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Interlacing properties and bounds for zeros of ${}_2\phi_1$ hypergeometric and little q-Jacobi polynomials

P Gochhayat · K Jordaan ·
K Raghavendar · A Swaminathan

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Abstract Let $\{p_n\}_{n=0}^{\infty}$, where p_n is a polynomial of degree n, be a sequence of polynomials orthogonal with respect to a positive probability measure. If $x_{1,n} < \cdots < x_{n,n}$ denotes the zeros of p_n while $x_{1,n-1} < \cdots < x_{n-1,n-1}$ are the zeros of p_{n-1}, the inequality

$$x_{1,n} < x_{1,n-1} < x_{2,n} < \cdots < x_{n-1,n} < x_{n-1,n-1} < x_{n,n},$$

known as the interlacing property, is satisfied. We use a consequence of a generalised version of Markov's monotonicity results to investigate interlacing properties of zeros of contiguous basic hypergeometric polynomials associated with little q-Jacobi polynomials and determine inequalities for extreme zeros of the above two polynomials. It is observed that the new bounds which are obtained in this paper give more precise upper bounds for the smallest zero of little q-Jacobi polynomials, improving previously known results by Driver and Jordaan [9], and in some cases, those by Gupta and Muldoon [17]. Numerical examples are given in order to illustrate the accuracy of our bounds.

Keywords Basic hypergeometric series · little q-Jacobi polynomials · zeros · bounds for zeros · interlacing of zeros · orthogonal polynomials

Mathematics Subject Classification (2000) 33C05 · 33C45 · 33D15

P Gochhayat
Dept of Mathematics and Applied Mathematics, University of Pretoria, 0002, South Africa
E-mail: pgochhayat@gmail.com

K Jordaan
Dept of Mathematics and Applied Mathematics, University of Pretoria, 0002, South Africa
E-mail: kjordaan@up.ac.za

K Raghavendar
Dept of Mathematics, Indian Institute of Technology, Roorkee-247 667, Uttarkhand, India
E-mail: raghavendar248@gmail.com

A Swaminathan
Dept of Mathematics, Indian Institute of Technology, Roorkee-247 667, Uttarkhand, India
E-mail: swamifma@iitr.ernet.in, mathswami@gmail.com

1 Introduction

The notion of basic hypergeometric series can be traced back to 1748, when Euler considered an infinite product of the form

$$\prod_{k=0}^{\infty}(1-q^{k+1})^{-1}$$

as a generating function for $p(n)$, the number of partitions of a positive integer n into positive integers (cf. [16, p. 239]). It was only 100 years later that a systematic treatment of basic hypergeometric series was attempted, when Heine [19] converted the simple observation that $\lim_{q\to 1}[(1-q^a)/(1-q)] = a, a \in \mathbb{C}$ into a theory of ${}_2\phi_1$ hypergeometric series analogous to that for ${}_2F_1$ hypergeometric series developed by Gauss. For more detail on this q-extension of hypergeometric series we refer to the books by Gasper and Rahman [16], Ismail [20] as well as the recent books by Koekoek, Lesky and Swarttouw [27] and Ernst [14].

Let $\{p_n\}_{n=0}^{\infty}$, where p_n is a polynomial of degree n, be a sequence of polynomials orthogonal with respect to a positive probability measure. Then p_n has n real, simple zeros which lie in the interior of the convex hull of the support of the measure and, if $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ denotes the zeros of p_n while $x_{1,n-1} < x_{2,n-1} < \cdots < x_{n-1,n-1}$ are the zeros of p_{n-1}, the inequality

$$x_{1,n} < x_{1,n-1} < x_{2,n} < \cdots < x_{n-1,n} < x_{n-1,n-1} < x_{n,n}, \tag{1}$$

called the interlacing of zeros, holds. A classical theorem due to Stieltjes (cf. [34, Theorem 3.3.3]) proves that, for $m < n$, there is at least one zero of p_n in between any two consecutive zeros of p_m, a property known as Stieltjes interlacing.

Interlacing of zeros of orthogonal polynomials was studied by several authors. Levit [30] was the first to study interlacing properties of zeros of Hahn polynomials from different orthogonal polynomials in 1967. In 1984, Askey [16] proved that the zeros of Jacobi polynomials $P_n^{\alpha,\beta}$ and $P_{n-1}^{\alpha+1,\beta}$ interlace and he conjectured that the zeros of $P_n^{\alpha,\beta}$ and $P_{n-1}^{\alpha+2,\beta}$ interlace. Further results were obtained in [6], [7], [11], [12] and [13] for the classical orthogonal polynomials such as Jacobi and Laguerre polynomials, for discrete orthogonal polynomials in [24] and for linear combinations of orthogonal polynomials in [4], [5], [26] and [33].

The mixed recurrence relations used to prove Stieltjes interlacing of the zeros of two polynomials from different sequences provide a set of points that can be applied as inner bounds for the extreme zeros of continuous and discrete orthogonal polynomials (cf. [8]). Classical methods to obtain bounds for zeros of orthogonal polynomials include the use of difference equations, Sturmian methods for differential equations (cf. [34]), Markov's theorem (cf. [31]), Obrechkov's theorem (cf. [32]) and the Hellmann-Feynmann theorem (cf. [20]).

Examples where these and other methods are applied to zeros of classical orthogonal polynomials of a discrete variable can be found in [29], [28], [1] and [2].

Ismail questioned whether similar techniques to those used for classical orthogonal polynomials could be used to prove inequalities satisfied by the zeros of q-orthogonal polynomials at the OPSFA 2007 conference in Marseille. In [25], mixed recurrence relations obtained from generating functions of Al-Salam-Chihara, q-Meixner-Pollaczek, q-ultraspherical and q-Laguerre polynomials were used to prove interlacing properties of the zeros for these q-orthogonal polynomials with shifted parameters. In the present sequel, we study interlacing properties of the zeros of little q-Jacobi polynomials using a consequence of the generalised version of Markov's monotonicity result (cf. [20, Thm. 7.1.1]) and the connection of little q-Jacobi polynomials with ${}_2\phi_1$ hypergeometric polynomials. The results obtained are applied to obtain inequalities satisfied by the extreme zeros of little q-Jacobi polynomials which improve inequalities obtained in [9] and, in some cases, also those in [17],

We will assume throughout this paper that $0 < q < 1$. Little q-Jacobi polynomials are discrete orthogonal on $(0, \infty)$ with respect to the weight function (cf. [16, 7.3.3], [3, p. 592], [27, 14.12.2])

$$w(x) = \frac{\alpha^x q^x (\beta q; q)_x}{(q; q)_x}, \qquad x = 0, 1, 2, \cdots. \tag{2}$$

which is positive for $0 < q < 1$, $\alpha q > 0$ and $\beta q < 1$. In order to let the moments exist, we impose $0 < \alpha q < 1$.

