frac{\rho I_p}{2}\int_0^l \left(q^2\dot{u}_1^2 + \dot{u}_2^2 - a_{11}q^2u_1'^2 - 2a_{12}qu_1'u_2 - c_{22}u_2^2 - a_{22}u_2'^2 + \frac{2}{\rho}q^2 f u_1 \right) dx \tag{13}$$

The new form of the boundary conditions (5) is:

$$B\mathbf{v}|_{\Gamma} = 0 \tag{14}$$

where

$$B = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\partial}{\partial x} \end{pmatrix} \tag{15}$$

is the boundary differential operator and $\Gamma = \{0, l\}$.

3 Free vibration of the Mindlin-Herrmann rod: the Sturm-Liouville problem

The objective of this section is to derive the solution of the Sturm-Liouville problem corresponding to Mindlin-Herrmann model. In fact the double orthogonality conditions of the obtained eigenfunctions (to be proven in section 4) will help to build the Green's functions for the model. Let us assume a harmonic vibration of the rod and seek the solution of Eq. (10) with $\mathbf{f} = \mathbf{0}$, in the form:

$$\mathbf{v} = \mathbf{V}(x)e^{i\omega t} \tag{16}$$

where $\mathbf{V}(x) = \begin{pmatrix} V_1(x) \\ V_2(x) \end{pmatrix}$, $i^2 = -1$ and ω is the angular frequency.
From formula (8) we set

$$\mathbf{v} = T\mathbf{u} \tag{17}$$

where $T = \begin{pmatrix} aS & 0 \\ 0 & bI_p \end{pmatrix}$and $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.
As $T\mathbf{u}(x,t) = \mathbf{V}(x)e^{i\omega t}$, this implies that $\mathbf{u}(x,t) = T^{-1}\mathbf{V}(x)e^{i\omega t}$ and thus we can set

$$\mathbf{U}(x) = T^{-1}\mathbf{V}(x) \text{ or } \mathbf{V}(x) = T\mathbf{U}(x), \tag{18}$$

where $\mathbf{U}(x) = \begin{pmatrix} U_1(x) \\ U_2(x) \end{pmatrix}$.
Substituting expression (16) into Eqs. (10) and (14) leads to the following Sturm-Liouville problem:

$$A\mathbf{V} + \omega^2\mathbf{V} = \mathbf{0} \tag{19}$$

with the associated mix boundary conditions for fixed ends and free shear force at the ends respectively,

$$B\mathbf{V}|_\Gamma = 0, \tag{20}$$

where A and B (total differential operator) are the same operator as above with $\frac{\partial}{\partial x}$ replaced by $\frac{d}{dx}$.

Remark 2 The Sturm-Liouville problem (19)-(20) is defined by a system of differential equations. Despite the fact that there is a lack of theory of general Sturm-Liouville (especially if operator A and B depend on the spectral parameter ω), it is sometimes possible to solve such a problem. Some examples of Sturm-Liouville problems for a system can be found in [24].

In this specific case, the solution $\mathbf{V}(x)$ can be sought in the form:

$$\mathbf{V}(x) = \mathbf{p}e^{\gamma x} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} e^{\gamma x} \tag{21}$$

where $\mathbf{p}$ is the non-zero constant vector amplitude and

$$\gamma = \gamma(\omega) = \alpha(\omega) + ik(\omega) \tag{22}$$

is the propagation coefficient, in which $\alpha(\omega)$ and $k(\omega)$ are the attenuation coefficient and wave number, respectively.
Substituting expression (21) into (16) leads to

$$\mathbf{v}(x,t) = \mathbf{p}e^{\gamma x + i\omega t}, \tag{23}$$

therefore $\mathbf{v}(x,t)$ is sought in the form of travelling wave. Substituting expression (21) into Eq. (19) yields a homogeneous system of two equations of two unknowns p_1 and p_2

$$\tilde{A}\mathbf{p} + \omega^2\mathbf{p} = \mathbf{0} \tag{24}$$

where

$$\tilde{A} = \begin{pmatrix} \gamma^2 a_{11} & \gamma a_{12} \\ -\gamma a_{12} & \gamma^2 a_{22} - c_{22} \end{pmatrix}. \tag{25}$$

The characteristic equation of the determination of the non-trivial solution of system (24) is

$$\gamma^4 a_{11}a_{22} + \gamma^2\left[a_{12}^2 + \omega^2\left(a_{22}+a_{11}\right) - c_{22}a_{11}\right] - \omega^2\left(c_{22}-\omega^2\right) = 0 \tag{26}$$

or

$$\gamma^4\vartheta + \gamma^2\beta - \zeta = 0 \tag{27}$$

where $\vartheta = a_{11}a_{22}$, $\beta = \beta(\omega) = a_{12}^2 + \omega^2\left(a_{22}+a_{11}\right) - c_{22}a_{11}$, $\zeta = \zeta\left(\omega\right) = \omega^2\left(c_{22}-\omega^2\right)$.
Solving Eq. (27) for γ^2 gives

$$\gamma_1^2 = \gamma_1^2\left(\omega\right) = \frac{\sqrt{\beta^2+4\vartheta\zeta}-\beta}{2\vartheta}, \quad \gamma_2^2 = \gamma_2^2\left(\omega\right) = \frac{-\sqrt{\beta^2+4\vartheta\zeta}-\beta}{2\vartheta}. \tag{28}$$

It is noticed that γ_1 is real and γ_2 is real or purely imaginary (in this case all the input energy contributes only to the activation of the lateral motion). The fundamental system of the solutions of Eq. (19) is:

$$\{e^{\gamma_1 x},\ e^{-\gamma_1 x},\ e^{\gamma_2 x},\ e^{-\gamma_2 x}\}. \tag{29}$$

