The Classification of Real Singularities Using Singular
Part I: Splitting Lemma and Simple Singularities

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Abstract. We present algorithms to classify isolated hypersurface singularities over the real numbers according to the classification by V.I. Arnold (Arnold et al., 1985). This first part covers the splitting lemma and the simple singularities; a second and a third part will be devoted to the unimodal singularities up to corank 2. All algorithms are implemented in the Singular library `realclassify.lib` (Marais and Steenpaß, 2012).

1. Introduction

Arnold et al. (1985) present classification theorems for singularities over the complex numbers up to modality 2 and for singularities over the real numbers up to modality 1, including complete sets of normal forms. For the complex case, they also give an algorithm how the type of a given singularity can be computed, called the "determinator of singularities" (cf. Arnold et al., 1985, ch. 16), but this question is left open for the real case. The goal of this paper, together with its subsequent parts, is to fill this gap. For this purpose, we present both, algorithms and an implementation thereof, for the classification of isolated hypersurface singularities up to modality 1 and corank 2 over the real numbers w.r.t. right equivalence.

We consider real functions with a critical point at the origin and critical value 0, i.e. functions in $\mathfrak{m}^2$, where $\mathfrak{m}$ denotes the ideal of function germs vanishing at the origin. Two function germs $f, g \in \mathfrak{m}^2 \subset \mathbb{R}[[x_1, \ldots, x_n]]$ are considered as right equivalent, denoted by $f \overset{r}{\sim} g$, if there exists an $\mathbb{R}$-algebra automorphism ϕ of $\mathbb{R}[[x_1, \ldots, x_n]]$ such that $\phi(f) = g$.

We have implemented all the algorithms presented here in the computer algebra system Singular (Decker et al., 2012). The implementation is freely available as a Singular library called `realclassify.lib` which relies on Singular's `classify.lib` to determine, for a given polynomial, the type in Arnold's classification over the complex numbers. The methods used in `classify.lib` will not be discussed in this paper. For more information in this regard, Krüger (1997) can be studied.

In Section 2, we introduce basic notions and methods which are frequently used for the algorithmic classification in the subsequent sections. We first give an overview of the different notions of equivalence in singularity theory and how they are related in Subsection 2.1. Thereafter we recall some basic results on the Milnor number and the determinacy in Subsections 2.2 and 2.3, and we also recall how these invariants can be computed. As a further prerequisite, we show that

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the homogeneous parts of lowest degree of two right equivalent functions factorize in the same way over $\mathbb{R}$ (Section 2.4, Proposition 8). We also show that in some cases, this factorization can even be carried out over $\mathbb{Q}$ which is important for the algorithmic aspect (Lemma 9).

Using the Splitting Lemma (Theorem 11), any function germ f over the real numbers with an isolated singularity at the origin can be written, after choosing a suitable coordinate system, as the sum of two functions of which the variables are disjoint. One of the functions, called the nondegenerate part of f, is a nondegenerate quadratic form and the other function, called the residual part of f, is an element of $\mathfrak{m}^3$. The number of variables in the residual part is equal to the corank of f, denoted by corank(f). In this paper we only consider germs with corank 0, 1 and 2. A version of the Splitting Lemma for singularities over $\mathbb{R}$ and a corresponding algorithm are discussed in Section 3.

In Arnold et al. (1985), the real singularities of modality 0 and 1 are classified up to stable equivalence into main types which split up into more subtypes depending on the sign of certain terms. Two functions are stably equivalent if they are right equivalent after the direct addition of nondegenerate quadratic forms. Hence after applying the Splitting Lemma, we only need to consider the residual part in order to compute the correct subtype. It can be easily seen that the subtypes are complex equivalent to a complex singularity type of the same name as its corresponding real main singularity type (see Table 1). In fact there is a bijection between the complex types of modality 0 and 1 and the real main types. Thus, if we can determine the complex type of a function germ, we only need to determine the correct subtype of the corresponding real main type. The classification of the residual part is given in Section 4, together with explicit algorithms for each singularity type.

2. Prerequisites

2.1. Equivalence. There are different notions for the equivalence of two power series in singularity theory:

Definition 1. *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$ and let $f, g \in \mathbb{K}[[x_1, \ldots, x_n]]$ be two power series.*