Little q-Jacobi polynomials were introduced by Hahn [18] and are of the form

$$p_n(\alpha, \beta; q; x) = {}_2\phi_1(q^{-n}, \alpha\beta q^{n+1}; \alpha q; q, xq), \tag{3}$$

where the ${}_2\phi_1$ hypergeometric series (cf. [16]) is defined by

$${}_2\phi_1(a, b; c; q, x) = \sum_{k=0}^{\infty} \frac{(a; q)_k (b; q)_k}{(q; q)_k (c; q)_k} x^k, \tag{4}$$

with

$$(a; q)_k = \begin{cases} 1; & k = 0, \\ \prod_{m=0}^{k-1} (1 - aq^m); & k \in \mathbb{N}. \end{cases} \tag{5}$$

The product (5) is referred to as the q-shifted factorial. It is assumed in (4) that $c \neq q^{-m}$ for $m = 0, 1, 2, \cdots$. Furthermore, we note that (cf. [27, 14.12.15])

$$\lim_{q \to 1^-} p_n(q^\alpha, q^\beta; q; x) = P_n^{\alpha,\beta}(1 - 2x)/P_n^{\alpha,\beta}(1),$$

where $P_n^{\alpha,\beta}(z)$ denotes Jacobi polynomials (cf. [27, 9.8.1]) and

$$\lim_{q \to 1^-} {}_2\phi_1(q^a, q^b; q^c; q, x) = {}_2F_1(a, b; c; x).$$

Here ${}_2F_1(a,b;c;z)$ is the Gaussian hypergeometric function.

Note that (cf. [16])

$$(a;q)_x = \frac{(a;q)_\infty}{(aq^x;q)_\infty} \qquad \text{for } x \in \mathbb{C} \text{ and } |q| < 1 \tag{6}$$

where

$$(a;q)_\infty = \prod_{m=0}^{\infty} (1 - aq^m),$$

which agrees with (5) for $x = 0, 1, 2, \cdots$ but holds for $x \in \mathbb{C}$ when $aq^x \neq q^{-k}$, $k = 0, 1, 2, \cdots$ and the principal value of q^x is taken (cf. [20]). This implies that the little q-Jacobi weight (2) is defined for general x and in particular for $x \in (0, \infty)$. We analyse inequalities satisfied by the zeros of different sequences of ${}_2\phi_1$ hypergeometric polynomials and little q-Jacobi polynomials in order to obtain an inner bound for the extreme zeros that improves the upper bound for the smallest zero of little q-Jacobi polynomials in [9] and, for certain parameter values, the bound due to Gupta and Muldoon (cf. [17]). Our method of proof requires a generalisation of a result on inequalities satisfied by the zeros of two polynomials orthogonal with respect to different weights (cf. [34, Thm. 6.12.2]). We prove this result and the contiguous relations satisfied by ${}_2\phi_1$ hypergeometric series that we need in the next section where we also discuss other preliminary material.

2 Preliminary results

The following generalised version of Markov's theorem, which describes the manner in which the zeros of a polynomial change as the parameter changes, is stated as an exercise in [15, Chap. 3, ex. 15] and can be applied to zeros of certain discrete orthogonal polynomials.

Lemma 1 *[20, Thm. 7.1.1] Let $\{p_n(x;\tau)\}_{n=0}^{\infty}$ be orthogonal with respect to $d\alpha(x;\tau)$,*

$$d\alpha(x;\tau) = \rho(x;\tau)d\alpha(x),$$

on an interval $I = (a,b)$ and assume that $\rho(x;\tau)$ is positive and has continuous first derivative with respect to τ for $x \in I$, $\tau \in T = (\tau_1, \tau_2)$. Furthermore, assume that

$$\int_a^b x^j \frac{\partial \rho(x;\tau)}{\partial \tau} d\alpha(x), \qquad j = 0, 1, 2, \cdots, 2n-1,$$

converge uniformly for every compact subinterval of T. Then the zeros of $p_n(x;\tau)$ are increasing (decreasing) function of τ, $\tau \in T$, if $\dfrac{\partial}{\partial \tau}\{\ln \rho(x,\tau)\}$ is an increasing (decreasing) function of x, $x \in I$.

We point out the following consequence which is analogous to [34, Thm 6.12.2].

Lemma 2 *Let $P_n(x,\tau)$ and $Q_n(x,\tau)$ be orthogonal with respect to $d\alpha(x;\tau)=\omega(x;\tau)d\alpha(x)$ and $d\beta(x;\tau)=\widetilde{\omega}(x;\tau)d\beta(x)$ respectively where $\omega(x;\tau)$ and $\widetilde{\omega}(x;\tau)$ are positive and continuous functions on $[a,b]$ and $\tau\in\mathbb{R}$.*
If the ratio $\dfrac{\widetilde{\omega}(x;\tau)}{\omega(x;\tau)}$ is an increasing function of x for $x\in[a,b]$, then

$$x_\nu < X_\nu, \qquad \nu=\{1,2,3,\cdots\},$$

where $x_1<x_2<\cdots<x_n$ and $X_1<X_2<\cdots<X_n$ are the zeros of $P_n(x,\tau)$ and $Q_n(x,\tau)$ respectively.

Proof For $x\in[a,b],\ \tau\in\mathbb{R}$, let $\rho(x;\tau,\mu)=(1-\mu)\omega(x;\tau)+\mu\widetilde{\omega}(x;\tau)$. Clearly $\rho(x;\tau,\mu)$ is positive and has a continuous partial derivative with respect to μ for $0<\mu<1$.

$$\begin{aligned}\frac{\frac{\partial\rho(x,\tau,\mu)}{\partial\mu}}{\rho(x,\tau,\mu)} &= \frac{\widetilde{\omega}(x;\tau)-\omega(x;\tau)}{(1-\mu)\omega(x;\tau)+\mu\widetilde{\omega}(x;\tau)}\\ &= \frac{1}{\mu}-\frac{1}{1-\mu+\mu\frac{\widetilde{\omega}(x;\tau)}{\omega(x;\tau)}}\end{aligned}$$

is an increasing function of x when $\dfrac{\widetilde{\omega}(x;\tau)}{\omega(x;\tau)}$ is an increasing function of x. Hence, Lemma 1 implies that, as μ increases, the zeros of polynomials associated with $\rho(x;\tau,\mu)$ increase.

Further, note that for the particular case $\mu=0$ and $\mu=1$

$$\rho(x;\tau,0)=\omega(x;\tau) \text{ and } \rho(x;\tau,1)=\widetilde{\omega}(x;\tau)$$

which shows that, as μ increases from 0 to 1, the zeros of polynomials associated with $\omega(x;\tau)$ are less than zeros of polynomials associated with $\widetilde{\omega}(x;\tau)$ and this completes the proof of the Lemma 2.