Thus

$$v_1(x) = c_1e^{\gamma_1 x} + c_2e^{-\gamma_1 x} + c_3e^{\gamma_2 x} + c_4e^{-\gamma_2 x} \tag{30}$$

and

$$v_2(x) = c_5 e^{\gamma_1 x} + c_6 e^{-\gamma_1 x} + c_7 e^{\gamma_2 x} + c_8 e^{-\gamma_2 x} \tag{31}$$

where c_5, c_6, c_7, c_8 are the constants which can be expressed in terms of c_1, c_2, c_3, c_4. Substituting Eqs. (30)-(31) into the first equation of the system (19) gives

$$\begin{array}{r} \{(a_{11}\gamma_1^2+\omega^2)c_1 + a_{12}\gamma_1 c_5\}e^{\gamma_1 x} + \{(a_{11}\gamma_1^2+\omega^2)c_2 - a_{12}\gamma_1 c_6\}e^{-\gamma_1 x} + \\ +\{(a_{11}\gamma_2^2+\omega^2)c_3 + a_{12}\gamma_2 c_7\}e^{\gamma_2 x} + \{(a_{11}\gamma_2^2+\omega^2)c_4 - a_{12}\gamma_2 c_8\}e^{-\gamma_2 x} = 0 \end{array} . \tag{32}$$

Since the exponential functions in Eq. (32) are linearly independent, one can assert that

$$\begin{array}{r} (a_{11}\gamma_1^2+\omega^2)c_1 + a_{12}\gamma_1 c_5 = 0 \\ (a_{11}\gamma_1^2+\omega^2)c_2 - a_{12}\gamma_1 c_6 = 0 \\ (a_{11}\gamma_2^2+\omega^2)c_3 + a_{12}\gamma_2 c_7 = 0 \\ (a_{11}\gamma_2^2+\omega^2)c_4 - a_{12}\gamma_2 c_8 = 0. \end{array} \tag{33}$$

Thus we obtain

$$\begin{array}{ll} c_5 = -\dfrac{a_{11}\gamma_1^2+\omega^2}{a_{12}\gamma_1}c_1, & c_6 = \dfrac{a_{11}\gamma_1^2+\omega^2}{a_{12}\gamma_1}c_2 \\ c_7 = -\dfrac{a_{11}\gamma_2^2+\omega^2}{a_{12}\gamma_2}c_3, & c_8 = \dfrac{a_{11}\gamma_2^2+\omega^2}{a_{12}\gamma_2}c_4 \end{array} \tag{34}$$

Remark 3 The relationship between c_1, c_2, c_3, c_4 and c_5, c_6, c_7, c_8 respectively can also be obtained by substituting Eqs. (30)-(31) into the second equation of the system (19). These relationships are derived in a form which is different from (34) but equivalent for γ_1 and γ_2 being solutions of (26).

After substituting (34) into (30), the boundary condition (20) for the functions v_1 and v_2 given in (29)-(30) results in the following homogeneous system for c_1, c_2, c_3 and c_4:

$$\begin{array}{r} c_1 + c_2 + c_3 + c_4 = 0 \\ (a_{11}\gamma_1^2+\omega^2)(c_1+c_2) + (a_{11}\gamma_2^2+\omega^2)(c_3+c_4) = 0 \\ c_1 e^{\gamma_1 l} + c_2 e^{-\gamma_1 l} + c_3 e^{\gamma_2 l} + c_4 e^{-\gamma_2 l} = 0 \\ (a_{11}\gamma_1^2+\omega^2)(c_1 e^{\gamma_1 l} + c_2 e^{-\gamma_1 l}) + (a_{11}\gamma_2^2+\omega^2)(c_3 e^{\gamma_2 l} + c_4 e^{-\gamma_2 l}) = 0 \end{array} . \tag{35}$$

The existence of a non trivial solution of the homogeneous system (35) requires that its coefficient matrix $D(\omega)$ is singular, that is,

$$|D(\omega)| = 0. \tag{36}$$

The transcendental equation (36) has many roots and can be solved in different ways. The numerical example in Section 6 use the method in [23] for approximating ω. Note that Eq. (36) is also the characteristic equation for the Sturm-Liouville problem (19)-(20). Hence the roots $\omega_n, n = 1, 2, \cdots$ of (36) are the eigenvalues of (19)-(20). The corresponding eigenfunctions are

$$v_{1n}(x, \omega_n) = c_1 e^{\gamma_1(\omega_n)x} + c_2 e^{-\gamma_1(\omega_n)x} + c_3 e^{\gamma_2(\omega_n)x} + c_4 e^{-\gamma_2(\omega_n)x} \tag{37}$$

and

$$\begin{aligned} v_{2n}(x, \omega_n) = & \frac{a_{11}\gamma_1^2(\omega_n) + \omega_n^2}{a_{12}\gamma_1(\omega_n)} (c_2 e^{-\gamma_1(\omega_n)x} - c_1 e^{\gamma_1(\omega_n)x}) \\ & + \frac{a_{11}\gamma_2^2(\omega_n) + \omega_n^2}{a_{12}\gamma_2(\omega_n)} (c_4 e^{-\gamma_2(\omega_n)x} - c_3 e^{\gamma_2(\omega_n)x}). \end{aligned} \tag{38}$$

The solution of the system (35) is obtained by choosing the value of one constant (c_4 for example) and the remaining three unknowns are obtained by the substitution method.

4 Orthogonalities of the eigenfunctions

In this section we prove two kinds of orthogonality properties of the eigenfunctions of the Sturm-Liouville problem (19)-(20) in $L_2(D)$ and $H_1(D) = W_2^1$ space respectively. The first orthogonality condition is well-known in the theory of spectral and Sturm-Liouville problems [20–22] and it is derived here just to show the similarity with the technique of getting the second orthogonality which is not well-documented in the literature. Both orthogonalities are the corner stone in our new approach of finding the analytical solution of the Mindlin-Herrmann model.