(1) *f and g are called* right equivalent*, denoted by $f \overset{r}{\sim} g$, if there exists a $\mathbb{K}$-algebra automorphism ϕ of $\mathbb{K}[[x_1, \ldots, x_n]]$ such that*

$$\phi(f) = g\,.$$

(2) *f and g are called* contact equivalent*, denoted by $f \overset{c}{\sim} g$, if there exist a $\mathbb{K}$-algebra automorphism ϕ of $\mathbb{K}[[x_1, \ldots, x_n]]$ and a unit $u \in \mathbb{K}[[x_1, \ldots, x_n]]^*$ such that*

$$\phi(f) = u \cdot g\,.$$

(3) *f and g are called* stably equivalent*, denoted by $f \overset{s}{\sim} g$, if there exist indices $k, l \in \{1, \ldots, n\}$ such that $f \in \mathbb{K}[[x_1, \ldots, x_k]]$, $g \in \mathbb{K}[[x_1, \ldots, x_l]]$, and the two power series become right equivalent after the addition of nondegenerate quadratic forms in the additional variables, i.e.*

$$\begin{aligned} & f(x_1, \ldots, x_k) \pm x_{k+1}^2 \pm \ldots \pm x_n^2 \\ \overset{r}{\sim}\ & g(x_1, \ldots, x_l)\ \pm x_{l+1}^2\ \pm \ldots \pm x_n^2\,. \end{aligned}$$

Remark 2. Note that right equivalence implies both contact and stable equivalence, but the converse statements are not true in general. For instance, $x_1^2 + x_2^2$ and $-x_1^2 - x_2^2$ are contact, but not right equivalent over $\mathbb{R}$.

This article and the SINGULAR library `realclassify.lib` both deal with the classification of the simple singularities w.r.t. *right* equivalence over $\mathbb{K} = \mathbb{R}$. We first use the Splitting Lemma and Algorithm 2 from Section 3 to get rid of the nondegenerate part. We can then apply the classification by Arnold et al. (1985) w.r.t. stable equivalence to the residual part in Section 4.

From the point of view of real algebraic geometry, a classification w.r.t. contact rather than right equivalence might be more interesting because it better reflects the local real geometry of a singularity. In the example from the remark above, $x_1^2+x_2^2$ and $-x_1^2-x_2^2$ both define a solitary point in the plane, as opposed to the two intersecting lines defined by $-x_1^2+x_2^2$ and $x_1^2-x_2^2$. But note that a classification w.r.t. right equivalence is only finer than one based on contact equivalence. Hence the shape of the local real geometry of a singularity can always be read off from its right equivalence class, given by its stable equivalence class together with the inertia index introduced in Theorem 11; the SINGULAR library `realclassify.lib` indeed also serves this purpose. For the simple singularities, it is moreover easy to see which of the right equivalence classes are contact equivalent.

2.2. The Milnor Number. We briefly recall the following well-known definition:

DEFINITION 3. *For $f \in \mathbb{R}[[x_1,\dots,x_n]]$ and $p \in \mathbb{A}^n_{\mathbb{R}}$, the* Milnor number *of f at p is defined as*

$$\mu(f,p) := \dim_{\mathbb{R}} \left(\mathbb{R}[[x_1-p_1,\dots,x_n-p_n]] \middle/ \left\langle \frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n} \right\rangle \right) \in \mathbb{N} \cup \{\infty\}\,.$$

If p is the origin, we simply write $\mu(f)$ instead of $\mu(f,p)$.

The Milnor number is known to be finite at isolated singularities (cf. Greuel et al., 2007, Chapter I, Lemma 2.3) and to be invariant under right equivalence (cf. Lemma 2.10 ibid.). It is thus an important tool for the classification of isolated singularities. We refer to Greuel et al. (2007) for more properties of this invariant.

There is a well-known algorithm for the computation of the Milnor number which is implemented in SINGULAR, see Greuel and Pfister (2008), pp. 526-528.

2.3. The Determinacy. In general, the singularities we deal with in this paper are defined by power series, but algorithmically, we want to work with polynomials. It is thus important for our algorithmic approach that any power series defining an isolated singularity is right equivalent to a polynomial which can be obtained from it by leaving out terms of sufficiently high order.

DEFINITION 4. *Let $f \in \mathbb{R}[[x_1,\dots,x_n]]$ be a power series.*

(1) *Let $f = \sum_{j=0}^{\infty} f_j$ be the decomposition of f into homogeneous parts f_j of degree j. For $k \in \mathbb{N}$, we define the k-*jet *of f as*

$$\operatorname{jet}(f,k) := \sum_{i=0}^{k} f_i\,.$$

In other words, the k-jet of f can be obtained from f by leaving out all terms of order higher than k.

(2) *f is called k-*determined *if*

$$\forall g \in \mathfrak{m}^{k+1}: \quad f \overset{r}{\sim} \operatorname{jet}(f,k) + g\,.$$

The determinacy is, just as the Milnor number, both invariant under right equivalence and finite for isolated singularities. We cite the following statement (cf. Greuel et al., 2007, Chapter I, Supplement to Theorem 2.23) due to its importance for the algorithmic approach and refer to Greuel et al. (2007) for further results regarding the determinacy:

PROPOSITION 5. *Let* $f \in \mathfrak{m} \subset \mathbb{R}[[x_1, \ldots, x_n]]$. *If*

$$\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle_{\mathbb{R}[[x_1,\ldots,x_n]]}$$

holds, then f *is* k*-determined.*

As a consequence of this, any power series f which has an isolated singularity at the origin is $(\mu(f)+1)$-determined (cf. Greuel et al., 2007, Chapter I, Corollary 2.24). But we can often compute a much better upper bound for the determinacy by using the above statement as in Algorithm 1.