Mixed relations between ${}_2\phi_1(q^\alpha,q^\beta;q^\gamma;q,x)$ and functions

$${}_2\phi_1(q^{\alpha\pm1},q^\beta;q^\gamma;q,x),\ {}_2\phi_1(q^\alpha,q^{\beta\pm1};q^\gamma;q,x) \text{ and } {}_2\phi_1(q^\alpha,q^\beta;q^{\gamma\pm1};q,x)$$

were given in [19] (see also [21], [22] and [23]). These relations are analogous to the standard contiguous relations for the Gaussian hypergeometric functions. Instead of a single shift in a single parameter, new relations can be obtained by either a double shift in a parameter or a single shift in two parameters. We will apply such contiguous relations to prove our results and list the relations we need in Lemma 3 using the notation

$$\begin{aligned}\phi_n &= {}_2\phi_1\left(q^{-n},b;c;q,xq\right) \quad , \quad \phi_n(bq)={}_2\phi_1\left(q^{-n},bq;c;q,xq\right),\\ \phi_n(\frac{b}{q}) &= {}_2\phi_1\left(q^{-n},\frac{b}{q};c;q,xq\right) \quad , \quad \phi_n(cq)={}_2\phi_1\left(q^{-n},b;cq;q,xq\right),\\ \phi_n(\frac{c}{q}) &= {}_2\phi_1\left(q^{-n},b;\frac{c}{q};q,xq\right) \quad \text{and} \quad \phi_n(bq,cq)={}_2\phi_1\left(q^{-n},bq;cq;q,xq\right),\end{aligned}$$

Lemma 3 *Let $n \in \mathbb{N}$ and $c \neq q^{-m}$ for $m = 0, 1, \ldots$. Then*

$$(i)\ \left(1-\frac{bq}{c}\right)\phi_n = bq\left(q^n - \frac{1}{c}\right)\phi_{n+1}(bq) + (1-bq^{n+1})\left(1-\frac{q^{-n+1}b}{c}x\right)\phi_n(bq) \tag{7}$$

$$(ii)\ (q^{-n}-b)\phi_n = q^{-n}(1-b)\phi_n(bq) - b(1-q^{-n})\phi_{n-1} \tag{8}$$

$$(iii)\ \left[b + q^{-n+1}\left(1-\frac{b}{c}-\frac{b}{q}\right)\right]\phi_n = q^{-n+1}\left(1-\frac{b}{c}\right)\phi_n\left(\frac{b}{q}\right) + b(1-q^{-n})\left(1-\frac{bq^{-n+1}}{c}x\right)\phi_{n-1} \tag{9}$$

$$(iv)\ (q^{-n+1}-c)\phi_n = q^{-n}(q-c)\phi_n\left(\frac{c}{q}\right) - c(1-q^{-n+1})\phi_{n-1} \tag{10}$$

$$(v)\ (q^{-n}bx - c)(1-c)\phi_n = q^{-n}x(b-c)\phi_n(cq) - c(1-c)\phi_{n+1} \tag{11}$$

$$(vi)\ (1-c)\phi_n - \phi_{n+1}(bq) = (q^{-n}-bq)x\phi_n(bq, cq) \tag{12}$$

$$(vii)\ \left[1+q-q^{-n}-\frac{q^{-n+1}}{c}+\frac{q^{-2n+1}}{c}\left(1-\frac{b}{q^{-n}}\right)x\right]\phi_n = q\left(1-\frac{q^{-n}}{c}\right)\phi_{n+1} + (1-q^{-n})\left(1-\frac{q^{-n}b}{c}xq\right)\phi_{n-1} \tag{13}$$

$$(viii)\ (c(1-b) - bq(1-cq^n))\phi_n = bq(cq^n-1)\phi_{n+1} + \frac{(1-b)}{q^n}(cq^n - bxq)\phi_n(bq) \tag{14}$$

$$(ix)\ (c-bq)\phi_n = [(c-bq) - bc(1-q^n)]\phi_n(bq) + b(1-q^n)(c-bq^{-n+2}x)\phi_{n-1}(bq) \tag{15}$$

Proof The contiguous relations can be verified by comparing coefficients of x^n. For example, the coefficient of x^n on the left-hand side of (12) can be written as

$$\begin{aligned} &q^n(1-c)\left(\frac{(q^{-n};q)_n(b;q)_n}{(c;q)_n(q;q)_n} - \frac{(q^{-n-1};q)_n(bq;q)_n}{(c;q)_n(q;q)_n}\right) \\ &= q^n(1-c)\frac{(q^{-n};q)_{n-1}(bq;q)_{n-1}}{(c;q)_n(q;q)_n}((1-q^{-1})(1-b) - (1-q^{-n-1})(1-bq^n)) \\ &= q^{n-1}(1-c)(q^{-n}-bq)(1-q^n)\frac{(q^{-n};q)_{n-1}(b;q)_{n-1}}{(c;q)_n(q;q)_n} \\ &= (q^{-n}-bq)\left[\frac{(q^{-n};q)_{n-1}(b;q)_{n-1}q^{n-1}}{(c;q)_{n-1}(q;q)_{n-1}}\right] \end{aligned} \tag{16}$$

and, since the expression in the square brackets of (16) is the coefficient of x^{n-1} in $\phi_n(bq, cq)$, this completes the proof of the contiguous relation (12). The proofs of the other relations follow in an analogous manner.

The following results concerning the interlacing of zeros have been proved, with some variations, in different contexts. They provide simple tools to study the zeros of polynomials and are useful in the context of our paper, so we summarise them here for the convenience of the reader.

Lemma 4 *(cf. [24]) Consider the finite or infinite interval (c, d) and let p_n and q_{n-1} be polynomials (not necessarily orthogonal) of degree n with real zeros $c < x_{1,n} < x_{2,n} < \cdots < x_{n,n} < d$ and degree $(n-1)$ with real zeros $c < x_{1,n-1} < x_{2,n-1} < \cdots < x_{n-1,n-1} < d$ respectively satisfying the interlacing property (1). Assume that a polynomial g satisfies*

$$g(x) = a(x)p_n(x) + b(x)q_{n-1}(x)$$

and denote the zeros of g with $deg(g) = i$ by $y_{1,i} < y_{2,i} < \cdots < y_{i,i}$. Let $k \in \{1, 2, \cdots, n-1\}$. If both $a(x)$ and $b(x)$ are continuous and have constant sign on (c, d), then all the zeros of g are real, simple and

(a) when g has degree n, either
  (i) $x_{k,n} < y_{k,n} < x_{k,n-1} < x_{k+1,n} < y_{k+1,n}$ or
  (ii) $y_{k,n} < x_{k,n} < x_{k,n-1} < y_{k+1,n} < x_{k+1,n}$

(b) when g has degree $n-1$, either
  (i) $x_{k,n} < y_{k,n-1} < x_{k,n-1} < x_{k+1,n}$ or
  (ii) $x_{k,n} < x_{k,n-1} < y_{k,n-1} < x_{k+1,n}$.

There are two possibilities for the ordering of the interlaced zeros in Lemma 4(a) and (b) respectively. When both $a(x)$ and $b(x)$ are non-zero constants and $q_{n-1} = p_{n-1}$, one may apply [5, Thm. 3] or [26, Thm. 5] to determine the specific arrangement of the zeros, while the monotonicity of the zeros is often useful when $a(x)$ and $b(x)$ are not constant.