Let $\mathbf{V}_n = \begin{pmatrix} V_{1n} \\ V_{2n} \end{pmatrix}$ and $\mathbf{V}_m = \begin{pmatrix} V_{1m} \\ V_{2m} \end{pmatrix}$ be two distinct eigenfunctions corresponding respectively to different eigenvalues ω_n and ω_m satisfying Eqs. (19)-(20)

$$A\mathbf{V}_n + \omega_n^2 \mathbf{V}_n = \mathbf{0}, \tag{39}$$

$$A\mathbf{V}_m + \omega_m^2 \mathbf{V}_m = \mathbf{0}. \tag{40}$$

Remark 4 The operator A is self-adjoint on the class of functions $C^2(D)$ satisfying boundary condition (20). This is discused further in Appendix B.

Multiplying Eqs. (39) and (40) by $\mathbf{V}_m$ and $\mathbf{V}_n$ (inner product) respectively gives

$$(A\mathbf{V}_n, \mathbf{V}_m) + (\omega_n^2 \mathbf{V}_n, \mathbf{V}_m) = 0, \tag{41}$$

$$(A\mathbf{V}_m, \mathbf{V}_n) + (\omega_m^2 \mathbf{V}_m, \mathbf{V}_n) = 0, \tag{42}$$

Subtracting Eq. (41) from Eq. (42) and using the self-adjoint property of the operator A, leads to

$$(\mathbf{V}_n, \mathbf{V}_m) = \int_0^l [V_{1n}V_{1m} + V_{2n}V_{2m}]dx = 0 \quad \text{for} \quad n \neq m \tag{43}$$

To express the orthogonality in terms of the original vector $\mathbf{U}$ (indirectly $\mathbf{u}$), we substitute formula (18) into Eq.(43), so the resulting equation gives the first orthogonality condition

$$(\mathbf{U}_n, \mathbf{U}_m)_1 = \int_0^l [SU_{1n}U_{1m} + I_p U_{2n}U_{2m}]dx = 0 \quad \text{for} \quad n \neq m \tag{44}$$

or

$$\int_0^l [SU_{1n}U_{1m} + I_p U_{2n}U_{2m}]dx = (\mathbf{U}_n, \mathbf{U}_m)_1 \delta_{nm} \tag{45}$$

where δ_{nm} is the Kronecker symbol.
The corresponding square norm can be written in the following form

$$\|\mathbf{U}_n\|_1^2 = \int_0^2 (SU_{1n}^2 + I_p U_{2n}^2)dx \tag{46}$$

In order to prove the second orthogonality, we multiply Eq. (39) and Eq. (40) by $\omega_m^2 \mathbf{V}_m$ and $\omega_n^2 \mathbf{V}_n$ (inner product) respectively leading to

$$(\omega_m^2 A\mathbf{V}_n, \mathbf{V}_m) + (\omega_n^2 \mathbf{V}_n, \omega_m^2 \mathbf{V}_m) = 0 \tag{47}$$

$$(\omega_n^2 A\mathbf{V}_m, \mathbf{V}_n) + (\omega_m^2 \mathbf{V}_m, \omega_n^2 \mathbf{V}_n) = 0 \tag{48}$$

Subtracting Eq. (47) from Eq. (48) and using the self-adjoint property of the operator A, gives

$$(A\mathbf{V}_n, \mathbf{V}_m) = \int_0^l A\mathbf{V}_n \cdot \mathbf{V}_m dx = 0. \tag{49}$$

Substituting expression (18), into (49) and multiplying the resulting equation by -1 yields

$$\int_0^l \left(-(aS)^2 a_{11} \frac{d^2U_{1n}}{dx^2} U_{1m} - (bI_p)^2 a_{22} \frac{d^2U_{2n}}{dx^2} U_{2m} - (abSI_p)a_{12} \frac{dU_{2n}}{dx} U_{1m} \right) dx + \int_0^l \left((abSI_p)a_{12} \frac{dU_{1n}}{dx} U_{2m} + (bI_p)^2 c_{22} U_{2n} U_{2m} \right) dx = 0 \tag{50}$$

Integrating by parts the first integral of Eq. (50), afterwards dividing the resulting equation by a^2SI_p and applying boundary conditions (20), leads to the second orthogonality condition

$$(\mathbf{U}_n, \mathbf{U}_m)_2 = \int_0^l \left(q^2 a_{11} U'_{1n} U'_{1m} + q a_{12} (U_{2n} U'_{1m} + U_{2m} U'_{1n}) + a_{22} U'_{2n} U'_{2m} + c_{22} U_{2n} U_{2m} \right) dx = 0 \quad \texttt{for}\ n \neq m \tag{51}$$

or

$$\int_0^l \left(q^2 a_{11} U'_{1n} U'_{1m} + q a_{12} (U_{2n} U'_{1m} + U_{2m} U'_{1n}) + a_{22} U'_{2n} U'_{2m} + c_{22} U_{2n} U_{2m} \right) dx = (\mathbf{U}_n, \mathbf{U}_m)_2 \delta_{nm} \tag{52}$$

where

$$(\mathbf{U}_n, \mathbf{U}_n)_2 = \|\mathbf{U}_n\|_2^2 = \int_0^l \left(q^2 a_{11} U'^2_{1n} + 2q a_{12} U_{2n} U'_{1n} + a_{22} U'^2_{2n} + c_{22} U^2_{2n} \right) dx \tag{53}$$

Remark 5 It is also possible to obtain the same kind of orthogonalities, by using any combination of two of the four boundary conditions (A14)-(A15) (see Appendix A).