Algorithm 1 Determinacy

Input: $f \in \mathbb{Q}[x_1, \ldots, x_n]$ with an isolated singularity at the origin
Output: an upper bound for the determinacy of f

$k := \text{MILNOR}(f) + 1$
$J := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) \subset \mathbb{Q}[x_1, \ldots, x_n]$
compute a standard basis G of $(\mathfrak{m}^2 J)$ w.r.t. a local monomial ordering $<$
for $(l = 1, \ldots, k-1)$ **do**
  if $(\text{NF}_<(\mathfrak{m}^{l+1}, G) = 0)$ **then**
    $k := l$
    break
return k

REMARK 6. In Algorithm 1, the for-loop computes the minimal $k \in \mathbb{N}$ such that the condition in Proposition 5 holds. This number is equal to the degree of the so-called highest corner of $\langle G \rangle = (\mathfrak{m}^2 J)$ (cf. Greuel and Pfister, 2008, Corollary A.9.7) and can thus also be computed by combinatorial means with the SINGULAR command `highcorner()` which is often much faster.

It is worth to note that the Milnor number of an arbitrary power series $f \in \mathbb{R}[[x_1, \ldots, x_n]]$ and the determinacy of a semi-quasihomogeneous power series $f \in \mathbb{R}[[x_1, \ldots, x_n]]$ do not change if we regard f as an element of $\mathbb{C}[[x_1, \ldots, x_n]]$. The same holds for the output of the corresponding algorithms presented here.

2.4. Results Regarding the Factorization of Homogeneous Polynomials over $\mathbb{R}$ and $\mathbb{Q}$.

DEFINITION 7. *Let* ϕ *be an* $\mathbb{R}$*-algebra automorphism of* $\mathbb{R}[[x_1, \ldots, x_n]]$. *For* $j \geq 0$ *we define the* j-jet *of* ϕ, *denoted by* ϕ_j, *to be the automorphism given by*

$$\phi_j(x_i) := \text{jet}(\phi(x_i), j+1) \quad \forall i = 1, \ldots, n\,.$$

The next result is in many cases a starting point for the algorithmic classification of the residual part, see Section 4. Given f and g with $f \overset{r}{\sim} g$, it can be used to determine ϕ_0 for some automorphism ϕ such that $\phi(f) = g$.

PROPOSITION 8. *Let* $f, g \in \mathbb{R}[[x_1, \ldots, x_n]]$ *be two power series with* $f \overset{r}{\sim} g$ *and* $k := \text{ord}(f) > 1$. *Let* ϕ *be an* $\mathbb{R}$*-algebra automorphism of* $\mathbb{R}[[x_1, \ldots, x_n]]$ *such that* $\phi(f) = g$.

If $\text{jet}(f, k)$ *factorizes as*

$$\text{jet}(f, k) = f_1^{s_1} \cdots f_t^{s_t}$$

in $\mathbb{R}[x_1,\ldots,x_n]$*, then* $\text{jet}(g,k)$ *factorizes as*

$$\text{jet}(g,k) = \phi_0(f_1)^{s_1}\cdots\phi_0(f_t)^{s_t}\,.$$

Proof. By assumption we have that $f = f_1^{s_1}\cdots f_t^{s_t} + f'$, where $f_1^{s_1}\cdots f_t^{s_t}$ is homogeneous of degree k and the order of f' is greater than k. We denote the higher order parts of ϕ by $\phi^* := \phi - \phi_0$. Since ϕ is a homomorphism, it follows that

$$\begin{aligned}\phi(f) &= \phi(f_1^{s_1}\cdots f_t^{s_t}) + \phi(f')\\ &= \phi_0(f_1^{s_1}\cdots f_t^{s_t}) + \phi^*(f_1^{s_1}\cdots f_t^{s_t}) + \phi(f')\end{aligned}$$

where $\phi_0(f_1^{s_1}\cdots f_t^{s_t})$ is homogeneous of degree k and both $\phi^*(f_1^{s_1}\cdots f_t^{s_t})$ and $\phi(f')$ are of order higher than k. Hence

$$\text{jet}(g,k) = \text{jet}(\phi(f),k) = \phi_0(f_1^{s_1}\cdots f_t^{s_t}) = \phi_0(f_1)^{s_1}\cdots\phi_0(f_t)^{s_t}\,.$$

□

Since we do not want to work with rounding errors nor field extensions in the implementation of the proposed algorithms, the above result would not be of much help for this purpose without the following result.

