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TRIPLED FIXED POINT RESULTS FOR T-CONTRACTIONS ON ABSTRACT METRIC SPACES

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Abstract. In this paper we introduce the notion of T-contraction for tripled fixed points in abstract metric spaces and obtain some tripled fixed point theorems which extend and generalize well-known comparable results in the literature. To support our results, we present an example and an applications to integral equations.

1. Introduction and Preliminaries

In 1922, Banach proved his famous fixed point theorem [5]. Afterward, many authors considered various definitions of contractive mappings and proved several fixed point theorems, which are extensions and generalizations of Banach's theorem (see, for example, [9, 13, 23]).

On the other hand, non-convex analysis has found some applications in optimization theory. Fixed point theory in K-metric and K-normed spaces was developed by Perov et al. [18], Mukhamadijev and Stetsenko [17] and others (we refer to a survey by Zabrejko [26]). The main idea consists in using an ordered Banach space instead of the set of real numbers, as the codomain for a metric. In 2007, Huang and Zhang [12] reintroduced such as spaces and defined cone metric spaces. Then, several fixed point results on cone metric spaces were obtained in [1, 2, 22] and references therein.

In 2009, Beiranvand et al. [6] defined T-contractions in metric spaces. Afterward, some related fixed point theorems were proved in [15]. Successively, Morales and Rajes [16] introduced T-Kannan and T-Chatterjea contractive mappings in cone metric spaces and studied the existence of fixed points for these mappings. Recently, Rahimi et al. [19, 21] proved fixed theorems for T-contractions involving two mappings on cone metric spaces.

Recently, Bhaskar and Lakshmikantham [8] introduced the concept of coupled fixed point in partially ordered metric spaces, starting a fruitful direction of research followed by many authors, also in the setting of ordered metric and ordered cone metric spaces; see [14, 24] and the references therein. Finally, Berinde and Borcut [7] introduced the notion of tripled fixed point (see also [3, 4]) and obtained results on the existence of tripled fixed points.

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In this paper we introduce the notion of T-contraction in tripled fixed point theory and prove some related results on abstract metric spaces. It is worth mentioning that our results do not rely on normality condition on cones involved therein. Our theorems extend, unify and generalize well-known results in the literature.

Following are some definitions and known results needed in the sequel.

Definition 1.1 ([11, 12])**.** Let E be a real Banach space and P a subset of E. Then P is called a cone if and only if

(a) P is closed, nonempty and $P \neq \{\theta\}$;
(b) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$;
(c) if $x \in P$ and $-x \in P$, then $x = \theta$.

Given a cone $P \subset E$, a partial ordering $\preceq$ with respect to P is defined by

$$x \preceq y \Longleftrightarrow y - x \in P.$$

We shall write $x \prec y$ to mean $x \preceq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in intP$, where $intP$ is the interior of P. If $intP \neq \emptyset$, the cone P is called solid. A cone P is called normal if there exists a number $K > 0$ such that, for all $x, y \in E$, we have

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P.

Definition 1.2. Let X be a nonempty set. Suppose that a mapping $d : X \times X \to E$ satisfies the following conditions:

(d1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric [12] or K-metric [26] on X and (X, d) is called a cone metric space [12] or K-metric space [26].

The concept of K-metric space is more general than that of metric space, in fact each metric space is a K-metric space where $X = \mathbb{R}$ and $P = [0, +\infty)$.

Definition 1.3 ([10])**.** Let (X, d) be a K-metric space, $\{x_n\}$ a sequence in X and $x \in X$. Then

(i) $\{x_n\}$ converges to x if, for every $c \in E$ with $\theta \ll c$ there exists an $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$;
(ii) $\{x_n\}$ is called a Cauchy sequence if, for every $c \in E$ with $\theta \ll c$ there exists an $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$;
(iii) a K-metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Definition 1.4 ([10])**.** Let (X, d) be a K-metric space, P a solid cone. A mapping $T : X \to X$ is said to be:

(i) sequentially convergent if the sequence $\{x_n\}$ in X is convergent, whenever $\{Tx_n\}$ is convergent;
(ii) subsequentially convergent if the sequence $\{x_n\}$ has a convergent subsequence, whenever $\{Tx_n\}$ is convergent;

(iii) continuous if for any sequence $\{x_n\}$ in X with $\lim_{n\to+\infty} x_n = x$ implies that $\lim_{n\to+\infty} Tx_n = Tx$.