5 Solution of the problem: Green's function

5.1 Solution of the Midlin-Herrmann model

Assume that the solution of the inhomogeneous system of the initial boundary problem (2)-(6) can be written as a Fourier series expansion with respect to the eigenfunction system $\left\{\begin{pmatrix} U_{1n} \\ U_{2n} \end{pmatrix}\right\}_{n=1}^{\infty}$ which is collinear to $\left\{\begin{pmatrix} V_{1n} \\ V_{2n} \end{pmatrix}\right\}_{n=1}^{\infty}$,

$$u_1(x,t) = \sum_{n=1}^{\infty} U_{1n}(x)\Phi_n(t), \quad u_2(x,t) = \sum_{n=1}^{\infty} U_{2n}(x)\Phi_n(t) \tag{54}$$

where the unknown function $\Phi_n(t)$ need to be determined.

Substituting expression (54) into the Lagrange functional (13) gives

$$\begin{aligned} L = &\frac{1}{2}\sum_{n=1}^{\infty} \dot{\Phi}_n^2 \rho \int_0^l \left\{SU_{1n}^2 + I_p S U_{2n}^2\right\} dx \\ &+ \sum_{n=1}^{\infty} \dot{\Phi}_n \dot{\Phi}_m \rho \int_0^l \left\{SU_{1n}U_{1m} + I_p U_{2n}U_{2m}\right\} dx \\ &- \frac{1}{2}\rho I_p \sum_{n=1}^{\infty} \Phi_n^2 \int_0^l \left\{c_{22}U_{2n}^2 + a_{22}(U'_{2n})^2 + q^2 a_{11}(U'_{1n})^2 + 2qa_{12}U'_{1n}U_{2n}\right\} dx \\ &- \rho I_p \sum_{n<m}^{\infty} \Phi_n \Phi_m \int_0^l \left\{c_{22}U_{2n}U_{2m} + a_{22}U'_{2n}U'_{2m} + q^2 a_{11}U'_{1n}U'_{1m}\right\} dx \\ &- \rho I_p \sum_{n<m}^{\infty} \Phi_n \Phi_m \int_0^l qa_{12}(U'_{1n}U_{2m} + U'_{1m}U_{2n})dx + \sum_{n=1}^{\infty} S\Phi_n \int_0^l f(x,t)U_{1n}dx \end{aligned} \tag{55}$$

Using orthogonality conditions (44) and (51) and their associated square norm formulas (46) and (53), equation becomes:

$$L = \sum_{n=1}^{\infty} L_n, \tag{56}$$

where

$$L_n = \frac{1}{2}\left\{\rho\dot{\Phi}_n^2 \|\mathbf{U}_n\|_1^2 - \Phi_n^2 \rho I_p \|\mathbf{U}_n\|_2^2 + 2\Phi_n S \int_0^l f(x,t)U_{1n}dx\right\}. \tag{57}$$

From the variational principle, the Lagrangian (56) satisfies the system of Euler-Lagrange differential equations (A.9) (see Appendix A), [25]. Here we assume that if (A.9) holds for L then L_n satisfy the following differential equation

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\Phi}_n}\right) + \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \Phi_n}\right) - \frac{\partial L}{\partial \Phi_n} = 0, \quad \text{for} \quad n = 1, 2 \ldots \tag{58}$$

Hence, we obtain the following time dependant ordinary differential equation

$$\ddot{\Phi}_n(t) + \Omega_n^2 \Phi_n(t) = f_n(t), \quad \text{for} \quad n = 1, 2 \ldots \tag{59}$$

where $\Omega_n = \dfrac{\sqrt{I_p}\|\mathbf{U}_n\|_2}{\|\mathbf{U}_n\|_1}$ and $f_n(t) = \dfrac{S}{\rho\|\mathbf{U}_n\|_1^2}\displaystyle\int_0^l f(x,t)U_{1n}dx$. The general solution of Eq. (59) is of the form

$$\Phi_n(t) = \Phi_n(0)\cos(\Omega_n t) + \frac{\dot{\Phi}_n(0)}{\Omega_n}\sin(\Omega_n t) + \frac{1}{\Omega_n t}\int_0^l f_n(\tau)\sin[\Omega_n(t-\tau)]d\tau. \tag{60}$$

To conveniently determine constants $\Phi_n(0)$ and $\dot{\Phi}_n(0)$, we need the initial conditions (6) which should also be expanded into Fourier series with respect to eigenfunctions system

$$\begin{aligned} g(x) = u_1(x,0) = \sum_{n=1}^{\infty}\Phi_n(0)U_{1n}(x),\ \ h(x) = \dot{u}_1(x,0) = \sum_{n=1}^{\infty}\dot{\Phi}_n(0)U_{1n}(x), \\ \phi(x) = u_2(x,0) = \sum_{n=1}^{\infty}\Phi_n(0)U_{2n}(x),\ \ \varphi(x) = \dot{u}_2(x,0) = \sum_{n=1}^{\infty}\dot{\Phi}_n(0)U_{2n}(x). \end{aligned} \tag{61}$$

Using the properties of the above expansion, orthogonality condition (44) and the norm formula (46), we can express $\Phi_n(0)$ and $\dot{\Phi}_n(0)$, as Fourier coefficients

$$\begin{aligned} \Phi_n(0) &= \frac{1}{\|\mathbf{U}_1\|_1}\int_0^l \left(SU_{1n}(x)g(x) + I_pU_{2n}(x)\phi(x)\right)dx, \\ \dot{\Phi}_n(0) &= \frac{1}{\|\mathbf{U}_1\|_1}\int_0^l \left(SU_{1n}(x)h(x) + I_pU_{2n}(x)\varphi(x)\right)dx. \end{aligned} \tag{62}$$