Lemma 9. *If* $f \in \mathbb{Q}[x,y]$ *is homogeneous and factorizes as*

$$\text{(i) } g_1^d \text{ or (ii) } g_1 g_2^d\,,$$

where $g_1, g_2 \in \mathbb{R}[x,y]$ *are polynomials of degree* 1 *and* $d > 1$*, then* f *factorizes as*

$$\text{(i) } a g_1'^d \text{ or (ii) } a g_1' g_2'^d\,,$$

respectively, where $g_1', g_2' \in \mathbb{Q}[x,y]$ *are polynomials of degree* 1 *and* $a \in \mathbb{Q}$.

Proof. (i) Let $f = (a_1x + a_2y)^d$, $a_1, a_2 \in \mathbb{R}$. Without loss of generality, suppose $a_1 \neq 0$. Then $f = a_1^d(x + \frac{a_2}{a_1}y)^d$. Since the coefficient of x^d in $f \in \mathbb{Q}[x,y]$ is a_1^d, we have $a_1^d \in \mathbb{Q}$ and therefore $(x+\frac{a_2}{a_1}y)^d \in \mathbb{Q}[x,y]$ which, by dehomogenization, leads to $(x + \frac{a_2}{a_1})^d \in \mathbb{Q}[x]$. Since $\mathbb{Q}$ is a perfect field it follows that $\frac{a_2}{a_1} \in \mathbb{Q}$. Thus $f = ag_1'^d$, where $a := a_1^d \in \mathbb{Q}$ and $g_1' = x + \frac{a_2}{a_1}y \in \mathbb{Q}[x,y]$.

(ii) Let $f = (a_1x + a_2y)(a_3x + a_4y)^d$, $a_1,\ldots,a_4 \in \mathbb{R}$. Suppose $a_1, a_3 \neq 0$. For the cases $a_1, a_4 \neq 0$, $a_2, a_3 \neq 0$ and $a_2, a_4 \neq 0$ the proofs are similar. We have $a_1a_3^d \in \mathbb{Q}$ analogously to part (i). Hence $(x + \frac{a_2}{a_1}y)(x + \frac{a_4}{a_3}y)^d \in \mathbb{Q}[x,y]$ which in turn implies $(x + \frac{a_2}{a_1})(x + \frac{a_4}{a_3})^d \in \mathbb{Q}[x]$. Since $\mathbb{Q}$ is a perfect field it follows that the roots of this polynomial are rational. Therefore $f = ag_1'g_2'^d$ with $a := a_1a_3^d \in \mathbb{Q}$, $g_1' := (x + \frac{a_2}{a_1}y) \in \mathbb{Q}[x,y]$, and $g_2' := (x + \frac{a_4}{a_3}y) \in \mathbb{Q}[x,y]$. □

3. The Splitting Lemma

Definition 10. *For* $f \in \mathbb{R}[[x_1,\ldots,x_n]]$*, we define the corank of* f*, denoted by* $\text{corank}(f)$*, as the corank of the Hessian matrix* $H(f)$ *at* $\mathbf{0}$*, i.e.*

$$\text{corank}(f) := \text{corank}(H(f)(\mathbf{0}))\,.$$

The following well-known theorem, called the Splitting Lemma, allows us to reduce the classification to germs of full corank or, algorithmically, to a polynomial contained in $\mathfrak{m}^3 \cap \mathbb{R}[x_1,\ldots,x_c]$ for a given input polynomial of corank c. We present a version for singularities over the real numbers, taking into account the signs of the squares.

Theorem 11. *If* $f \in \mathfrak{m}^2 \subset \mathbb{R}[[x_1,\ldots,x_n]]$ *has an isolated singularity and if its corank is* c*, then*

$$f \overset{r}{\sim} g - \sum_{i=c+1}^{c+\lambda} x_i^2 + \sum_{i=c+\lambda+1}^{n} x_i^2$$

with $g \in \mathfrak{m}^3 \cap \mathbb{R}[[x_1, \ldots, x_c]]$. g is called the residual part of f and λ is called the inertia index of f. Both λ and the right equivalence class of g are uniquely determined by f.

The following proof is based upon the proofs of Theorems 2.46 and 2.47 in Chapter I of Greuel et al. (2007).