Lemma 5 *(cf. [8, 10]) Let p_n and q_{n-1} be polynomials (not necessarily orthogonal) of degree n and $(n-1)$ respectively satisfying interlacing property (1) on the (finite or infinite) interval (c, d). Assume that a polynomial g_{n-2} of degree $(n-2)$ satisfies*

$$f(x)g_{n-2}(x) = a(x)p_n(x) + (x - A_n)q_{n-1}(x)$$

for some constant A_n where $f(x)$ and $a(x)$ are continuous and $f(x)$ has constant sign on (c, d). Then

(i) the $n-1$ real, simple zeros of $(x - A_n)g_{n-2}$ interlace with the zeros of p_n and A_n is an upper bound for the smallest, as well as a lower bound for the largest zero of p_n if g_{n-2} and p_n are co-prime;

(ii) if g_{n-2} and p_n are not co-prime,
a) they have 1 common zero that is equal to A_n and this common zero cannot be the largest or smallest zero of p_n;
b) the $n-2$ zeros of $g_{n-2}(x)$ interlace with the $n-1$ non-common zeros of p_n;
c) A_n is an upper bound for the smallest as well as a lower bound for the largest zero of p_n.

In the next section we state our main results and provide the proofs in Section 4.

3 Main Results

Theorem 1 *For* $0 < c < 1$ *and* $b < cq^{n-1}$*, the zeros of* ϕ_n *and* $\phi_{n+1}(bq)$ *are interlacing on* $(0, \infty)$.

Theorem 2 *Let* $0 < c < 1$ *and* $b < 0$*. Let* $0 < x_{1,n} < x_{2,n} < \ldots < x_{n,n} < \infty$ *and* $0 < X_{1,n} < X_{2,n} < \ldots < X_{n,n} < \infty$ *denote the zeros of* ϕ_n *and* $\phi_n(bq)$ *respectively. Then*

$$X_{k,n+1} < X_{k,n} < x_{k,n} < X_{k+1,n+1} \quad \text{for each } k = 1, 2, \ldots n. \tag{17}$$

Theorem 3 *For* $0 < c < 1$ *and* $b < 0$*, the zeros of* ϕ_n *and* ϕ_{n-1} *separate each other on* $(0, \infty)$.

Theorem 4 *If* $0 < c < 1$ *and* $b < 0$*, suppose that* $0 < x_{1,n} < x_{2,n} < \ldots < x_{n,n} < \infty$ *and* $0 < x_{1,n-1} < x_{2,n-1} < \ldots < x_{n-1,n-1} < \infty$ *denote the* n *and* $n-1$ *real zeros of* ϕ_n *and* ϕ_{n-1} *respectively. Further, let* $0 < y_{1,n} < y_{2,n} < \ldots < y_{n,n} < \infty$ *be the zeros of* $\phi_n(b/q)$*. Then* $x_{k,n} < y_{k,n} < x_{k,n-1}$ *for each* $k = 1, 2, 3, \ldots, n-1$ *and* $x_{n-1,n-1} < x_{n,n} < y_{n,n}$.

Theorem 5 *If* $0 < c < 1$ *and* $b < cq^{n-1}$*, suppose that* $0 < x_{1,n} < x_{2,n} < \ldots < x_{n,n} < \infty$ *and* $0 < x_{1,n-1} < x_{2,n-1} < \ldots < x_{n-1,n-1} < \infty$ *denote the* n *and* $n-1$ *real zeros of* ϕ_n *and* ϕ_{n-1} *respectively. Further, let* $0 < t_{1,n} < t_{2,n} < \ldots < t_{n,n} < \infty$ *be the zeros of* $\phi_n(c/q)$*. Then* $x_{k,n} < t_{k,n} < x_{k,n-1}$ *for each* $k = 1, 2, \ldots, n-1$ *and* $x_{n-1,n-1} < x_{n,n} < t_{n,n}$.

Theorem 6 *If* $0 < c < 1$ *and* $b < cq^n$*, suppose that* $0 < x_{1,n} < x_{2,n} < \ldots < x_{n,n} < \infty$ *and* $0 < x_{1,n+1} < x_{2,n+1} < \ldots < x_{n+1,n+1} < \infty$ *denote the* n *and* $n+1$ *real zeros of* ϕ_n *and* ϕ_{n+1} *respectively. Further, let* $0 < s_{1,n} < s_{2,n} < \ldots < s_{n,n} < \infty$ *be the zeros of* $\phi_n(cq)$*. Then* $x_{k,n+1} < s_{k,n} < x_{k,n} < x_{k+1,n+1}$ *for* $k = 1, 2, \ldots, n$.

Theorem 7 *If* $0 < c < 1$ *and* $b < cq^{n-1}$*, suppose that* $0 < x_{1,n} < x_{2,n} < \ldots < x_{n,n} < \infty$*,* $0 < X_{1,n+1} < X_{2,n+1} < \ldots < X_{n+1,n+1} < \infty$ *and* $0 < s_{1,n} < s_{2,n} < \ldots < s_{n,n} < \infty$ *denotes zeros of* ϕ_n*,* $\phi_{n+1}(bq)$ *and* $\phi_n(bq, cq)$ *respectively. Then* $X_{k,n+1} < s_{k,n} < x_{k,n} < X_{k+1,n+1}$ *for* $k = 1, 2, \ldots n$.

Theorem 8 *Let* $0 < c < 1$*,* $b < 0$ *and* $0 < t < 1$ *and suppose that* $0 < x_{1,n} < x_{2,n} < \ldots < x_{n,n} < \infty$*,* $0 < s_{1,n} < s_{2,n} < \ldots < s_{n,n} < \infty$ *and* $0 < r_{1,n} < r_{2,n} < \ldots < r_{n,n} < \infty$ *denote the real zeros of* ϕ_n*,* $\phi_n(bt)$ *and* $\phi_n(bq)$ *respectively. Then the following triple interlacing holds for* $k = 1, 2, \ldots, n-1$

$$r_{k,n} < s_{k,n} < x_{1,n} < r_{k+1,n} < s_{k+1,n} < x_{k+1,n}.$$

Theorem 9 *Let $0 < c < 1$ and $b < 0$, then*

(i) the zeros of ϕ_{n-2} together with the point A_n (B_n) interlace with the zeros of ϕ_n where

(a) $A_n = \frac{cq^{2n-3}\left(1+q-q^{-n+1}-\frac{q^{-n+2}}{c}\right)}{(bq^{n-1}-1)}$

(b) $B_n = \frac{cq^{n-2}}{b} + \frac{[c(b-1)+bq(1-cq^{n-1})][c-bq-cb(1-q^{n-1})]}{b(1-b)(c-bq)q^{-n+2}}$.