Substituting expression (62) into Eq. (60), we then substitute the resulting expression into Eq. (54) to obtain the solution of the problem for the longitudinal vibrations of the Mindlin-Herrmann isotropic, thick rod

$$\begin{aligned} u_1(x,t) = &\int_0^l Sg(\xi)\frac{\partial G_1(x,\xi,t)}{\partial t}d\xi + \int_0^l I_p\phi(\xi)\frac{\partial G_2(x,\xi,t)}{\partial t} \\ &+ \int_0^l Sh(\xi)G_1(x,\xi,t)d\xi + \int_0^l I_p\varphi(\xi)G_2(x,\xi,t)d\xi \\ &+ \frac{1}{\rho}\int_0^t\int_0^l f(x,t)G_1(x,\xi,t-\tau)d\tau d\xi \end{aligned} \tag{63}$$

$$
\begin{aligned}
u_2(x,t) = & \int_0^l Sg(\xi)\frac{\partial G_3(x,\xi,t)}{\partial t}d\xi + \int_0^l I_p\phi(\xi)\frac{\partial G_4(x,\xi,t)}{\partial t} \\
& + \int_0^l Sh(\xi)G_3(x,\xi,t)d\xi + \int_0^l I_p\varphi(\xi)G_4(x,\xi,t)d\xi \\
& + \frac{1}{\rho}\int_0^t\int_0^l f(x,t)G_3(x,\xi,t-\tau)d\tau d\xi, \qquad (64)
\end{aligned}
$$

where

$$
\begin{aligned}
G_1(x,\xi,t) &= \sum_{n=1}^{\infty}\left(\frac{U_{1n}(x)U_{1n}(\xi)\sin\Omega_n t}{\Omega_n\|\mathbf{U}_n\|_1^2}\right), \qquad (65) \\
G_2(x,\xi,t) &= \sum_{n=1}^{\infty}\left(\frac{U_{1n}(x)U_{2n}(\xi)\sin\Omega_n t}{\Omega_n\|\mathbf{U}_n\|_1^2}\right), \\
G_3(x,\xi,t) &= \sum_{n=1}^{\infty}\left(\frac{U_{2n}(x)U_{1n}(\xi)\sin\Omega_n t}{\Omega_n\|\mathbf{U}_n\|_1^2}\right), \\
G_4(x,\xi,t) &= \sum_{n=1}^{\infty}\left(\frac{U_{2n}(x)U_{2n}(\xi)\sin\Omega_n t}{\Omega_n\|\mathbf{U}_n\|_1^2}\right)
\end{aligned}
$$

are the Green's functions.

5.2 Examples

In this subsection we give a concise application of the method outlined above to two single mode models of longitudinal wave propagation

5.2.1 Classical model

Here $u_2 = 0$ $(\eta = 0)$ and $I_p = 0$ which means no lateral motion and all deformations are parallel to the neutral line. The equation of motion is

$$
\frac{\partial^2 u_1}{\partial t^2} - c^2\frac{\partial^2 u_1}{\partial x^2} = \frac{1}{\rho}f(x,t), \qquad (66)
$$

where $c = \sqrt{\frac{E}{\rho}}$ is the phase velocity of the rod. The solution of the initial boundary value problem (5)-(6) and (66) is given as follows

$$
\begin{aligned}
u_1(x,t) = & \int_0^l Sg(\xi)\frac{\partial G_1(x,\xi,t)}{\partial t}d\xi + \int_0^l Sh(\xi)G_1(x,\xi,t)d\xi \\
& + \frac{1}{\rho}\int_0^t\int_0^l f(x,t)G_1(x,\xi,t-\tau)d\tau d\xi \qquad (67)
\end{aligned}
$$

where G_1 is given by Eq. (65) in which

$$\|\mathbf{U}_n\|_1^2 = \int_0^l U_{1n}^2(x)dx, \ \|\mathbf{U}_n\|_2^2 = \int_0^l U_{1n}'^2(x)dx, \ \Omega_n = \frac{c\|\mathbf{U}_n\|_2}{\|\mathbf{U}_n\|_1} \tag{68}$$

5.2.2 Rayleigh-Bishop model

We assume that lateral deformation is proportional to the longitudinal strain, that is $u_2 = \eta u_1'$. The equation of motion is

$$S\left(\frac{\partial^2 u_1}{\partial t^2} - c^2\frac{\partial^2 u_1}{\partial x^2}\right) + I_p\eta^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u_1}{\partial t^2} + c_1^2\frac{\partial^2 u_1}{\partial x^2}\right) = \frac{1}{\rho}f(x,t), \tag{69}$$

where $c_1 = \sqrt{\frac{\mu}{\rho}}$ is the velocity of the shear wave (so-called S-wave) in the lateral direction. The solution of the initial boundary value problem (5)-(6) and (69) is given as follows

$$\begin{aligned} u_1(x,t) = S\int_0^l &\left(g(\xi)\frac{\partial G_1(x,\xi,t)}{\partial t} + h(\xi)G_1(x,\xi,t)\right)d\xi \\ &+I_p\eta^2\int_0^l\left(g'(\xi)\frac{\partial^2 G_1(x,\xi,t)}{\partial t\partial\xi} + h'(\xi)\frac{\partial G_1(x,\xi,t)}{\partial\xi}\right)d\xi \\ &+\frac{S}{\rho}\int_0^t\int_0^l f(x,t)G_1(x,\xi,t-\tau)d\tau d\xi \end{aligned} \tag{70}$$

where G_1 is given by Eq. (65) in which

$$\|\mathbf{U}_n\|_1^2 = \int_0^l \left(SU_{1n}^2(x) + I_p\eta^2 U_{1n}'^2(x)\right)dx, \quad \Omega_n = \frac{\|\mathbf{U}_n\|_2}{\sqrt{\rho}\|\mathbf{U}_n\|_1} \tag{71}$$

and

$$\|\mathbf{U}_n\|_2^2 = \int_0^l \left(SEU_{1n}'^2(x) + \mu I_p\eta^2 U_{1n}''^2(x)\right)dx. \tag{72}$$