Proof. The corank of the Hessian matrix of f at 0 is c, so by the theory of quadratic forms over $\mathbb{R}$ there is a transformation matrix T such that

$$T^t \cdot \tfrac{1}{2}H(f)(\mathbf{0}) \cdot T = \operatorname{diag}(0, \ldots, 0, -1, \ldots, -1, 1, \ldots, 1)\,.$$

Therefore the linear coordinate change $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n) \cdot T^t$ transforms the 2-jet of f into $\left(-\sum_{i=c+1}^{c+\lambda} x_i^2 + \sum_{i=c+\lambda+1}^{n} x_i^2\right)$ where λ is the inertia index of f. Applied to f, this transformation leads to

$$\begin{aligned} f^{(3)}(x_1, \ldots, x_n) &:= f((x_1, \ldots, x_n) \cdot T^t) \\ &= g_3 - \sum_{i=c+1}^{c+\lambda} x_i^2 + \sum_{i=c+\lambda+1}^{n} x_i^2 + \sum_{i=c+1}^{n} x_i \cdot h_i^{(3)} \end{aligned}$$

with $g_3 \in \mathfrak{m}^3 \cap \mathbb{R}[[x_1, \ldots, x_c]]$ and $h_i^{(3)} \in \mathfrak{m}^2$. The coordinate change $\phi^{(3)}$ defined by

$$\phi^{(3)}(x_i) := \begin{cases} x_i, & i = 1, \ldots, c, \\ x_i + \frac{1}{2}h_i^{(3)}, & i = c+1, \ldots, c+\lambda, \\ x_i - \frac{1}{2}h_i^{(3)}, & i = c+\lambda+1, \ldots, n, \end{cases}$$

yields

$$\begin{aligned} f^{(4)}(x_1, \ldots, x_n) &:= f^{(3)}(\phi^{(3)}(x_1, \ldots, x_n)) \\ &= g_3 + g_4 - \sum_{i=c+1}^{c+\lambda} x_i^2 + \sum_{i=c+\lambda+1}^{n} x_i^2 + \sum_{i=c+1}^{n} x_i \cdot h_i^{(4)} \end{aligned}$$

with $g_4 \in \mathfrak{m}^4 \cap \mathbb{R}[[x_1, \ldots, x_c]]$ and $h_i^{(4)} \in \mathfrak{m}^3$. Continuing in the same manner, the last sum will be of arbitrarily high order. It can be eventually left out because f is finitely determined as an isolated singularity. □

Since this proof is constructive, we can immediately derive Algorithm 2 from it.

4. The Real Classification of the Residual Part w.r.t. Stable Equivalence

Arnold et al. (1985) present independent classifications of the simple singularities over the complex and over the real numbers, using stable equivalence. We refer to the equivalence classes of the complex classification as *complex types*. In the classification over the real numbers, the simple singularities are divided into *main types* which split up into one or more *subtypes*. These subtypes differ from each other only in the sign of certain terms.

It is known that the modality does not decrease under complexification (Arnold et al., 1985, pp. 273-274). So by applying the algorithms for the complex classification to the real normal forms, it is easy to see that in modality 0, there is a one-to-one correspondence between the complex types and the real main types. The real classification can thus be seen as a refinement of the complex one. As we will see in the subsequent parts of this series of articles, the same holds true also in modality 1, but in both cases, this is not clear a priori and can only be deduced from the independently derived complex and real classifications. In fact, it is not

Algorithm 2 Algorithm for the Splitting Lemma

Input: $f \in \mathfrak{m}^2 \subset \mathbb{Q}[x_1, \ldots, x_n]$ and $k \in \mathbb{N}$ such that f is k-determined
Output: the corank c of f, the inertia index λ of f and $g \in \mathfrak{m}^3 \cap \mathbb{Q}[x_1, \ldots, x_c]$ such that

$$f \overset{r}{\sim} g - \sum_{i=c+1}^{c+\lambda} x_i^2 + \sum_{i=c+\lambda+1}^{n} x_i^2$$

compute a transformation matrix $T \in \mathbb{R}^{n \times n}$ such that

$$T^t \cdot \tfrac{1}{2} H(f)(\mathbf{0}) \cdot T = \operatorname{diag}(0, \ldots, 0, -1, \ldots, -1, 1, \ldots, 1) =: N$$

$c :=$ number of zeroes on the diagonal of N
$\lambda :=$ number of entries equal to -1 on the diagonal of N
$f^{(3)}(x_1, \ldots, x_n) := f((x_1, \ldots, x_n) \cdot T^t)$
for $(l = 3, \ldots, k)$ **do**
    write $f^{(l)}$ as

$$f^{(l)} = \sum_{j=3}^{l} g_j - \sum_{i=c+1}^{c+\lambda} x_i^2 + \sum_{i=c+\lambda+1}^{n} x_i^2 + \sum_{i=c+1}^{n} x_i \cdot h_i^{(l)}$$

    with $g_j \in \mathfrak{m}^j \cap \mathbb{Q}[x_1, \ldots, x_c]$ and $h_i^{(l)} \in \mathfrak{m}^{l-1}$
    $f^{(l+1)} := \phi^{(l)}(f^{(l)})$ where $\phi^{(l)}$ is defined by

$$\phi^{(l)}(x_i) := \begin{cases} x_i, & i = 1, \ldots, c, \\ x_i + \frac{1}{2} h_i^{(l)}, & i = c+1, \ldots, c+\lambda, \\ x_i - \frac{1}{2} h_i^{(l)}, & i = c+\lambda+1, \ldots, n. \end{cases}$$

$g := \sum_{j=3}^{k} g_j$
return c, λ, g

known whether the modality is preserved under complexification in general (Arnold et al., 1985, pp. 273-274).