Theorem 1.5 ([19, 20])**.** Let (X,d) be a complete K-metric space, P a solid cone and $T : X \to X$ a continuous and one to one mapping. Moreover, $f : X \to X$ be a mapping satisfying

$$\begin{aligned} d(Tfx, Tfy) \ &\preceq \ \alpha_1 d(Tx, Ty) + \alpha_2[d(Tx, Tfx) + d(Ty, Tfy)] \\ &\quad +\alpha_3[d(Tx, Tfy) + d(Ty, Tfx)], \end{aligned}$$

for all $x, y \in X$, where $\alpha_1, \alpha_2, \alpha_3 \geq 0$ with $\alpha_1 + 2\alpha_2 + 2\alpha_3 < 1$. Then

(i) for each $x_0 \in X$, $\{Tf^n x_0\}$ is a Cauchy sequence, (define the iterate sequence $\{x_n\}$ by $x_{n+1} = f^{n+1}x_0$);
(ii) there exists a $z_{x_0} \in X$ such that $\lim_{n\to+\infty} Tf^n x_0 = z_{x_0}$;
(iii) if T is subsequentially convergent, then $\{f^n x_0\}$ has a convergent subsequence;
(iv) there exists a unique $w_{x_0} \in X$ such that $fw_{x_0} = w_{x_0}$; that is, f has a unique fixed point;
(v) if T is sequentially convergent, then, for each $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to w_{x_0}.

Definition 1.6 ([25])**.** An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of a mapping $F : X \times X \times X \to X$ if $x = F(x, y, z)$, $y = F(y, z, x)$ and $z = F(z, x, y)$.

2. Main results

Definition 2.1. Let (X, d) be a K-metric space and $T : X \to X$ a mapping. A mapping $F : X\times X\times X \to X$ is said to be a T-contraction, if there exist $\alpha, \beta, \gamma \geq 0$, with $\alpha + \beta + \gamma < 1$, such that for all $x, y, z, x^*, y^*, z^* \in X$, we get

$$d(TF(x, y, z), TF(x^*, y^*, z^*)) \preceq \alpha d(Tx, Tx^*) + \beta d(Ty, Ty^*) + \gamma d(Tz, Tz^*). \tag{1}$$

Theorem 2.2. Let (X, d) be a complete K-metric space, P a solid cone and $F : X \times X \times X \to X$ a T-contraction, where $T : X \to X$ is a continuous and one to one mapping. Then

(i) $\{TF^n(x_0, y_0, z_0)\}$, $\{TF^n(y_0, z_0, x_0)\}$ and $\{TF^n(z_0, x_0, y_0)\}$ are Cauchy sequences for all $x_0, y_0, z_0 \in X$;
(ii) there exist $u_{x_0}, u_{y_0}, u_{z_0} \in X$ such that

$$\begin{aligned} \lim_{n\to+\infty} TF^n(x_0, y_0, z_0) \ &= \ u_{x_0} \ , \lim_{n\to+\infty} TF^n(y_0, z_0, x_0) = u_{y_0}, \text{ and} \\ \lim_{n\to+\infty} TF^n(z_0, x_0, y_0) \ &= \ u_{z_0}; \end{aligned}$$

(iii) if T is subsequentially convergent, then $\{TF^n(x_0, y_0, z_0)\}$, $\{TF^n(y_0, z_0, x_0)\}$ and $\{TF^n(z_0, x_0, y_0)\}$ have a convergent subsequence;
(iv) there exist uniques $w_{x_0}, w_{y_0}, w_{z_0} \in X$ such that

$$F(w_{x_0}, w_{y_0}, w_{z_0}) = w_{x_0}, \quad F(w_{y_0}, w_{z_0}, w_{x_0}) = w_{y_0}, \quad F(w_{z_0}, w_{x_0}, w_{y_0}) = w_{z_0};$$

that is, F has a unique tripled fixed point;

(v) if T is sequentially convergent, then, for all $x_0, y_0, z_0 \in X$, the sequence $\{TF^n(x_0, y_0, z_0)\}$ converges to $w_{x_0} \in X$, the sequence $\{TF^n(y_0, z_0, x_0)\}$ converges to $w_{y_0} \in X$ and the sequence $\{TF^n(z_0, x_0, y_0)\}$ converges to $w_{z_0} \in X$.