6 Numerical example and comparison of vibration theories

Here we consider an isotropic, thick, short rod, consisting of a cylindrical section made of a copper-based ($80\%Cu - 20\%Zn$ composition), whose characteristics are available in Table 1. The axial and lateral ends of the rod are fixed and free respectively. To solve the characteristic equation (36), we apply the method of bracketing root developed in [23], with the help of the mathematical

Table 1 The charateristics of the rod

Parameter	Symbol	Value	Unit
Modulus of elasticity	E	10^{11}	Nm^{-2}
Mass density	ρ	8.510^3	kgm^{-3}
Radius	r	0.50	m
Length	l	1	m
Area	A	0.7853	m^2
Poisson ratio	η	0.34	
Polar moment of inertia	I_p	0.098	m^4
Phase velocity	c	3426.971	ms^{-1}

software, *Mathcad* to implement and illustrate all the results. Results obtaining for Mindlin-Herrmann model will be compared to those of the single-mode Classical model.

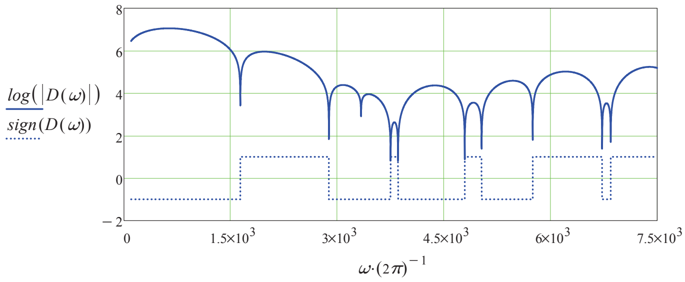

Fig. 1 Graph used to estimate the values of eigenfrequencies of the Mindlin-Herrman model

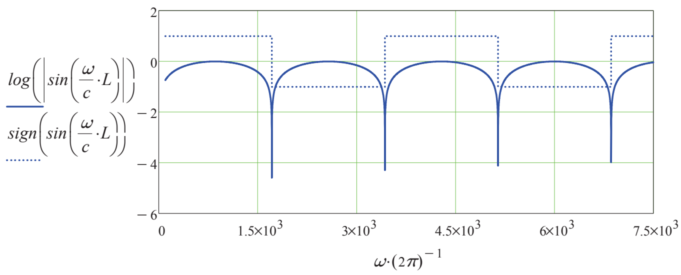

Fig. 2 Graph used to estimate the values of eigenfrequencies of the Classical model

In Fig. 1, only the downward spikes are informatives, in fact they indicate the solutions of the characteristic equation (36) or eigenvalues distribution except the third downward spike that shows only the change form a hyperbolic to a trigonometric function. This is well observed in Fig 3 at

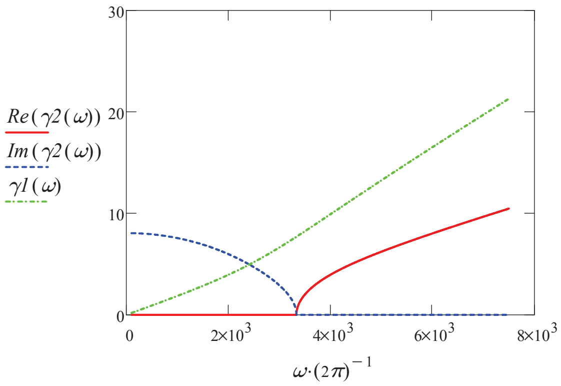

Fig. 3 Propagation coefficients of Mindlin-Herrmann model

$\omega(2\pi)^{-1} = 3.343 \times 10^3 Hz$ where the curve of $\mathcal{I}m\gamma_2(\omega)$ (dotted line) meets the solid line one $\mathcal{R}e\gamma_2(\omega)$. At that particular point the effect of the lateral attenuation coefficient ($\alpha_2 = \mathcal{I}m\gamma_2(\omega)$) vanishes and the lateral wavenumber ($k_2 = \mathcal{R}e\gamma_2(\omega)$) appears and increases substantially. Moreover the first, second, fifth, sixth, eighth and ninth downward spikes correspond to the first six axial vibration modes. The fourth, seventh and tenth downward spikes correspond to the first three pure transversal vibration modes. The first six and four

Table 2 Eigenfrequencies of Mindlin-Herrmann and Classical models

Axial mode of Mindlin-Herrmann model	Classical model	Unit
1.637×10^3	1.715×10^3	Hz
2.883×10^3	3.430×10^3	Hz
3.75×10^3	5.145×10^3	Hz
3.855×10^3	6.860×10^3	Hz
4.793×10^3		
6.726×10^3		

eigenfrequencies of the axial mode of Mindlin-Herrmann and Classical models are recorded in Table 2. It shows that the eigenfrequencies of the of the Classical theory are larger than those of the Mindlin-Herrmann theory which is in agreement with the comparison of vibration theories based on their frequency spectra (the dispersion curve of the classical theory is always above the curves of other rod vibration theories) done by Krawczuk *et .al* and Zak and Krawczuk [2,11]. Moreover in the same frequency range $[0, 7.500 \times 10^3]$ there are six axial eigenfrequencies associated with the Mindlin-Herrmann model and only four for the Classical model (see Fig 1 and 2). The latter observation is a proof that Mindlin-Herrmann theory is more accurate than Classical theory.