Both the real and complex normal forms of the simple singularities are listed in Table 1. From here onwards we will work with stable equivalence, cf. Definition 1(3). For all degenerate forms it is thus only necessary, after applying the Splitting Lemma, to consider their residual parts, i.e. germs in $\mathfrak{m}^3$. Note that the right equivalence class of a real singularity is given by its stable equivalence class together with its inertia index which can be computed using Algorithm 2.

Using the Singular library `classify.lib` (Krüger, 2012) for the complex classification and the one-to-one correspondence between the real main singularity types and the complex types, the algorithmic classification of a real germ boils down to determining to which of the corresponding subtypes the germ is equivalent. For the singularity types E_7 and E_8, there is nothing left to do because each of these types has only one real subtype. The rest of the cases is considered one by one in the following subsections.

Throughout the rest of this article we write f for the given input polynomial, g for its residual part which can be obtained by applying the Splitting Lemma, and c for the corank of f. We also assume that f, and thus g, is a polynomial over $\mathbb{Q}$. With these notations, g is a polynomial in c variables.

4.1. A_1. If $c = 0$, then f is of complex type A_1. The residual part in this case is $g = 0$, even though Table 1 assigns the normal form x^2 to this type for formal reasons. As a consequence, all the real singularities of main type A_1 are

Table 1. Real normal forms of singularities of modality 0.

	Complex normal form	Normal forms of real subtypes	Equivalences	Values of k
A_k	x^{k+1}	$+x^{k+1}$ (A_k^+)	$A_k^+ \overset{r}{\sim} A_k^-$ for even k	$k \geq 1$
		$-x^{k+1}$ (A_k^-)		
D_k	$x^2y + y^{k-1}$	$x^2y + y^{k-1}$ (D_k^+)	-	$k \geq 4$
		$x^2y - y^{k-1}$ (D_k^-)		
E_6	$x^3 + y^4$	$x^3 + y^4$ (E_6^+)	-	-
		$x^3 - y^4$ (E_6^-)		
E_7	$x^3 + xy^3$	$x^3 + xy^3$	-	-
E_8	$x^3 + y^5$	$x^3 + y^5$	-	-

stably equivalent and their right equivalence class is completely determined by their inertia index λ.

4.2. $A_k, k > 1$. If $c = 1$, then the singularity is of complex type A_k for some $k > 1$. Over the real numbers, this type splits up into the subtypes A_k^+ and A_k^- if k is odd. Furthermore g is a univariate polynomial in this case, say $g \in \mathbb{Q}[x]$. The value of k is given by the order of g minus 1 because $\pm x^{k+1}$ and g are right equivalent and thus have the same order.

Note that if k is even, then $A_k^+ \overset{r}{\sim} A_k^-$ and we have only one real subtype which we denote by A_k. Let k be odd. Then the sign of the singularity type is determined by the sign of the coefficient of x^{k+1}. This follows since Proposition 8 implies $\text{jet}(g, k+1) = \pm(\phi_0(x))^{k+1} = \pm(\alpha x)^{k+1}$, where $\phi(\pm x^{k+1}) = g$, $\alpha \in \mathbb{R}$, and the sign depends on the singularity type. Since $k+1$ is even and $\alpha \in \mathbb{R}$, ϕ does not change the sign of the coefficient of x^{k+1}. We use Algorithm 3, after applying the Splitting Lemma in case $c = 0$ or $c = 1$.

Algorithm 3 Algorithm for the case A_k

Input: $f \in \mathbb{Q}[x_1, \dots, x_n]$ of complex singularity type A_k, the output polynomial g after applying Algorithm 2, and the corank c of f

Output: the real singularity type of f, i.e. A_k, A_k^+ or A_k^-, $k \in \mathbb{N}$

```
if c = 0 then
    type := A_1
if c = 1 then
    k := ord(g) − 1
    if k is even then
        type := A_k
    else
        s := coefficient of x^{k+1} in g
        if s > 0 then
            type := A_k^+
        else
            type := A_k^-
return type
```

. For the rest of the paper we turn our attention to singularities of corank 2. In these cases $0 \neq g \in \mathfrak{m}^3$ is a polynomial in two variables, say $g \in \mathbb{Q}[x, y]$. Using the Singular library `classify.lib`, we determine the complex singularity type and thus the real main singularity type of g, or equivalently f. The purpose of the remaining algorithms in the paper is to determine the correct real subtype of g, or equivalently f. We now consider each complex type, or equivalently every real main type, separately.