(ii) Let $x_{1,n}$ and $x_{n,n}$ denote the smallest and largest zero of ϕ_n, then

$$0 < x_{1,n} < B_n < A_n < x_{n,n}.$$

Corollary 1 *Let $0 < \alpha q < 1$ and $\beta < 0$.*

(i) The zeros of $p_{n-2}(\alpha, \beta q^2; q; x)$ together with the point $\mathcal{A}_n$ ($\mathcal{B}_n$) interlace with the zeros of $p_n(\alpha, \beta; q; x)$ where

(a) $\mathcal{A}_n = \frac{\alpha q^{2n-2}\left(-\frac{q^{1-n}}{\alpha}-q^{1-n}+q+1\right)}{\alpha\beta q^{2n}-1}$

(b) $\mathcal{B}_n = \frac{1}{\beta q^2} + \frac{[\alpha\beta q^{n+1}-1+\beta q^{n+1}(1-\alpha q^n)][1-\beta q^{n+1}-\alpha\beta q^n(1-q^{n-1})]}{\beta q^2(1-\alpha\beta q^{n+1})(1-\beta q^{n+1})}$.

(ii) Let $w_{1,n}$ and $w_{n,n}$ denote the smallest and largest zero of $p_n(\alpha, \beta; q; x)$, then

$$0 < w_{1,n} < \mathcal{B}_n < \mathcal{A}_n < w_{n,n}.$$

Remark 1 A different bound for the extreme zeros of $p_n(\alpha, \beta; q; x)$,

$$w_{1,n} < \mathcal{C}_n = \frac{q^n\left(\alpha(\alpha+1)\beta q^{2n} + \alpha(\beta+1)(-(q+1))q^n + (\alpha+1)q\right)}{(\alpha\beta q^{2n}-1)(\alpha\beta q^{2n}-q^2)} < w_{n,n},$$

is given in [9, 4.10] for $0 < \alpha q < 1$ and $0 < \beta q < 1$. We note that $\mathcal{C}_n$ is an inner bound for the extreme zeros of $p_n(\alpha, \beta; q; x)$ also when $0 < \alpha q < 1$ and $\beta < 1$ and therefore $\mathcal{C}_n$ can be compared to $\mathcal{A}_n$ for negative values of β. Since the difference

$$\mathcal{C}_n - \mathcal{A}_n = \frac{\alpha\beta(q+1)q^{2n-2}\left(q^{n-1}-1\right)\left(1-\alpha q^{n-1}\right)}{(\alpha\beta q^{2n}-1)(\alpha\beta q^{2n-2}-1)} > 0$$

for $0 < \alpha < 1$, $\beta < 0$, we conclude that $w_{1,n} < \mathcal{B}_n < \mathcal{A}_n < \mathcal{C}_n < w_{n,n}$ when $0 < \alpha < 1$ and $\beta < 0$ and it is clear that the new bound $\mathcal{B}_n$ is the more precise upper bound for the smallest zero of $p_n(\alpha, \beta; q; x)$.

In [17], Gupta and Muldoon give another another upper bound

$$\mathcal{D}_n = \frac{(q+1)q^{n-1}\left(1-\alpha q^2\right)(1-\alpha q)}{(1-q)\left(q^n(\alpha q(\beta(q(\alpha q-2)-1)-q-2)+1)+\alpha\beta q^{2n+1}\left(\alpha q^2+1\right)+\alpha q^2+1\right)} \tag{18}$$

for the smallest zero $z_{1,n}$ of the different q-Jacobi polynomial $p_n(\alpha, \beta; q; z(1-q))$, $0 < \alpha q < 1$, $\beta < 1$, which cannot immediately be compared to the upper bounds $\mathcal{A}_n$, $\mathcal{B}_n$ and $\mathcal{C}_n$ for the smallest zero $w_{1,n}$ of $p_n(\alpha, \beta; q; x)$. However, the following corollary will enable us to do a comparison.

Corollary 2 *Let $0 < \alpha q < 1$ and $\beta < 0$.*

(i) The zeros of $p_{n-2}(\alpha, \beta q^2; q; z(1-q))$ together with the point $\mathcal{E}_n$ interlace with the zeros of
$p_n(\alpha, \beta; q; z(1-q))$ where

$$\mathcal{E}_n = \frac{\alpha q^{2n-2}\left(-\frac{q^{1-n}}{\alpha} - q^{1-n} + q + 1\right)}{(1-q)\left(\alpha\beta q^{2n} - 1\right)}. \tag{19}$$

(ii) If $z_{1,n}$ and $z_{n,n}$ denote the smallest and largest zero of $p_n(\alpha, \beta; q; z(1-q))$ respectively, then

$$z_{1,n} < \mathcal{E}_n < z_{n,n}.$$

Remark 2 Comparing the new bound $\mathcal{E}_n$ to the bound $\mathcal{D}_n$ given by Gupta and Muldoon in [17], it is clear from the numerical examples given in Table 1 for specific values of $0 < \alpha q < 1$ and $\beta < 0$, that the choice of the parameters determines whether $\mathcal{D}_n$ or $\mathcal{E}_n$ is the better upper bound for the smallest zero $z_{1,n}$ of $p_n(\alpha, \beta; q; z(1-q))$.

Table 1 Comparison of upper bounds for the smallest zero of $p_n = p_n(\alpha, \beta; q; z(1-q))$ when $n = 8$ for different values of q, α and β.

Values of q, α and β	$q = 0.9$, $\alpha = 0.1$ $\beta = -0.4$	$q = 0.1$, $\alpha = 0.05$ $\beta = -0.4$	$q = 0.9$, $\alpha = 0.95$ $\beta = -10.4$
Smallest zero of p_n	3.39968	1.10494×10^-7	0.0370522
Bound $\mathcal{E}_n$ in (19)	4.79109	1.16667×10^-7	1.83607
Bound $\mathcal{D}_n$ in (18)	5.29965	1.2149×10^-7	0.0507134

The difference between the two bounds $\mathcal{D}_n$ and $\mathcal{E}_n$ can be simplified as

$$\begin{aligned} &\mathcal{D}_n - \mathcal{E}_n \\ &= \frac{q^{n-1}}{1-q}\left(\frac{\alpha\left(1-q^{n-1}\right) + (1-\alpha q^n)}{\alpha\beta q^{2n} - 1} + \frac{(q+1)(1-\alpha q)\left(1-\alpha q^2\right)}{\mathcal{F}_n}\right), \end{aligned} \tag{20}$$

where

$$\begin{aligned} \mathcal{F}_n = &\left(1-\alpha q^{n+1}\right) + q^n(1-\alpha q) + \alpha q^2\left(1-q^n\right) \\ &-\alpha\beta q^{n+1}\left(q\left(1-\alpha q^{n+1}\right) + (1-q^n) + q(1-\alpha q)\right). \end{aligned}$$

For $0 < \alpha q < 1$ and $\beta < 0$, all the denominator and numerator terms in (20) are positive except for the denominator $\alpha\beta q^{2n} - 1$ in the second fraction on the right hand side. Hence, for $0 < \alpha q < 1$ and $\beta < 0$, $\mathcal{D}_n - \mathcal{E}_n > 0$ if and only if

$$\frac{(q+1)(1-\alpha q)\left(1-\alpha q^2\right)}{\mathcal{F}_n} > \frac{\alpha\left(1-q^{n-1}\right)+(1-\alpha q^n)}{1-\alpha\beta q^{2n}}$$

or, equivalently,

$$\beta < \frac{1}{\alpha q^{2n}}\left(\frac{\mathcal{F}_n(\alpha-\alpha q^{n-1}-q^n+1)}{(q+1)(1-\alpha q)\left(1-\alpha q^2\right)}\right) = \mathcal{G}_n.$$

We conclude that the bound (18) due to Gupta and Muldoon is a better upper bound for the smallest zero of $p_n(\alpha,\beta;q;z(1-q))$ compared to the new bound (19) if and only if $0 < \alpha q < 1$ and $\beta < \min\{0,\mathcal{G}_n\}$.