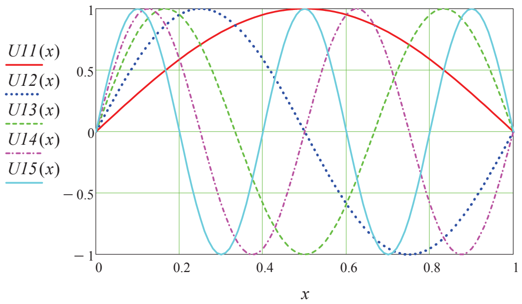

Fig. 4 The axial eigenfunctions associated with the first five axial eigenvalues

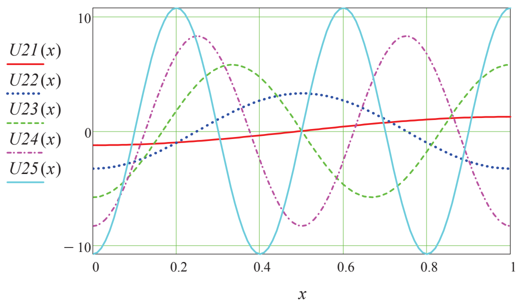

Fig. 5 The transversal eigenfunctions associated with the first five axial eigenvalues

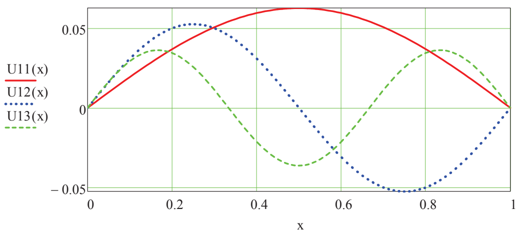

Fig. 6 The transversal eigenfunctions associated with the first three lateral eigenvalues

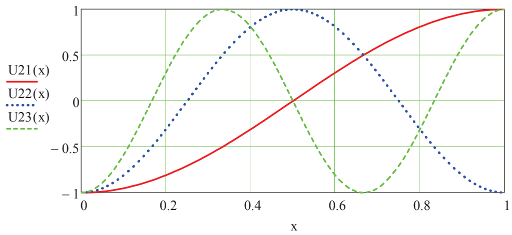

Fig. 7 The Axial eigenfunctions associated with the first three lateral eigenvalues

The graph of the axial eigenfunctions of the Mindlin-Herrmann theory, showing the shape of the longitudinal wave propagation at different modes is plotted in Fig 4. It is important to emphasise that the behaviours of the axial wave propagation (as illustrate in Fig 4) in Mindlin-Herrmann theory are comparable to those of Classical theory but with different eigenfrequencies. Fig 5 shows the shape of the lateral waves displacement at different axial angular frequencies (eigenvalues) caused by the effect of the axial wave propagation. In fact, to realise zero displacement in the axial direction at $x = 0, l$, we need to apply substantial forces. Hence, the stresses at these points are quite large and according to Hooke's Law, stresses are proportional to strains and in this particular case $u_2 = \eta u_1'$ (see Eq. (1)). It is clearly visible in Fig 4 and 5 that both the axial and lateral wave vibrate in opposition and alternatively between their absolute maximum values and nodes. The behaviours of the pure transversal wave displacement associated with the first three eigenvalues due to input energy and the deformations produce in the axial direction by the effect of the lateral displacements are illustrated in Fig 6 and 7 respectively.

Remark 6 In practice the Mindlin-Herrmann model can be applied for relatively long (slender) rods. In this example we apply it to short rod (that is, diameter is comparable to the length) to illustrate both effects of radial and longitudinal vibrations and compare the spectrum with those of the classical model for rod with the same characteristics.

7 Conclusion

The general analytical solution in terms of Green's functions of the Mindlin-Herrmann model for longitudinal vibration of an isotropic rod with constant cross-section is presented here. The model is a system of two hyperbolic equations and is derived by applying the energy method and the Hamilton's variational principle in the process of which boundary conditions are obtained. The self-adjoint property of the operator of the corresponding Sturm-Liouville operator is used to prove two types of orthogonality conditions of the associated eigenfunctions. The method of multiple orthogonalities for vibration problems is used to derive the exact solution for the model. A numerical simulation of the eigenfunctions is considered to show the harmonics visible as distinct spikes, which provide an insight into the mechanism that generates the entire signal inside the rod. Although the solution technique presented in this paper is based on Mindlin-Herrmann model, the same approach is used to tackle free and forced vibration problems of thick and short rods based on Rayleigh-Love and Rayleigh-Bishop models.

Appendix A Derivation of the Mindlin-Herrmann Model

The aim here is to show how to derive the equations of motion and the associated boundary conditions using the Hamilton variational principle.

Using Eq. (1) the strains are obtained as follows

$$\begin{array}{c} \varepsilon_{xx} = \frac{\partial u}{\partial x} = u_1', \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} = u_2, \quad \varepsilon_{zz} = \frac{\partial v}{\partial z} = u_2 = \varepsilon_{yy} \\ \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = y u_2', \quad \varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0, \quad \varepsilon_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = z u_2'. \end{array} \tag{A.1}$$

The stresses of the rod are

$$\begin{array}{l} \sigma_{xx} = (\lambda + 2\mu)\varepsilon_{xx} + \lambda(\varepsilon_{yy} + \varepsilon_{zz}) = (\lambda + 2\mu)u_1' + 2\lambda u_2 \\ \sigma_{yy} = (\lambda + 2\mu)\varepsilon_{yy} + \lambda(\varepsilon_{xx} + \varepsilon_{zz}) = 2(\lambda + \mu)u_2 + \lambda u_1' \\ \sigma_{zz} = (\lambda + 2\mu)\varepsilon_{zz} + \lambda(\varepsilon_{xx} + \varepsilon_{yy}) = 2(\lambda + \mu)u_2 + \lambda u_1' = \sigma_{yy} \\ \sigma_{xy} = \mu\varepsilon_{xy} = \mu y u_2', \quad \sigma_{yz} = \mu\varepsilon_{yz} = 0, \quad \sigma_{zx} = \mu\varepsilon_{zx} = \mu z u_2', \end{array} \tag{A.2}$$

Kinetic energy is as follows:

$$K = \frac{\rho}{2}\int_0^l \int_{(s)} \left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) ds dx \tag{A.3}$$

Substituting expression (1) into Eq. (6) leads to

$$K = \frac{\rho}{2} \int_0^l \left(S\dot{u}_1^2 + \dot{u}_2^2 I_p \right) dx, \tag{A.4}$$

The arguments in all functions are sometimes omitted for simplicity.