Proof. Let define

$$\begin{aligned} D : X^3 \times X^3 \to P \quad &; \quad D((x_1, y_1, z_1), (x_2, y_2, z_2)) = d(x_1, x_2) + d(y_1, y_2) + d(z_1, z_2); \\ \mathfrak{F} : X^3 \to X^3 \quad &; \quad \mathfrak{F}(x, y, z) = (F(x, y, z), F(y, z, x), F(z, x, y)); \\ \mathfrak{T} : X^3 \to X^3 \quad &; \quad \mathfrak{T}(x, y, z) = (Tx, Ty, Tz). \end{aligned}$$

Then (X^3, D) is a complete K-metric space, and $\mathfrak{T}$ is continuous and one-to-one. It is clear that $(x, y, z) \in X^3$ is a tripled fixed point of F if, and only if, it is a fixed point of $\mathfrak{F}$. Suppose that $F^n(x, y, z)$ throughout the text (which is not properly defined) means exactly $\mathfrak{F}^n(x, y, z)$. Let $k = \alpha + \beta + \gamma < 1$. Therefore,

$$\begin{aligned} &D(\mathfrak{T}\mathfrak{F}(x, y, z), \mathfrak{T}\mathfrak{F}(x^*, y^*, z^*)) \\ &= D(\mathfrak{T}(F(x, y, z), F(y, z, x), F(z, x, y)), \mathfrak{T}(F(x^*, y^*, z^*), F(y^*, z^*, x^*), F(z^*, x^*, y^*))) \\ &= D((TF(x, y, z), TF(y, z, x), TF(z, x, y)), (TF(x^*, y^*, z^*), TF(y^*, z^*, x^*), TF(z^*, x^*, y^*))) \\ &= d(TF(x, y, z), TF(x^*, y^*, z^*)) + d(TF(y, z, x), TF(y^*, z^*, x^*)) \\ &\quad + d(TF(z, x, y), TF(z^*, x^*, y^*)) \\ &\preceq [\alpha d(Tx, Tx^*) + \beta d(Ty, Ty^*) + d(Tz, Tz^*)] \\ &\quad + [\alpha d(Ty, Ty^*) + \beta d(Tz, Tz^*) + \gamma d(Tx, Tx^*)] \\ &\quad + [\alpha d(Tz, Tz^*) + \beta d(Tx, Tx^*) + d(Ty, Ty^*)] \\ &= (\alpha + \beta + \gamma) d(Tx, Tx^*) + (\alpha + \beta + \gamma) d(Ty, Ty^*) + (\alpha + \beta + \gamma) d(Tz, Tz^*) \\ &= k(d(Tx, Tx^*) + d(Ty, Ty^*) + d(Tz, Tz^*)) \\ &= kD((Tx, Ty, Tz), (Tx^*, Ty^*, Tz^*)) \\ &= kD(\mathfrak{T}(x, y, z), \mathfrak{T}(x^*, y^*, z^*)). \end{aligned}$$

The proof further follows by Theorem 1.5 (taking $\alpha_1 = k < 1$ and $\alpha_2 = \alpha_3 = 0$). This completes the proof. □

Corollary 2.3. Let (X, d) be a complete K-metric space, P a solid cone, and $T : X \to X$ a continuous and one to one mapping. If $F : X \times X \times X \to X$ satisfies

$$d(TF(x, y, z), TF(x^*, y^*, z^*)) \preceq \frac{k}{3}[d(Tx, Tx^*) + d(Ty, Ty^*) + d(Tz, Tz^*)], \tag{2}$$

for all $x, y, z, x^*, y^*, z^* \in X$, where $k \in [0, 1)$, then the conclusions of Theorem 2.2 hold true.

Proof. The thesis follows easily from Theorem 2.2, by putting $\alpha = \beta = \gamma = k/3$ in (1). □

Corollary 2.4. Let (X, d) be a complete K-metric space and P a solid cone. If $F : X \times X \times X \to X$ satisfies

$$d(F(x, y, z), F(x^*, y^*, z^*)) \preceq \alpha d(x, x^*) + \beta d(y, y^*) + \gamma d(z, z^*), \tag{3}$$

for all $x, y, z, x^*, y^*, z^* \in X$, where $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$, then, F has a unique tripled fixed point.

Proof. The thesis follows easily from Theorem 2.2, by putting $T = I_x$, where I_x is the identity mapping on X. □

Corollary 2.5. Let (X, d) be a complete K-metric space and P a solid cone. If $F : X \times X \times X \to X$ satisfies

$$d(F(x, y, z), F(x^*, y^*, z^*)) \preceq \frac{k}{3}[d(x, x^*) + d(y, y^*) + d(z, z^*)], \tag{4}$$

for all $x, y, z, x^*, y^*, z^* \in X$, where $k \in [0, 1)$, then F has a unique tripled fixed point.