4 Proof of Main Results

Proof of Theorem 1. Little q-Jacobi polynomials have the following basic hypergeometric representation from (3)

$$p_n(\alpha,\beta;q;x) = {}_2\phi_1(q^{-n},b;c;q,z), \tag{21}$$

where $b=\alpha\beta q^{n+1},\ c=\alpha q$ and $z = xq$. From the orthogonality it follows that the zeros of little q-Jacobi polynomial are real and positive for $0<\alpha<1/q$ and $\beta < 1/q$. These conditions are equivalent to $0<c<1$ and $b<cq^{n-1}$. Therefore, for $0<c<1$ and $b<cq^{n-1}$, the zeros $z=xq$ of ${}_2\phi_1(q^{-n},b;c;q,z)$ lie in $(0,\infty)$. Furthermore, since the zeros of $p_n^{(\alpha,\beta)}$ and $p_{n+1}^{(\alpha,\beta)}(x)$ are interlacing and

$$\begin{aligned}p_{n+1}(\alpha,\beta;q;x) &= {}_2\phi_1(q^{-n-1},\alpha\beta q^{n+2};\alpha q;q,xq)\\ &= {}_2\phi_1(q^{-(n+1)},\alpha\beta q^{n+1}q;\alpha q;q,xq) = \phi_{n+1}(bq),\end{aligned}$$

we deduce that the zeros of ϕ_n and $\phi_{n+1}(bq)$ also separate each other. □

Throughout the following proofs we will denote a step function with unit jumps at $x=0,1,2,\cdots$ by $\alpha(x)$.

Proof of Theorem 2. For $0<c<1$ and $b<cq^{n-1}$ it follows from (21) and (2) that the polynomial ϕ_n is orthogonal with respect to the weight function $w_1(x) = \dfrac{(\frac{b}{c}q^{-n+1};q)_x}{(q;q)_x}c^x$ for $x=0,1,2,\cdots$ which, using (6), can be represented as

$$\begin{aligned}w_1(x) &= \frac{(\frac{b}{c}q^{-n+1};q)_\infty}{(\frac{b}{c}q^{x-n+1};q)_\infty}\frac{(qq^x;q)_\infty}{(q;q)_\infty}c^x\ \alpha(x)\\ &= \omega_1(x;b)\ \alpha(x)\end{aligned}$$

when $x\in(0,\infty)$, $0<c<1$ and $b<cq^{n-1}$ where $\alpha(x)$ denotes a step function with unit jumps at $x=0,1,2,\cdots$ as previously mentioned. Similarly,

$\phi_n(bq)$ is orthogonal with respect to $w_2(x) = \omega_2(x;b)\ \alpha(x)$ where $\omega_2(x;b) = \dfrac{(\frac{b}{c}q^{-n+2};q)_x}{(q;q)_x}c^x$ for $x \in (0,\infty)$, $0 < c < 1$ and $b < cq^{n-1}$. Now

$$\frac{d}{dx}\left(\frac{\omega_1(x;b)}{\omega_2(x;b)}\right) = \frac{d}{dx}\left(\frac{c-bq^{-n+1}}{c-bq^{-n+1+x}}\right) = \frac{bq^{n-1}q^x(c-bq^{n-1})\ln q}{(c-bq^{1-n+x})^2}$$

is positive for $b < 0,\ 0 < c < 1$, hence the ratio $\dfrac{\omega_1(x;b)}{\omega_2(x;b)}$ is an increasing function of x for $x \in (0,\infty)$. Since $\omega_1(x;b)$ and $\omega_2(x;b)$ are positive, continuous functions for $0 < c < 1$, $b < 0$, $x \in (0,\infty)$ it follows from Lemma 2 that $X_{k,n} < x_{k,n}$ for each $k = 1,2,\ldots,n$ which proves one part of assertion (17). Next, from Theorem 1, we know that ϕ_n and $\phi_{n+1}(bq)$ have interlaced zeros.

Rewriting (7) we obtain

$$\phi_n(bq) = \frac{bq(q^n - \frac{1}{c})}{(bq^{n+1}-1)(1-\frac{q^{-n+1}bx}{c})}\phi_{n+1}(bq) + \frac{(1-\frac{bq}{c})}{(1-bq^{n+1})(1-\frac{q^{-n+1}bx}{c})}\phi_n.$$

Since both $\frac{bq(q^n-\frac{1}{c})}{(bq^{n+1}-1)(1-\frac{q^{-n+1}bx}{c})}$ and $\frac{(1-\frac{bq}{c})}{(1-bq^{n+1})(1-\frac{q^{-n+1}bx}{c})}$ are continuous and have constant sign in $(0,\infty)$, application of Lemma 4(b)(i), yields

$$X_{k,n+1} < X_{k,n} < x_{k,n} < X_{k+1,n+1} \qquad \text{for } k = 1,2,\cdots,n.$$

This completes the proof of assertion (17). □

Proof of Theorem 3. By Theorem 2 we have that the zeros of ϕ_n and $\phi_n(bq)$ separate each other as described in (17) for $0 < c < 1$ and $b < 0$. Evaluating (8) at consecutive zeros $x_{k,n}$ and $x_{k+1,n},\ k = 0,1,2,\cdots,n-1$, of ϕ_n we deduce that

$$\phi_{n-1}(x_{k,n})\phi_{n-1}(x_{k+1,n}) < 0$$

which implies that ϕ_{n-1} has an odd number of zeros between any two consecutive zeros of ϕ_n. Since ϕ_{n-1} has at most $n-1$ zeros and there are $n-1$ intervals where they must lie, it follows that the $n-1$ real simple zeros of ϕ_{n-1} interlace with the n real simple zeros of ϕ_n. □