Strain energy is as follows:

$$P = \frac{\rho}{2} \int_0^l \int_{(s)} \left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\varepsilon_{xy} + \sigma_{yz}\varepsilon_{yz} + \sigma_{zx}\varepsilon_{zx} \right) dsdx \tag{A.5}$$

where σ_{ij}, ε_{ij} are given by Eqs.(A.1) and (A.2). Hence

$$P = \frac{1}{2} \int_0^l \left\{ S \left[(\lambda + 2\mu) u_1'^2 + 4\lambda u_2 u_1' + 4(\lambda + \mu) u_2^2 \right] + \mu I_p u_2'^2 \right\} dx. \tag{A.6}$$

Let W be the work done by the distributed force $f = f(x,t)$

$$W = \int_0^l \int_{(s)} f u_1 dsdx = \int_0^l f u_1 S dx. \tag{A.7}$$

The Lagrangian is as follows:

$$\begin{aligned} L &= K - P + W \\ &= \frac{1}{2} \int_0^l \left\{ \rho S \dot{u}_1^2 + \rho I_p \dot{u}_2^2 - (\lambda + 2\mu) S u_1'^2 - 4\lambda S u_1' u_2 - 4(\lambda + \mu) S u_2^2 \right\} dx \\ &\quad + \frac{1}{2} \int_0^l \left\{ 2 f u_1 S - \mu I_p u_2'^2 \right\} dx \end{aligned} \tag{A.8}$$

Applying the Hamiltonian principle to the Lagrange functional Eq. (A.8) we obtain the system of equations of motion in general form:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_1}\right) + \frac{d}{dx}\left(\frac{\partial L}{\partial u_1'}\right) - \frac{\partial L}{\partial u_1} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_2}\right) + \frac{d}{dx}\left(\frac{\partial L}{\partial u_2'}\right) - \frac{\partial L}{\partial u_2} = 0 \end{cases} \tag{A.9}$$

and the associated boundary conditions in general form is as follows

$$u_1|_{x=0,l} = 0, \; u_2|_{x=0,l} = 0 \quad \text{for fixed ends} \tag{A.10}$$

or

$$\frac{\partial L}{\partial u_1'}|_{x=0,l} = 0, \; \frac{\partial L}{\partial u_2'}|_{x=0,l} = 0 \quad \text{for free ends.} \tag{A.11}$$

We assume that the rod has a constant cross-section and that parameters such as λ, μ, S and I_P are constants. Hence the explicit form of Eq. (A.9) is as follows

$$\begin{cases} \rho \ddot{u}_1 - (\lambda + 2\mu) u_1'' - 2\lambda u_2' = f(x,t) \\ \rho I_p \ddot{u}_2 + 4(\lambda + \mu) S u_2 - \mu I_p u_2'' + 2S\lambda u_1' = 0 \end{cases} \tag{A.13}$$

with the following corresponding boundary conditions

$$u_1|_{x=0,l} = 0,\ u_2|_{x=0,l} = 0 \quad \text{for fixed ends} \tag{A.14}$$

or

$$(\lambda + 2\mu)u_1' + 2\lambda u_2|_{x=0,l} = 0,\ \mu I_p u_2'|_{x=0,l} = 0 \quad \text{for free ends.} \tag{A.15}$$

Appendix B Analysis of the operator of the Sturm-Liouville problem

In this section our goal is to determine the nature of the operator A of the Sturm-Liouville problem.

We firstly prove that A is self-adjoint on the class of functions $C^2(D)$ satisfying boundary condition (20). Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. We define the scalar product as follow:

$$(\mathbf{u}, \mathbf{v}) = \int_0^l \mathbf{u} \cdot \mathbf{v} dx \tag{B.1}$$

$$\begin{aligned} (A\mathbf{u}, \mathbf{v}) &= \int_0^l A\mathbf{u} \cdot \mathbf{v} dx \\ &= \int_0^l \left(a_{11} \frac{d^2 u_1}{dx^2} v_1 + a_{12} \frac{du_2}{dx} v_1 \right) dx \\ &\quad + \int_0^l \left(-a_{12} \frac{du_1}{dx} v_2 + a_{22} \frac{d^2 u_2}{dx^2} v_2 - c_{22} u_2 v_2 \right) dx \end{aligned} \tag{B.2}$$

Integrating twice and once by part the terms with the second and first derivative respectively of the expression (B.2) and applying boundary conditions (20) we obtain

$$\begin{aligned} (A\mathbf{u}, \mathbf{v}) &= \int_0^l \left(a_{11} u_1 \frac{d^2 v_1}{dx^2} - a_{12} u_2 \frac{dv_1}{dx} \right) dx \\ &\quad + \int_0^l \left(a_{12} u_1 \frac{dv_2}{dx} + a_{22} u_2 \frac{d^2 v_2}{dx^2} - c_{22} u_2 v_2 \right) dx \\ &= \int_0^l u_1 \left(a_{11} \frac{d^2 v_1}{dx^2} + a_{12} \frac{dv_2}{dx} \right) dx \\ &\quad + \int_0^l u_2 \left(-a_{12} \frac{dv_1}{dx} + a_{22} \frac{d^2 v_2}{dx^2} - c_{22} v_2 \right) dx \\ &= \int_0^l \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} . A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} dx \\ &= (\mathbf{u}, A\mathbf{v}) \end{aligned} \tag{B.3}$$

Equality (B.3) shows that the operator is self-adjoint which means that all the eigenvalues of the Sturm-Liouville problem (19)-(20) are real.

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