4.3. D_4. The normal form of the complex singularity type D_4 is $x^2y + y^3$, which splits up into x^2y+y^3 (D_4^+) and x^2y-y^3 (D_4^-) in the real case. The two cases can be distinguished by factorization; the details are carried out in Algorithm 4. Since the determinacy of D_4 is 3, it suffices to look at the 3-jet. The number of factors of the 3-jet over $\mathbb{R}$ is an invariant of the real subtype which is 1 in the case D_4^+ and 3 for D_4^-.

However, using the Singular command `factorize` in order to determine the number of factors is problematic because the factorization over $\mathbb{R}$ differs from those over $\mathbb{Q}$ and $\mathbb{C}$ in some cases. As an alternative, we dehomogenize the 3-jet and count the number of real roots of the resulting univariate polynomial which is exactly the same as the number of factors of the 3-jet over $\mathbb{R}$.

If we want to dehomogenize the 3-jet via $x \mapsto x$, $y \mapsto 1$ without reducing its degree, we first have to make sure that the coefficient of x^3 is non-zero. It is easy to check that this is achieved by lines 2 to 13 of Algorithm 4. For the implementation in Singular, we used the library `rootsur.lib` (Tobis, 2012) to count the number of real roots of a univariate polynomial.

Algorithm 4 Algorithm for the case D_4

Input: $g \in \mathfrak{m}^3 \subset \mathbb{Q}[x, y]$ of complex singularity type D_4
Output: the real singularity type of g, i.e. D_4^+ or D_4^-
1: $h := \text{jet}(g, 3)$
2: $s_1 :=$ coefficient of x^3 in h
3: $s_2 :=$ coefficient of y^3 in h
4: **if** $(s_1 = 0)$ **then**
5:     **if** $(s_2 \neq 0)$ **then**
6:         swap the variables x and y in h
7:     **else**
8:         $t_1 :=$ coefficient of x^2y in h
9:         $t_2 :=$ coefficient of xy^2 in h
10:         **if** $(t_1 + t_2 \neq 0)$ **then**
11:             apply $x \mapsto x$, $y \mapsto x + y$ to h
12:         **else**
13:             apply $x \mapsto x$, $y \mapsto 2x + y$ to h
14: apply $x \mapsto x$, $y \mapsto 1$ to h
15: $n :=$ number of real roots of h
16: **if** $(n < 3)$ **then**
17:     **return** D_4^+
18: **else**
19:     **return** D_4^-

Remark 12. Geometrically, the dehomogenization in Algorithm 4 corresponds to blowing the 3-jet up at the origin plus choosing a chart. Since the 3-jet is homogeneous, blowing-up always yields three lines in the complex case. In the real case, however, we get either one or three lines depending on their position w.r.t. the real subspace in the complex picture. All the lines lie in the chosen chart because the coefficient of x^3 is non-zero.

4.4. $D_k, k > 4$. For the cases D_k with $k > 4$, the complex normal form is $x^2y + y^{k-1}$. It splits up into $x^2y + y^{k-1}$ (D_k^+) and $x^2y - y^{k-1}$ (D_k^-) for each k over the reals. We use the following two results from Siersma (1974, p. 35) to distinguish between the two cases:

LEMMA 13. *A singularity of type D_k^+ or D_k^- is $(k-1)$-determined.*

LEMMA 14. *Let $j \geq 4$. Then there exists a polynomial $R \in \mathfrak{m}^{j+1} \subset \mathbb{R}[[x, y]]$ such that*

$$x^2y + a_0x^j + a_1x^{j-1}y + \ldots + a_jy^j \overset{r}{\sim} x^2y + a_jy^j + R, \quad a_0, \ldots, a_j \in \mathbb{R},$$

using the $\mathbb{R}$-algebra automorphism

$$\begin{aligned} x &\mapsto x + p_1, \text{ where } p_1 = -\frac{1}{2}(a_1x^{j-2} + \ldots + a_{j-1}y^{j-2}), \\ y &\mapsto y + p_2, \text{ where } p_2 = -a_0x^{j-2}. \end{aligned}$$

By Lemma 13, the determinacy of a singularity of main type D_k is $k-1$. Therefore we only need to consider the $(k-1)$-jet of g in this case. By Proposition 8, the 3-jet of g factorizes as $\text{jet}(g, 3) = g_1^2g_2$ over $\mathbb{R}$, where g_1 and g_2 are homogeneous polynomials of degree 1. Note that Lemma 9 ensures that this factorization can be carried out even over $\mathbb{Q}$. We can thus transform g into a polynomial of the form

$$x^2y + \text{terms of degree higher than 3}$$

by applying the automorphism defined by $g_1 \mapsto x$, $g_2 \mapsto y$ to g.