Proof of Theorem 4. The weight function for $\phi_n(b/q)$ on $(0,\infty)$ is $w_0(x) = \omega_0(x;b)\alpha(x)$, where
$\omega_0(x;b) = \dfrac{(\frac{b}{c}q^{-n};q)_x}{(q;q)_x}c^x$ for $x \in (0,\infty),\ 0 < c < 1,\ b \leq cq^{n-1}$. Since

$$\frac{d}{dx}\left(\frac{\omega_0(x;b)}{\omega_1(x;b)}\right) = \frac{d}{dx}\left(\frac{c-bq^{-n}}{c-bq^{-n+x}}\right) = \frac{bq^{-n+x}(c-bq^{-n})\ln q}{(c-bq^{-n+x})^2}$$

is positive when $b < 0$ and $0 < c < 1$, the ratio $\dfrac{\omega_0(x;b)}{\omega_1(x;b)}$ is an increasing function of x in $(0,\infty)$. Since $\omega_1(x;b)$ and $\omega_0(x;b)$ are positive, continuous functions for $0 < c < 1$, $b < 0$, $x \in (0,\infty)$ from Lemma 2 we can conclude that

$$x_{k,n} < y_{k,n} \quad \text{for } k = 1,2,\ldots,n. \tag{22}$$

Next, from the contiguous relation (9) and application of Lemma 4(a)(i) together with (22), we have

$$x_{k,n} < y_{k,n} < x_{k,n-1} \qquad \text{for each } k = 1,2,3,\ldots,n-1$$

and

$$x_{n-1,n-1} < x_{n,n} < y_{n,n}.$$

Hence the proof of Theorem 4 is completed. □

Proof of Theorem 5. $\phi_n(c/q)$ is orthogonal with respect to the weight function $w_3(x) = \omega_3(x;b)\alpha(x)$, where $\omega_3(x;b) = \dfrac{(\frac{b}{c}q^{-n+2};q)_x}{(q;q)_x}(\frac{c}{q})^x$, for $x \in (0,\ \infty)$, $0 < c < 1$ and $b < cq^{n-1}$. Since

$$\frac{d}{dx}\left(\frac{\omega_3(x;b)}{\omega_1(x;b)}\right) = \frac{d}{dx}\left(\frac{(c - bq^{-n+x+1})}{(c - bq^{-n+1})q^x}\right) = \frac{-cq^{-x}\ln q}{(c - bq^{-n+1})},$$

which is positive for $b < cq^{n-1}$ and $0 < c < 1$, the ratio $\dfrac{\omega_3(x;b)}{\omega_1(x;b)}$ is an increasing function of x in $(0,\infty)$. Since $\omega_1(x;b)$ and $\omega_3(x;b)$ are positive, continuous functions for $x \in (0,\infty)$, $0 < c < 1$, and $b < cq^{n-1}$, it follows from Lemma 2 that

$$x_{k,n} < t_{k,n} \qquad \text{for } k = 1,2,\ldots,n. \tag{23}$$

Further, using Lemma 4(a)(i) and the contiguous relation (10) together with (23), we have

$$x_{k,n} < t_{k,n} < x_{k,n-1} \quad \text{for all } k = 1,2,\ldots,n{-}1 \quad \text{and} \quad x_{n-1,n-1} < x_{n,n} < t_{n,n}.$$

□

Proof of Theorem 6. For $x \in (0,\ \infty)$, $0 < c < 1$ and $b < cq^n$, $\phi_n(cq)$ is orthogonal with respect to the weight function $w_4(x) = \omega_4(x;b)\alpha(x)$, where $\omega_4(x;b) = \dfrac{(\frac{b}{c}q^{-n};q)_x}{(q;q)_x}(cq)^x$. Since

$$\frac{d}{dx}\left(\frac{\omega_4(x;b)}{\omega_1(x;b)}\right) = \frac{d}{dx}\left(\frac{(c - bq^{-n})q^x}{c - bq^{-n+x}}\right) = \frac{cbq^{n+x}(cq^n - b)\ln q}{(cq^n - bq^x)^2}$$

is negative when $b < cq^n$ and $0 < c < 1$, the ratio $\dfrac{\omega_4(x;b)}{\omega_1(x;b)}$ is an decreasing function of x on $(0,\infty)$. Since $\omega_1(x;b)$ and $\omega_4(x;b)$ are positive, continuous functions for $x \in (0,\infty)$, $0 < c < 1$ and $b < cq^n$, from Lemma 2 we can conclude that $s_{k,n} < x_{k,n}$ for $k = 1,2,\ldots,n$. Further, rewriting the contiguous relation (11) as

$$B(x)\phi_n(x) + C(x)\phi_{n+1}(x) = \phi_n(cq;x)$$

with

$$B(x) = \frac{(q^{-n}bx - c)(1-c)}{q^{-n}(b-c)x}$$

and

$$C(x) = \frac{c(1-c)}{q^{-n}(b-c)x},$$

since both $B(x)$ and $C(x)$ are continuous and have constant sign in $(0,\infty)$, we see by Lemma 4(b)(i) that all zeros of $\phi_n(cq)$ are real, simple and the zeros satisfy

$$x_{k,n+1} < s_{k,n} < x_{k,n} < x_{k+1,n+1} \quad \text{for } k = 1,2,\cdots,n.$$

□

Proof of Theorem 7. $\phi_n(bq,cq)$ is orthogonal with respect to the weight function $w_5(x) = \omega_5(x;b)\alpha(x)$, where $\omega_5(x;b) = \dfrac{(\frac{b}{c}q^{-n+1};q)_x}{(q;q)_x}cq^x$, for $x \in (0,\infty)$, $0 < c < 1$ and $b < cq^{n-1}$. The ratio $\dfrac{\omega_1(x;b)}{\omega_5(x;b)} = 1/q^x$, which is an increasing function of x on $(0,\infty)$. Since $\omega_1(x;b)$ and $\omega_5(x;b)$ are positive, continuous functions for $x \in (0,\infty)$, $0 < c < 1$ and $b < cq^{n-1}$, from Lemma 2 we can conclude that $s_{k,n} < x_{k,n}$ for $k = 1,2,\ldots,n$. Next, for $b < 0$ and $x > 0$ it follows from the contiguous relation (12) together with Lemma 4(b)(i) and Theorem 1, for $k = 0,1,2,\cdots,n$ that

$$X_{k,n+1} < s_{k,n} < x_{k,n} < X_{k+1,n+1}.$$

□

Proof of Theorem 8 $\phi_n(bt)$ is orthogonal with respect to the weight function $w_6(x) = \omega_6(x;b)\alpha(x)$, where $\omega_6(x;b) = \dfrac{(\frac{bt}{c}q^{-n+1};q)_x}{(q;q)_x}c^x$, for $x \in (0,\infty)$, $0 < c < 1$, $0 < t < 1$ and $bt < cq^{n-1}$. Consider the ratio

$$H(x) = \frac{\omega_1(x;b)}{\omega_6(x;b)} = \frac{(\frac{b}{c}q^{-n+1};q)_x}{(\frac{bt}{c}q^{-n+1};q)_x} = \frac{(c-bq^{-n+1})(c-bq^{-n})\cdots(c-bq^{-n+x})}{(c-tbq^{-n+1})(c-tbq^{-n})\cdots(c-tbq^{-n+x})}.$$

Taking logarithmic differentiation of $H(x)$ with respect to x we obtain

$$\frac{H'(x)}{H(x)} = \frac{-bq^{-n+x}\ln q}{c-bq^{-n+x}} + \frac{tbq^{-n+x}\ln q}{c-tbq^{-n+x}},$$

which on further simplification gives

$$\begin{aligned}H'(x) &= \left(\frac{(t-1)bcq^{-n+x}\ln q}{(c-bq^{-n+x})(c-tbq^{-n+x})}\right)\times H(x)\\ &= \frac{(t-1)bcq^{-n+x}\ln q(c-bq^{-n+1})(c-bq^{-n})\cdots(c-bq^{-n+x})}{(c-bq^{-n+x})(c-tbq^{-n+x})(c-tbq^{-n+1})(c-tbq^{-n})\cdots(c-tbq^{-n+x})}.\end{aligned}$$

Since $0<c<1$, $b<cq^{n-1}$ and $bt<cq^{n-1}$, $H'(x)>0$ if and only if either both b and $(t-1)$ are positive or both are negative. Since $(t-1)$ is negative for all $t\in(0,1)$, the above inequality holds true for $b<0$.