Proof. Result follows from Corollary 2.3, taking $T = I_X$. □

Example 2.6. Let $X = [0, 1]$ and $E = C^1_{\mathbb{R}}[0, 1]$ endowed with the order induced by $P = \{\phi \in E : \phi(t) \geq 0 \text{ for } t \in [0, 1]\}$. Define $d : X\times X \to E$ by $d(x, y)(t) = |x-y|2^t$, for all $x, y \in X$.
Clearly, (X, d) is a complete K-metric space with a cone having nonempty interior. Next, define the mappings $F : X \times X \times X \to X$ and $T : X \to X$ by

$$Tx = \frac{x}{2}, \text{ for all } x \in X \text{ and } F(x, y, z) = \frac{x + y + z}{6}, \text{ for all } x, y, z \in X.$$

Then F satisfies the contractive condition (2) for $k = 1/2$; that is,

$$d(TF(x, y, z), TF(u, v, w)) \preceq \frac{1}{6}[d(Tx, Tu) + d(Ty, Tv) + d(Tz, Tw)],$$

for all $x, y, z, u, v, w \in X$. Consequently, Corollary 2.3 applies to F, which has a unique tripled fixed point; that is $(0, 0, 0)$.

3. Applications

Let $C([0, T], \mathbb{R})$ be the set of continuous functions defined in $[0, T]$, where $T > 0$. Consider the metric given by

$$d(u, v) = \sup_{t\in[0,T]} |u(t) - v(t)|, \text{ for all } u, v \in \mathbb{R}.$$

Note that $(C([0, T], \mathbb{R}), d)$ is a complete metric space.

Now, we study the existence and uniqueness of solution to an integral equation, by using Corollary 2.5. Precisely, we consider the equation

$$x(t) = \int_0^T k(t, s)(f(s, x(s)) + g(s, x(s)) + h(s, x(s)))\, ds + a(t), \quad t \in [0, T]. \tag{5}$$

Now, we state and prove the following theorem.

Theorem 3.1. *Assume that the following conditions hold:*

(i) $k \in C([0, T] \times [0, T], \mathbb{R})$ *such that* $\sup_{s,t\in[0,T]} |k(t, s)| = M < \frac{1}{T}$;
(ii) $a \in C([0, T], \mathbb{R})$;
(iii) $f, g, h \in C([0, T] \times \mathbb{R}, \mathbb{R})$;
(iv) *for all* $x_i, y_i, z_i \in C([0, T], \mathbb{R})$, *where* $i = 1, 2$, *and* $t \in [0, T]$ *we have*

$$|f(t, x_1(t)) - f(t, x_2(t))| + |g(t, y_1(t)) - g(t, y_2(t))| + |h(t, z_1(t)) - h(t, z_2(t))|$$
$$\leq \frac{1}{3}(|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)| + |z_1(t) - z_2(t)|).$$

Then, the integral equation (5) *has a unique solution.*

Proof. Consider the mapping $F : C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})\times C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ defined by

$$F(x,y,z)(t) = \int_0^T k(t,s)(f(s,x(s)) + g(s,y(s)) + h(s,z(s)))\ ds + a(t), \quad t \in [0,T].$$

It is easy to show that (x,y,z) is a solution of (5) if and only if (x,y,z) is a tripled fixed point of F. To establish the existence of such a point, we will use Corollary 2.5. In fact, by condition (iv), we have easily

$$\begin{aligned}&|F(x_1,y_1,z_1)(t) - F(x_2,y_2,z_2)(t)| \\ &\le \int_0^T |k(t,s)|\ \frac{1}{3}(|x_1(s) - x_2(s)| + |y_1(s) - y_2(s)| + |z_1(s) - z_2(s)|)\ ds \\ &\le \frac{1}{3}\left(\int_0^T |k(t,s)|\ ds\right)(d(x_1,x_2) + d(y_1,y_2) + d(z_1,z_2)),\end{aligned}$$

for all $x_i, y_i, z_i \in C([0,T],\mathbb{R})$, where $i = 1,2$ and $t \in [0,T]$. By (i), it follows that

$$d(F(x_1,y_1,z_1), F(x_2,y_2,z_2)) \le \frac{MT}{3}(d(x_1,x_2) + d(y_1,y_2) + d(z_1,z_2)),$$

for all $x_i, y_i, z_i \in C([0,T],\mathbb{R})$, where $i = 1,2$. Then, condition (4) of Corollary 2.5 is satisfied with $k = MT < 1$ and hence, applying Corollary 2.5, we obtain the existence of a unique tripled fixed point of F; that is, the integral equation (5) has a unique solution. □

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