We now systematically consider the terms of each degree $3 < j < k$. By applying the transformations in Lemma 14, for each j, the only term of total degree j which possibly remains is a_jy^j. This term vanishes for $j < k-1$ and it does not vanish for $j = k-1$, otherwise g is not of complex type D_k. Thus, after applying these transformations, we can write g as $g = x^2y + \alpha y^{k-1}$ with $\alpha \neq 0$. Clearly if $\alpha > 0$ then $x^2y + \alpha y^{k-1} \overset{r}{\sim} x^2y + y^{k-1}$ and if $\alpha < 0$ then $x^2y + \alpha y^{k-1} \overset{r}{\sim} x^2y - y^{k-1}$.

Algorithm 5 Algorithm for the case D_k, $k > 4$

Input: $g \in \mathfrak{m}^3 \subset \mathbb{Q}[x, y]$ of complex singularity type D_k, $k \in \mathbb{N}$, $k > 4$
Output: the real singularity type of g, i.e. D_k^+ or D_k^-

$k := \mu(g)$
$h := \text{jet}(g, k-1)$
factorize $\text{jet}(h, 3)$ as $h_1^2h_2$, where h_1 and h_2 are linear
apply $h_1 \mapsto x$, $h_2 \mapsto y$ to h
for $(j = 4, \ldots, k-1)$ **do**
    if $(\text{jet}(h, j) - x^2y \neq 0)$ **then**
        write $\text{jet}(h, j) - x^2y$ as $a_0x^j + a_1x^{j-1}y + \cdots + a_jy^j$, $\quad a_0, \ldots a_j \in \mathbb{Q}$
        apply $x \mapsto x - \frac{1}{2}(a_1x^{j-2} + \cdots + a_{j-1}y^{j-2})$, $y \mapsto y - a_0x^{j-2}$ to h
        $h := \text{jet}(h, k-1)$
write h as $h = x^2y + \alpha y^{k-1}$, $0 \neq \alpha \in \mathbb{Q}$
if $(\alpha > 0)$ **then**
    return D_k^+
else
    return D_k^-

4.5. E_6. In this case, whose complex normal form is $x^3 + y^4$, we have that either $g \overset{r}{\sim} x^3 + y^4$ (E_6^+) or $g \overset{r}{\sim} x^3 - y^4$ (E_6^-). Therefore there exists an $\mathbb{R}$-algebra automorphism ϕ of $\mathbb{R}[[x,y]]$ such that $g = (\phi(x))^3 + (\phi(y))^4$ or such that $g = (\phi(x))^3 - (\phi(y))^4$. Since the coefficients of x^3 and y^3 in g cannot both be zero, we can ensure that the coefficient of x^3 is non-zero by swapping the variables if necessary. Now, by Proposition 8 and Lemma 9, the 3-jet of g factorizes as $c(g_1)^3$ with $c \in \mathbb{Q}$ and $g_1 = b_0x + b_1y \in \mathbb{Q}[x,y]$, $b_0 \neq 0$. By applying $x \mapsto \frac{x - b_1y}{b_0}$, $y \mapsto y$ to g, we can thus assume without loss of generality that ϕ_0 is of the form $\phi_0(x) = c'x$, $\phi_0(y) = d_0x + d_1y$ with $c', d_0, d_1 \in \mathbb{R}$. Since ϕ is an automorphism, we have that $d_1 \neq 0$. Hence

$$(\phi(y))^4 = d_1^4y^4 + (\text{terms of degree 4 and higher, not of the form } \alpha y^4,\ \alpha \in \mathbb{R})\,.$$

If we can show that $(\phi(x))^3$ does not contain a term of the form αy^4, $\alpha \in \mathbb{R}$, then we can determine whether g is of type E_6^- or E_6^+ by considering the sign of the coefficient of the monomial y^4. A simple calculation yields

$$\begin{aligned} \mathrm{jet}((\phi(x))^3, 4) - \mathrm{jet}((\phi(x))^3, 3) &= 3(\phi_0(x)^2)(\phi_1(x) - \phi_0(x)) \\ &= 3(c'x)^2(\phi_1(x) - \phi_0(x))\,, \end{aligned}$$

which means that $(\phi(x))^3$ does not have any term of the form αy^4, $\alpha \in \mathbb{R}$.

Algorithm 6 Algorithm for the case E_6

Input: $g \in \mathfrak{m}^3 \subset \mathbb{Q}[x,y]$ of complex singularity type E_6
Output: the real singularity type of g, i.e. E_6^+ or E_6^-

$h := \mathrm{jet}(g, 3)$
$s :=$ coefficient of x^3 in h
if $(s = 0)$ **then**
  swap the variables x and y
factorize h into linear factors over $\mathbb{Q}[x,y]$, with a factor $g_1 = b_0x + b_1y$
apply $x \mapsto \frac{x - b_1y}{b_0}$, $y \mapsto y$ to g
$d :=$ coefficient of y^4 in g
if $(d > 0)$ **then**
  return E_6^+
else
  return E_6^-

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