Therefore $H(x)$ is an increasing function of x on $(0,\infty)$ when $b<0$. Since $\omega_1(x;b)$ and $\omega_6(x;b)$ are positive, continuous functions for $0<c<1$, $b<0$, $x\in(0,\infty)$ from Lemma 2 we can conclude that

$$s_{k,n} < x_{k,n} \qquad \text{for } k=1,2,\ldots,n \tag{24}$$

Next, consider the ratio

$$H_1(x)=\frac{\omega_6(x;b)}{\omega_2(x;b)}=\frac{(\frac{bt}{c}q^{-n+1};q)_x}{(\frac{b}{c}q^{-n+2};q)_x}=\frac{(c-tbq^{-n+1})(c-tbq^{-n})\cdots(c-tbq^{-n+x})}{(c-bq^{-n+2})(c-bq^{-n+1})\cdots(c-bq^{-n+x+1})}.$$

Using a similar argument as for $H(x)$ above, we observe that $H_1'(x)$ positive if

$$cbq^{-n+x}\ln q(1-qt)(c-bq^{-n+1})(c-bq^{-n})\cdots(c-bq^{-n+x}) \;>\; 0.$$

Which is holds true for $0<bt<cq^{n-x}$ and $x\in(0,\infty)$. Therefore, $H_1(x)$ is an increasing functions of x in $(0,\infty)$ for $0<bt<cq^{n-x}$. Since $\omega_2(x;b)$ and $\omega_6(x;b)$ are positive, continuous functions for $0<c<1$, $b<0$, $x\in(0,\infty)$ it follows from Lemma 2 that

$$r_{k,n} < s_{k,n} \qquad \text{for } k=1,2,\ldots,n. \tag{25}$$

Combining (24) and (25) with Theorem 2 the result follows for $0<c<1$, $b<0$ and $0<t<1$. □

Proof of Theorem 9.

(i)(a) Replacing n by $n-1$, the contiguous relation (13) may be written as

$$\begin{aligned}&\frac{-c(1-q^{-n+1})}{q^{-2n+3}(bq^{n-1}-1)}\left(1-\frac{q^{-n+2}b}{c}x\right)\phi_{n-2}\\ &\qquad\qquad = \frac{(c-q^{-n+1})}{q^{-2n+2}(bq^{n-1}-1)}\phi_n+(x-A_n)\phi_{n-1}.\end{aligned}\tag{26}$$

Therefore, by Lemma 5, since $\left(1-\frac{q^{-n+2}b}{c}x\right)$ does not change sign on $(0,\infty)$ when $0<c<1$ and $b<0$, the zeros of ϕ_{n-2} together with the point A_n interlace with the zeros of ϕ_n and A_n is an upper bound for the smallest as well as lower bound for the largest zero of ϕ_n. □

(i) (b) Substituting the expression for $\phi_n(bq)$ in (14) into (15) we obtain

$$bq(1-cq^n)\phi_{n+1} = (c(b-1)+bq(1-cq^n))\phi_n \\ + \frac{(1-b)(c-bxq^{-n+1})}{((c-bq)-cb(1-q^n))}((c-bq)\phi_n - b(1-q^n)(c-bq^{-n+2}x)\phi_{n-1}(bq)),$$

which on replacement of n by $n-1$ and further simplification yields

$$(x-\frac{cq^{n-2}}{b})(x-\frac{cq^{n-3}}{b})\phi_{n-2}(bq) = \frac{[c-bq-bc(1-q^{n-1})](cq^{n-1}-1)}{(1-b)(1-q^{n-1})b^2q^{-2n+4}}\phi_n \\ -\frac{(c-bq)}{(1-q^{n-1})b^2q^{-n+3}}(x-B_n)\,\phi_{n-1}),$$

The result follows by Lemma 5 that the zeros of $\phi_{n-2}(bq)$ together with the point B_n interlace with the zeros of ϕ_n and B_n is an upper bound for smallest as well as lower bound for largest zero of ϕ_n.

(ii) The quantity

$$A_n - B_n = \frac{bq^{n-3}(q^n-q)(cq^n-q^2)(-bq+c(1+b(q^n-1)))}{(b-1)(c-bq)(q-bq^n)}$$

is positive for all $n = 0, 1, \cdots$ when $0 < c < 1$ and $b < 0$.

□

Proof of Corollary 1. The result follows from the proof of Theorem 9 by proper transformation of the polynomials. Therefore, we sketch the outline of Corollary 1(i)(a) and omit the detail of (i)(b) and (ii).

(i)(a) Replacing $c = \alpha q$, $b = \alpha\beta q^{n+1}$ in (26) and using the connection (21) between basic hypergeometric and little q-Jacobi polynomials we have

$$\frac{\alpha q^{2(n-1)}\left(1-q^{1-n}\right)\left(1-\beta q^2 x\right)}{1-\alpha\beta q^{2n}} p_{n-2}(\alpha, \beta q^2; q; x) \\ = \frac{q^{2n-2}\left(\alpha q - q^{1-n}\right)}{\alpha\beta q^{2n}-1} p_n(\alpha, \beta; q; x) + (x-\mathcal{A}_n)p_{n-1}(\alpha, \beta q; q; x). \quad (27)$$

Since the zeros of ϕ_n and ϕ_{n-1} interlace, so do also those of $p_n(\alpha, \beta; q; x)$ and $p_{n-1}(\alpha, \beta q; q; x)$. Therefore applying Lemma 5, the zeros of $p_{n-1}(\alpha, \beta q^2; q; x)$ together with the point $\mathcal{A}_n$ interlace with the zeros of $p_n(\alpha, \beta; q; x)$. □

Proof of Corollary 2.

Substituting x by $z(1-p)$ we can write (27) as

$$\frac{\alpha q^{2(n-1)}\left(1-q^{1-n}\right)\left(1-\beta q^2(1-q)z\right)}{(1-q)\left(1-\alpha\beta q^{2n}\right)} p_{n-2}(\alpha, \beta q^2; q; z(1-q)) \\ = (z-\mathcal{D}_n)p_{n-1}(\alpha, \beta q; q; z(1-q)) + \frac{q^{2n-2}\left(\alpha q - q^{1-n}\right)}{(1-q)\left(\alpha\beta q^{2n}-1\right)} p_n(\alpha, \beta; q; z(1-q)).$$

It follows from Lemma 5 that, if $z_{1,n} < z_{n,n}$ denote the extreme zeros of $p_n(\alpha, \beta; q; z(1-q))$, we have $z_{1,n} < \mathcal{D}_n < z_{n,n}$. □

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